Transportation
Library
TA
1
.A526
V.113

TRANSACTIONS

OF THE

AMERICAN SOCIETY

OF

CIVIL ENGINEERS

(INSTITUTED 1852)

VOLUME 113

1948

Edited by the Secretary, under the direction of the Committee on Publications.

Reprints from this publication, which is copyrighted, may be made on condition that the full title of paper, name of author, and page reference are given.

NEW YORK
PUBLISHED BY THE SOCIETY

1948

Copyright 1948, by the AMERICAN SOCIETY OF CIVIL ENGINEERS.

NOTE.—The Society is not responsible for any statement made or opinion expressed in its publications.

CONTENTS

PAPERS AND DISCUSSIONS

(Alphabetical Indexes Begin on Page 1633)

No. PAGE

2329 **MISSISSIPPI RIVER CUTOFFS**
By **Gerard H. Matthes** 1

Discussion:
By W. E. ELAM 16
C. L. HALL 17
H. D. VOGEL 18
HARRY N. PHARR 20
LYTLE BROWN 21
ANSON MARSTON 22
E. J. WILLIAMS, JR., AND EUGENE A. GRAVES 24
GERARD H. MATTHES 27

2330 **A RATIONAL EXPLANATION OF COLUMN BEHAVIOR**
By **Frederick L. Ryder** 40

Discussion:
By R. ROBINSON ROWE 71
GEORGE WINTER 71
ROBERT M. MAINS 73
CHI-TEH WANG 74
ERNEST L. ROBINSON 76
FREDERICK L. RYDER 77

2331 **CLEANING AND GROUTING OF LIMESTONE FOUNDATIONS, TENNESSEE VALLEY AUTHORITY: A SYMPOSIUM** 79

FORT LOUDOUN DAM.
By **Arthur L. Pauls** 82

DOUGLAS DAM.
By **T. F. Taylor** 102

Discussion:
By BERLEN C. MONEYMAKER 120
C. E. BLEE 123
ROBERT A. LAURENCE 124
JAMES B. WARD 126
J. C. NOWELL 128
JOHN F. PARTRIDGE 130
PORTLAND P. FOX 137

No. Page

2332 STABILITY OF SOIL SLOPES
By Ek-Khoo Tan.......... 139

Discussion:
By Paul Baumann.......... 159
Gregory P. Tschebotarioff.......... 163
Jacob Feld.......... 164
Earl M. Buckingham.......... 167
K. B. Hirashima.......... 169
Donald M. Burmister.......... 171
Ek-Khoo Tan.......... 178

2333 EXPERIMENTAL OBSERVATIONS ON GROUTING SANDS AND GRAVELS.
By Alfred Machis.......... 181

Discussion:
By James B. Hays.......... 206
William S. Foster.......... 206
Frederick J. Smith.......... 208
Alfred Machis.......... 212

2334 THEORY OF INELASTIC BENDING WITH REFERENCE TO LIMIT DESIGN.
By Alexander Hrennikoff.......... 213

Discussion:
By George Winter.......... 248
D. C. Drucker.......... 248
Michael R. Horne.......... 250
E. P. Popov.......... 258
Bernard A. Vallerga.......... 261
Alexander Hrennikoff.......... 264

2335 REFLECTIONS ON STANDARD SPECIFICATIONS FOR STRUCTURAL DESIGN
By Alfred M. Freudenthal.......... 269

Discussion:
By I. Oesterblom.......... 288
John M. Hayes.......... 289
Alfred M. Freudenthal.......... 290

2336 THIN-SHELLED DOMES LOADED ECCENTRICALLY
By Walter C. Voss, Dean Peabody, Jr., Howard R. Staley, and Albert G. H. Dietz 293

2337 UNDERGROUND CONDUITS—AN APPRAISAL OF MODERN RESEARCH.
By M. G. Spangler.......... 316

Discussion:
By Wilson V. Binger.......... 346
Jacob Feld.......... 347
Zdeněk Bažant, Jr.......... 349
Anson Marston.......... 351
George E. Shafer.......... 354
E. F. Kelley.......... 364
Bailey Tremper.......... 365
M. G. Spangler.......... 368

Trans.
W.J. Emmons
1-7-49
Gift

No. PAGE

2338 SUBSIDENCE OF THE TERMINAL ISLAND-LONG BEACH AREA, CALIFORNIA
By **Frederic R. Harris and Eugene H. Harlow** 375

Discussion:
By James Gilluly and U. S. Grant 397
Jacob Feld 399
Hyde Forbes 400

2339 RIVER INFILTRATION AS A SOURCE OF GROUND WATER SUPPLY
By **Raphael G. Kazmann** 404

Discussion:
By C. V. Youngquist 421
Raphael G. Kazmann 422

2340 NEW FORMULAS FOR STRESSES IN CONCRETE PAVEMENTS OF AIRFIELDS
By **H. M. Westergaard** 425

Discussion:
By Robert Horonjeff 440
Evan P. Bone 441
H. M. Westergaard 443

2341 BEAM DEFLECTIONS BY SECOND AND THIRD MOMENTS.
By **Hsu Shih-Chang** 445

Discussion:
By A. Floris 465
Robert B. B. Moorman 466
Frank J. McCormick 469
William A. Conwell 469
Hsu Shih-Chang 470

2342 DEVELOPMENT AND HYDRAULIC DESIGN, SAINT ANTHONY FALLS STILLING BASIN.
By **Fred W. Blaisdell** 483

Discussion:
By Donald E. Blotcky 521
M. M. Culp 529
Paul Baumann 530
A. J. Peterka 534
Louis M. Laushey 539
J. H. Douma 545
Robert F. Ewald 550
Fred W. Blaisdell 551

2343 FORECASTING PRODUCTIVITY OF IRRIGABLE LANDS.
By **W. C. Muldrow** 562

Discussion:
By Arthur F. Johnson 567
H. W. Van Loo 568
Will H. Noble 568
Charles Kirby Fox 570
M. R. Lewis 571
Harry F. Blaney 573
R. L. Parshall 575
Howard N. Magness 576

No. PAGE

2344 CONTINUOUS FRAME ANALYSIS BY ELASTIC SUPPORT ACTION.
By J. Charles Rathbun and C. W. Cunningham 577
Discussion:
By L. J. Mensch 614
Frederick S. Merritt 616
I. Oesterblom 621
Thomas C. Kavanagh 623
A. Floris 625
Tao King 626
Leroy A. Beaufoy 630
A. A. Eremin 634
Robert B. B. Moorman 635
Eduardo Agramonte 636
Stephen J. Fraenkel and Robert L. Janes 643
Phil M. Ferguson 644
Ralph W. Stewart 645
Maurice Barron 649
Edwin H. Gaylord 660
William A. Conwell 666
Harry Garfinkel 667
J. Charles Rathbun and C. W. Cunningham 676

2345 STABILITY OF THIN CYLINDRICAL SHELLS IN TORSION.
By R. G. Sturm 681
Discussion:
By S. B. Batdorf and Manuel Stein 706
Glenn Murphy 707
L. H. Donnell 709
R. L. Moore 713
R. G. Sturm 716

2346 CONSOLIDATION OF FINE-GRAINED SOILS BY DRAIN WELLS
By Reginald A. Barron 718
Discussion:
By Kenneth S. Lane 743
Philip Keene 747
Walter Kjellman 748
Reginald A. Barron 751

2347 INFLUENCE LINES FOR CONTINUOUS STRUCTURES BY GEOMETRICAL COMPUTATION.
By Dean F. Peterson, Jr. 755
Discussion:
By Ralph E. Spaulding 772
Ralph W. Stewart 772
Yi-Mai Yao 776
Otto Gottschalk 779
Dean F. Peterson, Jr. 783

2348 DRAINAGE OF AIRPORT SURFACES—SOME BASIC DESIGN CONSIDERATIONS.
By Stifel W. Jens 785
Discussion:
By H. M. Williams 810
W. I. Hicks 813
E. J. A. Gain 817
E. F. Brater 819
Carl F. Izzard 825
Stifel W. Jens 831

No. Page

2349 LEAST SQUARES ADJUSTMENT OF TRIANGULATION NET BETWEEN GEODETIC STATIONS.

By **E. F. Coddington and O. C. J. Marshall** 837

Discussion:

By Julius L. Speert 855
C. H. Ney 856
H. W. Hemple 863
Lansing G. Simmons 865
E. F. Coddington and O. C. J. Marshall 867

2350 THE GENERAL PRINCIPLES OF HIGHWAY TRANSITION CURVE DESIGN.

By **John Joseph Leeming** 868

Discussion:

By Carl F. Meyer 881
Ralph L. Fisher 884
William R. Welty 888
Elmer R. Haile, Jr. 889
Charles M. Noble 893
John Joseph Leeming 896

2351 CLASSIFICATION AND IDENTIFICATION OF SOILS.

By **Arthur Casagrande** 901

Discussion:

By Ralph E. Fadum 931
James H. Stratton 933
Donald J. Belcher 937
J. A. Haine and J. W. Hilf 941
Jacob Feld 946
Kenneth S. Lane 949
George F. Sowers 959
René S. Pulido y Morales 960
Raymond F. Dawson 963
D. F. Glynn 694
L. F. Cooling, A. W. Skempton, and R. Glossop 966
Milton Vargas 970
Donald M. Burmister, ... 971
M. G. Spangler 978
E. W. Lane 984
B. K. Hough, Jr. 985
Arthur Casagrande 987

2352 ESTIMATING DATA FOR RESERVOIR GATES.

By **Frank L. Boissonnault** 992

Discussion:

By Theodore B. Rights 1012
George R. Latham 1015
D. A. Buzzell 1016
A. E. Niederhoff 1017
Joseph R. Bowman 1018
Frank L. Boissonnault 1024

No. PAGE

2353 FRICTION COEFFICIENTS IN A LARGE TUNNEL.

By G. H. Hickox, A. J. Peterka, and R. A. Elder 1027

Discussion:

By Weston Gavett 1047
W. R. Barrows 1048
Hunter Rouse 1049
Karl R. Kennison 1053
E. J. K. Chapman 1057
Julian Hinds 1059
William P. Creager and Stephen H. Haybrook 1063
G. S. Tapley 1064
J. N. Bradley and S. P. Wing 1066
G. H. Hickox, A. J. Peterka, and R. A. Elder 1071

2354 ANALYSIS OF STEPPED-COLUMN MILL BENTS.

By Daniel S. Ling 1077

Discussion:

By Ralph E. Spaulding 1099
E. I. Fiesenheiser 1100
Walter J. Gray 1102
L. J. Mensch 1103
Arthur R. Graves 1107
Erik Sollid 1108
A. Floris 1112
Charles W. Dohn 1114
Daniel S. Ling 1119

2355 TESTS OF TIMBER STRUCTURES FROM GOLDEN GATE INTERNATIONAL EXPOSITION.

By Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program 1123

Discussion:

By James H. Carr, Jr. 1238
Howard J. Hansen 1240
E. G. Stern 1242
Melvin W. Jackson 1234
Charles Mackintosh 1244
Sidney Novick 1246
Frank J. Hanrahan 1248
Walter J. Ryan 1250
Walter M. Pratt 1251
Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program 1254

2356 INVESTIGATION OF DRAINAGE RATES AFFECTING STABILITY OF EARTH DAMS.

By F. H. Kellogg 1261

Discussion:

By Harry R. Cedergren 1285
Ross M. Riegel 1293
T. W. Lambe 1294
W. L. Shannon 1302
F. H. Kellogg 1307

No. PAGE

2357 APPLICATION OF GEOLOGY TO TUNNELING PROBLEMS.
By Ernest E. Wahlstrom 1310

Discussion:
By BERLEN C. MONEYMAKER 1322
ROBERT S. MAYO 1325
F. A. NICKELL 1326
R. H. KEAYS 1329
JACOB FELD 1332
THOMAS W. FLUHR 1334
PORTLAND P. FOX 1335
HYDE FORBES 1336
ROGER RHOADES, A. B. REEVES, AND W. H. IRWIN 1345
ERNEST E. WAHLSTROM 1348

2358 VARIATION OF COEFFICIENTS OF SIMULTANEOUS LINEAR EQUATIONS.
By Bernard L. Weiner 1349

Discussion:
By T. F. HICKERSON 1375
A. FLORIS 1377
FREDERICK S. MERRITT 1378
I. F. MORRISON 1379
SIGMUND ROOS 1381
BERNARD L. WEINER 1382

2359 THE SOCIETY'S FUTURE AND ITS ECONOMIC LIMITATIONS: ANNUAL PRESIDENTIAL ADDRESS.
By R. E. Dougherty 1391

MEMOIRS OF DECEASED MEMBERS

Honorary Members

	Page
Francis Trenholm Crowe	1397
George Watson Kittredge	1403
Luigi Luiggi	1406

Members

Frank Hooker Alfred	1414
Edward Anderberg	1418
Rupert Andrew Anderegg	1421
Frank Alexander Barbour	1422
Walter Lacy Boyd	1424
Paul Lemon Brockway	1426
Ora Bundy	1428
Arthur Gray Butler	1430
Benjamin Lucien Campbell	1433
Edward Emery Carpenter	1433
John Taylor Chambers	1436
Cleveland Beach Coe	1437
Clarence Knight Conard	1439
John Rose Wilson Davis	1442
Orville Vosbury Derr	1445
Richard Griffith Develin	1447
Elbert Hermon Dresser	1449
Clifford George Dunnells	1450
John Conrad Fitterer	1452
James Benton French	1453
Pierce Powers Furber	1457
James Edwin Gibson	1458
Edward Godfrey	1461
Alfred Cookman Gregory	1462
Heinrich Eduard Gruner	1464
Julius Reed Hall	1466
Leslie Standish Hall	1468
Borden Baker Harris	1470
Herbrand Harvey	1472
Philip Walter Henry	1475
Chester James Hogue	1480
Henry Macy Jones	1481

	PAGE
Marion Reed Kays	1483
Joseph Grant Kelley	1487
John Edward Kemp	1488
Richard Smith Lea	1491
William Schurman Lea	1493
George Allen Lee	1495
Carlos Lobo	1496
James Button Clyde Lockwood	1497
Charles Harvey MacCulloch	1499
George Alfred Maney	1500
Wayne Dickson Maxwell	1504
Thomas Mero McClure	1505
Robert Lee McCormick	1508
William Naylor McDonald	1509
Benjamin McKeen	1511
John Thomas Lamar McNew	1512
Rudolph Phillip Miller	1517
Charles Adam Mixer	1524
Robert Regnault Nace	1526
Charles Edward Parry	1527
Harry Nelson Pharr	1529
Henry Cyrus Porter	1533
De Witt Pawling Pugh	1535
Reinaldo Ramirez	1536
Allan Appleton Robbins	1537
William Joseph Rowland	1539
Joshua Stewart Sawyer	1540
Francis Joseph Seery	1541
Valentine Bernard Siems	1543
Sidney Montefiore Siesel	1544
Joseph Emory Sirrine	1546
Leonard Sewall Smith	1548
Jesse Baker Snow	1550
Isaac Joshua Stander	1555
Homer Reed Stanford	1557
Henry Arthur Van Alstyne	1559
Albion Noyes Van Vleck	1563
William Bernard Ward	1566
William Godfrey Wegner	1567
George Grover Wickline	1568
Arthur Jacob Widmer	1571
Clement Clarence Williams	1573

PAGE
Albert William Zesiger.................................. 1576
Roy Richard Zipprodt.................................. 1579

Associate Members

Ernest Samuel Alderman.................................. 1580
Robert Morrison Andrew.................................. 1581
Murray Chase Ayers.................................. 1582
Thomas George Gordon Beck.................................. 1584
Royall Douglas Bradbury.................................. 1585
Harry Custer Diesem.................................. 1587
James Dougan.................................. 1588
Douglass Hewitt Ferry.................................. 1590
Alexander Sylvester Forster.................................. 1592
Magnus Lawrence Hydle.................................. 1593
Roy Wrenshaw Jablonsky.................................. 1594
Roland Winthrop Lefavour.................................. 1598
Paul Wright Longsdorf.................................. 1599
Burton Lowther.................................. 1601
John Joseph McGinley.................................. 1603
William Canon Muldrow.................................. 1604
Lemuel Bynum Ogilvie.................................. 1605
Arthur George Ready.................................. 1607
John George Schmidt.................................. 1609
Edward Simeon Skillin.................................. 1610
Jerome Branch Stocking.................................. 1612
Robert Raymond Tway.................................. 1614
Benjamin Weiss.................................. 1615

Juniors

William George Clark.................................. 1616
Ludlow Vanderburg Clark Deichler.................................. 1618
Hugh Joseph Noonan.................................. 1621
Bernard Wick Shrubb.................................. 1622
Dean Edwin Swift.................................. 1623
William Fraser Tompkins, Jr.................................. 1624
John Merlin Wilson.................................. 1627

Affiliates

Frederick Snare.................................. 1628
Maximilian Toch.................................. 1630

ERRATA

Transactions, Vol. 111, 1946

Page 251, Eq. 41b should read: “$x_{r3} = \frac{\cosh\ (a\ x_3)}{a \sinh\ (a\ x_3)} = \frac{\coth\ (a\ x_3)}{a}$.....(41*b*).”

Transactions, Vol. 112, 1947

Page 1304, Eq. 108, change “J^2” to “ρ^2” in two places.

Page 1304, line 3 after Eq. 108, change “J = polar moment of inertia” to “ρ = polar radius of gyration.”

Page 1305, line 13, change “J^2” to “ρ^2.”

Page 1305, Eq. 111*a*, change “J^2” to “ρ^2.”

Page 1307, Fig. 14(*c*), change “Eccentric Load” to “Concentrated Load.”

Page 1308, line 7, after, the first word, “zero,” add “in evaluating n.”

Page 1311, line 26, after “of n and m/j” add “(such as those given by Eqs. 112, 116, and 121, and exhibited in Fig. 14).”

Page 1311, lines 27 and 28, “K, a_1, I_y K, and $I^2{}_y$ h^2 $1/s\sqrt{I_y\ R}$” should read “K, a_1, I_y K, $I^2{}_y$ h^2, and $\frac{\sqrt{I_y\ K.''}}{S}$.”

Page 1312, line 9, after “the top flange, $H = d/2$” add “(Values of n and m/j have been computed from Eqs. 112 and 116; they may also be determined approximately from Figs. 14(*a*) and 14(*b*)).”

Transactions, Vol. 113, 1948

Page 449, in Theorem III, transpose “(M''_O)” forward in the sentence, to follow “equal to the second moment”; and transpose “($M''_{O,N}$)” forward in the sentence, to follow “plus the second moment.”

Page 453, line following Eq. 30; change “M'''_{AB}” to “$M'''_{AB,B}$.”

Page 463, in the line preceding Eq. 71*b* change “$M_{\overline{CB},C}$” to “$M_{\overline{BC},C}$.”

Page 463, in line 11, change “AB” to “BC.”

Page 782, in line 33, change “$S_n + 1$” to “S_{n+1}.”

PAPERS AND REPORTS OF COMMITTEES PUBLISHED IN PROCEEDINGS, ASCE, FEBRUARY, 1947, TO OCTOBER, 1948, INCLUSIVE, NOT PUBLISHED IN TRANSACTIONS, VOL. 113

	Number of *Proceedings*
THE MARINE OPERATING PROBLEMS, PANAMA CANAL, AND THE SOLUTION. By Miles P. DuVal	February, 1947
SEA LEVEL PLAN FOR PANAMA CANAL. By J. G. Claybourn	February, 1947
MISSISSIPPI RIVER BRIDGE AT DUBUQUE, IOWA. By R. N. Bergendoff and Josef Sorkin	June, 1947
PROBLEMS AND CONTROL OF DECENTRALIZATION IN URBAN AREAS: A SYMPOSIUM. By John M. Picton, and Russell H. Riley	November, 1947
EXPERIENCES WITH PREDETERMINING PILE LENGTHS. By William W. Moore	November, 1947
OPERATION OF THE CONOWINGO HYDROELECTRIC PLANT. By Robert E. Turner	November, 1947
DETERMINATION OF POSITION AND AZIMUTH BY SIMPLE AND ACCURATE METHODS. By T. F. Hickerson	November, 1947
A CENTROIDAL METHOD OF RIGID-FRAME ANALYSIS. By S. J. Bell	November, 1947
EXPERIMENTAL DETERMINATION OF VIBRATION CHARACTERISTICS OF STRUCTURES. By Samuel J. Loring	December, 1947
MATRIX ANALYSIS OF PIN-CONNECTED STRUCTURES By Pei-ping Chen	December, 1947
UNUSUAL DESIGN PROBLEMS—SECOND TACOMA NARROWS BRIDGE. By Charles E. Andrew	December, 1947
ANALYSIS OF FRAMES WITH ELASTIC JOINTS. By Ralph W. Stewart	December, 1947
THE SIGNIFICANCE OF PORE PRESSURE IN HYDRAULIC STRUCTURES. By L. F. Harza	December, 1947
ELASTIC FOUNDATIONS ANALYZED BY THE METHOD OF REDUNDANT REACTIONS. By Zusse Levinton	December, 1947
LATERAL EARTH PRESSURES ON FLEXIBLE RETAINING WALLS: A SYMPOSIUM. By William H. Smith; Gregory P. Tschebotarioff; Edward R. Ward, John R. Bayliss, and Philip P. Brown; Harris Epstein; L. A. Palmer; J. C. Gebhard; and L. C. Coxe	January, 1948
SHEARING STRESS DISTRIBUTION IN BOX GIRDERS WITH MULTIPLE WEBS. By A. S. Neiman	February, 1948
HIGHWAY BRIDGE FLOORS: A SYMPOSIUM. By F. E. Richart, N. M. Newmark, and C. P. Siess	March, 1948

Number of *Proceedings*

PANAMA CANAL—THE SEA-LEVEL PROJECT: A SYMPOSIUM.
By James H. Stratton; Ralph P. Johnson and Sydney O. Steinborn; F. S. Brown; J. S. Meyers and E. A. Schultz; C. A. Lee and C. E. Bowers; J. E. Reeves and E. H. Bourquard; Wilson V. Binger and Thomas F. Thompson; A. Casagrande and W. L. Shannon; and J. J. Rose, F. L. Dye, W. B. Watson, and L. T. Crook............ April, 1948

THE RELATIONSHIP BETWEEN PILE FORMULAS AND LOAD TESTS.
By Robert D. Chellis............ May, 1948

STABILITY OF THIN-WALLED COMPRESSION MEMBERS.
By H. Dudley Wimer, Jr............ May, 1948

VALUATION AND DEPRECIATION OF PUBLIC UTILITY PROPERTY.
By Maurice R. Scharff............ June, 1948

REVIEW OF SLOPE PROTECTION METHODS: REPORT OF THE SUBCOMMITTEE ON SLOPE PROTECTION OF THE COMMITTEE ON EARTH DAMS OF THE SOIL MECHANICS AND FOUNDATIONS DIVISION... June, 1948

INDUSTRIAL WASTES: A SYMPOSIUM.
By Don E. Bloodgood, and F. W. Mohlman............ September, 1948

ADJUSTMENT OF THE "SHORAN" TRIANGULATION.
By Chung-jui Chu............ September, 1948

UNUSUAL DESIGN PROBLEMS, HARLAN COUNTY DAM.
By Louis G. Feil............ September, 1948

EXPANSION JOINT PRACTICE IN HIGHWAY CONSTRUCTION.
By A. A. Anderson............ September, 1948

A SIMPLIFIED METHOD OF ANALYZING SUSPENSION BRIDGES.
By Ling-hi Tsien............ September, 1948

FLEXURAL CONSTANTS OF HAUNCHED BEAMS BY AREA COMPUTATION.
By Fang-Yin Tsai............ September, 1948

DYNAMIC INSTABILITY OF TRUSS-STIFFENED SUSPENSION BRIDGES UNDER WIND ACTION.
By Friedrich Bleich............ October, 1948

ADVANCES IN SEWAGE TREATMENT AND PRESENT STATUS OF THE ART: FOURTH PROGRESS REPORT OF THE COMMITTEE OF THE SANITARY ENGINEERING DIVISION ON SEWERAGE AND SEWGAE TREATMENT............ October, 1948

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2329

MISSISSIPPI RIVER CUTOFFS

By Gerard H. Matthes,[1] Hon. M. ASCE

With Discussion by Messrs. W. E. Elam, C. L. Hall, H. D. Vogel, Harry N. Pharr, Lytle Brown, Anson Marston, E. J. Williams, Jr., and Eugene A. Graves, and Gerard H. Matthes

Synopsis

Natural and artificial cutoffs on the Lower Mississippi River in the 50-mile stretch north of Vicksburg, Miss., are described with respect to their effect on river shortening and flood-stage lowering. Historical and physical facts are given which explain why, during the period from 1884 to 1932, natural cutoffs were dreaded and engineering measures were taken to prevent their occurrence, thus permitting the river to become considerably lengthened. The paper describes the unforeseen circumstances which caused a cutoff at Yucatan Bend to occur unexpectedly during the low-water season of 1929; and it relates how the satisfactory manner in which the cutoff developed unassisted gave impetus to the launching of a program of river shortening by artificial cutoffs, the first of which, across Diamond Point, was opened on January 8, 1933. The technique adopted avoided making cutoffs across narrow necks, except where conditions made this necessary, and also avoided the pitfalls of European practice. Since that date fourteen additional cutoffs have been constructed, making a total of sixteen. In connection with other forms of channel rectification and dredging operations, these cutoffs have shortened the Lower Mississippi a total of 170 miles between Memphis, Tenn., and Baton Rouge, La., in an original 680-mile river length, or 25%. The appreciable lowering of flood stages which resulted from the cutoffs has saved vast outlays in levee construction. The cutoff data presented are from the records of the Mississippi River Commission.

Historical

In early times, the Lower Mississippi River, operating without restraint, made cutoffs at the rate of from about thirteen to fifteen in the course of a century. These cutoffs usually occurred during very high stages when water

---

Note.—Published in January, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] New York, N. Y.; formerly Head Engr., Mississippi River Comm., and Director, U. S. Waterways Experiment Station, Vicksburg, Miss.

scoured out channels across the narrow necks of overdeveloped bends. Individual shortenings of the river channel so produced ranged between 6 miles and as many as 21 miles. An assumption of fifteen cutoffs and an average net shortening of 12 miles per cutoff would represent a total shortening of 180 miles in a century, or 1.8 miles a year. As applied to the 840-mile stretch of river between Cairo, Ill., and Baton Rouge (cutoffs have not occurred below the latter point, see Fig. 1) 1.8 miles per year, or any value like it, seems totally inadequate to compensate for the habitual lengthening of the river, which is known to be more nearly on the order of 4 or 5 miles per year. Thus, other shortening processes must account for the difference. The most obvious of these is the channel shortening effected by chute developments across point bars. These developments result from the gradual enlargement of shallow swales (which began during high stages) into channels deep enough to flow water even at low stages and, in time, the swales become main-river channels. Chute developments have been numerous; but, because they were mostly slow processes and therefore not as spectacular as natural cutoffs, they have attracted little attention. Appreciable river shortenings have resulted from them, however. In most instances, the gradual enlargement of chutes has been conducive to producing better channels than have resulted from quick-acting natural cutoffs. The latter, because of their disruptive nature, as a rule were followed by rapid and drastic channel changes often involving major property losses and inconvenience to navigation.

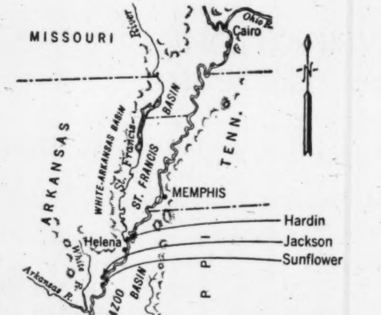

MISSOURI
Ohio R.
Cairo
River
N
TENN.
BASIN
ARKANSAS
WHITE-ARKANSAS BASIN
St. Francis
ST. FRANCIS
MEMPHIS
Hardin
Helena
Jackson
Sunflower
White R.
Arkansas R.
YAZOO BASIN
MISSISSIPPI
Caulk
Arkansas City
Greenville
Tarpley
Ashbrook
Leland
Worthington
Sarah
Ouachita R.
Boeuf R.
Yazoo River
Willow Point
Marshall
TENSAS BASIN
Vicksburg
Big Black River
Diamond
LOUISIANA
Black R.
Natchez
Yucatan
Rodney
Giles
Old River
Glasscock
Scale in Miles
0 20 40 60 80 100
Atchafalaya R.
BATON ROUGE
Lake Pontchartrain
NEW ORLEANS
ATCHAFALAYA BASIN
GULF OF MEXICO

FIG. 1.—LOWER MISSISSIPPI RIVER

In addition to cutoffs and chute developments, river shortenings have resulted from bend flattening. This phenomenon has been of minor importance but is of interest in that it dispels the popular belief that river bends always tend to assume greater, and eventually excessive, curvature.

One cannot but marvel at the consistency with which these natural processes have operated to maintain the over-all length in the Cairo-to-Baton Rouge stretch about the same through the long course of time: 884 miles according to maps compiled from 1820–1825 survey data; 833 miles according to the first Mississippi River Commission survey of 1881–1882; 842 miles according to the 1916 survey; and 846 miles according to the 1929–1930 survey.

When, on January 8, 1933, Maj.-Gen. Harley B. Ferguson, M. ASCE, then President of the Mississippi River Commission, opened the first pilot cut across Diamond Point (Fig. 1), the erstwhile policy of "no cutoffs at any price" maintained by the Mississippi River Commission over a period of 48 years went into the discard. The events which originally caused the Commission in 1884 to adopt its "no cutoff" stand briefly were these: During the 1870's three natural cutoffs occurred above Memphis which produced drastic changes in the river's alinement, wiped large tracts of agricultural land literally off the map, and interfered seriously with navigation. In the 1870's also, Centennial Cutoff near Vicksburg came into existence (see Fig. 2), partly through human activity; and finally in 1884 another cutoff occurred near Water Proof, La., where nature again was aided by the hand of man. Collectively these five

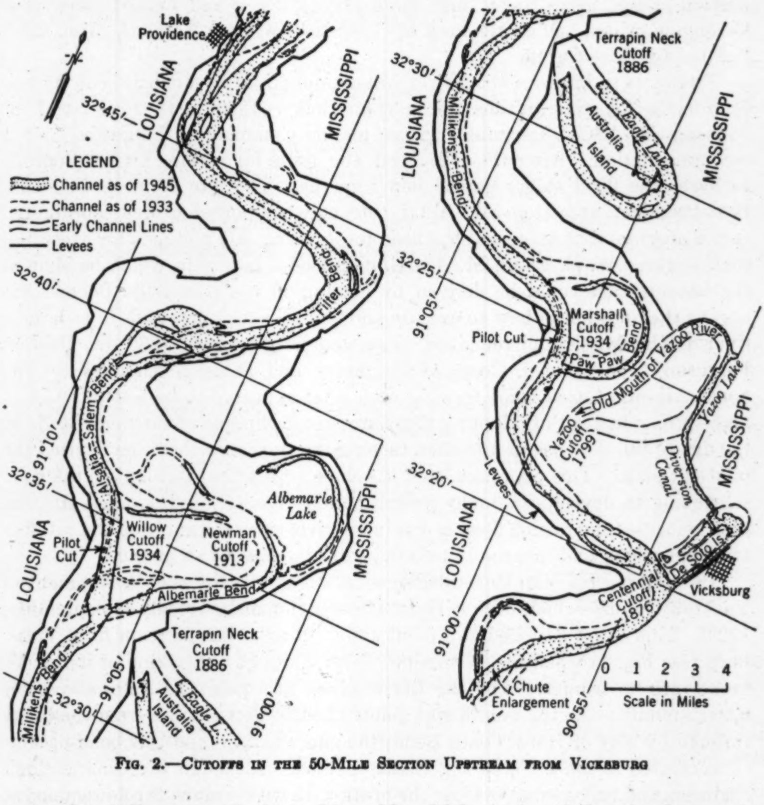

Lake Providence
LOUISIANA
MISSISSIPPI
32°45'
LEGEND
Channel as of 1945
Channel as of 1933
Early Channel Lines
Levees
32°40'
Filter Bend
Salem Bend
Alsatia
91°10'
32°35'
Albemarle Lake
Willow Cutoff 1934
Newman Cutoff 1913
Pilot Cut
Albemarle Bend
LOUISIANA
MISSISSIPPI
Terrapin Neck Cutoff 1886
Millikens Bend
32°30'
91°05'
Australia Island
Eagle L.
91°00'
Terrapin Neck Cutoff 1886
32°30'
LOUISIANA
Millikens Bend
Australia Island
Eagle Lake
MISSISSIPPI
32°25'
91°05'
Marshall Cutoff 1934
Pilot Cut
Paw Paw Bend
Old Mouth of Yazoo River
Yazoo Lake
Yazoo Cutoff 1799
Diversion Canal
MISSISSIPPI
32°20'
Levees
LOUISIANA
De Soto Is.
Centennial Cutoff 1876
Vicksburg
91°00'
0 1 2 3 4
Scale in Miles
Chute Enlargement
90°55'

FIG. 2.—CUTOFFS IN THE 50-MILE SECTION UPSTREAM FROM VICKSBURG

cutoffs shortened the river an aggregate of 50 miles within one decade. The engineers of the Mississippi River Commission were powerless to direct or control the river's rampaging.

The Commission's policy to permit no further natural cutoffs to occur found justification in the fact that the Commission in those days had neither the funds nor the dredging equipment at its command with which to make artificial cutoffs or to control their development. Moreover, natural cutoffs across narrow necks of land as a rule produced only temporary lowerings of flood stages and induced successions of channel changes extending upstream as well as downstream. To prevent impending cutoffs from taking place, the Commission caused dikes to be built on Ashbrook Neck and Leland Neck near Greenville, Miss., and on the neck of Cowpen Point near Natchez, Miss., later isolated by Giles Cutoff.

Proposals to improve the Lower Mississippi River by artificial cutoffs had been made from time to time dating as far back as 1850; in 1882 the late J. B. Johnson, M. ASCE, assistant engineer for the Commission, submitted such a recommendation. After the 1927 flood, the desire for making artificial cutoffs to lower the flood stages gained new impetus. The late John F. Coleman, Past-President and Hon. M. ASCE, strongly supported a well-coordinated cutoff program, but expressed the need for devising safe methods for executing such works. W. E. Elam, M. ASCE, discussed a paper[2] in which he showed the benefits that could be derived by cutting off the Greenville Bends—but always the question of how to execute and control such operations in as large a river as the Mississippi remained unanswered. Not until 1930 did General Ferguson, then Colonel, Corps of Engineers, and Division Engineer for the South Atlantic Coast, come forward with what appeared to be a sound plan of procedure. Instead of adopting the European technique of making cutoffs in the dry to full dimensions and then turning the river into them, he devised the pilot-cut plan. This plan permitted the slope adjustment incident to the river shortening to develop gradually over a period of years, and avoided creating high velocities and raising flood stages in the river downstream from the cutoff—two evils which had invariably resulted from the European method.

In adopting his plan General Ferguson was guided by the development of Yucatan Cutoff—which cut off Hard Times Bend unexpectedly in September, 1929. This cutoff occurred at a point about 40 miles downstream from Vicksburg (see Fig. 1) when the Mississippi River, through the caving of its banks, broke into its tributary, the Big Black River, at a point 2.5 miles above the latter's mouth. As the mouth was about 12 miles downstream from the break through by way of Hard Times Bend, the entire fall around this bend became concentrated in the 2.5-mile Big Black channel. Although the Commission's engineers had endeavored to close the break with willow mats, the flow gained so much headway that it could not be stopped. There resulted what subsequent events have shown to be the model cutoff on the Lower Mississippi. After two flood seasons the crooked little valley of Big Black River had become transformed into a wide and nearly straight cutoff channel fit for navigation. By

---

[2] *Transactions*, ASCE, Vol. 93, 1929, p. 937.

April, 1932, it was carrying 60% of the river's flow; it was more than 100 ft deep in places; and this great depth gave so ample a cross section that the rate of fall was almost the same as in the river above and below the cutoff. Consequently, velocities were moderate and offered no obstacles to navigation. Concurrently with this development, deterioration of the old channel around Hard Times Bend had progressed, through deposition of sand, until, by November 22, 1933, during a low-river stage, the entire flow was passing through Yucatan Cutoff and the writer was able on that date to walk dry-shod across the severed bend channel at its upper end.

The Yucatan incident, aside from being extraordinary as a natural avulsion (occurring during low-river, instead of high-river, stages) set a new precedent in cutoff history. It proved that a narrow channel a mile or two in length was superior, as a cutoff route, to a short cut across a narrow neck of land. Yucatan Cutoff developed without any assistance from the engineers, and has not upset the river either upstream or downstream in a detrimental way. Such channel changes as did result were not appreciably different from those ordinarily witnessed in the river's meandering.

The program of cutoffs inaugurated in 1933 has followed this precedent wherever conditions permitted it. It was aimed at shortening the river without, however, straightening it. Mild curvature was held to be essential for preserving a deep navigable channel as well as for effecting such channel stabilization as might be consistent with the meandering nature of an alluvial river. The main objective, however, was the lowering of flood stages; and, in this respect, the cutoff program has made an extraordinary contribution to the control of floods in the Lower Mississippi River. More than that, it was so planned as to make it incumbent on the waters of the Mississippi River to perform the major part of the work of excavating new channels and of filling abandoned ones. The program has effected stupendous savings in the construction costs of the levee system.

NATURAL CUTOFFS

Six cutoffs that have occurred in the 50-mile reach upstream from Vicksburg afford a unique opportunity to analyze this subject (see Fig. 2)—Centennial (1876), Yazoo (1799), Terrapin Neck (1886), Newman (1913), Marshall (1934), and Willow (1934). The first four were cut by nature whereas the remaining two were created by man. For the most part, the paper is concerned with the latter two—Marshall and Willow cutoffs.

Centennial Cutoff.—In 1876, at Vicksburg (see Fig. 2), the Centennial Cutoff moved the river away from Vicksburg and created De Soto Island, which still remains a part of the State of Louisiana. Although this is a natural cutoff, the process was aided artificially.

Yazoo Cutoff.—Only a small part of the original Yazoo Bend, which was cut off in 1799, remains as Yazoo Lake.

Terrapin Neck Cutoff.—This action, in 1886, eliminated 16 miles of the original river channel, creating Eagle Lake in the process. Australia Island, within the loop formed by the lake, remains as a part of the State of Louisiana.

Newman Cutoff.—This development was not a cutoff in the true sense, but a natural chute development caused by the enlargement of Albemarle Chute in 1913.

MARSHALL CUTOFF

Marshall Cutoff extends across Marshall Point around which the river formerly flowed in an almost perfect semicircular bend known as Paw Paw Bend. This bend had become subject to chronic bank caving and in 1932 threatened to effect a junction with the abandoned channel of the Yazoo River. It became necessary either to revet 3 miles of Paw Paw Bend or to make a cutoff. The decision was in favor of the cutoff.

The method of construction consisted in first excavating the high ground with dragline machines and then finishing the pilot cut down to a depth of 29 ft below mean low water with cutter-head dredges. Four draglines began operations on October 19, 1933, and worked a total of 80 machine days. On December 17, 1933, the high ground yardage removed was 1,841,000 cu yd. This cut was 6,000 ft long. Between November 12, 1933, and February 20, 1934, a cutter-head dredge worked 101 days removing 2,950,000 cu yd to open a pilot cut across the huge sand bar at the lower end for a distance of 4,400 ft. The cost of excavating this part of the pilot cut instead of trusting the river to wash out a channel of its own through the sand bar has been amply justified by the excellent performance of the cutoff channel in enlarging itself in the exact place it was designed to occupy. During February and March, 1934, a dredge worked 20 days removing 551,000 cu yd. The width of the completed pilot cut was 250 ft on the bottom, which represented the full swing that the dredges could make at one operation. The side slopes were left as steep as the soil would stand.

When the pilot cut was opened on March 12, 1934, the over-all length was 13,600 ft. However, to compute the amount by which the river was being shortened, it was necessary to add to the actual length of the pilot cut the distance to the axis of the main river at each end. This computation gave the navigable length of the cutoff as 3.1 miles. Deducting this length from the 7.3 miles around Paw Paw Bend made the net shortening 4.2 miles. The fall in the cutoff, at time of opening, was 2.2 ft or about 0.7 ft per mile. The river was not high and the resulting rush of water was not damaging, but the bed and banks of the pilot cut began to erode immediately, causing much earth to be entrained. Rising river stages augmented the flow through the cut; and, on March 27, 15 days after the opening, 60,500 cu ft per sec, or 8% of the flow in the Mississippi River, was passing through the cutoff. By April 10 the flow had increased to 84,000 cu ft per sec, or 10% of the river discharge. When river stages fell, the percentage of flow in the cutoff decreased. On May 28, at a low-river stage, only 18,600 cu ft per sec, or 7% of the river discharge, was passing and this percentage prevailed throughout the low-water season. Table 1(*a*) is a condensed record of how the discharge capacity increased during the course of time as the cutoff channel enlarged itself from a pilot cut 250 ft wide to a sizable river. While this enlargement progressed, the river channel in Paw Paw Bend shoaled and contracted without any aid from man.

TABLE 1.—PERFORMANCE STATISTICS, ARTIFICIAL CUTOFFS ON THE MISSISSIPPI RIVER

(a) MARSHALL CUTOFF

Date of observation (1)	Gage height[a] (ft) (2)	Discharge in Cutoff: Cu ft per sec (3)	Discharge in Cutoff: Percentage[b] (4)
1934:			
March 27	30.9	60,500	8
April 9	33.3	84,000	10
May 28	8.0	18,600	7
August 13	3.4	10,900	7
1935:			
February 9	33.8	133,000	15
May 22	40.2	269,000	25
July 5	45.8	404,000	31
September 20	14.2	179,000	49
October 23	1.9	86,000	61
December 16	14.1	190,000	50
1936:			
March 12	27.7	315,000	42
April 23	42.6	517,000	42
May 29	12.5	229,000	66
September 1	3.1	83,900	88
December 18	10.4	229,000	67
1937:			
January 26	41.1	608,000	50
February 17	55.1	940,000	47
July 21	10.6	322,000	99
August 10	6.2	247,000	100
During low water	....	....	100
1938:			
January 4	20.7	487,000	84
April 20	41.1	757,000	62
June 8	34.1	704,000	70
October 5	9.7	350,000	100
During low water	....	....	100
1939:			
March 10	43.4	901,000	61
May 5	42.8	954,000	70
August 28	6.5	292,000	100
During low water	....	....	100
1940:			
(No record)	....	....	....
1941:			
April 10	8.3	333,000	93
May 3	24.5	628,000	86
July 3	12.3	408,000	97
October 15	16.9	552,000	92
During low water	....	....	100
1942:			
January 14	12.8	416,000	92
April 28	36.2	911,000	82
During low water	....	....	100
1943:			
January 19	35.2	1,091,000	91
March 17	16.2	498,000	94
April 9	35.4	1,088,000	87
During low water	....	....	100
1944:			
May 23	43.7	1,124,000	84
During low water	....	....	100
1945:			
June 2	38.3	1,012,000	88

(b) WILLOW CUTOFF

Date of observation (1)	Gage height[a] (ft) (2)	Discharge in Cutoff: Cu ft per sec (3)	Discharge in Cutoff: Percentage[b] (4)
1934:			
April 10	33.9	68,000	8
June 16	4.8	3,080	2
August 14	3.3	5,940	4
November 13	4.3	13,000	7
December 17	18.6	55,100	12
1935:			
January 26	27.1	114,000	15
March 29	42.0	298,000	26
April 30	46.8	367,000	29
June 29	45.6	433,000	33
August 16	14.4	155,000	42
October 22	1.9	73,000	52
December 17	13.6	167,000	46
1936:			
May 2	43.9	499,000	42
June 4	10.4	181,000	60
August 12	−1.0	108,000	85
September 2	−3.1	93,760	97
November 16	17.7	306,000	63
1937:			
January 27	41.9	785,000	59
February 24	55.2	1,101,000	56
July 1	20.1	427,000	81
October 9	−0.9	146,000	99
December 30	15.8	424,000	86
1938:			
April 21	41.4	865,000	70
June 9	34.5	817,000	79
September 15	2.6	204,000	100
During low water	....	....	100
1939:			
March 8	43.3	1,159,000	80
August 1	7.1	304,000	100
During low water	....	....	100
1940:			
(no record)	....	....	....
1941:			
April 12	8.6	358,000	100
May 1	25.5	707,000	92
During low water	....	....	100
1942:			
January 2	11.1	447,000	100
April 29	36.1	964,000	88
During low water	....	....	100
1943:			
January 22	34.6	1,037,000	93
During low water	....	....	100
1944:			
May 25	42.2	1,059,000	88
During low water	....	....	100
1945:			
May 31	38.4	1,105,000	96

[a] On the Vicksburg Canal gage.

[b] Percentage of the total river discharge.

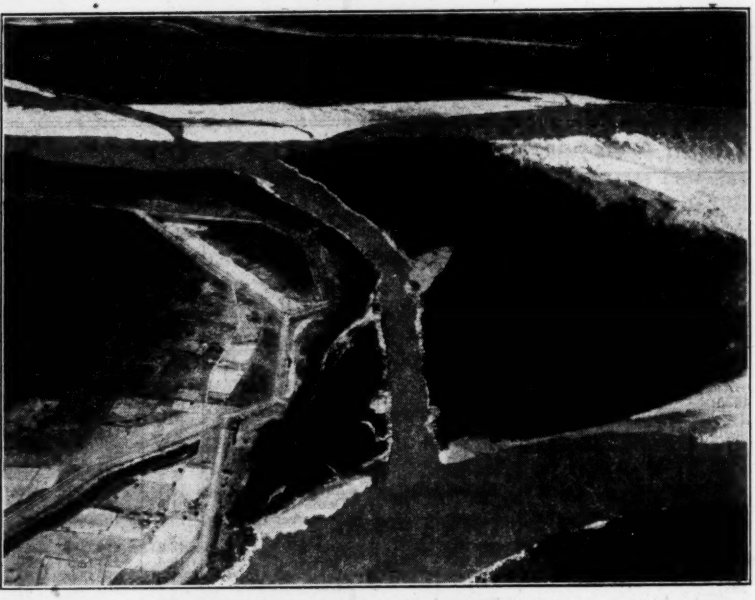


FIG. 3.—MARSHALL CUTOFF FACING UPSTREAM, LOW WATER, OCTOBER 25, 1935

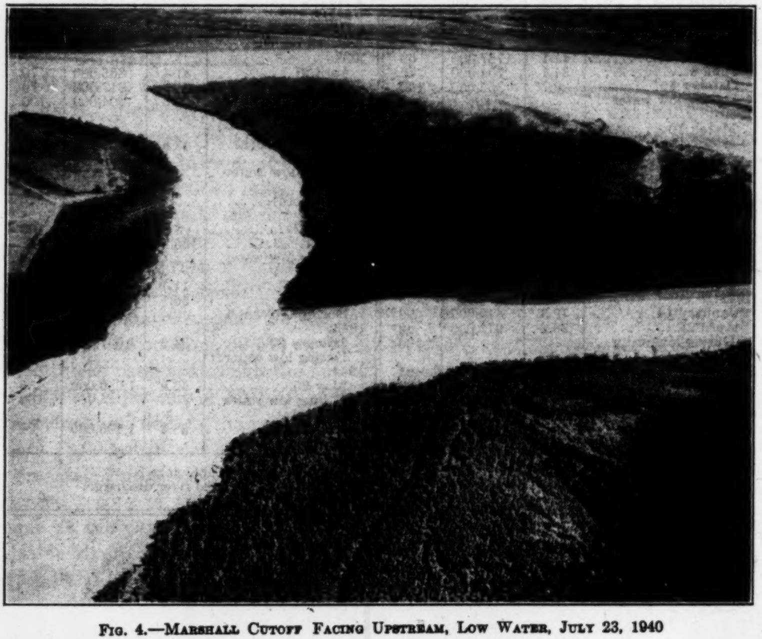


FIG. 4.—MARSHALL CUTOFF FACING UPSTREAM, LOW WATER, JULY 23, 1940

Marshall Cutoff, throughout, has been a well-behaved stretch of channel and has required very little assistance. Following the high-river stages in 1934, the upper end of the pilot cut silted for a short distance to the approximate elevation of mean low water. A pump barge removed 405,000 cu yd in 12 days, thereby restoring the cutoff to a depth of 25 ft below mean low water. During 1935, 355,000 cu yd of additional shoaling required removal. Following this cleaning, the cut continued to develop (see Fig. 3) and no further dredging has been done. A view of this cutoff as it appeared on July 23, 1940, is shown in Fig. 4.

WILLOW CUTOFF

About 1913 Willow Point (Fig. 2) was more than 7 miles long. It stuck out on the map "like a sore thumb." It had been lengthening itself year after year because of the caving habits of Albemarle Bend. During the great flood of 1913 the river made itself a new channel across the middle of Willow Point by washing out a shallow swale known as Albemarle Chute. The new river bend which thus came into existence became known as Newman Cutoff. The abandoned Albemarle Bend channel became a lake. Although the river shortened itself in this process nearly 5 miles, Newman Cutoff, according to modern terminology, amounts to a chute development and should not be dignified by the name "cutoff." The shortening was not destined to last long, for the new bend location was not favorable and it appeared that inevitably the bend would migrate again and Willow Point would probably lengthen as much as before. It was deemed advisable, therefore, to locate Willow Cutoff at a place where the resulting new channel would have no opportunity to migrate for a long time to come.

The lower end of Willow Cutoff overlaps the old Terrapin Neck Cutoff which divorced Eagle Lake in 1866. The combined effect of the two cutoffs has been to shorten the river 23.7 miles since 1866.

The method of constructing Willow Point Cutoff, in general, was similar to that employed on Marshall Cutoff, except that the greater length called for more equipment. From November 19, 1933, to January 28, 1934, seven dragline machines were operated for 322 machine days in excavating 2,392,000 cu yd, thereby lowering land elevations along the route of the pilot cut some 21 ft over a length of 13,000 ft. Between December 26, 1933, and May 15, 1934, hydraulic dredges removed 5,101,000 cu yd thereby extending the pilot cut to a depth of 14 ft below mean low water over a distance of 5,800 ft at the upper end, and to depths ranging from 10 ft to 15 ft below mean low water over other stretches. At the lower end of the cut a wide flat sand bar lay between the high bank through which the pilot channel had been dug and the river. A low slough separated this bar from higher ground and it was feared that the new cutoff channel might tend to follow this slough. It became necessary, therefore, to dredge through the bar a distance of 5,800 ft, thus making the total low-water length of the cutoff more than 20,000 ft.

Opened on April 8, 1934, nearly a month after the opening of Marshall Cutoff, on a medium high-river stage, on April 10, Willow Cutoff discharged 68,000 cu ft per sec, representing 8% of the total for the river. During the

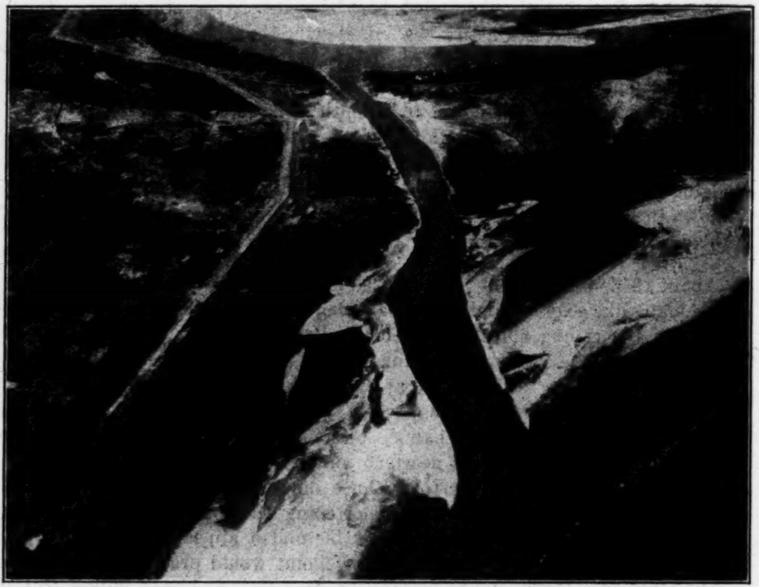


FIG. 5.—WILLOW CUTOFF FACING UPSTREAM, LOW WATER, OCTOBER 25, 1935

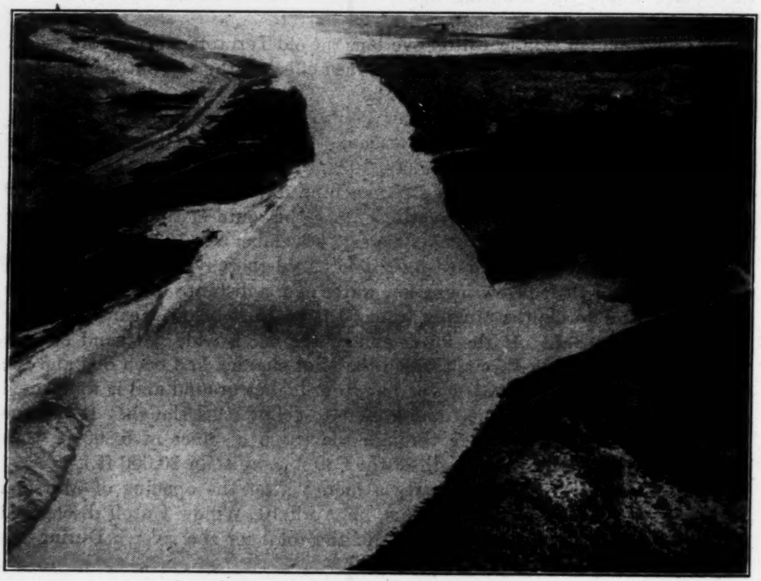


FIG. 6.—WILLOW CUTOFF FACING UPSTREAM, LOW WATER, JULY 23, 1940

low-river stages that followed in June, lack of depth caused the percentages to fall as low as 2% and 4%; but, after dredging, these percentages increased appreciably during increasing river stages. The development of the discharge capacity of Willow Cutoff is best gleaned from the data in Table 1(*b*).

As was the case for Marshall Cutoff some supplementary dredging was required soon after the cutoff began to function. It was necessary to cut the bottom to 40 ft below mean low water at the center and to 22 ft at the ends. A view of this cutoff photographed on October 25, 1935, is shown in Fig. 5. Later, in June and July, 1938, a dredge was returned to the cutoff to remove a tough clay deposit in the bottom at its upper end, which had resisted the cutting action of the current and retarded development. On one other occasion, in 1936, a dredge was operated to enlarge and shape up the entrance.

In 1933, prior to making the pilot cut, the distance around Willow Point was 13.1 miles measured from the head to the foot of the pilot cut. After the 1937 flood the cutoff channel had a length of 4.7 miles measured between the same points as in 1933, indicating a net river shortening of 8.4 miles. By 1939 distances had changed so as to make the shortening nearer 7.7 miles.

At the crest of the 1934 high water, shortly after the pilot cut began to function, the hydraulic slope in the cut was 1.18 ft per mile. During the crest of the great 1937 flood, this slope had flattened to 0.53 ft per mile. In 1939 the flattening had become complete, the slope at the crest of the flood being only 0.36 ft per mile—practically the same as that of the Mississippi River in that latitude. Previous to 1937, velocities in the cutoff during high stages ran as much as from 7 to 9 ft per sec, and boats at such times preferred to travel via the bend when headed upstream. Except for these first 3 years Willow Cutoff has been easy to navigate by commercial tows. Fig. 6 is a view of this cutoff as of July 23, 1940.

TABLE 2.—EFFECT OF CUTOFFS ON THE 50-MILE LENGTH OF THE RIVER SHOWN IN FIG. 2

Year	Interval (years)	LENGTH, IN MILES		Remarks
		River	Lengthening	
1765		71±		
	55		7	Only one cutoff, the Yazoo, occurred in this period. The net effect is not known, but is presumed to have been more than 10 miles.
1820		68		
	62		11	In this interval Terrapin Neck Cutoff shortened the river 16 miles and Centennial Cutoff, 6 miles—a total of 22 miles.
1882		57		
	34		6.8	Albemarle Chute (known as Newman Cutoff) shortened the river nearly 5 miles.
1916		58.8		
	13		0.7	No cutoffs occurred in this interval.
1929		58.1		
	16		2.7	Marshall and Willow cutoffs in 1934 shortened the river nearly 12 miles.
1945		48.8		

RIVER SHORTENING

The effect of these six cutoffs on the length of the stretch of river in Fig. 2 raised many questions, because the sum total of the shortenings amounted to more than the total present equivalent length of river. In the course of the 80 years under consideration the lengthening processes due to river meandering offset in part the total net shortenings effected by the six cutoffs (see Table 2). For this purpose the nearly 50-mile reach from Vicksburg to Lake Providence, La., had to be considered as no statistics were available for individual parts of it.

RIVER STEEPENING

Such steepening as was caused in the Mississippi River by the development of Marshall and Willow cutoffs, and by the cutoffs below, was slight and at most temporary. The river soon re-established its natural slopes in this section as may be seen from Table 3.

TABLE 3.—FLOOD-CREST SLOPES IN THE REACHES ABOVE CUTOFFS

Year	(a) UPSTREAM FROM WILLOW CUTOFF TO LOWER END OF OPPOSUM CHUTE		(b) UPSTREAM FROM MARSHALL CUTOFF TO LOWER END OF WILLOW CUTOFF	
	Distance, in miles	Slope, in ft per mile	Distance, in miles	Slope, in ft per mile
1929	33.62	0.333	14.45	0.339
1933	33.62	0.301	14.45	0.359
1935	30.46	0.334	7.65	0.340
1937	31.98	0.319	7.65	0.315
1939	30.25	0.341	7.38	0.355

FLOOD-STAGE LOWERING

The flood-stage lowering effected in the river immediately upstream from Willow Cutoff in April, 1936 (two years after the cutoff began to function), amounted to nearly 2 ft for about a bankfull flow. Marshall Cutoff similarly caused a reduction of about 1 ft. These data are based on the reduction in fall observed within the cutoffs.

Actually, the total amount of flood-stage lowering in this part of the river is on the order of that observed on the Vicksburg gage—that is, from about 7 ft to 8 ft, varying with stage conditions. The total lowering is transmitted upstream from the five cutoffs below Vicksburg, and represents a cumulative effect to which these cutoffs and various dredging operations have contributed, minus losses caused by the inability of the stream bed to lower itself in places. Upstream from Vicksburg seven cutoffs (including Marshall and Willow), plus channel improvements, increase the stage lowering progressively until on the Arkansas City, Ark., gage their cumulative effect amounts to a decrease of from 12 ft to 13 ft in flood stages as compared to equivalent river discharges in 1933.

Flood-stage reductions have been of inestimable benefit to backwater farmers; they have released millions of dollars for raising levees; and, by dispensing with a floodway on the west side of the river, they have saved a large area of productive lands from needless overflow. Furthermore, floodwater travel has become accelerated. In the reach from Arkansas City to Natchez alone, the time of travel of flood crests has been shortened several days.

The cutoff program was inaugurated in 1932 as part of the work of stabilizing the navigable channel of the Mississippi River. The first artificial cutoff to be opened was at Diamond Point below Vicksburg on January 8, 1933. The cutoff program progressed rapidly, as may be noted from Table 4. By 1939, thirteen river bends, totaling 156.5 miles, had been abandoned between the mouth of the Arkansas River and Baton Rouge. The aggregate length of the thirteen new channels across the necks amounted to 40.7 miles, thus accounting for a net shortening of 115.8 miles, measured at bankfull stage, in what formerly was a 462-mile river distance.

In 1946 there were sixteen cutoffs in operation, three having been constructed in 1941 and 1942 between the mouth of the Arkansas River and Memphis. The sixteen cutoffs together have shortened the Mississippi River 151.9 miles between Memphis and Baton Rouge; but, in addition, other river improvements (notably chute enlargements accelerated by dredging operations) have created further shortenings, decreasing the total length 170 miles between Memphis and Baton Rouge. This value represents the length by which the low-water navigation channel between these two cities has been shortened. In 1932 the river distance between Memphis and Baton Rouge was about 680 miles.

TABLE 4.—CUTOFFS ON THE MISSISSIPPI RIVER BETWEEN MEMPHIS, TENN., AND BATON ROUGE, LA.

(Location and Distances as Determined by Approved Center Line and Scaled Mid-Mean Low Water Mileage)

Name	Opened	Location of upper end (Mile AHP[a])	Distance (in Miles) as of Date Cutoff Was Made			Length of dredged cut (in ft)	Fall Across Neck (In Ft)	
			Across neck	Around bend	Net shortening		At high water 1929	At mean low water[b]
Hardin	March 18, 1942	676.0	1.86	18.82	16.96	5,900	4.0[c]	8.1
Jackson	April 26, 1941	624.4	2.40	11.06	8.66	11,300	4.5[d]	3.6
Sunflower	February 16, 1942	621.9	2.43	12.87	10.44	10,300	4.0[d]	4.3
Caulk	May 13, 1937	568.5	2.0	17.2	15.2	4,400	2.9[e]	5.8[f]
Ashbrook	November 19, 1935	542.0	1.9	13.3	11.4	4,530	5.4	2.8
Tarpley	April 21, 1935	535.0	3.6	12.2	8.6	13,000	1.8	3.4
Leland	July 8, 1933	531.7	1.4	11.2	9.8	....[g]	4.3	3.4
Worthington	December 25, 1933	507.0	3.8	8.1	4.3	17,600	3.7	1.6
Sarah	March 23, 1936	498.2	3.2	8.5	5.3	12,600	2.2	3.0
Willow	April 8, 1934	457.7	4.7	12.4	7.7	22,000	4.0	3.7
Marshall	March 12, 1934	443.9	3.1	7.3	4.2	13,600	2.4	2.2
Diamond	January 8, 1933	420.0	2.6	14.6	12.0	9,175	2.2	4.2
Yucatan	Fall of 1929	404.4	2.6	12.2	9.6	....[g]	3.7	2.8
Rodney	February 29, 1936	385.4	4.1	9.9	5.8	13,000	2.7	2.0
Giles	May 25, 1933	363.8	2.9	14.0	11.1	10,000	4.6	2.8
Glasscock	March 26, 1933	342.1	4.8	15.6	10.8	20,800	3.2	3.4
Total	....	....	47.4	199.2	151.8	168,205	55.6	57.1

[a] "Above Heads of Passes," the starting point for river distances established in 1942. [b] As of date of opening cutoff. [c] At bankfull high water in 1937. [d] At bankfull high water in 1940. [e] In 1937 the high water fall was 4.7 ft. [f] Actual fall on August 28, 1936 was 10.6 ft, at low water. [g] Natural cutoff; no artificial channel was dredged.

In view of the undesirable aftereffects which have repeatedly been experienced at artificial cutoffs made on other rivers, concern has been expressed regarding the wisdom of launching an extensive cutoff program on the Mississippi River. Prior to the beginning of the present development, a study was made of the difficulties and liabilities experienced at cutoffs elsewhere, and a clear understanding was gained regarding the pitfalls to be avoided. In the first place, it became clear that the river should not be straightened unduly. To straighten it deliberately in time would have tended to convert the Mississippi River into a shallow stream 3 or more miles wide, interspersed with many islands and bars separated by interlacing channels. Such a "braided" river would have impaired navigability to the point of forcing its abandonment; yet a perfectly straight channel alinement from Cairo to the Gulf of Mexico has not been without its proponents.

The conception also existed that water flowing through a cutoff raised stages downstream by "piling-up." Instances were cited where this had been actually witnessed in early days when the river broke through a narrow neck of land during a high stage. It was important, therefore, to avoid inviting these "piling-up" effects and this result was readily accomplished through the adoption of the pilot-cut principle as previously explained.

Another "bête-noire" in cutoff procedure—namely, loss of valley storage—had to be avoided as much as possible. The retirement of a series of large river bends by cutoffs in time obviously would rob the river of millions of acre-feet of useful valley storage, and would tend to raise flood stages at points downstream. This problem was solved by leaving the retired bend-way channels wide open—thus permitting floodwaters to enter them, flow around the old bends, and flood the land back to the levees as in the past. The preservation of this valley storage, although diminished somewhat by a reduction in flood stages, proved to be a wise forethought on the part of General Ferguson. The valley storage remaining available still is vast and constitutes an important asset in the general plan. At cutoffs made as early as 1933 or 1934, which have since become full-size Mississippi River channels, water still continues to flow through the abandoned bend-way channels during high stages and may continue to do so for an indefinite number of years.

SUMMARY

The cutoff history of the 50-mile reach described was selected as the topic of the paper because it brings out strikingly the vast difference between the art of cutoff procedure as originated by General Ferguson, and nature's unrestrained art. It is the difference between a conservative, gradual process, at all times under complete control, and a cataclysmic way of producing a new channel in a matter of days.

Marshall and Willow cutoffs are examples of safe engineering practice applicable to large meandering rivers, but furnish no criterion for problems of this kind in small streams. The success and simplicity of the work at Marshall and Willow cutoffs must not be taken, however, as representative of all Mississippi River cutoff operations. In many instances, adverse physical conditions—some of them natural, others man made—called for considerable departure from the methods and principles employed at Marshall and Willow, and in some cases these departures affected, in large part, the cutoff project as a whole.

The original plan contemplated opening the principal cutoffs downstream from Vicksburg ahead of those planned at points upstream. This was for the purpose of developing, in the lower river, adequate capacity for transporting the increased sediment load that was expected to result from upstream operations. The plan of procedure has remained fundamental, but it was marred in a few instances by unforeseeable events. Thus, Leland Cutoff, at the foot of the famous Greenville Bends, had to be constructed more than a year ahead of schedule because of an unexpected break in Leland Neck during the summer of 1933. Adventitious as regards both time and place, Leland Cutoff forced a complete revision to be made of the original rectification plan for the Greenville Bends. Moreover, Leland Cutoff had severed an extremely narrow neck, which was contrary to the principles laid down. In the revised plan it became necessary to place a cutoff at the head of the Greenville Bends across Ashbrook Neck—another perilously slender neck. Complex problems have resulted from this situation which in turn have called for large expenditures in dredging and revetment work. Another cutoff that had to be constructed a year ahead of schedule was located across Worthington Point. Here, a rich agricultural area on the east bank had demanded relief from the rapid bank caving in Kentucky Bend, and hope had been entertained that the cutoff might alleviate this threat by causing the river to shift its main channel over to the west bank.

Contrary to plan, also, was the exceedingly slow development of the cutoff farthest downstream—Glasscock, the second artificial cutoff to be opened. In 1938, 5 years after it was opened, despite repeated dredging, Glasscock Cutoff was carrying only one third of the total river discharge. Resistance to erosion on the part of the thick mantle of claylike alluvium which prevails in the southern part of the valley was responsible for retarding its development. Agreeably contrary to expectation, however, little or none of the sand liberated at upstream cutoffs and at other rectification operations appears to have affected Glasscock Cutoff. It is presumed that this sand, in large part, was deposited by the river in the abandoned bends which purposely had been left open to serve as sand depositories.

In conclusion it is desired to emphasize that the principal objective of the cutoff program has been to lower flood stages, and that the lowerings that have been so effected have fully met the original estimates made in 1930. In future years, when river bends lengthen into new loops, additional cutoffs will doubtless be required to maintain the flood-stage lowerings effective as of 1946. The rate at which such new cutoffs will be needed of course will depend upon the rate at which river lengthening takes place, and cannot be forecast reliably. The 1944 plan[3] of stabilizing riverbanks by applying suitable revetments to resist bank caving, should be of material assistance in postponing the time when additional cutoffs will be needed in the stretch of river shortened by the sixteen cutoffs listed in Table 4.

ACKNOWLEDGMENT

The photographs in this paper were taken by the U. S. Army Air Corps and were made available, for use in this paper, by the Mississippi River Commission.

---

[3] "New Project for Stabilizing and Deepening Lower Mississippi River," by Charles Senour, *Transactions*, ASCE, Vol. 112, 1947, p. 277.

DISCUSSION

W. E. Elam,[4] M. ASCE.—As complete a summary of the Mississippi River cutoffs as could be desired is contained in this paper. However, the political situation involved was not revealed as much as merited.

Two great needs brought about this development in flood control plans. Protection from floods, beginning at New Orleans, La., about 1717 and gradually working northward, was the first factor and was undertaken solely by state and local interests. Flood control remained a local responsibility until about 1917, when the first Flood Control Act was passed by Congress and the federal government accepted about 50% of the burden of levee construction. The great flood of 1927 brought the realization that flood control was a national problem and responsibility.

Navigation is the second factor and had its inception in 1879, when the Mississippi River Commission was set up by Congress to foster and develop navigation. Flood control was not considered to be a function of the Commission. Local interests were left with this part of the problem; they willingly accepted the responsibility until recurring disasters demonstrated that the federal government had to solve the problem or this great alluvial valley would be destroyed as a national asset.

The effort by two agencies, with different objectives, resulted in confusion and failure to attain flood protection. Navigation interests caused the Mississippi River Commission to be established and probably were opposed to cutoffs, believing that lengthening the river would produce a sluggish river, which would best serve the needs of navigation. This could have been the reason the Commission adopted the policy of preventing cutoffs. Reports and papers supporting this policy have added confusion to an already complicated problem. Even the official reports after the 1927 flood specifically rejected cutoffs as aids to flood control. Local authorities followed the lead of the official reports and also condemned cutoffs. Too much credit cannot be given Maj.-Gen. Lytle Brown, M. ASCE, formerly Chief of Engineers, and Maj.-Gen. H. B. Ferguson, M. ASCE, formerly President of the Mississippi River Commission, for adding cutoffs to the flood control plan when the entire valley officially opposed their use. This one feature made flood control possible.

Only one small part of the problem was covered in the discussion of the Greenville Bends[2] mentioned by Mr. Matthes. However, the conclusions in that discussion broadened the scope of the paper considerably. An article in *Engineering News-Record*[5] went into the matter more fully and considered the problem of the entire river from Cairo, Ill., to the Gulf of Mexico. This same subject is also covered in a book.[6]

The conclusion that the river tends to return to a certain length may not be strictly true. The function of the river is to carry water to the sea. The Mississippi has always brought along soil, sand, and gravel which fills the alluvial valley to a gradual slope from the upper end to the Gulf of Mexico. This material is shoved out of the main channel or deposited in slack water to form bars, generally on the concave side of bends. Bars thus formed on the points cause the caving of opposite banks. Some of these caving bends above and below a point tend to approach each other and eventually to produce a natural cutoff. As Mr. Matthes has stated, some of these proved disastrous. However, if these same cutoffs had been laid out intelligently, they could have been beneficial.

---

[4] Chf. Engr., Mississippi Levee Dist., Greenville, Miss.

[5] "Flood Control Through Slope Correction," by W. E. Elam, *Engineering News-Record*, June 28, 1928, p. 996.

"Speeding Floods to the Sea," by W. E. Elam, Hobson Book Press, New York, N. Y., 1946.

There is the possibility that man could well select a length of river that would produce a maximum flood control benefit with a minimum interference with navigation. Data to do this are probably in existence. Recognition should more often be made of the facts that the Mississippi River carries water, silt, sand, and gravel and that all these move irresistibly together and all must be taken into account if the caving bank situation is to be mastered. With the mass of reports and writings about the Mississippi, framed about the old theory that cutoffs would surely breed disaster, it is little wonder that much confusion still exists and progress is being halted by old theories based on erroneous assumptions. The Army Engineers and the Mississippi River Commission probably will eventually clarify this situation since these groups seem best qualified to work out a balanced long time program. They are on the right track, but many details yet remain to be fitted into the picture.

Mr. Matthes' estimate of the lowering of flood stages due to the cutoffs is about as accurate as any. There is no way to estimate the effect accurately. Flow conditions have been so altered that even the rates of runoff and of discharge are not comparable. What many engineers call "valley storage" is a part of the discharge area where the velocity is checked in the wide overbank channel. Valley storage is beneficial. Storage in the backwater areas is detrimental. The vast volume of water withdrawn into these areas in advance of the maximum discharge merely served to hold the river stationary at these points longer. If another flood formed in the valley in a week or ten days, as frequently happened, higher stages resulted than would have if the water had been kept out of the backwater areas.

The river seems to have done a much better job than expected in transporting and moving the detritus out of the channel. It is possible that the increase in velocity kept much of this material from dropping in unwanted places. It is noted that, in certain reaches, such as Sessions, or Francis (mile 604 to 612 "Above Heads of Passes"), there is a tendency for an excess of detritus to clog the channel and to cause excessive caving. Probably the best treatment for this condition would be to shorten the river, where possible, at these points. Thus, the carrying capacity of the river should become more uniform and at the same time the old channel should absorb a lot of the excess detritus. It may be that more shortening has been effected than is necessary; but, as Mr. Matthes states, gradual lengthening rapidly will compensate for this condition. It is far better this way.

C. L. Hall,[7] M. ASCE.—The reasons that led to creation of artificial cutoffs on the Lower Mississippi River are clearly stated in this paper, which also well describes the methods of accomplishing these channel changes. There seems to be nothing to criticize in the paper—as far as it goes.

---

[7] Col., Corps of Engrs., Beach Erosion Board, Washington, D. C.

Mr. Matthes states (see heading, "River Steepening") that "it became clear that the river should not be straightened unduly" and also that "a 'braided' river would have impaired navigability." Cutoffs are essentially instruments for straightening the river, although perhaps not unduly. Obviously, however, the tendency of the actions taken is toward "braiding" and the consequent impairment to navigability must be resisted by increased maintenance. Since the beneficial effect of the reduction of floods caused by the cutoffs is purely economic in character, there is no way of determining the net benefit except by comparing the value of the flood reduction secured with the additional costs of maintenance. Any flood control works made unnecessary, or any maintenance costs reduced in amount by the new scheme, constitute, of course, part of the value of flood reduction.

A judgment on the utility of the plan therefore requires answers to three questions:

(1) How much are costs of channel maintenance increased by cutoffs?
(2) What flood control works are made unnecessary by the plan?
(3) To what extent is levee maintenance reduced?

There is nothing in the paper to show that these questions have even been propounded: There is certainly no sign of an answer to them. Until the questions are answered, it is probable that the members of the profession will reserve opinion on the question of cutoffs.

H. D. Vogel,[8] **M. ASCE.**—Several reasons have been outlined for the earlier, uncompromising attitude of the Mississippi River Commission with respect to cutoffs. Principal among these was the fear engendered by observations on cutoffs that had occurred naturally during high water periods. However, it should also be recalled that, until the Flood Control Act of 1928, funds appropriated by Congress were too limited and restricted as to use to do the research and the investigational and experimental work necessary to find a solution to the problem. The only money available in any amount was that appropriated for the benefit of navigation; nevertheless, the existing levees had to be protected against incursions by the river at its bends. Revetments could be placed as aids to navigation; hence, revetments were placed and bends were held.

Plans for the United States Waterways Experiment Station were made in the summer of 1929 at Memphis, Tenn. By December of that year it was decided to build the station in Vicksburg, Miss. Early in December, as director-elect of the station, the writer visited Yucatan Point with the late Gen. T. H. Jackson, M. ASCE, then President of the Mississippi River Commission, who decided to desist from attempts to prevent the upper bend of the Mississippi River from cutting through into the channel of the Big Black River. Such a cutoff would develop slowly and would not endanger any levees. By permitting the break-through to occur, the Big Black River would act as a natural pilot channel across the neck of Yucatan Point to the lower bend. The cutoff of the Mississippi River into the Big Black River occured between December 25, 1929, and January 1, 1930. This, the first cutoff on the Mississippi River deliberately made by man (for the decision to withhold action against it constituted action for it), occurred quietly and without attendant catastrophe.

---

[8] Col., Corps of Engrs., U. S. Army, Washington, D.C.

Encouraged by observations on the Yucatan Cutoff, General Jackson ordered that a model of the Greenville Bends be built and tested by the U. S. Experiment Station as one of its first two studies. The Greenville Bends model was built during the summer of 1930 and operated throughout the succeeding fall and winter.

Results of tests with this model are indicated in the following excerpt from an article in *Engineering News-Record:*[9]

> "The [Greenville Bends] model showed a maximum lowering of 2.2 ft. due to a cutoff at Tarpley Neck, the influence being felt for a distance of 45 miles above with no change below. This was in almost direct contradiction to the old theory, offered first in this country by Humphreys and Abbott, that stages are increased below a cutoff by the same degree they are decreased above.
>
> "The model gave no indication of detrimental effects due to cutoff, but on the other hand showed a slight tendency toward improved conditions at Leland Neck. The reduced head above Leland Dike, produced by a cutoff at Tarpley, relieved the strain on that structure to some degree and resulted in lowered velocities through it. It is to be expected that Chicot Point will will be severely attacked by increased currents through the cutoff; but if erosion occurs, it will be along the line of a shallow chute now existent, and revetment of the east bank will resist attack. Erosional adjustments that follow will produce a sinuous channel in due course of time."

Encouraged by results of the experiments with models of the Greenville Bends, General Jackson began plans for an artificial cutoff across Diamond Point; and comprehensive model tests of an extensive river reach were ordered in the spring of 1932. Actual field work on the cutoff was begun during the following summer, shortly before General Jackson was relieved by Maj.-Gen. Harley B. Ferguson, M. ASCE, as President of the Mississippi River Commission. General Ferguson pushed the work on Diamond Point with such energy that the final break-through was effected on January 8, 1933.

General Ferguson, also, immediately upon taking office as President of the Mississippi River Commission, ordered comprehensive studies directed toward the creation of a whole series of cutoffs throughout the length of the lower river. Mr. Matthes, who came to Vicksburg with General Ferguson, has told the story from this point.

In commenting upon the work begun by General Jackson, there is no desire or intent to detract from the accomplishments of General Ferguson. The final result—control of the Mississippi—is important enough to provide undying glory for both. General Jackson pioneered the way, overcoming prejudices and fears of long standing; General Ferguson, with indomitable courage, applied the lessons developed by General Jackson's tests to the river as a whole. Mr.

---

[9] "Application of Model Research to Mississippi Flood Problems," by Herbert D. Vogel, *Engineering News-Record*, July 16, 1931, p. 87.

Matthes, in characteristic fashion, has declined to describe his own role in this great engineering development. It was he, however, who stood always beside General Ferguson when he was criticized for going too far and too fast. It was he who interpreted many of the laboratory results and put them to practical application. His present paper comprises a valuable contribution both to engineering knowledge and to the Mississippi River history.

Harry N. Pharr,[10] M. ASCE.—A statement of the purpose and a clear description of the effect of the cutoffs made on the Mississippi River are presented in this paper.

The conception that the effect of a cutoff is to raise the stage downstream arises from the fact that, as the stage rises in a stream of varying slopes, the water surface tends toward a uniform slope. Until the cutoff in its process of development regulates the disarranged alinement, slope, and section of the stream channel, the velocity of the current immediately below the cutoff is somewhat retarded, the impetus of the stream is diminished, and the tendency there is to increase the stage. This effect extends a short distance downstream to about the point where the regular alinement, slope, and section of the stream are reached. In a series of cutoffs within the influence of each other this result is modified except at the cutoff farthest downstream. The effect of the decrease in valley storage, because of stage reduction resulting from the cutoff, on the increase in stream discharge below the cutoff diminishes with the usual occurrence of slight daily rises and increase in duration of stage near the flood crest.

Immediately above its location the effect of a cutoff is to increase the velocity of the stream, to lower the stage, and, in some situations, to increase the tendency to cave the concave bank. Below bankfull stage the action of drawing the current away from the concave bank and across the toe of the opposite point bar is not effective in some situations beyond the influence of the cutoff, as at Hardin Cutoff (676 Above Heads of Passes), made in 1942. In Walnut Bend, the accelerated rate of caving along the concave bank necessitated installation of bank revetment, as the current was not materially deflected across the opposite point bar despite occasional dredging in an effort to assist in accomplishing that result.

James B. Miles was among the early advocates of cutoffs to reduce flood heights in the Lower Mississippi River. Following the disastrous flood of 1897, he proposed a plan to the Nelson Subcommittee of the Committee on Commerce of the United States Senate which made an examination, inquiry, and report on the problems of flood control and improvement of the Mississippi River.[11] Mr. Miles stated:

> "I find there are 26 places at which cutoffs could be made. Of these about 16 are such as could be made at but little cost or time * * *. The distance across these necks or narrow places is from one to three miles, except one of five miles. The cost would be in digging a ditch or canal across these necks. Some would require large and some small ditches * * *. The ditch or canal should be dug so as to turn the current of the river above as

---

[10] Civ. Engr., Member, Mississippi River Comm., War Dept., Memphis, Tenn. (Mr. Pharr died on November 3, 1947.)

[11] *Senate Report No. 1433*, 55th Cong., 3d Session, December 15, 1898, pp. 296–301.

> directly as possible into the channel or current below. About all the cutoffs have been made where the neck was narrowest. This, in most instances, has thrown the current against the bank on the opposite side so as to cause great destruction. * * * I do not claim that the river can be made straight but I claim that the distance can be reduced * * *. The distance from Cairo to the Gulf could be shortened by cutoffs some 170 miles. * * * The cutoffs between Arkansas River (mouth) and Greenville should lower the water at the Arkansas nearly or quite 10 feet."

The plan which Mr. Miles advocated is somewhat similar to the plan executed. He was not a professional engineer, but he was a student of the river. It is remarkable that, 50 years ago, he selected the exact number of cutoffs and estimated both the exact mileage reduction in length of river and about the amount of reduction in stage as that of the plan executed.

To the reasons given by the author as to why natural cutoffs were not permitted to occur during the period from 1884 to 1929 should be added the fact that, previous to the first Flood Control Act of 1917, the appropriations by Congress of funds for the improvement of the Mississippi River were generally restricted to works in aid of navigation; works solely for the purpose of flood protection were prohibited. Levees had been constructed as auxiliaries to the plan of channel improvement. This resulted in the adoption of a policy of stabilization rather than a change of channel location, and any benefit to flood control resulting from works designed to aid navigation was considered incidental. Since 1917 Congress has authorized works in aid of flood control as well as works for the improvement of navigation. The principal objective of the cutoffs is the control of floods and any resulting improvement aiding navigation is usually incidental.

LYTLE BROWN,[12] M. ASCE.—The idea of making artificial cutoffs on the Mississippi River is an old one and has been put forward in the past by various people, mostly by those who had but little experience with the river.

In 1859, two officers of the Corps of Engineers, U. S. Army (Capt. A. A. Humphreys, Hon. M. ASCE, and Lt. Henry L. Abbot) made the first comprehensive study of the lower river. These two officers were among the most distinguished that the Corps of Engineers has ever furnished to the United States. Their massive report for many years was regarded as a standard in river hydraulics.[13] It contains an elaborate discussion of cutoffs based on both analysis and on the observation of natural cutoffs. The conclusion reached was unfavorable to cutoffs, but was cautiously stated, and ended with the remark that further discussion was not necessary since it was not practicable to make artificial cutoffs. At that time it was not practicable because the dragline on the hydraulic dredge had not then been developed. Gen. U. S. Grant tested the practicability of this theory opposite Vicksburg, Miss., in 1863.

The fear of injuring navigation by either artificial or natural cutoffs in 1859, when navigation was of supreme importance, made experiments with cutoffs a risky business. Experience with natural cutoffs and the authoritative report of Messrs. Humphreys and Abbot made a policy against cutoffs unshakable until 1933.

---

[12] Maj. Gen., U. S. Army (Retired), Franklin, Tenn.

[13] "Report on the Physics and Hydraulics of the Mississippi River," by A. A. Humphreys and Henry L. Abbot, *Professional Paper No. 13*, Corps of Engrs., U. S. Army, 1876.

The plan for flood control on the Mississippi following the flood of 1927 (sometimes called the Jadwin Plan) contemplated no cutoffs; and, since that plan received the sanction of Congress, radical departures from it were frought with danger.

From 1929 until 1933 there existed among some of the authorities on the river an attitude of hostility toward anything that might in any way alter the Jadwin Plan. This attitude tended toward discouraging further study of new attacks on the problem of flood control. Col. (later Maj.-Gen.) Harley B. Ferguson, M. ASCE, an officer well known to the writer (then Chief of Engineers, U. S. Army) as an able engineer willing and capable of developing a new attack on the flood control problem or other hydraulic problems, was accordingly appointed President of The Mississippi River Commission in the hope that he would not be crippled by adherence to any hard and fast ideas. General Ferguson was the first and only responsible man who ever brought to the Chief of Engineers the serious proposition to make artificial cutoffs on the Mississippi River. Whatever credit is due for a courageous effort to lower the height of floods on the confined waters of the Mississippi is due to Maj.-Gen. Harley B. Ferguson. There are many who project an idea where danger is involved but there are few with the courage to give it effect and to assume the responsibility. According to the writer's knowledge, Mr. Matthes' paper cannot be questioned in any important particular.

ANSON MARSTON,[14] PAST-PRESIDENT AND HON. M. ASCE.—The writer is keenly interested in this paper on "Mississippi River Cutoffs." Early in 1932, Maj-Gen. Lytle Brown, M. ASCE, then Chief of Engineers, appointed a three-man Mississippi River Engineering Board of Review to restudy and to report upon the "Jadwin Plan" for Mississippi River flood control, which had been adopted by Congress after the great 1927 flood. The Board members were the then Brig-Gen. Harley B. Ferguson, M. ASCE, President; the then Col. George B. Spalding, U. S. Division Engineer at St. Louis, Mo.; and the writer, as a civilian engineer.

It should be emphasized that General Ferguson is entitled to full credit for originating in 1932, and later as President of the Mississippi River Commission for constructing, the extensive and highly successful cutoff program so well described by Mr. Matthes. That the previous Mississippi River Commission policy and much engineering professional opinion had been hostile to cutoffs was demonstrated in the 1929 ASCE Symposium[15] on flood control. Therein Col. Edward H. Schulz, M. ASCE, United States Corps of Engineers, long engaged in Mississippi flood control work, stated: "Cutoffs * * * would be a detriment to the river and are decidedly inadvisable." This had been the long continued policy of the Mississippi River Commission and was supported by prevailing engineering opinion.

John F. Coleman, Past-President and Hon. M. ASCE, stated:

---

[14] Dean Emeritus of Eng., Iowa State College, Ames, Iowa.

[15] *Transactions*, ASCE, Vol. 93, 1929, pp. 655–969.

"For the reasons stated, the writer is of the earnest conviction that levees, with auxiliary works for the protection of banks from caving, will afford the only certain method of controlling the floods of the Mississippi River."

However, in a preceding discussion,[16] Mr. Coleman indicated an open mind on trying out cutoffs "gradually," and "with caution." W. E. Elam, M. ASCE, assistant engineer, Mississippi Levee Board, and a few others advocated trying out cutoffs, but it remained for General Ferguson to begin in 1932 to design, and in 1933 to construct, the cutoffs shown in Fig. 1.

One of the early acts of General Brown's Board of Review in 1932 was to ask Brig.-Gen. T. H. Jackson, M. ASCE, then President of the Mississippi River Commission, for suggestions concerning the work. He said nothing to the Board to indicate that he was in favor of making any cutoffs to improve the river. The Yucatan Cutoff (1929) into the Black River Channel has been mentioned (see heading, "Historical") as demonstrating how a properly designed cutoff could improve the river. However, it appears that General Jackson "desisted" from efforts to close the break only after efforts to do so by willow mats failed.

By that time it had become evident that there was danger that the river would break also through the Ashbrook and Leland Greenville Necks (Miss.) in spite of the Commission's costly preventive dikes. In 1933, the river did break the Leland Neck; the writer rode through the break on a government launch down a deep and wide raging torrent within less than 48 hours after it happened.

Construction of the model of the Greenville bends was a wise move, so successful that before the Board of Review concluded its work in 1933 the model was extended to include the whole distance from the Arkansas River to Old River. However, getting ready to quit losing fights to prevent Greenville bend cutoffs was quite different from starting a planned program to improve the river by making many cutoffs. In 1933, it was the writer's privilege during a conference in Washington, D. C., with General Brown, to advise him to approve General Ferguson's plans for the other Greenville bend cutoffs, shown in Fig. 1.

The Board of Review concluded its work in 1933, by which time the cutoff program described in the paper was well under way. Evidently its success had been all that was forecast. The advantage to navigation of shortening the river 170 miles between Memphis (Tenn.) and Baton Rouge (La.) is obvious. It also is obvious that the cutoffs have materially improved the flood-carrying capacity of the river in that reach, although the exact extent is not obvious.

Also, at each cutoff two channels—the cutoff and the still open channel around the old bend—are now available to carry the flow. The advantage of having the bends open to get rid of silt is obvious, but in the past such bends eventually have silted full, or at least have become blocked off from the river by silt. It would be of interest for the author to add some discussion of this matter to his valuable paper. Are there any actual measurements proving continuing scour of the river's bed in the reaches north of Old River?

---

[16] *Engineering News-Record*, June 28, 1928.

There are some other questions on which the author doubtless can give valuable fact or opinion information. Table 1 shows that the greatest flood to June, 1945, in the Mississippi River reaches from Arkansas City (Ark.) to Vicksburg (Miss.) since the cutoffs therein were made occurred in 1937, and was about 2,000,000 cu ft per sec. It is stated[17] that the 1927 maximum flood flow at Arkansas City was 2,662,000 cu ft per sec, and a still larger future flood is possible. It would be of great interest to readers if the author would add information on the following three points:

1. What, in the author's judgment, is the present maximum flood flow in the Arkansas City-Vicksburg Mississippi River channel reaches which will not dangerously decrease the adjacent levee freeboards?
2. When the Board of Review concluded its work in the fall of 1933 it was of the opinion that, in addition to the completion of the cutoff program, an adequate floodway west of the river from below Arkansas City to Old River would be necessary to insure that part of the Mississippi Valley from flood damage by a future "super flood" of the magnitude then thought possible; it was hoped that the future developments of reservoirs above Arkansas City on the Mississippi River and its tributaries (including Arkansas River) might in time have some measurable value in reducing floods. What is the author's current opinion on this matter?
3. Fig. 1 indicates a well-developed Atchafalaya River channel from Old River to Grand Lake, although in 1932 such channel existed only part way south of Old River. Does this mean that General Ferguson's big dredges have succeeded in extending such channel clear to Grand Lake?

In conclusion, the writer wishes to say that the author is entitled to much credit for his own indispensable services in helping to plan and develop the entire admirable Mississippi River cutoff program described in the paper.

E. J. Williams, Jr.,[18] and Eugene A. Graves,[19] Assoc. Members, ASCE. —An insight into the planning and execution of the cutoffs on the Lower Mississippi River, the results as far as high-water stage reductions are concerned, the technique employed, and the pitfalls to be avoided are all presented in this interesting paper. The main objective of the cutoffs, as stated by the author, was to lower flood heights. In this they have been highly successful. Effects on low-water flow have been slight, but perhaps warrant consideration.

The low-water profile (Fig. 7) is composed of a series of alternately steep and flat slopes, which become more accentuated at lower stages. Steep, low-water slopes generally occur at the foot of shallow crossings, whereas pools between the crossings are generally so deep that very little slope is required to handle the low-water flow. The least depth available varies greatly from year to year, and in a random manner. This is more or less the nature of the river, which normally builds up its crossings during periods of high water and scours them during falling stages and low water in a varying degree, depending on a number of conditions cataloged hereafter. The slopes at the crossings are variable as they depend on: How much the crossings were built up during high stages; the rapidity of the fall; the length of crossing; the toughness of material on the bottom; the amount of dredging; and, possibly, other factors.

---

[17] *Transactions*, ASCE, Vol. 93, 1929, p. 707 and p. 874.

[18] Engr., Mississippi River Comm., Vicksburg, Miss.

[19] Engr., Mississippi River Comm., Vicksburg, Miss.

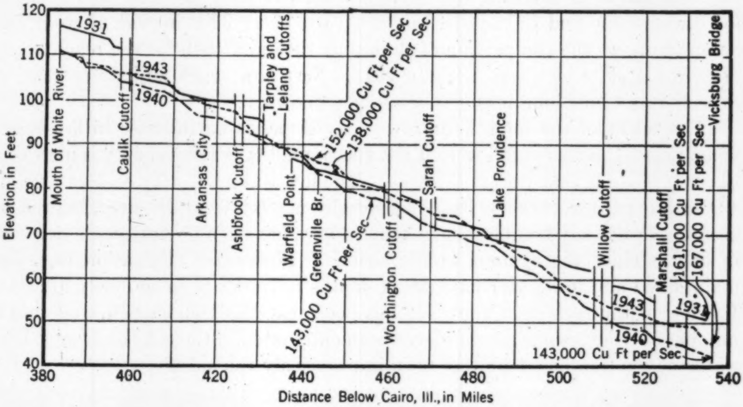


FIG. 7.—COMPARISON OF LOW-WATER PROFILES BEFORE AND AFTER CUTOFFS ON MISSISSIPPI RIVER; MILEAGE BASED ON 1937 SURVEY, BUT MEASURED THROUGH ALL CUTOFFS

The easily erodible crossings are lowered first and a large part of the fall transferred to the ones which do not erode so easily, thus making dredging necessary to assist the natural action of the river at these points. During the low-water periods immediately following the creation of each cutoff, it was necessary to perform more than the normal amount of dredging at the crossings in the upstream reach affected by the cutoff. The excess dredging requirements

TABLE 5.—LOW-WATER SLOPES OBSERVED BEFORE AND AFTER CUTOFFS

Cutoff designation	Opening date	Distance (mile)[a]	FALL (FEET)				SLOPE (FOOT PER MILE)				DISCHARGE[b] (1,000 CU FT PER SEC)			
			Oct. 2, 1931	Dec. 21, 1937	Oct. 18, 1940	Nov. 11, 1944	Oct. 2, 1931	Dec. 21, 1937	Oct. 18, 1940	Nov. 11, 1944	Oct. 2, 1931	Dec. 21, 1937	Oct. 18, 1940	Nov. 11, 1944
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
Marshall....	3-12-34	8.4	3.5	3.2	1.8	2.7	0.42	0.38	0.21	0.32	138	135	121	178
Willow......	4- 8-34	37.6	11.7	19.6	18.8	19.8	0.31	0.52	0.50	0.53	138	130	121	177
Sarah.......	3-23-36	4.9	1.8	4.0	2.9	2.3	0.37	0.82	0.59	0.47	138	130	115	179
Worthington.	12-25-33	23.2	6.7	6.2	11.5	9.8	0.29	0.27	0.50	0.42	138	130	115	180
Tarpley.....	4-21-35	4.4	1.0	2.7	1.3	3.2	0.23	0.61	0.30	0.73	138	127	112	181
Ashbrook....	11-19-35	24.2	5.9	9.4	8.7	7.5	0.24	0.39	0.36	0.31	138	127	112	182
Caulk.......	5-13-37	35.0	8.6	....	11.6	12.1	0.25	....	0.33	0.35	138	....	110	182
Average.....	....	....	....	....	....	....	(0.30)	(0.50)	(0.40)	(0.45)	....	....	....	....

[a] Distance from head of cutoff to foot of next cutoff above, measured by 1930 low-water mileage for the 1931 conditions and by 1940 low-water mileage for the 1937 and 1944 conditions. [b] Approximate discharge.

reduced rapidly from year to year, however, as the local drawdown occasioned by the cutoff was gradually distributed over a considerable reach of the river.

Table 5 shows, by comparison, the changes in slopes in the reaches immediately above the heads of the cutoffs. The slopes above the cutoffs have naturally retained some increase, but there is a tendency for the river to adjust the low-water slope and to improve the steep reaches. This is evident from Fig. 8, which shows the comparison of slopes for 1944 and 1946 for the reach above Hardin Cutoff, which was made in 1942. Just how much adjustment can be expected and how long it will take are not known.

The result of the increase in low-water slopes is an increase in low-water velocities. It does not follow that the maximum low-water velocity is increased but it is possible that such is the case. The low-water velocity seldom exceeds 4 ft or 5 ft per sec, however, and is generally much lower; also dredging can be used to correct any troublesome reach. There has been no noticeable decrease in depths since the opening of the cutoffs. Low-water navigation has been maintained and low-water navigation depths have not been seriously affected. In view of the facts that the river has been materially shortened, and that a number of troublesome reaches have been eliminated in the old bend ways, low-water navigation has definitely been improved by the cutoffs.

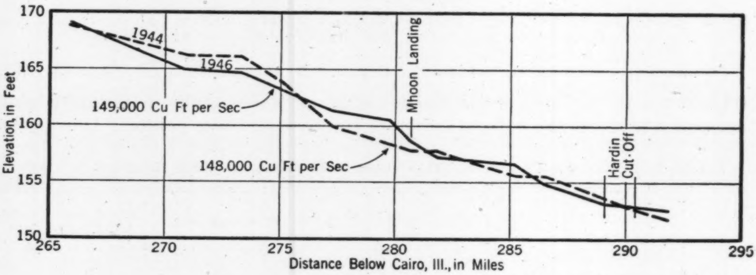


Fig. 8.—Low-Water Profiles Above Hardin Cutoff to Show Slope Adjustment After Cutoff Development

Loss of valley storage tends to increase flood flows below the cutoffs, as stated by Mr. Matthes. Leaving the old bend ways open saves only a small part of the storage, as lowerings in the backwaters, where large areas have been unwatered, have reduced the available flood storage by a considerable amount. Computations indicate that there is some increase in peak flow as a result of the cutoffs. However, cutoffs below the backwater areas combined with other developments, such as the increased capacity of the Atchafalaya River, in Louisiana, have more than compensated for the increased flow and no resultant stage increase is apparent.

In time of high water the greater depth provides over the wide crossings a large and efficient cross section for flow. The high-water slopes are more uniform than the low-water slopes, being largely independent of those channel features which predominate during low water. For this reason, during high water the lowering effect may extend a considerable distance upstream immediately after the development of the cutoff. To show any progressive changes in high-water slopes, a base year preceding the opening of each cutoff was selected and the slope for this year has been compared with the slopes for years after the opening of the cutoffs.

High-water slope data, in the form of increments of slope over the slope for the base year, are summarized in Table 6 according to the number of years intervening between the opening of the cutoff and the year in question. It will be noted that the increments shown are, in each case, the total increase since the base year and not the annual increase. Practical difficulties have precluded obtaining strictly comparable slope data over a long period of years. This has resulted in the omission of a number of cutoffs from the tabulation, which shows that the average increase in high-water slope resulting from the cutoffs is about 0.1 ft per mile. The average slope for these reaches before the cutoffs was about 0.3 ft per mile and after the cutoffs, about 0.4 ft per mile.

TABLE 6.—INCREASE IN HIGH-WATER SLOPES IN REACH ABOVE EACH CUTOFF FOLLOWING OPENING OF CUTOFF[a]

Cutoff designation	NUMBER OF YEARS FOLLOWING OPENING OF CUTOFF											
	1	2	3	4	5	6	7	8	9	10	11	12
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
Hardin	0.18	0.17	0.11	....	....	....	....	....	....	....	....	....
Jackson	0.04	0.03	0.04	....	....	....	....	....	....	....	....	....
Caulk	....	0.10	....	....	....	0.12	0.11	0.05	....	....	....	....
Ashbrook	0.14	....	0.14	....	....	....	0.19	0.21	0.21	....	....	....
Worthington	....	....	0.04	....	0.07	....	....	....	0.09	0.15	0.12	....
Willow	....	....	0.00	....	0.04	....	....	....	0.09	0.06	0.04	....
Diamond	....	....	....	0.10	....	0.09	....	....	....	0.12	0.11	0.15
Rodney	0.06	....	0.08	....	....	....	0.09	0.07	0.05	....	....	....
Giles	....	....	....	0.05	....	0.08	....	....	....	0.11	0.07	0.07
Average[b]	(0.10)	(0.10)	(0.07)	(0.08)	(0.06)	(0.10)	(0.13)	(0.11)	(0.11)	(0.11)	(0.08)	(0.11)

[a] Increment in slope, in feet per mile, for year shown over slope for base year preceding opening of the particular cutoff. [b] Average total increase to year shown: Average total increase for 12-yr period, 0.096 ft per mile.

GERARD H. MATTHES,[20] **HON. M. ASCE.**—The discussions of this paper have served to widen the scope of the original subject matter and have been valuable in focusing attention on certain important phases of the cutoff problems not covered in the paper, as well as in calling for additional information.

Early advocates of cutoffs appear to have been numerous; among them were laymen who had become thoroughly acquainted with the Mississippi River and its modus operandi. The late Mr. Pharr, in his discussion, called attention to the plan proposed by James B. Miles in 1897. It showed a remarkably clear conception of what might be accomplished by making twenty-six cutoffs. An even earlier proposal, noteworthy for revealing the problems and trends of thought of that time, appeared in the *Weekly Vicksburg Whig* of December 28,

---

[20] New York, N. Y.; formerly Head Engr., Mississippi River Comm., and Director, U. S. Waterways Experiment Station, Vicksburg, Miss.

1859. It evidently was prompted by the occurrences of the great flood of 1858, which was followed by two more floods in 1859. It is quoted in full:

> "Cut-Offs—To the Members of the Mississippi Legislature—Gentlemen: —The subject of a general system of cutoffs on the Mississippi River, is rapidly gaining interest throughout the whole swamp country. In making laws upon this subject, the great object is to adopt a general system and not legislate for any special locality. The statute of Mississippi making it a penal offense to make cutoffs would be advocated upon the ground, that an isolated cutoff disconnected with the general system will do a great injury to the planters on either bank of the river—immediately below—nine-tenths of swamp planters are perfectly willing for such a law to remain on the statute books of both Mississippi and Louisiana—provided the legislatures of the two states make laws for a general system of cutoffs to commence below, and made whenever practicable along the river throughout the two states. I defy anyone to bring forward the names of twenty respectable swamp planters who would oppose such a measure. CATO
>
> Madison Parish, Louisiana, Dec. 19, 1859."

These two proposals show that the fruitlessness of making local changes in the river's course had been recognized early. The necessity for making continuous any corrective treatment by channel shortening was obvious. However sound, and however well accepted by many engineers, the magnitude of any comprehensive plan required deferment because of a lack of suitable dredging equipment and the great risks involved. Moreover, as pointed out by the late Mr. Pharr and by Major-General Brown, prior to 1917 federal appropriations were restricted to works in aid of navigation; flood control by use of cutoffs and levees had not then been recognized as a proper federal function. It seems appropriate to mention here that the first powerful dredge built specifically for use on the Mississippi River was designed by a member of the Mississippi River Commission, the late Henry Flad, Past-President, ASCE, who personally supervised its construction (1891–1892).[21]

Dimming recollections of the events that took place fifteen years ago apparently have caused conflicting statements to creep into some of the discussions. For the sake of historic accuracy, an effort is made herein to set divergent opinions at rest by reference to the writer's personal diaries, which fix dates and events beyond controversy. These show that the first comprehensive study of the cutoffs on the Lower Mississippi River dates back to October 24, 1930. The writer, then employed at the United States District Engineer's Office in Norfolk, Va., in charge of surveys[22] (navigation, water power, flood control, and irrigation) of certain rivers in Virginia and North Carolina, was called by Maj.-Gen. Harley B. Ferguson, M. ASCE, then Colonel and Division Engineer, to the South Atlantic Division Office at Norfolk to assist him with starting this study. It was aimed at effecting major flood-stage reductions in the Lower Mississippi River by cutoffs, and also contemplated stabilization of the new channel by the revetment of banks.

General Ferguson's conception was a bold one and without precedent for such a large meandering river. Fundamentally, however, his plan was sound, as it was aimed at preserving hydraulic gradients so as not to exceed those prevailing in stretches where the river was known to possess an "orderly channel." This term was applied by General Ferguson to designate a channel having mild curvature, free from islands and back channels, and whose shifting tendencies involved no excessive bank caving and bar building. Fundamentally sound, also, was that part of his plan which called for promoting the lowering of the river bed (and therefore the low-water profile) sufficiently, so that low-water depths needed by navigation would not become reduced. In these respects General Ferguson set up criteria which, so far as is known to the writer, had never before been promulgated in projects contemplating river shortening by cutoffs. Certainly, none of the earlier advocates of cutoffs for the Mississippi River had ever insisted on, or even suggested, the importance of lowering the river bed as an integral part of any cutoff program, or of keeping water surface slopes within the range of natural slopes.

---

[21] "Early Presidents of the Society," *Civil Engineering*, July, 1937, p. 529.

[22] *House Document No. 308*, 39th Cong., 1st Session, 1926.

He recognized that the energy developed by a body of water flowing at the rate of 1,000,000 cu ft per sec (which at that time represented the bankfull discharge of the river) on gradients of only from ¼ ft to ⅓ ft fall per mile was of tremendous magnitude; any work planned without regard to maintaining proper slopes for both bed and water surface could create conditions which, once out of hand, might well become uncontrollable. Therefore, his criteria amounted (physically) to the necessity of reconstructing the river channel at a lower elevation. To this end he planned his operations with a view to enlisting to the fullest possible extent the river's own energy in enlarging the cutoff channels and also in degrading its bed, assisting this with adequate dredging wherever clay beds or highly compacted silts resisted erosion. In 1930 a great deal of thought was devoted by him to the types of dredges that would be particularly suited to these phases of work, and by 1932 two special dredges were being outfitted. Profiles of the river bed had been prepared to study the characteristics of the succession of pools and crossing bars typical of the river.

Subsequent events have since proved the correctness of General Ferguson's judgment in regard to the ability of the Lower Mississippi River to erode its bed where it is composed of sand, and to alter its curvature in response to slight changes in hydraulic gradient. It is deemed pertinent to state that the river actually has shown itself sensitive to changes in gradient as slight as 0.01 ft per mile. This had been perceived when the early office computations in Norfolk were made. Slope figures from the start were carried out to three decimal places, and the practice developed there was generally adopted later in the office of the President of the Mississippi River Commission. The cutoff program from its inception required continuous study of slopes throughout the period that General Ferguson was connected with it. To implement this study, General Ferguson, after he became President of the Mississippi River Commission, ordered gages maintained at 5-mile intervals throughout the many hundred miles of river affected by his operations. The cost and labor involved in maintaining these gages (all referenced to sea level datum), the reading of them twice a week or more often, and the computing of the slopes represented a large investment that was felt indispensable in order that nothing be left to chance.

The chronology shows that by November 22, 1930, General Ferguson had completed for submission to the Board of Rivers and Harbors a 38-page memorandum report recommending cutoffs in the 370-mile stretch between the mouths of White River and Old River. It included provisions for extensive revetting of banks which were deemed in need of protection against the effects of the increased energy released by slope steepening. This report received consideration by the Board of Rivers and Harbors at its sessions on December 2, 1930, and January 5 and February 3, 1931 (which the writer was required to attend); on the latter date the board approved General Ferguson's report, including his recommendation for undertaking a certain amount of experimental dredging aimed at expediting the lowering of the river bed.

At this time there had developed among engineers two distinct schools of thought—those who favored cutoffs, but who felt that extended flood-stage lowerings could result only when a reasonable amount of bed lowering could be brought about, and those who opposed cutoffs, because they felt the river would not lower its bed acceptably, and that, therefore, any flood-stage lowering effected by a cutoff would soon dwindle to zero upstreamward in accordance with customary backwater computation. Col. E. H. Schulz, M. ASCE, a member of the board, went to considerable trouble to prove this limitation mathematically to the board at its session on January 5. Also, in this second school of thought were many of the engineers—both military and civil—stationed on the Mississippi River. Finally, many engineers insisted that the meandering habits of the river would soon restore the river to its former length and thus nullify all stage lowerings effected.

General Brown, then Chief of Engineers, United States Army, recognized the importance of reducing the height of floods by cutoffs, but felt that any channel improvements so accomplished should be maintained through adequate bank protection works. His attitude is clearly stated in an article[23] wherein he commented on the economic height of the levees that had been reached, and expressed the need for reducing flood heights as follows:

> "Our hope is to solve this problem satisfactorily. There are three phases of it that require careful consideration as well as physical research. They are: (1) Economical bank protection to stabilize the levee system and improve the channel conditions for navigation; (2) storage reservoirs; and (3) increased channel capacity by the elimination of bends. Of these three phases of the question, (1) and (3) are interdependent, (3) not being practicable unless (1) is solved."

The soundness of his logic has since been amply demonstrated, particularly that contained in the last sentence. Bank protection works have constituted an important phase of the cutoff project from the start. Under the direction of Maj.-Gen. Max C. Tyler, M. ASCE (then Brigadier-General), who succeeded General Ferguson in 1939 as President of the Mississippi River Commission, the cutoff work was extended farther upstream and considerable effort was devoted to developing better forms of bank protection. This effort was resumed under his successor, Maj.-Gen. R. W. Crawford, and today forms an important assignment receiving study at the Waterways Experiment Station (Vicksburg, Miss.).

---

[23] "Flood Control Work on the Mississippi," by Lytle Brown, *Scientific Monthly*, June, 1930, p. 499.

In marked contrast with the objectives of Generals Brown and Ferguson in 1931 was the attitude toward any plan involving cutoffs on the part of the late Gen. T. H. Jackson, M. ASCE, then President of the Mississippi River Commission. In an article,[24] under the heading "Channel Shortening," General Jackson recited all the disadvantages of rectification by cutoffs, but he did not so much as mention the benefits of the stage lowering that could be derived from it, concluding with:

> "In flood-control plans of the past the cutoff has played an unimportant role, and there is little prospect of it playing a more important one in the future."

Colonel Vogel evidently labored under a misapprehension when he stated in his discussion that "General Jackson pioneered the way [for the Mississippi River cutoff program], overcoming prejudices and fears of long standing * * *" and that General Ferguson "* * * applied the lessons developed by General Jackson's tests to the river as a whole." The writer can testify to the fact that General Ferguson, upon taking office at Vicksburg on June 15, 1932, found practically his entire staff strongly prejudiced against cutoffs, and, in addition, this prejudice had become extended to include many state officials and engineers of levee boards, as stated by Mr. Elam in his discussion. It is necessary also to correct certain misapprehensions regarding the cutoff experiments made at the Waterways Experiment Station in 1931. These had been ordered by General Jackson in a memorandum, dated January 2, 1931, in furtherance of a directive to him from General Brown, dated December 29, 1930.

These experiments have been referred to as helpful to the cutoff program. Aside from having demonstrated that cutoffs did not increase river stages downstream from them, the results obtained from the models were far from encouraging, and the summary of the report thereon was adverse to undertaking most of the cutoffs proposed, except for those at the Greenville Bends. Made in solid concrete models, and incapable of simulating bed and bank erosion, these experiments did little more than confirm what had previously been arrived at by that school of thought which clung to backwater computations and assumed that no changes in river cross sections would take place. In this respect these early experiments ignored the very issues that were basic to General Ferguson's plan of channel rectification and bank stabilization, particularly in regard to the importance of promoting bed erosion. Some use was made of sand and sawdust in these tests to determine where deposits were likely to form, but no thought appears to have been given to investigating the effect of lowering the river bed.

The tenor of the Waterways Experiment Station findings are best gaged by quoting from a report[25] as follows:

> "Item *e*: Any cutoff across Ashbrook Neck should be studiously avoided because of its probable effects on upstream velocities and downstream

---

[24] "Flood Control on Alluvial Rivers with Particular Reference to Flood Control on the Mississippi River," by T. H. Jackson, *Engineering-News Record*, January 15, 1931, pp. 107–108.

[25] "Experiment to Determine the Effects of Proposed Dredged Cut-Offs in the Mississippi River," *Report No. I*, U. S. Waterways Experiment Station, Vicksburg, Miss., April 15, 1932, pp. 13–14.

> directions of currents. * * * Cutoffs at Caulk and Tarpley Necks can do little material good in reducing stages where desired, and although no objection can be seen to letting Caulk cut through by itself, there seems no valid reason for encouraging it.
>
> "Item *f*: The cutoffs between Vicksburg and Lake Lee would be valueless and extremely expensive to dredge. No further thought should be given to them. [This relates to Marshall, Willow and Sarah cutoffs.]
>
> "Item *g*: A cutoff at Diamond Point would have several good features and no apparent bad ones. Further tests will definitely indicate the advisability of assisting, or allowing it to occur naturally. As to the other three below Vicksburg: Yucatan is already accomplished; a cutoff at Natchez [later called Giles] would be of dubious value in an already fairly stable reach; and one at Esperance [later called Glasscock] of no value at all."

In the same report[26] the model of the Greenville Bends is stated to have produced a "Maximum lowering of 9.7 feet at mile 442.9" (which was then the head of the bends) as the result of simulating fully developed cutoffs across Leland, Tarpley, and Ashbrook necks. In contrast with this an observed lowering of more than 13 ft was actually realized at that point during the great flood of 1937, and this has since increased somewhat. Similar lowerings were registered also on the Arkansas City (Ark.) gage 6 miles farther upstream in 1937. This is illustrative of the marked increase in stage reduction effected wherever bed erosion and channel adjustment were made to play their parts, assisted where necessary by dredging.

Again, Caulk, Tarpley, Giles, and Glasscock cutoffs, contrary to model results, have proved effective. Glasscock, although greatly hampered in its early development due to the tenacious character of the soil which resisted erosion to a marked degree, has caused flood stages to become lowered all of 3 ft at Natchez, Miss., 18 miles upstream, corresponding to an increase in flood discharge of 200,000 cu ft per sec for rising river conditions. In regard to the Marshall and Willow cutoffs, which the report called valueless and extremely expensive to dredge and not worthy of further thought, it is only necessary to refer to the description in the paper of the case with which the pilot cuts were made (considering their lengths), and the satisfactory results obtained. Where the laboratory results showed a flood-stage lowering of a scant 3 ft at Vicksburg, General Ferguson brought about a lowering of 7 ft (which later increased to 10 ft) by lowering the bed profile along Racetrack Towhead by dredging.

The foregoing may indicate that these early experiments were of no help to General Ferguson, being in essence entirely at variance with the latter's conception of executing the work. It was not until after General Ferguson had taken office as President of the Commission, on June 15, 1932, that the Experiment Station received instructions to prepare river models with erodible beds. A series of studies of Mississippi River problems were undertaken that summer with erodible bed models.

In further reference to Yucatan Cutoff (an occurrence long considered inevitable), the writer's mention of its having been unexpected related to the fact that the break was caused by low-water caving at a time when no one was aware of it. Not an engineer, surveyman, or pilot of the Engineer Department was present to witness it. It was many days after the break that its occurrence became known to officialdom. The exact date of the occurrence of this remarkable natural avulsion has never been established.

---

[26] "Experiment to Determine the Effects of Proposed Dredged Cut-offs in the Mississippi River," *Report No. 1*, U. S. Waterways Experiment Station, Vicksburg, Miss., April 15, 1932, p. 40.

It devolved upon General Brown to find a man to carry out the cutoff program together with all that was consistently and inseparably connected with it in the way of channel improvement and bank protection. Illuminating in this connection is General Brown's appraisement of the qualifications such a man should possess, as stated by him: "There are many who project an idea where danger is involved but there are few with the courage to give it effect and to assume the responsibility." He found a man who combined these qualities in General Ferguson, who not only had projected the idea, but possessed the indomitable nerve to tackle such a stupendous job and to see it through. General Ferguson stuck to his task more than seven years, until his retirement on August 15, 1939.

The location for a cutoff across Diamond Point, the first of the artificial cutoffs, was determined by General Ferguson on June 17, 1932, two days after he assumed office. In the presence of the writer he drew the center line and the width of the right of way on a map, and dictated a directive "to have work initiated this summer with a view to acquiring right-of-way and completing a cut by dredging before next high water" (from the writer's personal diary dated June 17, 1932). A conference was held on that date in his office to discuss procedure. Those present were: Col. J. N. Hodges, District Engineer, New Orleans, La., in whose district the cutoff was to be made; his military assistant, the late Lt. J. P. Dean, United States Corps of Engineers; and Maj. P. S. Reinecke of the Commission staff. If any surveys or borings had been made previously, as mentioned by Colonel Vogel, these did not come to the writer's attention.

The writer is indebted to Dean Marston for presenting the work of the Mississippi River Engineering Board of Review on which he served. Credit is due the Board of Review for having given the practicability of the cutoff program very careful consideration. Its recommendations were in large measure responsible for the promptness with which authority for the work was granted by the Chief of Engineers. The Board of Review was appointed by General Brown by Special Orders No. 16, dated March 1, 1932, by order of the Secretary of War in furtherance of a resolution by the Committee on Flood Control of the House of Representatives, dated January 28, 1932, requesting that the Chief of Engineers

> "* * * examine and review the present status and condition of the works now in progress as authorized by the Flood Control Act of May 15, 1928 * * * with a view of determining if changes or modifications should be made in relation to the project and its final execution."

The Board of Review was directed by General Brown to determine, among other things,

> "* * * (d) the practicability and desirability of increasing the discharge capacity of the main leveed channel in flood time, and at the same time improving conditions for navigation. Measures for protection against flood waters that must of necessity escape from the main leveed channel of the river as modified by any conclusions as to channel enlargement."

These items had not previously been specifically authorized, except as covered under the broad head of channel stabilization. In addition to the three Board of Review members mentioned by Dean Marston, the Special Orders designated Lieutenant Dean to serve as "Recorder and Executive." Lieutenant Dean's services, due to his familiarity with the valley and its river problems, proved of inestimable assistance to the Board of Review on its inspection trips. The writer accompanied the board as civilian assistant to General Ferguson.

The Board of Review convened at Memphis (Tenn.) on March 9, 1932, at the office of the District Engineer, the then Maj. Brehon Somervell, Hon. M. ASCE, and almost immediately started on its first trip down the Mississippi River on board the steamer *Inspector*. The flood of that year was past the crest, and the river had dropped 20 ft, permitting an excellent view of bank and revetment conditions. In the lower valley, however, stages were still very high, and General Brown was at New Orleans to study the river while it was at danger stage. The trip of the Board of Review extended down the main river to the mouth of Old River, thence through Old River down the Atchafalaya River. It included visits to two crevasses in the Atchafalaya Basin levees, the closure of which was in progress. After a visit to the Bonnet Carré spillway the trip ended at New Orleans on March 12, 1932. A second inspection trip, from May 8 to May 18, 1932, started from Cairo (Ill.) and was made partly by steamer and partly by automobile with much attention being given to problems in the St. Francis, Tensas, and Atchafalaya basins. The proposed enlargement of the Atchafalaya River to which Dean Marston referred was given considerable study on that occasion.

Answering Dean Marston's question regarding the development of the proposed enlargement as the result of the dredging operations performed, the Atchafalaya has been provided with two main channels emptying into Grand Lake by fairly direct routes. By 1940 these channels had become efficient outlets gradually replacing the previous labyrinth of bayous, thereby greatly improving flood-water outflow. The upper 56 miles of the original Atchafalaya River channel have been widened and deepened considerably through erosion by flood flow aided by supplementary dredging in places where tough bottom materials resisted erosion. The materials so scoured out in large part came to rest in the shallow waters of Grand Lake and are rapidly filling it. Two main outlets have been provided through these deposits—one by way of the old channel past Morgan City, La., and the other through a dredged channel into Wax Lake, and thence into the Gulf of Mexico. The discharge capacity of the Atchafalaya River proper has become greatly increased. During the flood of 1937 its maximum discharge confined between levees was in excess of 450,000 cu ft per sec. During the flood of 1945 discharge measurements showed further increases, the maximum flow being in excess of 600,000 cu ft per sec.

In regard to the question of leaving open the abandoned bend channels of the Mississippi River to serve as depositories for sediment and as high-water carriers and also for valley storage—these advantages are, of course, greatest and do the most good during the early development of a cutoff, but thereafter diminish in time. The resulting effects on the river are well described in the discussions by Mr. Pharr and by Messrs. Williams and Graves, and need not be restated, except to add that the Mississippi River is amply capable of transporting the entire load after the storage capacities of the bend ways become exhausted. The river is not a heavy sediment carrier compared to other western rivers; in fact, expressed in terms of parts per million of suspended load, it carries on an average only one tenth as much as the Missouri River.

Dean Marston's further requests for data are answered as follows: The planned flood-carrying capacity of the river between levees in the Arkansas City-Vicksburg reach now is 3,065,000 cu ft per sec. This is the maximum flood-crest flow estimated as possible of occurrence in that reach and was officially adopted as the revised project flow by the act of August 19, 1941. The capacity of 3,065,000 cu ft per sec is consistent with a similar maximum of 2,450,000 cu ft per sec provided for between Cairo and Arkansas City, and with a maximum of 3,000,000 cu ft per sec in the latitude of Angola (La.). The 3,065,000 value is appreciably larger than the 2,662,000 cu ft per sec, representing the estimated maximum confined flood-crest flow in 1927. The act of 1941 also provided for increases of freeboard on the main-line levees, which had become facilitated by the increases in discharge capacity. These increases made possible the further abandonment of the proposed floodways between the Arkansas River and the Red River. The 1941 standard levee requirements provided for a widening of the levee crowns to 25 ft to serve as roadways. Although much yardage remains to be placed to meet fully all these requirements, the height of the main-line levees in the Arkansas City-Vicksburg reach is now sufficient to take care of the new project flood flow.

In order to present as concisely as possible the picture of the development in channel capacity that has taken place since the first artificial cutoff was made in 1933, the approximate bankfull discharges at key points have been listed, year by year, in Table 7. The figures were estimated by inspection of the stage discharge relations for rising flood stages only, as these afforded the only meaningful comparison. The point is that falling flood stages, because of the variability they exhibit in their stage discharge relations, are not useful for comparative purposes. Falling stages, moreover, are characterized by momentary changes in discharge, corresponding to changes in the rate at which the crossing

TABLE 7.—INCREASES IN BANKFULL DISCHARGE CAPACITY AND REDUCTION IN FLOOD STAGE FOR RISING RIVER

Gage	Bank-full stage (ft)	DISCHARGE CAPACITY (MILLION CU FT PER SEC)								Total flood-stage lowering (ft)
		1932	1935	1937	1939	1943	1945	1946	Total increase	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Columbus, Ky.	43.0	....	1.00	1.03	....	1.10	1.10	....	0.10	1.5
Memphis, Tenn.	34.0	....	....	1.05	1.05	1.10	1.17	1.25	0.20	4
Helena, Ark.	41.0	....	0.94	0.96	0.98	1.00	1.10	1.20	0.26	5
Arkansas City, Ark.	44.0	1.00	1.12	1.65	....[a]	1.68	1.85	....[a]	0.85	15
Vicksburg, Miss.	42.0	1.00	1.17	1.30	1.38	1.45	1.55	....[a]	0.55	10
Natchez, Miss.	46.0	....	1.30	1.30	1.31	1.50	....	....[a]	0.20	3

[a] Flood crest did not reach top of banks.

bars are being built up, characteristic of rivers flowing in mobile beds. In interpreting Table 7 it is important to keep in mind that for falling bankfull stages the channel capacity is much less than shown, the reductions ranging from about 200,000 cu ft per sec to as much as 250,000 cu ft per sec, in some instances.

Statistically, flood-stage lowerings should be predicated on crest stages, but since no two floods are of the same magnitude in regard to crest discharge, a direct comparison is seldom feasible. In Table 8 an attempt has been made to compare the approximate increases in flood-crest stages that occurred in the past due to confinement by levees with the approximate flood-stage lowerings that have resulted since 1932 from the cutoff and channel improvement program.

The lowering of the river bed is in an entirely different category, being caused by scour, which in turn is affected by increased slope and lessened depth during floods—two factors which partly offset each other's effects. Since the bed of a meandering stream is a succession of deeps and shoals, it gives the low-water profile a step-like appearance quite unlike the smoothly curving high-

TABLE 8.—COMPARISON OF EARLY INCREASES WITH RECENT REDUCTIONS IN FLOOD-CREST STAGES

INCREASES IN FLOOD-CREST STAGES DUE TO LEVEE CONFINEMENT PRIOR TO 1932				Reductions in flood-crest stages effected since 1932
Place (1)	Year (2)	Crest stage (ft) (3)	Total increase (ft) (4)	(5)
New Orleans, La.	1849	15.2		Bonnet Carre spillway keeps stages below 20 ft
	1851	15.4		
	1922	21.27	6	
Baton Rouge, La.	1849	34.93		Effect of Atchafalaya River enlargement and future use of floodways not known
	1850	34.53		
	1851	34.53		
	1927	47.80	13	
Angola, La.	1851	46.4		
	1867	46.3		
	1927	57.5	11	
Natchez, Miss.	1770	48.0		3 ft
	1815	48.6		
	1927	56.65	8	
Vicksburg, Miss.	1828	46.38		10 ft
	1927	58.40	12	
Arkansas City, Ark.	1880[a]	45.10		14 ft to 15 ft
	1927	60.4	15	
Helena, Ark.	1850	42.81		5 ft
	1927	56.75	14	
Memphis, Tenn.	1828	32.86		4 ft
	1844	33.16		
	1850	33.56		
	1927	45.8	13	
Columbus, Ky.	1844	42.0		1½ ft
	1858	42.9		
	1927	51.00	8 to 9	

[a] No record prior to 1880.

water profile. Available data show that the low-water lowering, although not as large in amount or as uniform as the high-water lowering, extends equally far upstream. Both eventually run out, and in each case the new over-all slope is steeper than the original slope before cutoffs were initiated. Table 5, supplied by Messrs. Williams and Graves, who have given this phase detailed study, shows the increases in low-water slopes that have resulted. Supplementing their data by reference to actual discharge measurements, it appears that the low-water plane for a uniform discharge of 150,000 cu ft per sec has been lowered as follows: Columbus, less than 1 ft; Memphis, about 3 ft; Helena, about 2¼ ft; Arkansas City, more than 3 ft; and Vicksburg, all of 10 ft.

As pointed out by Messrs. Williams and Graves, there has been no noticeable decrease in the low-water depths on the crossing bars since the opening of the cutoffs; what is more significant, low-water navigation has not been seriously inconvenienced. In fact, the depths on the crossing bars, throughout the length of river affected, have improved as the result of the work done, and in 1941 minimum depths of 12 ft were found to exist over most of the bars. At the few shallower crossings 12-ft depths were readily maintained by making cuts with dredges of the "dust-pan" type. In consequence, the previously prescribed 9-ft project depth between Cairo and Baton Rouge was authorized to be increased by an act of Congress on December 22, 1944, to 12 ft, leaving the 300-ft navigable width unaltered. Below Baton Rouge the old 35-ft project depth for sea-going vessels has remained in effect.

The continuity of the river-bed profile in the reaches north of Old River (to answer Dean Marston's inquiry) shows a decided improvement over that existing previous to 1933. The largest humps in the old profile have been eliminated, and the step-like succession of pools and crossings has become a gradually ascending series which possesses a considerable degree of regularity. An example of the nature of the adjustment that takes place is shown in Fig. 8. The new low-water profile shown by the heavy black line tends to approach, in the matter of spacing of pools and crossing shoals, the 5-mile interval, which in the early studies was found to be a characteristic of the river bed in its "orderly" reaches. A thin rubber band stretched over the 1946 profile in Fig. 8 will show that it averages a straight line, which is normal for a 25-mile distance.

The probable effect on the Lower Mississippi River of the flood control reservoirs in the Arkansas River watershed and on other Mississippi River tributaries is at present unpredictable. This is the writer's opinion, requested by Dean Marston. These reservoirs, however great their number eventually may become, are designed for the express purpose of providing protection to the lives and property in the valleys of the tributaries on which the dams are built, or are to be built. Operation by any coordinated plan still remains to be worked out, more particularly in regard to timing the release of the impounded floodwaters. The huge model of the Mississippi River watershed under construction by the Waterways Experiment Station at a site near Clinton, Miss., is designed specifically for shedding much-needed light on this complex problem. Furthermore, it is the opinion of the writer that the safety provided by the existing project for the Lower Mississippi River, although it may benefit incidentally from the operation of these reservoirs, is not dependent on such operation.

In answer to Colonel Hall's request for information concerning the economic aspects of the cutoff project, a segregation of costs would not be helpful because of the close interrelation of the several phases of the work. A statement follows which describes what the cutoff project has accomplished, what additional work it has entailed, and what additional improvements it has made possible that previously had remained unattainable.

River Channel.—The shortening by cutoffs and chute enlargements has eliminated excessive curvature in a river length of about 500 miles without straightening the channel unduly, or creating any tendency toward channel braiding. It has eliminated annual maintenance on 199.2 miles of abandoned bend channels, and has increased channel discharge capacity and reduced flood stages as shown in Table 7. The river bed has been lowered so as to leave low-water depths unreduced. It has made possible channel improvements by dredging which previously were not warranted economically. The amount of dredging involved runs into large quantities but has produced permanent results essential, in most cases, to insuring the full measure of benefit derivable from the cutoffs. Further operations of this kind will reduce progressively as channel stabilization advances. The yardage excavated in the pilot cuts is a small item in relation to the yardage eroded by the river in the enlargement of the pilot cuts to full-sized river channels. Likewise, the yardage dredged for improving the river channel is a fraction of what the river has removed additionally.

Banks and Revetments.—Annual maintenance on 59.3 miles of existing revetments rendered inactive in abandoned bends has been largely eliminated. On the other hand, channel deepening upstream from cutoffs, in addition to the increased force of the current due to local profile steepening, has been responsible for local bank caving and the sliding and destruction of several revetments. Further trouble of this kind must be expected until hydraulic gradients become stabilized and the attack on the banks affected ceases. In this connection it seems pertinent to emphasize again the fact that as yet no adequate type of revetment has been devised for use on the Lower Mississippi River.

Control of Floods.—Major benefits have resulted from the lowering of flood stages. Only a meager conception of the magnitude of these lowerings is afforded by reference to Table 7. It is best conveyed by stating that they have made practicable of attainment the protection against the greatest flood estimated as likely to occur without having to raise the entire levee system. In the large backwater areas, where the most fertile soils are to be found, the reductions in the frequency, duration, and depths of flooding that have resulted have insured better returns from the land and have increased land values. Along the levees, the dangers from underseepage and landside boils have been lessened as flood heights have been reduced and the passage of flood crests has been expedited. In two respects no gains have been made—namely, the total length of the main-line levees has not become reduced by the channel shortenings; and the bank caving above newly made cutoffs or by readjustments in channel alinement will continue to require setting back levees, in places, until a less vulnerable type of revetment is evolved. Perhaps the greatest economic benefit that has been derived from the reduction of flood stages is the ability to dispense with a floodway on the west side of the valley between the Arkansas River and the Old River. This has removed the threat of inundation from many thousands of acres of valuable farming land and from large stands of hardwood timber. Finally, from the viewpoint of the citizens of Greenville, the removal of the river from their door by cutting off the Greenville Bends ranks as the outstanding benefit that has accrued to any community in the valley.

Navigation.—Distinct benefits have accrued to commercial navigation. Shortening the river has reduced the time of travel at ordinary stages appreciably, especially for ascending tows, despite the slight increase in current velocities due to slope steepening. Low-water depths are better than before cutoffs were made. Time lost at points where rapid channel changes were in progress has been a temporary inconvenience. Only during flood stages is there some increase in running time, but this is offset in part by the shortened duration of high flood stages. Thus summarized, a fair conclusion must be that the benefits by far outweigh the disadvantages, and, as a federal project, the investment is economically sound.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2330

A RATIONAL EXPLANATION OF COLUMN BEHAVIOR

BY FREDERICK L. RYDER,[1] ESQ.

WITH DISCUSSION BY MESSRS. R. ROBINSON ROWE, GEORGE WINTER, ROBERT M. MAINS, CHI-TEH WANG, ERNEST L. ROBINSON, AND FREDERICK L. RYDER

SYNOPSIS

The general method of approach is to analyze, mathematically, the elastic behavior of the simple ideal column, and then to apply the results, with suitable modifications, to the practical column. The basic differential equation is first set up, and then is integrated to yield the fundamental column formula relating axial force, moment of inertia, modulus of elasticity, length, and center deflection (Section 1).

For a given P/A-value, the variation of center deflection with the L/r-ratio is studied, for conditions above the critical point. The projected length corresponding to any value of center deflection is calculated, thus affording an approximate idea of the deflected column shape. A chart relates values of P/A, L/r, projected length, and center deflection. Finally, the exact shape of a typical ideal column is computed, and is shown to be similar for all ideal columns having the same relative center deflection (Section 2).

It is next shown that, above the critical point, the basic differential equation is satisfied if the column is (*a*) straight, or (*b*) deflected to the position dictated by the fundamental column formula. The stability of the column in these two positions, and in fact in any position, is analyzed by a detailed study of the internal moments caused by deflection and the external moments caused by the axial force. It is shown that only one position of the column is stable—the position corresponding to the fundamental column formula. The analysis also re-emphasizes the well-known fact that at the critical point the deflection is indeterminate (Section 3).

For conditions below the critical point, a chart relates the factor of safety of an ideal column to the P/A-value and the L/r-value. The differences between the practical and the ideal column are next discussed; and the analytical

---

NOTE.—Published in March, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Development Engr., Shirgun Corp., New York, N. Y.

solution for the deflection and stresses of a column having end eccentricity, initial deflection, and transverse loading is obtained. The "step-by-step" method for the numerical solution of practical column problems is then presented in detail and is applied to two typical problems (Section 4). Stresses and factor of safety in a column loaded above the critical point are discussed (Section 5).

INTRODUCTION

Most of the familiar and widely used column formulas are essentially based on the results of tests. They are convenient for predicting column action under certain important conditions, but they are not intended to present the rationale of column behavior. For example, some questions that cannot be answered by ordinary column formulas are:

A. Why do columns buckle suddenly at the critical load?
B. What are the actual stresses in the column?
C. How do columns act above the critical load?

These questions, and others of similar interest, can be answered by an a priori mathematical analysis, without recourse to test results. In fact, a complete solution of the elastic behavior of a simple column can be deduced mathematically from the given statical condition of the column. It is the writer's purpose to trace the mathematical steps leading to this solution and to interpret the results in terms most useful to the average engineer.

As a supplement to the theoretical discussion, the powerful and universally adaptable step-by-step method of solving practical column problems is included in the argument on actual stress and factor of safety.

The scope of this paper, then, is twofold:

1. To explain the mechanism of column behavior by mathematical analysis; and
2. To present a widely applicable method for the solution of practical column problems.

Notation.—The letter symbols in this paper are defined where they first appear, in the text or by diagram, and are assembled alphabetically in the Appendix. Discussers are requested to adapt their notation to this form.

1. THE FUNDAMENTAL DIFFERENTIAL EQUATION AND ITS INTEGRATION

The column shown in Fig. 1(*a*) is assumed to be perfectly straight initially, and is of homogeneous material having the modulus of elasticity, E. The cross section and therefore the moment of inertia, I, are constant. The resultant of the load P acts through the centroid of the cross section, thus insuring that P does not produce moment at the ends of the column. The length of the column, measured along the curve, equals L.

The column is shown in a transversely deflected position, δ being the deflection at the center. At any point, such as A, the column feels the moment:

$$M = P\,y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

According to familiar theory in the strength of materials, the moment at any point of an initially straight member equals:

$$M = -E\,I \times \text{(curvature)} \ldots\ldots\ldots\ldots\ldots\ldots (1b)$$

For present purposes, curvature is advantageously considered the "rate of change of slope per unit of length along the curve." Referring to the triangle of infinitesimals, Fig. 1(*b*):

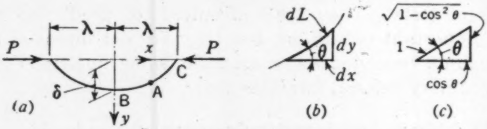


FIG. 1

$$\text{(Curvature)} = \frac{d\theta}{ds} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1c)$$

in which ds is the differential of length of the curve; and θ is the angle of slope or

$$\theta = \tan^{-1}\frac{dy}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Combining Eqs. 1:

$$P\,y = -E\,I \times \frac{d\theta}{ds} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Eq. 3 is the fundamental differential equation of the elastic curve of the column, and forms the basis for the present analysis.

Certain assumptions and omissions are implicit in Eq. 1*b*:

a. Only the curvature caused by bending moment is considered in Eq. 1*b*. With the column in a deflected position, the curvature will be somewhat affected by shearing forces. However, the curvature caused by shear is usually very small compared to the curvature caused by moment, and will therefore be neglected.

b. Eq. 1*b* is not strictly true for a member with appreciable curvature.[3] The error is not important, however, if the ratio ρ/c remains larger than about 10, in which ρ is the radius of curvature and c is the distance from the neutral axis to the extreme fibers. For small deflections this error is negligible. Large deflections, and consequently large curvatures, will affect the results slightly.

c. The modulus of elasticity, E, is assumed to be constant, necessitating the maximum stress in the column to be less than the proportional limit of the material.

---

[3] "Advanced Mechanics of Materials," by Fred B. Seely, John Wiley & Sons, Inc., New York, N. Y., 1932, Chapter VII.

Eq. 1*b* is often used in the simplified form:

$$M = -EI\frac{d^2y}{dx^2} \qquad (4)$$

in which $\frac{d^2y}{dx^2}$ is made equal to the curvature. The true expression for the curvature in terms of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ (see any calculus text) is:

$$(\text{Curvature}) = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} \qquad (5)$$

The curvature is $\frac{d^2y}{dx^2}$ only when $\frac{dy}{dx} = 0$—that is, when the member is perfectly straight. Therefore, Eq. 4 is not valid when the member has an appreciable deflection. The simplified Eq. 4 yields limited and imperfect results in the case of an ideal column, but is basically important in the solution of problems involving the practical column.

The first step in the integration of Eq. 3 is the elimination of one of its three variables. By Fig. 1(*b*),

$$ds = \frac{dy}{\sin\theta} \qquad (6a)$$

Substitution in Eq. 3 yields:

$$Py = -EI\frac{d\theta}{dy}\sin\theta \qquad (6b)$$

In Eq. 6*b*, if—

$$k^2 \equiv \frac{P}{EI} \qquad (6c)$$

—it follows that

$$k^2\,y\,dy = -\sin\theta\,d\theta \qquad (6d)$$

Integrating Eq. 6*d*,

$$\frac{k^2y^2}{2} + C = \cos\theta \qquad (7)$$

in which C is an arbitrary constant. If $y = \delta$ then $\theta = 0$ (by symmetry), and $\cos\theta = 1$. Substituting these values in Eq. 7:

$$C = 1 - \frac{k^2\delta^2}{2} \qquad (8a)$$

Rewriting Eq. 7:

$$\cos\theta = 1 - \frac{k^2}{2}(\delta^2 - y^2) \qquad (8b)$$

Eq. 8*b*, the result of the first integration, expresses a relation between the angle of slope (θ), the deflection at the center (δ), and the deflection (y) at any point. The object of the next integration is to eliminate θ, an unknown, and to introduce L, the total length of the deflected column, into the equations. By similarity (Figs. 1(b) and 1(c)):

$$\frac{ds}{1} = \frac{dy}{\sqrt{1 - \cos^2 \theta}} = \frac{dy}{\sqrt{(1 + \cos \theta)\,(1 - \cos \theta)}} \quad \ldots\ldots\ldots\ldots (9a)$$

Using Eq. 8b:

$$ds = \frac{dy}{\sqrt{\left[2 - \frac{k^2}{2}(\delta^2 - y^2)\right]\left[\frac{k^2}{2}(\delta^2 - y^2)\right]}} \quad \ldots\ldots\ldots\ldots (9b)$$

Eq. 9b may be simplified by writing:

$$y \equiv \delta \cos \phi \quad \ldots\ldots\ldots\ldots (10a)$$

$$dy = -\,\delta \sin \phi \, d\phi \quad \ldots\ldots\ldots\ldots (10b)$$

and

$$\delta^2 - y^2 = \delta^2 - \delta^2 \cos^2 \phi = \delta^2 \sin^2 \phi \quad \ldots\ldots\ldots\ldots (10c)$$

Then

$$ds = \frac{-\,\delta \sin \phi \, d\phi}{\sqrt{\left(2 - \frac{k^2\,\delta^2}{2}\sin^2 \phi\right)\left(\frac{k^2\,\delta^2}{2}\sin^2 \phi\right)}} = \frac{d\phi}{k\sqrt{1 - \frac{k^2\,\delta^2}{4}\sin^2 \phi}} \quad \ldots (11)$$

The minus sign disappears in the numerator of Eq. 11 because $\sqrt{\delta^2}$ is selected as $-\,\delta$. Writing:

$$q \equiv \frac{k\,\delta}{2} \quad \ldots\ldots\ldots\ldots (12a)$$

it follows that

$$k\,ds = \frac{d\phi}{\sqrt{1 - q^2 \sin^2 \phi}} \quad \ldots\ldots\ldots\ldots (12b)$$

The integral of the right-hand term of Eq. 12b is known as an elliptic integral.[3] It can be evaluated only between definite limits. Points B and C in Fig. 1(a) are selected as the lower and upper limits, respectively.

Applying Eq. 10a at point B where $y = \delta$, it is found that $\cos \phi = \frac{y}{\delta} = \frac{\delta}{\delta} = 1$, from which $\phi = 0$. Likewise, at point C, $y = 0$, $\cos \phi = \frac{0}{\delta} = 0$, and $\phi = \frac{\pi}{2}$. The arc lengths s are 0 and $L/2$ at points B and C, respectively. With these quantities in Eq. 12b:

$$k \int_0^{L/2} ds = \frac{k\,L}{2} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - q^2 \sin^2 \phi}} \equiv K \quad \ldots\ldots\ldots\ldots (13)$$

Tabular values of K for different values of q have been prepared by B. O. Peirce.[4]

---

[3] "Advanced Calculus," by E. B. Wilson, Ginn & Co., Boston, Mass., 1912, p. 507.

[4] "A Short Table of Integrals," by B. O. Peirce, Ginn & Co., Boston, Mass., 3d Ed., 1929, p. 121.

Eq. 13, the fundamental column formula, expresses a relation between q, k, and L. For example, if $k^2 \equiv \frac{P}{E\,I}$ and L are known, then it becomes possible to solve for $q \equiv \frac{k\,\delta}{2}$, and hence δ can be ascertained. The implications of Eq. 13, with regard to practical engineering problems, are treated in Section 2.

2. Column Deflections Under Various Conditions

A pressing question confronts the engineer in regard to the behavior of the column: "What is the relationship of the deflection to the load P and to the physical constants E, I, and L?" With the aid of the fundamental column formula, Eq. 13, it is possible to answer this question by plotting δ/L, the relative deflection at the center, against $L/\bar{r}$, the ratio of the length to the minimum radius of gyration of the cross section. The ratio $L/\bar{r}$, of course, is the accepted criterion of stiffness for a column. (Incidentally, $\bar{r}$ should be regarded as a measure of the efficacy of distribution of the elements of cross-sectional area of the column, in so far as resistance to buckling is concerned.)

TABLE 1.—Center Deflection and Projected Length of a Simple Column

$\sin^{-1} q$ (degrees) (1)	K[a] (2)	q (3)	$L/\bar{r} = 245\,K$ (4)	$\delta/L = q/K$ (5)	J[a] (6)	$\lambda/L = \frac{J}{K} - \frac{1}{2}$ (7)
0	1.57	0	384	0	1.57	0.50
30	1.69	0.5	415	0.296	1.47	0.37
60	2.16	0.866	529	0.401	1.21	0.06
75	2.77	0.966	676	0.35	1.08	−0.11
89	5.43	1.0	1,330	0.185	1.00	−0.316
90	∞	1.0	∞	0	1.00	−0.50

[a] "A Short Table of Integrals," by B. O. Peirce, Ginn & Co., Boston, Mass., 3d Ed., 1929, p. 121.

To introduce $\bar{r}$ into Eq. 6c, substitute $I = A\,\bar{r}^2$ and solve for k:

$$k = \frac{\sqrt{\frac{P}{A}}}{\bar{r}\sqrt{E}} \quad \text{.........................} (14)$$

in which P/A is familiar to engineers as the direct compressive unit stress caused by P when the column is straight.

Consider a steel column with $E = 30{,}000{,}000$ lb per sq in. and $P/A = 2{,}000$ lb per sq in. Then, by Eq. 14:

$$k = \frac{\sqrt{2{,}000}}{\bar{r}\,\sqrt{30{,}000{,}000}} = \frac{1}{122.5\,\bar{r}} \quad \text{................} (15a)$$

Referring to Eq. 13: $K = \frac{k\,L}{2} = \frac{L}{2 \times 122.5\,\bar{r}}$, or

$$\frac{L}{\bar{r}} = 245\,K \quad \text{.......................} (15b)$$

By solving Eqs. 12a and 13 to eliminate k:

$$\frac{\delta}{L} = \frac{q}{K} \quad \text{...........................} (16)$$

Eq. 16 makes it possible to compile Cols. 1 to 5 of Table 1. The value of K corresponding to $\sin^{-1} q = 90°$ does not appear in Mr. Peirce's table;[4] but it can be derived easily by using Eq. 13, thus: $K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - 1 \sin^2 \phi}} = \int^{\pi/2} \sec \phi \, d\phi$ $= \frac{1}{2}\left[\left(\log \frac{1 + \sin \phi}{1 - \sin \phi}\right)\right]_0^{\pi/2} = \frac{1}{2} \log \frac{\frac{1+1}{1+0}}{\frac{1-1}{1-0}} = \infty$. The values of δ/L are plotted against those of L/r in Fig. 2. To assist in the interpretation of this curve, some idea should be formed as to the actual shape of the column corresponding to various plotted points. This is possible by solving for the different values of λ/L, in which λ is half the projection of the column on the x-axis (see Fig. 1(a)). Having found λ/L, and knowing δ/L from the curve of Fig. 2, the shape of the column, between the end points and the center, can be sketched. It is possible to derive the exact shape of the entire column (see Section 2 under "Exact Shape of the Deflected Column"), but this is not necessary for present purposes.

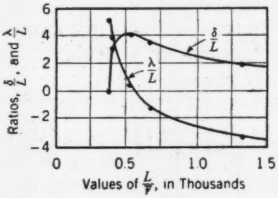


FIG. 2

Referring to Eq. 8b, and noting the similarity between the triangles of Figs. 1(b) and 1(c), the quantity x is introduced into the equations by writing:

$$\frac{dx}{ds} = \cos \theta = 1 - \frac{k^2}{2}(\delta^2 - y^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (17a)$$

To eliminate s (see Fig. 1(b)):

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \ldots\ldots\ldots\ldots (17b)$$

Substituting Eq. 17b in Eq. 17a: $\frac{dx}{dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = \cos \theta$;

$$\frac{dy}{dx} = \sqrt{\frac{1}{\cos^2 \theta} - 1} = \sqrt{\frac{1 - \cos^2 \theta}{\cos^2 \theta}}; \quad \text{or} \quad \frac{dy \cos \theta}{\sqrt{(1 + \cos \theta)(1 - \cos \theta)}} = dx.$$

In other words,

$$dx = \frac{\left[1 - \frac{k^2}{2}(\delta^2 - y^2)\right] dy}{\sqrt{\left[2 - \frac{k^2}{2}(\delta^2 - y^2)\right]\left[\frac{k^2}{2}(\delta^2 - y^2)\right]}} \quad \ldots\ldots\ldots\ldots (18)$$

Substituting from Eqs. 10 and 12a, $dx = \frac{(1 - 2 q^2 \sin^2 \phi)(- \delta \sin \phi \, d\phi)}{\sqrt{k^2 (1 - q^2 \sin^2 \phi)(\delta^2 \sin^2 \phi)}}$; or (selecting $\sqrt{\delta^2} = -\delta$):

$$k\,dx = \frac{(1 - 2\,q^2 \sin^2 \phi)\,d\phi}{\sqrt{1 - q^2 \sin^2 \phi}} \qquad (19a)$$

Eq. 19*a* can be integrated in terms of elliptic integrals, but first some manipulation is necessary. In Eq. 19*a*, the quantity in parentheses can be written: $2\,(1 - q^2 \sin^2 \phi) - 1$, from which:

$$k\,dx = \frac{2\,(1 - q^2 \sin^2 \phi)\,d\phi}{\sqrt{1 - q^2 \sin^2 \phi}} - \frac{d\phi}{\sqrt{1 - q^2 \sin^2 \phi}} \qquad (19b)$$

Integrating Eq. 19*b* from point B to point C, Fig. 1(*a*):

$$\int_0^{\lambda} k\,dx = k\,\lambda = 2\int_0^{\pi/2} \sqrt{1 - q^2 \sin^2 \phi}\,d\phi - \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - q^2 \sin^2 \phi}} \qquad (19c)$$

In other words,

$$k\,\lambda = 2\,J - K \qquad (19d)$$

in which J and K are elliptic integrals defined by the last two terms of Eq. 19*c*. Substituting $k = \frac{2\,K}{L}$ (see Eq. 13) in Eq. 19*d*, and manipulating:

$$\frac{\lambda}{L} = \frac{J}{K} - \frac{1}{2} \qquad (20)$$

Values of J corresponding to certain values of q are listed in Col. 6, Table 1. The value of J for $\sin^{-1} q = 90°$ can be obtained as follows:

$$J = \int_0^{\pi/2} d\phi\,\sqrt{1 - 1\sin^2 \phi} = \int_0^{\pi/2} d\phi\,\cos\phi = \sin\frac{\pi}{2} = 1 \qquad (21)$$

The values of λ/L corresponding to certain values of q, and hence corresponding to certain values of δ/L and $L/\hat{r}$ are shown in Col. 7, Table 1, and are plotted in Fig. 2. Using these values the center and end points of the column can be located on a scale drawing, and rough sketches of the column can thus be drawn for various values of δ/L and $L/\hat{r}$, as shown in Table 2.

The sketches are largely self-explanatory. Case 6 corresponds to a limp string, for which the effective value of $\hat{r}$ is zero.

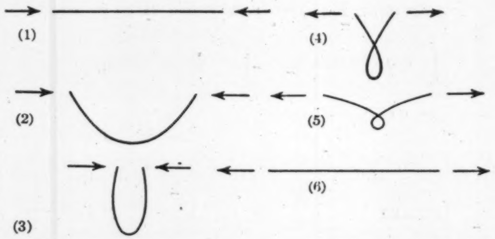
It should be remembered that the results illustrated by Table 2 do not hold true if the stress exceeds the elastic limit of the material.

Graphical Solution for δ/L and λ/L.—To simplify numerical work, curves will be derived by which the values of δ/L and λ/L can be obtained quickly if the values of E, P/A, and $L/\hat{r}$ are known.

The method used to derive the curve of δ/L versus $L/\hat{r}$ in Fig. 2 can readily be extended to derive the curves of δ/L versus P/A for various values of $L/\hat{r}$. Also, by Fig. 2, the curve of λ/L versus δ/L can be drawn. (The latter curve will presently be shown to be independent of the dimensions of the column.) These curves, shown in Fig. 3, can be used to obtain practically all the information that may be desired about the column. For example, with a given column, if a load P is assumed, δ/L can be determined; hence, the maximum moment,

$P\,\delta$, and the maximum bending stress, $\frac{P\,\delta}{Z}$, can be evaluated (Z is the section modulus). Also, with δ/L determined, the value of λ/L can be selected immediately, and the general shape of the column can be approximated.

TABLE 2.—IDEAL COLUMN SHAPES UNDER VARIOUS CONDITIONS
(E = 30,000,000 LB PER SQ IN. AND P/A = 2 KIPS PER SQ IN.)

Case	$\frac{L}{r}$	$\frac{\delta}{L}$	$\frac{\lambda}{L}$	Column shape
1......	384	0	0.50	
2......	415	0.296	0.37	
3......	529	0.401	0.06	
4......	676	0.35	−0.11	
5......	1,330	0.185	−0.316	
6......	∞	0	−0.50	

In selecting a value of λ/L corresponding to a particular value of δ/L, the value of λ/L should lie to the left of the crest of the λ/L-versus-δ/L curve if the value of δ/L lies to the left of the crest of the δ/L-versus-P/A curve, and vice versa.

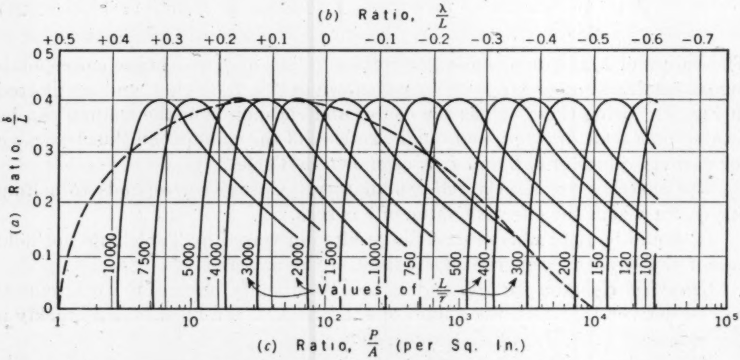


FIG. 3.—GRAPHICAL SOLUTION FOR δ/L AND λ/L

The curves of Fig. 3 are for steel (E = 30,000,000 lb per sq in.). For other values of E, the following relationship should be used:

$$\left(\frac{P}{A}\right)_{\text{indicated}} = \frac{30{,}000{,}000}{E} \times \left(\frac{P}{A}\right)_{\text{actual}} \qquad \text{(22)}$$

The use of Fig. 3 can be illustrated by a numerical example. Consider a machine member of spring bronze ($E = 15{,}000{,}000$ lb per sq in.), 24 in. long and 0.1 in. in diameter, D, loaded according to the conditions of Fig. 1(a) with $P = 1.25$ lb. The problem is to find the approximate shape of the elastic curve and the maximum stress. The slenderness ratio is $L/r = \frac{L}{D/4} = \frac{24}{0.1/4} = 960$; and $P/A = \frac{P}{\pi\, D^2/4} = \frac{4 \times 1.25}{\pi\,(0.1)^2} = 160$ lb per sq in. From Fig. 3, $\delta/L = 0.15$ ± and $\lambda/L = 0.46$. The shape of the column corresponds roughly to the shape shown as case 2, Table 2. The maximum bending stress is equal to $\frac{P\,\delta}{Z} = \frac{P \times \delta/L \times \mathrm{L}}{\pi\, D^3/32} = \frac{32 \times 1.25 \times 0.15 \times 24}{\pi\,(0.1)^3} = 46{,}000 \pm$ lb per sq in.

In this case the direct stress, $P/A = 160$ lb per sq in., is negligible compared to the maximum bending stress.

Exact Shape of the Deflected Column.—To illustrate the completeness of this mathematical solution the exact shape of the elastic curve of a typical column can be derived by finding the values of y/L corresponding to various values of x/L, as x (see Fig. 1(a)) is varied between the values of 0 and λ.

First, integrate Eq. 19b between points B and A, Fig. 1(a), letting $\phi = \beta$ (see Eqs. 10) at point A, as follows:

$$k\int_0^x dx = k\,x = 2\int_0^\beta d\phi\,\sqrt{1-q^2\sin^2\phi} - \int_0^\beta \frac{d\phi}{\sqrt{1-q^2\sin^2\phi}} \equiv 2\,G - H \quad \text{. . (23a)}$$

In other words,

$$k\,x = 2\,G - H \quad \text{. . . . . (23b)}$$

in which G and H are elliptic integrals. From Eq. 13, $k = 2\,K/L$—which value, substituted in Eq. 23b, yields:

$$\frac{x}{L} = \frac{2\,G - H}{2\,K} \quad \text{. . . . . (24)}$$

Assume that the physical constants are such that $q \equiv \frac{k\,\delta}{2} = \sin 75°$. Then, by Table 1, $K = 2.77$. The constants G and H can be evaluated for different values of β by tables,[5] and the corresponding values of x/L can be determined by Eq. 24. Eq. 16 yields $\frac{\delta}{L} = \frac{q}{K} = \frac{\sin 75°}{2.77} = 0.35$. Therefore, the values of y/L corresponding to the various values of β can be determined from Eqs. 10 by writing:

$$\frac{y}{L} = \frac{\delta}{L}\cos\beta = 0.35\cos\beta \quad \text{. . . . . (25)}$$

The computations and the elastic curve are presented in Table 3. Note that the relationship between x/L and y/L depends only on q. This fact leads to an important conclusion, derived as follows:

By referring to Eqs. 13 and 16, it can be seen that q is specified if $\delta/L = q/K$ is specified. Hence, the shape of the column is independent of the column dimensions and depends only on the value of δ/L. (Since there are two load conditions corresponding to each value of δ/L (see Fig. 3), it is necessary to specify whether the column is loaded below or above the crests shown in Fig. 3.)

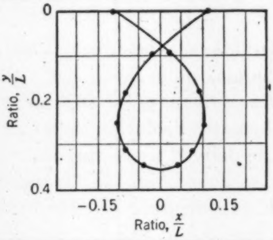
---

[5] "A Short Table of Integrals," by B. O. Peirce, Ginn & Co., Boston, Mass., 3d Ed., 1929, pp. 122 and 123.

3. Equilibrium Positions of a Loaded Column

Several difficulties now require attention. First, the fundamental column formula, Eq. 13, is obviously not valid for values of P/A less than the value corresponding to $\delta/L = 0$ in Fig. 3. (The values of P/A and L/r corresponding to $\delta/L = 0$ are termed "critical" values.) For values of P/A below the critical, the solution $y = 0$ is first tested. Substituting the trial solution—

TABLE 3.—Exact Shape of the Deflected Column ($q = \sin 75°$)

β	G	H	$\frac{x}{L}$	$\frac{y}{L}$	Elastic curve
0.......	0	0	0	0.35	
15.......	0.259	0.264	0.046	0.338	
30.......	0.502	0.548	0.0822	0.3025	
45.......	0.713	0.873	0.099	0.247	
60.......	0.881	1.28	0.0866	0.1745	
75.......	0.999	1.87	0.0231	0.0904	
90.......	1.08	2.77	−0.11	0	

$$y = 0 \quad \text{(26)}$$

—in the fundamental differential Eq. 3, and noting that $\frac{d\theta}{ds}$, the curvature, must equal zero if y is zero everywhere: $P \times 0 \equiv -E\,I \times 0$.

Then, the solution $y = 0$ must be considered a possible solution of Eq. 3 under all admissible conditions. In other words, the column may remain perfectly straight for all values of P/A, whether above or below the critical. Summing up the solutions, Eqs. 13 and 26, the conclusion follows that, with values of P/A less than the critical, the column must remain straight; whereas with values of P/A greater than the critical, the column either can remain straight, or can assume the form dictated by the curves of Fig. 3 and illustrated in Table 2.

It is possible to identify the critical values in this paper with Euler's well-known critical point. By Table 1, the critical value corresponds to $q = 0$, for which value Eq. 13 becomes

$$\frac{k\,L}{2} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - 0}} = \frac{\phi}{2} \quad \text{(27a)}$$

Substituting Eq. 6c in Eq. 27a, $\frac{L}{2}\sqrt{\frac{P}{E\,I}} = \frac{\pi}{2}$; or

$$P = \frac{\pi^2\,E\,I}{L^2} \quad \text{(27b)}$$

which is Euler's familiar expression for critical loading of a simple column. The meaning of the word "critical" will be discussed in detail presently.

As demonstrated, when the value of P/A exceeds the critical, there are two widely different solutions for the shape of the column, one corresponding to a perfectly straight column and one to a shape similar to those sketched in Table 2. Do both of these solutions represent a stable condition? To prove mathematically what is already known from experience—that the straight column is unstable, and the bent column stable—proceed as follows:

In any deflected position (see Fig. 1(*a*)), not necessarily an equilibrium position, the column feels certain internal moments as a result of its curvature. Similarly, the load P applies certain external moments at all points of the deflected column. If, at all points, the internal moment equals the external moment, then the column will be in equilibrium; if the internal moment is greater than the external moment, the curvature will tend to decrease and the column will straighten; and, if the internal moment is less than the external moment, the curvature will tend to increase and the column will assume a sharper bend.

To understand the meaning of the stability of the column, suppose that a loaded column is slightly displaced from its equilibrium position by small transverse forces and that the transverse forces are then removed. If, with the column in the displaced position, the difference between internal and external moments at all points is such as to tend to restore the column to its original position, then the column is stable; otherwise, it is unstable. To simplify the analysis, compare the internal and external moments at the center of the column only. It will then be possible to judge the stability of the column as a whole by inspection.

In evaluating the internal and external moments at the center for different values of δ/L, the internal and external moments at the center are analyzed in units of inch-pounds per unit of cross-sectional area and per unit of length, $\frac{M_i}{A\,L}$ and $\frac{M_e}{A\,L}$, respectively.

First, for a particular value of δ/L, by Fig. 1(*a*):

$$\frac{M_i}{A\,L} = \frac{P'}{A} \times \frac{\delta}{L} \qquad (28a)$$

in which P'/A is the value of P/A which would hold the column in equilibrium, with δ/L equal to the value chosen. Substituting Eq. 14 in Eq. 13,

$$K = \frac{L\sqrt{P'/A}}{2\,r\sqrt{E}} \qquad (28b)$$

from which

$$\frac{P'}{A} = \frac{4\,K^2\,E}{(L/r)^2} \qquad (28c)$$

Substituting Eq. 28*c* in Eq. 28*a*:

$$\frac{M_i}{A\,L} = \frac{4\,K^2\,E}{(L/r)^2} \times \frac{\delta}{L} \qquad (28d)$$

Also, by Fig. 1(*a*):

$$\frac{M_e}{A\ L} = \frac{P}{A} \times \frac{\delta}{L} \qquad \text{(28e)}$$

in which P is the actual load applied to the column.

Consider a typical column subject to various values of the loading, P/A, with $E = 30{,}000{,}000$ lb per sq in.; $L/r = 384$; and $P/A = 1{,}000$, 2,000 (critical), 3,000, and 8,000 lb per sq in. By Eq. 28*d*: $\frac{M_i}{A\ L} = \frac{4\ K^2 \times 30{,}000{,}000}{(384)^2} \times \frac{\delta}{L} = 815\ K^2 \times \frac{\delta}{L}$.

Using methods similar to those for preparing Table 1, the values of $\frac{M_i}{A\ L}$ corresponding to certain values of $\frac{\delta}{L}$ can be tabulated, as shown in Table 4.

TABLE 4.—VALUES OF $\frac{M_i}{A\ L}$

$\text{Sin}^{-1}\,q$ (degrees)	$\delta/L = q/K$	K	$\frac{M_i}{A\ L}$
0..........	0	1.57	0
30..........	0.296	1.69	690
60..........	0.401	2.16	1,520
75..........	0.35	2.77	2,200
89..........	0.185	5.43	4,450
90..........	0	∞	∞

The curve of $\frac{M_e}{A\ L}$ versus $\frac{\delta}{L}$ will be a straight line passing through the origin. Also, if $\frac{\delta}{L} = 0.4$, then by Eq. 28*e* the values of $\frac{M_e}{A\ L}$ corresponding to the different values of P/A will be as follows:

$\frac{P}{A}$	$\frac{M_e}{A\ L}$
1,000..................	400
2,000..................	800
3,000..................	1,200
8,000..................	3,200

The values of $\frac{M_i}{A\ L}$ and $\frac{M_e}{A\ L}$ can now be plotted against those of $\frac{\delta}{L}$, as in Fig. 4. The stability of the column, as revealed by these curves, can be studied as follows:

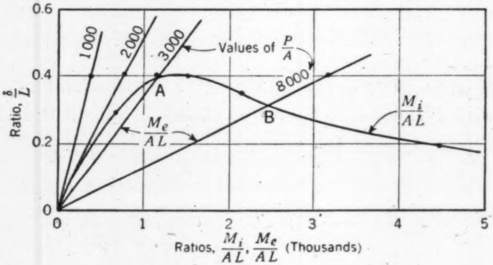


FIG. 4

First, suppose that $\frac{P}{A} = 1$ kip per sq in., a value that is less than the critical. Assume that the column is in the straight position, $y = 0$. If transverse forces are applied to cause the column to take a shape approximately corresponding to some value of δ/L and if these forces are then removed, at the instant of removal of the forces $\frac{M_i}{A\ L}$ will always be larger than $\frac{M_e}{A\,L}$, according to Fig. 4, and the column will return to the position $y = 0$. Therefore, with P/A less than the critical value, the column can have only one possible shape—perfectly straight. This result agrees with previous conclusions in this paper.

Next, suppose that $\frac{P}{A} = 3$ kips per sq in., a value that is greater than the critical, and which will cause the column to be in equilibrium at a value of δ/L corresponding to point A, Fig. 4. The shape of the column at this value of δ/L corresponds roughly to case 3, Table 2. If, by transverse forces, the column is slightly displaced in such a direction as to increase δ/L, then $\frac{M_i}{A\,L}$ will be larger than $\frac{M_e}{A\,L}$; and, by comparison with case 3, Table 2, it is seen that the column will return to the equilibrium position as soon as the transverse forces are withdrawn. On the other hand, if the column is displaced in such a direction as to decrease δ/L, then $\frac{M_i}{A\,L}$ will be smaller than $\frac{M_e}{A\,L}$; and, by comparison with case 3, Table 2, it is seen that the column will again return to the equilibrium position when the transverse forces are withdrawn. Point A, therefore, is a point of stable equilibrium.

If $P/A = 8$ kips per sq in., equilibrium point B will lie on the descending section of the $\frac{M_i}{A\,L}$-versus-$\frac{\delta}{L}$ curve of Fig. 4. The shape of the column at point B corresponds roughly to case 4, Table 2. Point B can be proved to be a point of stable equilibrium by the methods just used to investigate point A, but the details of proof will be left to the reader.

Consider that P/A equals either 3 kips per sq in. or 8 kips per sq in., and that the column is in the perfectly straight position. At any value of δ/L less than the equilibrium value at either point A or point B, $\frac{M_e}{A\,L}$ will be larger than $\frac{M_i}{A\,L}$. Therefore, the slightest disturbance of the column will suffice to cause a sudden large deflection corresponding to point A or point B. (The deflection cannot proceed past point A or point B, for reasons discussed in the preceding paragraphs.) This sudden large deflection is called "buckling." It is familiar to engineers, but its mathematical and physical reasons are generally not clearly understood.

Finally, let $P/A = 2$ kips per sq in., the critical value. The curves of $\frac{M_i}{A\,L}$ and $\frac{M_e}{A\,L}$ appear to be tangent at the origin. If this is true (and it is indeed true), then at any point close to the origin $\frac{M_i}{A\,L}$ and $\frac{M_e}{A\,L}$ are nearly equal. Therefore, the deflection of the column must be indeterminate. (This conclusion is proved in an entirely different manner in Section 4.)

The conclusions of this section can be summarized as follows:

(1) At values of P/A below the critical, an ideal column must remain perfectly straight.

(2) At values of P/A above the critical, the column may either remain perfectly straight or may assume a shape dictated by Fig. 3 and illustrated by Table 2. If the column is straight, the slightest displacement will suffice to cause the transverse deflection to change suddenly from zero to the deflection corresponding to the the curves of Fig. 3.

(3) At values of P/A close to the critical, the deflection is indeterminate, and may assume any fairly small value.

It remains to prove the assumption that, at the critical load, the $\frac{M_i}{A\,L}$-versus-$\frac{\delta}{L}$ curve is tangent to the $\frac{M_e}{A\,L}$-versus-$\frac{\delta}{L}$ curve at $\frac{\delta}{L} = 0$. By Eq. 28*e*, the the slope of the $\frac{M_e}{A\,L}$-versus-$\frac{\delta}{L}$ curve at $\frac{\delta}{L} = 0$ equals:

$$\frac{d\left(\frac{M_e}{A\,L}\right)}{d\left(\frac{\delta}{L}\right)} = \frac{d\left(\frac{P_e}{A} \times \frac{\delta}{L}\right)}{d\left(\frac{\delta}{L}\right)} = \frac{P_e}{A} \qquad (29a)$$

in which P_e is the critical load. By Eq. 28*a*, the slope of the $\frac{M_i}{A\,L}$-versus-$\frac{\delta}{L}$ curve equals:

$$\frac{d\left(\frac{P'}{A} \times \frac{\delta}{L}\right)}{d\left(\frac{\delta}{L}\right)} = \frac{P'}{A} + \frac{\delta}{L} \times \frac{d\left(\frac{P'}{A}\right)}{d\left(\frac{\delta}{L}\right)} \qquad (29b)$$

Since at the critical point $\frac{P'}{A} = \frac{P_e}{A}$ and $\frac{\delta}{L} = 0$, then, by comparing Eqs. 29, it may be seen that the curves of $\frac{M_i}{A\,L}$ and $\frac{M_e}{A\,L}$ versus $\frac{\delta}{L}$ are indeed tangent at the origin.

This result leads to a further interesting conclusion in regard to the shape of the curves of Fig. 3 in the region close to the critical point: If $\frac{M_i}{A\,L}$ and $\frac{M_e}{A\,L}$ are equal near the critical point, then δ/L must be indeterminate, as has been stated; if δ/L is indeterminate, it is obvious that $\frac{d\left(\frac{P}{A}\right)}{d\left(\frac{\delta}{L}\right)}$ must be zero at the critical point. This conclusion agrees with the shape of the curves of Fig. 3.

4. Actual Stress and Factor of Safety Below the Critical Point

All structural columns, and most machinery columns, are designed to operate below the critical point. In these columns, the appreciable transverse deflection that occurs at the critical point constitutes failure. In the case of relatively short columns, the direct stress P/A reaches the yield point and causes failure before the load reaches the critical. The factor of safety, N, must therefore be based either on P_c, or on P_y (the load that produces a direct stress equal to σ_y, the yield-point stress)—whichever governs.

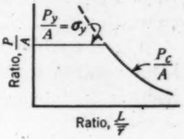


FIG. 5

In the case of steel columns, with E and σ_y taken at 30,000,000 lb per sq in. and 30,000 lb per sq in., respectively, the values of P_y/A and P_c/A may be plotted against the values of $L/\bar{r}$ (see Fig. 5). The ratio P_y/A $=$ a constant $= \sigma_y$, whereas P_c/A may be derived from Euler's formula by writing $P_c = \dfrac{\pi^2 E I}{L^2} = \dfrac{\pi^2 E A \bar{r}^2}{L^2}$; or

$$\frac{P_c}{A} = \frac{\pi^2 \times 30{,}000{,}000}{\left(\frac{L}{\bar{r}}\right)^2} = \frac{296{,}000{,}000}{\left(\frac{L}{\bar{r}}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots (30)$$

The dotted section of the curve in Fig. 5 would be reached only after failure had already occurred, caused by a direct stress greater than σ_y. Neglecting this section, one may proceed to redraw the remainder of the curve for different factors of safety, simply dividing the values of P_y/A and P_c/A by the assumed values of N. In Fig. 6, if σ_y is not equal to 30,000 lb per sq in., or, if E is some value other than 30,000,000 lb per sq in., $N_y \propto \sigma_y$ and $N_c \propto E$.

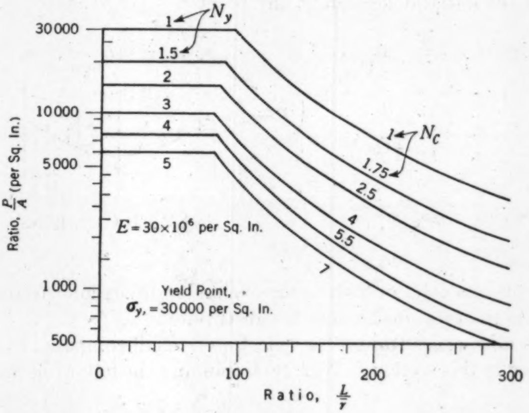


FIG. 6.—FACTORS OF SAFETY OF SIMPLE IDEAL COLUMNS (If $\sigma_y \neq 30{,}000$ LB PER SQ IN. OR $E \neq 30{,}000{,}000$ LB PER SQ IN., THEN $N_y \propto \sigma_y$ AND $N_c \propto E$)

It seems fair to assign larger values of N to the part of each curve based on P_c than to the part based on P_y, since failure at the critical point is much more violent and complete than is failure caused by the direct stress reaching the value σ_y. If the quantity $N - 1$ is considered as a "margin of safety," it seems desirable that:

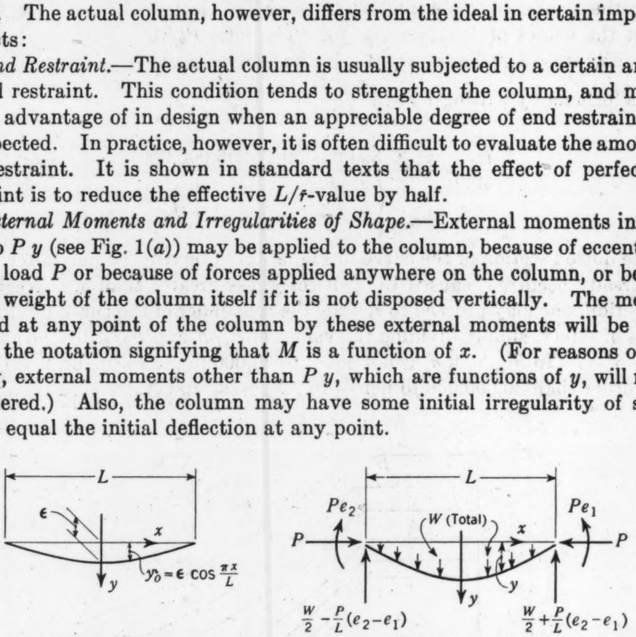
$$N_c - 1 = 1.5\,(N_y - 1) \qquad (31)$$

in which the subscripts c and y denote whether the factor of safety is based on P_c or P_y. Accordingly, the values of N that are indicated for the straight and curved parts of each of the curves of Fig. 6 were chosen to agree with Eq. 31.

Fig. 6 enables the designer to determine the factor of safety of any simple, ideal column loaded below the critical point. According to the analysis in Section 3, the stress in such a column is nothing more or less than the direct stress. The actual column, however, differs from the ideal in certain important respects:

End Restraint.—The actual column is usually subjected to a certain amount of end restraint. This condition tends to strengthen the column, and may be taken advantage of in design when an appreciable degree of end restraint may be expected. In practice, however, it is often difficult to evaluate the amount of end restraint. It is shown in standard texts that the effect of perfect end restraint is to reduce the effective L/r-value by half.

External Moments and Irregularities of Shape.—External moments in addition to $P\,y$ (see Fig. 1(a)) may be applied to the column, because of eccentricity of the load P or because of forces applied anywhere on the column, or because of the weight of the column itself if it is not disposed vertically. The moment exerted at any point of the column by these external moments will be called $M(x)$, the notation signifying that M is a function of x. (For reasons of simplicity, external moments other than $P\,y$, which are functions of y, will not be considered.) Also, the column may have some initial irregularity of shape. Let y_o equal the initial deflection at any point.


L
ε
x
y
$y_0 = \epsilon \cos \frac{\pi x}{L}$
L
Pe_2
W (Total)
x
Pe_1
P
P
y
y
$\frac{W}{2} - \frac{P}{L}(e_2 - e_1)$
$\frac{W}{2} + \frac{P}{L}(e_2 - e_1)$

(a) INITIAL SHAPE (b) LOADED CONDITIONS

FIG. 7

An example of a column such as the one under discussion is shown in Fig. 7, in which W is the total load uniformly distributed.

Only columns whose transverse deflection is small compared to their length are considered in this section. With such columns, the following simplifications are possible:

(a) The curvature may be expressed by $\frac{d^2y}{dx^2}$ (see Section 1, Eq. 4).

(b) The projected length of the column on the x-axis equals the developed length.

Eq. 4 can be extended to mean that M, the moment at any point, is equal to $-\,E\,I$ times the difference between the final and initial curvatures of the column. Then, since the initial curvature of the column is equal to $\frac{d^2y_o}{dx^2}$, Eq. 4 can be rewritten:

$$M = P\,y + M(x) = -\,E\,I\left(\frac{d^2y}{dx^2} - \frac{d^2y_o}{dx^2}\right) \ldots\ldots\ldots\ldots (32)$$

Letting $k^2 \equiv \frac{P}{E\,I}$, as before, Eq. 32 can be written:

$$\frac{d^2y}{dx^2} + k^2\,y = \frac{d^2y_o}{dx^2} - \frac{M(x)}{E\,I} \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

Eq. 33 is a linear differential equation of the second order with constant coefficients, and is treated in every standard work on differential equations. For the column shown in Fig. 7, the solution of Eq. 33 works out to be:

$$\begin{aligned} y = {} & \left(\frac{W}{P\,L\,k^2} + \frac{e_2 + e_1}{2}\right)\frac{\cos k\,x}{\cos \frac{k\,L}{2}} - \frac{e_2 - e_1}{2} \\ & \times \frac{\sin k\,x}{\sin \frac{k\,L}{2}} + \frac{\epsilon\,\pi^2}{\pi^2 - k^2\,L^2}\cos\frac{\pi\,x}{L} + \frac{W\,x^2}{2\,L\,P} \\ & + \frac{e_2 - e_1}{L}\,x - \frac{W}{P}\left(\frac{L}{8} + \frac{1}{L\,k^2}\right) - \frac{e_2 + e_1}{2} \ldots\ldots\ldots\ldots (34) \end{aligned}$$

Knowing y, the moment M can be determined by Eq. 32. Hence, the maximum bending stress at any section, $\sigma_b = \frac{M\,c}{I}$ (in which c is the distance from the neutral axis to the extreme fibers), can be calculated. The total stress is:

$$\sigma_t = \sigma_b + \frac{P}{A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35a)$$

and its maximum value can be determined by setting $\frac{d\sigma_t}{dx} = 0$, according to the routine methods of the differential calculus. The factor of safety may be defined as the ratio between the external loading producing the yield-point stress σ_y and the actual external loading, and may be determined by trial.

If $e_2 = e_1 = e$, and $\epsilon = W = 0$, then, by Eqs. 34 and 32, it can readily be shown that the maximum bending stress equals:

$$\sigma_b = E\,k^2\,e\,c\sec\frac{k\,L}{2} \ldots\ldots\ldots\ldots\ldots\ldots (35b)$$

Eq. 35*b* is the well-known secant formula.

Again, if $e_2 = e_1 = W = 0$, then, by Eqs. 34 and 32, the maximum bending stress equals:

$$\sigma_b = \frac{E\,\epsilon\,c\,\pi^2\,k^2}{\pi^2 - k^2\,L^2} \ldots\ldots\ldots\ldots\ldots\ldots (35c)$$

Eq. 35*c* should prove useful in many practical cases. Most structural members are not fabricated perfectly straight, but have a camber. The actual shape of the column is usually roughly similar to a cosine curve, and the value of ϵ can be determined easily by stretching a string between the ends of the member. By substituting the highest probable value of ϵ in Eq. 35*c*, the highest probable bending stress caused by initial camber may be determined.

Eq. 35*c*, the camber formula, should be carefully distinguished from Eq. 35*b*, the secant formula, which gives the bending stress due to eccentricity of the load *P*.

It is possible to derive Euler's formula for critical loading from Eq. 34 because, if $e_2 = e_1 = W = \epsilon = 0$, then:

$$y = \frac{\epsilon\,\pi^2}{\pi^2 - k^2 L^2} \qquad (36)$$

If ϵ is reduced to zero and $k^2 = \frac{\pi^2}{L^2}$, then $y = \frac{0}{0}$, which is indeterminate, and $P = \frac{\pi^2 E I}{L^2}$, which constitutes Euler's condition.

In the case just mentioned, if $\epsilon = 0$ and $k^2 \neq \frac{\pi^2}{L^2}$, then Eq. 34 reduces to $y = 0$. Therefore, the simplified differential Eq. 4, from which Eq. 34 was derived, fails to give a satisfactory solution in the case of an ideal column. For that reason it was necessary to use the more exact, if more difficult, Eq. 3 in developing the theory of the ideal column.

The Step-by-Step Method.—The formal solution of Eq. 33, tedious under the best of conditions, becomes prohibitively complicated in certain cases, especially when *I* is variable or $M(x)$ is discontinuous. In such cases it is usually more practical to use some form of numerical integration. The step-by-step method of numerical integration, a mathematical weapon of long standing, has certain distinct advantages in connection with the solution of column problems:

(*a*) It is not a trial-and-error method, but leads directly to the solution.

(*b*) It is universally applicable to all columns whose relative deflection is small, enabling the designer to solve widely different types of column problems with only minor changes of technique.

(*c*) It may be used to obtain a rough solution fairly quickly or to obtain an accurate solution at the expense of more numerical work, as desired.

In the step-by-step method the column is divided into a number of intervals, not necessarily equal. All variable quantities are considered to be constant during each interval. The change, Δ, in any quantity in each interval is obtained numerically, and the value of any quantity at the end of any interval is obtained by adding Δ to the value of the quantity at the beginning of the interval.

Applying this process of reasoning to Eq. 32 and letting the order of differentiation be denoted by primes $\left(\text{for example, } y'' \doteq \frac{d^2y}{dx^2}\right)$:

$$y'' = y''_o - \frac{M}{E I} \qquad (37)$$

Integrate Eq. 37 from x = A to an indefinite point x = B, considering that the values of y and of its derivatives are known at point A. Then:

$$y'_B - y'_A = (y'_o)_B - (y'_o)_A - \int_A^B \frac{M\,dx}{E\,I} \ldots\ldots\ldots\ldots (38a)$$

Now, if the interval is small, $\frac{M}{E\,I}$ can be considered to be practically constant between points A and B. Then:

$$\Delta y' = \Delta y'_o - \frac{i\,M}{E\,I} \ldots\ldots\ldots\ldots\ldots\ldots (38b)$$

in which $i = x_B - x_A$.

Interpreted physically, Eq. 38*b* means that the change of slope of the elastic curve between point A and point B equals the change of slope of the unloaded column, plus the change of slope caused by moment—the latter term being equal to $-\frac{i\,M}{E\,I}$.

Next, integrate Eq. 38*a* over the interval, once more letting $(y'_o)_B = y'_o$ and $y'_B = y'$, remembering that $(y'_o)_A$ and y'_A are constant:

$$y_B - y_A - i\,y'_A = (y_o)_B - (y_o)_A - i\,(y'_o)_A - \int_A^B \int_A^B \frac{M\,dx\,dx}{E\,I} \ldots (39)$$

If $x_B = x$ and $\frac{M}{E\,I}$ is a constant during the interval, the last term of Eq. 39 can be expanded as follows:

$$\int_A^B \int_A^B \frac{M\,dx\,dx}{E\,I} = \frac{M}{E\,I} \int_A^B (x - x_A)\,dx = \frac{M}{E\,I}\left(\frac{x^2}{2} - x_A\,x\right)_A^B$$

$$= \frac{M}{E\,I}\left(\frac{x^2_B}{2} - \frac{x^2_A}{2} - x_A\,x_B + x^2_A\right) = \frac{M}{2\,E\,I}(x_B - x_A)^2 = \frac{i^2\,M}{2\,E\,I} \ldots (40)$$

Thus, Eq. 39 becomes:

$$\Delta y = \Delta y_o - i\,(y'_o)_A + i\,y'_A - \frac{i^2\,M}{2\,E\,I} \ldots\ldots\ldots\ldots (41)$$

Then, in physical terms, the change of deflection over an interval equals the change of deflection of the unloaded column, plus the change of deflection produced by the slope $y'_A - (y'_o)_A$ acting over the length i, plus the change of deflection caused by moment—the last term being equal to $-\frac{i^2\,M}{2\,E\,I}$.

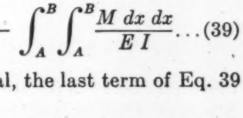


FIG. 8

Eqs. 38*b* and 41 are the keys to the method. The actual technique of procedure will be illustrated by a numerical example. The column shown in Fig. 8 is a steel rod 1 in. in diameter. The unloaded deflection may be expressed by

$$y_o = \epsilon \cos \frac{\pi\,x}{L} \ldots\ldots\ldots\ldots\ldots\ldots (42)$$

Numerical values are as follows: $E = 30{,}000{,}000$ lb per sq in.; $I = 0.0491$ in.^{4}; $A = 0.785$ in.^{2}; $L = 30$ in.; $\epsilon = 0.1$ in.; $P = 2{,}700$ lb; and $e = 1$ in.

The column will be divided into 5-in. intervals, starting from the left end. Considering Eq. 38*b*, it is clear that y'_o and $\frac{i\,M}{E\,I}$ must be evaluated. The first of these terms can be calculated from the known initial deflection of the column, as follows: $y_o = \epsilon \cos \frac{\pi x}{L}$, $y'_o = -\frac{\epsilon \pi}{L} \sin \frac{\pi x}{L} = -0.01048 \sin (6\,x)^\circ$.

If $(y'_o)_n$ represents the value of y'_o at the beginning of the *n*th interval, then:

$$\Delta y'_o = (y'_o)_{n+1} - (y'_o)_n \qquad (43)$$

The calculations leading to the values of $\Delta y'_o$ are tabulated in Cols. 1, 2, 3, and 4 of auxiliary Table 5(*a*), and the values of $\Delta y'_o$ are listed in Col. 15 of main Table 5(*b*).

In cases where $\Delta y'_o$ cannot be derived analytically, it can always be obtained graphically, from the known initial deflection of the column.

Next consider the quantity $\frac{i\,M}{E\,I}$:

$$\frac{i\,M}{E\,I} = \frac{i\,M(x)}{E\,I} + \frac{i\,M(y)}{E\,I} \qquad (44)$$

in which $M(x)$ is a function of x and constants, and $M(y)$ is a function of y alone. Moment $M(x)$ can be calculated immediately, by the same methods that are used to calculate the moment at any point of an ordinary beam, whereas the calculation of $M(y)$ at any point must be deferred until the value of y at that point is known.

A word in regard to sign conventions will be in order at this point. If x is positive to the right, and y is positive downward, as in Fig. 8, then moments acting to the left of any section are positive if they are clockwise, and moments acting to the right of the section are positive if they are counterclockwise. This condition will be found to be in accordance with previous work in this paper. By Fig. 8:

$$M(x) = \frac{P\,e}{L}\left(\frac{L}{2} + x\right) = \frac{P\,e}{2} + \frac{P\,e\,x}{L} \qquad (45a)$$

Then:

$$\frac{i\,M(x)}{E\,I} = \frac{5}{30 \times 10^6 \times 0.0491}\left(\frac{2{,}700}{2} + \frac{2{,}700\,x}{30}\right) = 0.00459 + 0.000306\,x. \qquad (45b)$$

In calculating the value of $\frac{i\,M(x)}{E\,I}$, the value of x at the middle of the interval will be used. The value of $\frac{i\,M(x)}{E\,I}$ so obtained will be a good approximation of the average value of $\frac{i\,M(x)}{E\,I}$ over the entire interval. (The subscript i will be used where necessary to denote quantities which are taken at the middle of an interval.) The calculations for determining $\frac{i\,M(x)}{E\,I}$ are tabulated in Cols. 8, 9, and 10, Table 5.

The next step is to compute the value of $\frac{i\,M(y)}{E\,I}$. By Fig. 8, $M(y) = P\,y$, or

$$\frac{i\,M(y)}{E\,I} = \frac{5}{30 \times 10^6 \times 0.0491} \times 2{,}700\,y = 0.00918\,y \ldots\ldots (46a)$$

An approximate value y_i (the value of y at the middle of any interval) can be determined for use in Eq. 46*a*; thus:

$$y_i = y + \frac{i}{2}\,y' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (46b)$$

in which the quantities without subscripts are taken at the beginning of the interval. From Fig. 8, at the beginning of the first interval, $y = 0$ (Col. 23) and $y' = \theta$ (Col. 18), in which θ is defined as the angle of slope at the left end of the column. Then $\frac{i\,y'}{2} = 2.5\,\theta$ (Col. 24), and $y_i = 0 + 2.5\,\theta$ (Col. 12). By Eq. 46*a*, $\frac{i\,M(y)}{E\,I} = 0.00918 \times 2.5\,\theta = 0.02295\,\theta$ (Col. 11). Adding Cols. 10 and 11: $\frac{i\,M}{E\,I} = 0.000766 + 0.02295\,\theta$ (Col. 16). By Eq. 38*b*, $\Delta y' = -\,0.002166 - 0.02295\,\theta$ (Col. 17). Adding Cols. 17 and 18, the value of y' at the beginning of the second interval is obtained; thus: $y' = -\,0.002166 + 0.97705\,\theta$.

It will be necessary to carry along the unknown, θ, in the computations until some known condition of the column makes a solution possible. For the present, θ will be treated exactly as any other of the values.

The next step is the utilization of Eq. 41. The value of Δy_o is equal to:

$$(y_o)_{n+1} - (y_o)_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47a)$$

and may be obtained analytically or graphically. In the present case,

$$y_o = 0.1 \cos\,(6\,x)^\circ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47b)$$

The values of y_o and Δy_o are listed in Cols. 5 and 6, Table 5. The values of y'_o have already been derived (Col. 4). Since $i = 5$, the values of $-\,i\,y'_o$ (Col. 7) can be obtained. Then the values of $-\,\frac{\Delta y_o}{i\,y'_o}$ are listed in Col. 20.

The first value of $i\,y' = 5\,y'$ can be obtained from Col. 18, and is shown in Col. 19. Also, the value of $-\,\frac{i^2\,M}{2\,E\,I} = -\,2.5 \times \frac{i\,M}{E\,I}$ can be obtained from Col. 16, and is shown in Col. 21. Referring to Eq. 41, Δy is equal to $-\,0.004316 + 4.9426\,\theta$ (Col. 22). Adding this value to the first value of y (Col. 23), the value of y at the beginning of the second interval is finally obtained; thus: $y = -\,0.004316 + 4.9426\,\theta$.

The reader will have noted that care was exercised in grouping and arranging the columns properly for easy numerical manipulation. The arrangement actually chosen was not based on any general rules, but was built up after a few trial calculations to suit the needs of this particular problem.

The calculations will now be traced through the second interval. Knowing y (Col. 23) and y' (Col. 18) from previous calculations, the values of y_i (Eq. 46*b* and Col. 12) and of $\frac{i\,M(y)}{E\,I}$ (Eq. 46*a* and Col. 11) can be found. Cols. 10 and 11 are added to find $\frac{i\,M}{E\,I}$ (Col. 16). By Eq. 38b, $\Delta y'$ (Col. 17) is found. From previous computation, $y' = -\,0.002166 + 0.97705\,\theta$ (Col. 18). Adding this value to the second value of $\Delta y'$, $y' = 0.008212 + 0.9094\,\theta$ at the beginning of the third interval.

The second values of $i\,y'$ (Col. 19) and $-\frac{i^2\,M}{2\,E\,I}$ (Col. 21) are easily obtained from Cols. 18 and 16, respectively. By Eq. 41 $\Delta y = -\,0.02513 + 4.716\,\theta$ (Col. 22). Adding this value to the second value of y results in: $y = -\,0.029446 + 9.659\,\theta$ at the beginning of the third interval (Col. 23).

TABLE 5.—COMPUTATIONS

x (1)	$6\,x$ (2)	Sin $(6\,x)^0$ (3)	y'_o (4)	y_o (5)	Δy_o (6)	$-i\,y'_o$ (7)	x_i (8)	$0.000306\,x_i$ (9)	$\frac{i\,M(x)}{E\,I}$ (10)	$\frac{i\,M(y)}{E\,I}$ (11)	y_i (12)	y (13)
(*a*) AUXILIARY TABLE												
−15	−90	−1	0.01048	0	0.05	−0.0524	−12.5	−0.003824	0.000766	$0.02295\,\theta$	$2.5\,\theta$	0
−10	−60	−0.866	0.00908	0.05	0.0366	−0.0454	− 7.5	−0.002295	0.002295	-0.000089 $+0.0677\,\theta$	$-\ 0.009726$ $+\ 7.386\,\theta$	0.115
− 5	−30	−0.5	0.00524	0.0866	0.0134	−0.0262	− 2.5	−0.000765	0.003825	-0.000459 $+0.1096\,\theta$	$-\ 0.049976$ $+11.929\,\theta$	0.205
0	0	0	0	0.1	−0.0134	0	2.5	0.000765	0.005355	-0.001228 $+0.146\,\theta$	$-\ 0.133756$ $+15.930\,\theta$	0.245
5	30	0.5	−0.00524	0.0866	−0.0366	0.0262	7.5	0.002295	0.006885	-0.002433 $+0.176\,\theta$	$-\ 0.265076$ $+19.200\,\theta$	0.225
10	60	0.866	−0.00908	0.0500	−0.05	0.0454	12.5	0.003824	0.008414	-0.004025 $+0.198\,\theta$	$-\ 0.438246$ $+21.59\,\theta$	0.142
15	90	1	−0.01048	0	..	..	..	..	..	..	..	..

Thus, the calculations are carried along until, finally, $y = -\,0.540026 + 22.29\,\theta$ at $x = 15$; but, according to Fig. 8, y at this point equals zero. Therefore, $0 = -\,0.540026 + 22.29\,\theta$, or $\theta = 0.0242$.

With this value of θ substituted in the computations, any numerical values desired can be determined. For example, the numerical values of y can be computed, and are shown in Col. 13. For the sake of comparison, the values of

y_e in Col. 14 are the values of y obtained by the "exact" equation (Eq. 34). The correspondence is remarkably close, in spite of the fact that the calculations were performed on an ordinary 10-in. slide rule. The discrepancy is less than 1%, an insignificant amount in engineering calculations.

With the value of θ known, the values of $\frac{i\,M}{E\,I}$ (Col. 16) can be determined. The maximum moment is found to occur at the right end of the column, and is found equal simply to $P\,e = 2{,}700$ in.-lb. The maximum bending stress is $\sigma_b = \frac{M\,c}{I} = \frac{2{,}700 \times 0.5}{0.0491} = 27{,}500$ lb per sq in. The direct stress, P/A, equals $2{,}700/0.785 = 3{,}500$ lb per sq in.; and the total stress equals $\sigma_t = \sigma_b + \frac{P}{A} = 31{,}000$—say, 30,000 lb per sq in.

Using a factor of safety of $N = 4$ based on a yield point of 30,000 lb per sq in., the allowable load is found equal to $P_a = \frac{P}{4} = \frac{2{,}700}{4} = 675 \pm$ lb in which P_a is the allowable load.

FOR THE COLUMN IN FIG. 8

y_e (14)	$\Delta y'_e$ (15)	$\frac{i\,M}{E\,I}$ (16)	$\Delta y'$ (17)	y' (18)	$i\,y'$ (19)	$\frac{\Delta y_e}{-i\,y'_e}$ (20)	$-\frac{i^2\,M}{2\,E\,I}$ (21)	Δy (22)	y (23)	$\frac{i\,y'}{2}$ (24)	x (1)
	(*b*) MAIN TABLE										
0	-0.00140	0.000766 $+0.02295\,\theta$	-0.002166 $-0.02295\,\theta$	θ	$5\,\theta$	-0.0024	-0.001916 $-0.0574\,\theta$	-0.004316 $+4.9426\,\theta$	0	$2.5\,\theta$	-15
0.114	-0.00384	0.002206 $+0.0677\,\theta$	-0.006046 $-0.0677\,\theta$	-0.002166 $+0.97705\,\theta$	-0.01082 $+4.885\,\theta$	-0.0088	-0.00551 $-0.169\,\theta$	-0.025130 $+4.716\,\theta$	$-\ 0.004316$ $+\ 4.9426\,\theta$	-0.005410 $+2.443\,\theta$	-10
0.203	-0.00524	0.003366 $+0.1096\,\theta$	-0.008606 $-0.1096\,\theta$	-0.008212 $+0.9094\,\theta$	-0.04105 $+4.545\,\theta$	-0.0128	-0.00841 $-0.274\,\theta$	-0.06226 $+4.271\,\theta$	$-\ 0.029446$ $+\ 9.659\,\theta$	-0.02053 $+2.270\,\theta$	$-\ 5$
0.244	-0.00524	0.004127 $+0.146\,\theta$	-0.009367 $-0.146\,\theta$	-0.016818 $+0.7998\,\theta$	-0.0842 $+4.00\,\theta$	-0.0134	-0.01032 $-0.365\,\theta$	-0.10792 $+3.635\,\theta$	$-\ 0.091706$ $+13.930\,\theta$	-0.04205 $+2.00\,\theta$	0
0.224	-0.00384	0.004452 $+0.176\,\theta$	-0.008292 $-0.176\,\theta$	-0.026185 $+0.654\,\theta$	-0.1310 $+3.270\,\theta$	-0.0104	-0.01112 $-0.440\,\theta$	-0.15252 $+2.830\,\theta$	$-\ 0.199626$ $+17.565\,\theta$	-0.06545 $+1.635\,\theta$	5
0.141	-0.00140	0.004389 $+0.198\,\theta$	..	-0.034477 $+0.478\,\theta$	-0.1723 $+2.390\,\theta$	-0.0046	-0.01098 $-0.495\,\theta$	-0.18788 $+1.895\,\theta$	$-\ 0.352146$ $+20.395\,\theta$	-0.0861 $+1.195\,\theta$	10
..	..	..	..	..	..	..	..	..	$-\ 0.540026$ $+22.290\,\theta$	..	15

This result can be compared with the result yielded by one of the rough formulas commonly used by engineers in designing columns subject to both axial load and moment. Such a formula is:

$$\left(\frac{P}{A}\right)_a = \frac{P}{A} + \frac{M\,c}{I} \quad \text{.....................} (48)$$

in which P/A is the direct stress, M is the maximum moment with the column at its initial deflection, and $\left(\frac{P}{A}\right)_a$ is the direct stress that would be allowed for the column if M were zero. Noting that $\frac{L}{r} = 120$, the proper value of $\left(\frac{P}{A}\right)_a$ to yield a factor of safety of $N = 4$ is 5,150 lb per sq in., by Fig. 6. The value of M can be shown to be a maximum at the right end of the column, where it is equal to $P\,e = P$. Then, by Eq. 48: $5{,}150 = \frac{P}{0.785} + \frac{P \times 0.5}{0.0491}$, or $P = 450$ lb.

Then, for a factor of safety of $N = 4$, the allowable value $P = 450$ lb, as obtained by the approximate formula (Eq. 48), is 33% smaller than the allowable value $P = 675$ lb, as obtained by the relatively accurate step-by-step method. Clearly, formulas which are of the same order of approximation as Eq. 48 are unsatisfactory in close design. They are very useful, however, as first approximations.

The versatility of the step-by-step method can be demonstrated by applying it to a very different type of problem. The column shown in Fig. 9 is fixed at its lower end and free at its upper end. The lower part, 200 in. long, is made of steel pipe (outside diameter, 4 in. and inside diameter, 3 in.). The upper part, 250 in. long, is made of steel pipe (outside diameter, 3 in. and inside diameter, 2 in.). A wind force, of 30 lb per square foot of projected area, acts transversely. A load P acts at the top with an eccentricity of e in. The distributed weight of the column acts vertically.

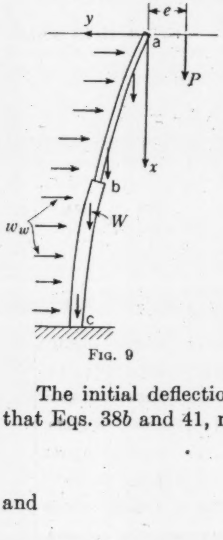


FIG. 9

The initial deflection of the column is assumed to be practically zero, so that Eqs. 38*b* and 41, respectively, reduce to:

$$\Delta y' = -\frac{i\,M}{E\,I} \qquad (49a)$$

and

$$\Delta y = i\,y'_A - \frac{i^2 M}{2\,E\,I} \qquad (49b)$$

For illustrative purposes the column is divided into only four intervals. Let $i = 125$ in. between points a and b, and 100 in. between points b and c. Also, let the wind force be represented by w_w pounds per linear inch, and assume that W, the weight of the column in each interval, is concentrated at the middle of the interval. (This last simplification is convenient rather than necessary; a closer approximation to the true condition could readily be obtained, if desired.)

The axes x and y have been chosen to agree with the adopted sign conventions. By rotating Fig. 9 through 90° counterclockwise, it can be seen that this column is really one half of a symmetrical column of the conventional type, with the left end at point a and the center at point c. Begin the computation at point a, where $y = 0$ and $y' = \theta$; then progress along the column as far as point c. At this point the knowledge that $y' = 0$ will permit a solution for θ. Knowing the value of θ, it is then possible to evaluate y, y', M, etc., at the various points of the column.

The numerical values are as follows: $P = 1{,}000$ lb; $e = 10$ in.; and $E = 30{,}000{,}000$ lb per sq in.

From point a to point b: $A = 3.93$ in.^{2}; $I = 3.19$ in.^{4}; $w_w = \dfrac{30 \times 3}{144} = 0.625$ lb per in.; $i = 125$ in.; $W = i\,A \times$ weight per cubic inch $= 125 \times 3.93 \times 0.2834 = 140$ lb; and $\dfrac{i}{E\,I} = 1.305 \times 10^{-6}$.

From point b to point c: $A = 5.5$ in.^{2}; $I = 8.59$ in.^{4}; $w_w = \dfrac{30 \times 4}{144} = 0.833$ lb per in.; $i = 100$ in.; $W = 100 \times 5.5 \times 0.2834 = 156$ lb; and $\dfrac{i}{E\,I} = 0.388 \times 10^{-6}$.

Also:

$$M(x)\Big]_a^b = P\,e + \frac{w_w\,x^2}{2} = 10{,}000 + 0.3125\,x^2 \dots\dots\dots\dots (50a)$$

and

$$M(x)\Big]_b^c = 10{,}000 + 250 \times 0.625\,(x - 125) + 0.833\,\frac{(x - 250)^2}{2}$$
$$= 0.4165\,x^2 - 52\,x + 16{,}500 \dots\dots\dots\dots (50b)$$

The expression for the value of $M(y)$ at the middle of the nth interval is derived as follows, remembering that the subscript i refers to values taken at the middle of any interval: $[M(y)]_n = P(y_i)_n + W_{n-1}\,[(y_i)_n - (y_i)_{n-1}] + W_{n-2}\,[(y_i)_n - (y_i)_{n-2}] + \cdots$; or

$$[M(y)]_n = P(y_i)_n + (y_i)_n \sum_1^{n-1} W - \sum_1^{n-1} W\,y_i \dots\dots\dots\dots (51)$$

Adding the quantity $0 = W_n(y_i)_n - W_n(y_i)_n$ to the right-hand side of Eq. 51:

$$M(y) = P\,y_i + y_i \sum W - \sum W\,y_i \dots\dots\dots\dots (52)$$

in which the quantities under the summation signs are to be taken from 1 to n.

The calculations are shown in auxiliary Table 6(a) and in main Table 6(b). Cols. 1 through 4, Table 6(a), lead to the values of $M(x)$ listed in Col. 11. To evaluate $M(y)$, it is first necessary to evaluate y_i. Knowing that $y = 0$ (Col. 19) and $y' = \theta$ (Col. 15) at $x = 0$, it is possible to obtain the first value of $y_i = y + \frac{i}{2}\,y'$ (Cols. 20 and 10). Next, since the different values of W are known, it is easy to tabulate the values of $\sum W$ (Col. 6). Then, the first values of $y_i\,(\sum W)$ (Col. 8), $W\,y_i$ (Col. 5), and $\sum (W\,y_i)$ (Col. 9) can be listed. Also, the first value of $P\,y_i$ can be listed (Col. 7). Then, by Eq. 52, the first value of $M(y)$ can be calculated (Col. 12). By adding this value to $M(x)$ (Col. 11), the first values of M (Col. 13) and of $\Delta y' = \dfrac{-\,i\,M}{E\,I}$ (Eq. 49a and Col.

14) can be listed. The latter value is added to the first value of y' (Col. 15) to determine the second value of y'.

Next, the value of Δy is calculated by Cols. 16 and 17 (see Eq. 49*b*), and is listed in Col. 18. This value is added to the first value of y (Col. 19) to obtain the second value of y. The second value of $y_i = y + \frac{i}{2} y'$ can be computed using the second values of y and y'. The computations then proceed as before.

Finally, at $x = 450$, y' is found to equal $-0.0695 + 0.4106\,\theta$. Since this quantity is known to be equal to zero, θ can be evaluated, and is found to equal 0.1692. Then, this value of θ may be substituted to yield the values of y, y', and M at the various points, as desired. These values may be plotted to obtain values at points other than those used in the computations. The details of this part of the procedure are omitted. Some of the numerical results are as follows:

TABLE 6.—COMPUTATIONS

x (1)	x_i (2)	$0.3125\, x_i^2$ (a) (3)	$-52\, x_i$ (4)	$W y_i$ (5)	ΣW (6)	$P y_i$ (7)	$y_i\,(\Sigma W)$ (8)	$\Sigma(W y_i)$ (9)	y_i (10)	$M(x)$ (11)
(*a*) AUXILIARY TABLE										
0	62.5	1,222	..	$8{,}750\,\theta$	140	$62{,}500\,\theta$	$8{,}750\,\theta$	$8{,}750\,\theta$	$62.5\,\theta$	11,220
125	187.5	10,980	..	-256.5 $+24{,}800\,\theta$	280	$-1{,}832$ $+177{,}300\,\theta$	-513 $+49{,}600\,\theta$	-256.5 $+33{,}550\,\theta$	-1.832 $+177.3\,\theta$	20,980
250	300	37,500[a]	$-15{,}600$	-976 $+39{,}400\,\theta$	436	$-6{,}253$ $+252{,}300\,\theta$	$-2{,}730$ $+110{,}100\,\theta$	$-1{,}232$ $+72{,}950\,\theta$	-6.253 $+252.3\,\theta$	38,400
350	400	66,600[a]	$-20{,}800$	$-1{,}772$ $+48{,}000\,\theta$	592	$-11{,}370$ $+307{,}600\,\theta$	$-6{,}730$ $+182{,}000\,\theta$	$-3{,}004$ $+121{,}000\,\theta$	-11.37 $+307.6\,\theta$	62,300
450	..	..	..	.. ..	..	.. ..	.. ..	.. ..		..

[a] For $x = 250$ and 350,

Item	Description	Quantity
1	Deflection at point a, in inches	40.7
	Bending Stress, in Pounds Per Square Inch, at:	
2	Point b	31,000
3	Point c	29,000
	Percentage of the Total Bending Stress at Point c, for the Moment:	
4	Acting on the undeflected column	62.0
5	Caused by load P, disregarding e	32.5
6	Caused by the weight of the column	5.5
7	Total bending stress (%)	100.0

These numerical values should be considered illustrative rather than accurate, since only four intervals were used in the calculations.

General Remarks.—It is necessary to carry along an unknown in the calculations. In both of the numerical examples selected, this unknown has been θ, the value of y' at a certain point. In some types of problems, however, it is advantageous to choose for the unknown a value of y or of M. Then, with the aid of Eqs. 38*b* and 41, the computations are developed in much the same way as in the illustrative problems of the paper. Finally, a point is reached at which the value of y', y, or M is known, enabling the designer to evaluate the unknown. With the value of the unknown determined, it becomes possible to evaluate y', y, or M at any of the tabulated points.

The computations should be carefully grouped to save time and mental effort. The proper arrangement is determined by the needs of the individual problem.

FOR THE COLUMN IN FIG. 9

$M(y)$ (12)	M (13)	$\Delta y' = \frac{-i\,M}{E\,I}$ (14)	y' (15)	$i\,y'$ (16)	$\frac{-0.5\,i^2\,M}{E\,I}$ (17)	Δy (18)	y (19)	$0.5\,i\,y'$ (20)	x (1)
				(*b*) MAIN TABLE					
62,500 θ	11,220 +62,500 θ	−0.01465 −0.0816 θ	θ	125 θ	−0.916 −5.10 θ	−0.916 +119.9 θ	0	62.5 θ	0
−2,088 +193,350 θ	18,890 +193,350 θ	−0.02465 −0.2525 θ	−0.01465 +0.9184 θ	−1.83 +114.8 θ	−1.542 −15.79 θ	−3.372 +99.0 θ	−0.916 +119.9 θ	−0.916 +57.4 θ	125
−7,751 +289,450 θ	30,650 +289,450 θ	−0.0119 −0.1122 θ	−0.03930 +0.6659 θ	−3.93 +66.59 θ	−0.595 −5.61 θ	−4.525 +60.98 θ	−4.288 +218.9 θ	−1.965 +33.35 θ	250
−15,100 +368,600 θ	47,200 +368,600 θ	−0.01832 −0.1431 θ	−0.0512 +0.5537 θ	−5.12 +55.37 θ	−0.916 −7.160 θ	−6.036 +48.21 θ	−8.813 +279.9 θ	−2.56 +27.65 θ	350
.. ..	.. ..	.. ..	−0.0695 +0.4106 θ	.. ..	.. ..	.. ..	−14.849 +328.1 θ	.. ..	450

these values are 0.4165 x^2_i.

The method is not self-checking. A procedure that will afford a check on the numerical work and that will at the same time give an indication of the accuracy of the method is to work through the computations first with a small number of intervals and then with a larger number of intervals. A comparison will reveal numerical errors quickly. Also, since the results generally converge very quickly toward exactness as the number of intervals is increased, the difference between the two sets of results gives an approximate indication of the error caused by choosing a finite number of intervals.

The step-by-step method can be applied directly to ordinary beam problems. In that case $M(y) = 0$, and the total moment at any point can be written immediately, thus greatly simplifying the procedure.

5. ACTUAL STRESS AND FACTOR OF SAFETY ABOVE THE CRITICAL POINT

Columns loaded above the critical point, of course, would never be used as structural members; but sometimes they occur in the form of springlike members in various types of machinery.

Only one such column—axially loaded at the ends with negligible end restraint, eccentricity, and irregularity of shape—will be treated. The direct stress will be neglected, since it is generally small compared to the bending stress. With a given column and a known load, δ can be determined immediately by Fig. 3, and the maximum bending stress, $\sigma_b = \frac{P\,\delta\,c}{I}$, can be evaluated readily. The factor of safety, N, may be defined as $\frac{P_y}{P}$, in which P_y is the load producing a bending stress equal to the yield stress. The safety factor may be determined by trial.

The difference between the ideal and the actual column is less marked above, than below, the critical point because, above the critical point, the large deflection is only little affected by minor eccentricities and irregularities of shape whereas, below the critical point, any small eccentricity or irregularity causes the column to depart from its ideally straight position and thus introduces an appreciable bending stress.

6. SUMMARY

The integration of the fundamental differential equation of the simple, ideal column leads to two solutions, one corresponding to a perfectly straight form and one corresponding to a certain deflected shape. Below the critical point, only the first of these solutions exists; consequently, the column must remain straight. Above the critical point both solutions exist, but the first is unstable and the second is stable; consequently, if the column is in the straight condition, the slightest disturbance will suffice to throw it into the deflected position.

With the column straight, the stress consists only of the direct stress P/A. With the column deflected, the stress is chiefly caused by the moment exerted by the applied load at the various points of the column.

In the actual column, end restraint, eccentricity of load, and irregularity of shape often reach appreciable values. If the load is less than the critical, deflections are small, and the elastic curve of the column can be represented accurately by a linear differential equation of the second order with constant coefficients. The solution of this equation is single valued and continuous within the practical range; hence, the shape of the actual column must vary continuously with change of load. In a strict sense, therefore, there is no critical point for an actual column. The sharp mathematical discontinuity that occurs in the critically loaded ideal column is leveled off, to a greater or less extent, by the irregularities of the practical column.

If the irregularities of the actual column are known to be small, the factor of safety at any load may be determined very simply by Fig. 6. Since even small irregularities may have an appreciable effect on the column, it is recommended that the values of Fig. 6 be modified according to judgment, to suit the known or expected irregularities.

The formal solution of the differential equation of the practical column is prohibitively tedious in many cases. Under such conditions, the versatile step-by-step method of numerical integration is suggested.

Above the critical point, the column is not greatly affected by minor irregularities. The numerical solution of columns loaded above the critical point is greatly facilitated by the curves of Fig. 3.

APPENDIX. NOTATION

The following letter symbols, adopted for use in the paper and for the guidance of discussers, conform essentially to American Standard Letter Symbols for Mechanics of Solid Bodies (ASA—Z10.3—1942), prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942:

A = area of cross section;
C = an arbitrary constant in Eq. 7;
c = distance from the neutral axis of a beam to the extreme fibers;
D = diameter;
E = modulus of elasticity;
e = eccentricity of axial load on a column;
G = an elliptic integral in Eq. 23*b*;
H = an elliptic integral in Eq. 23*b*;
I = rectangular moment of inertia;
i = the length of an interval in Eq. 38*b* $= x_B - x_A$;
J = an elliptic integral in Table 1, defined by Eq. 21;
K = an elliptic integral defined by Eq. 13;
k = a substitution constant $= \sqrt{P/(E\,I)}$ in Eq. 6*c*;
L = the total length of a deflected column;
M = bending moment:
    M_i and M_e = moments resulting from internal effects (M_i) and external effects (M_e); they are denoted "internal moments" and "external moments," respectively;
    $M(x)$ and $M(y)$ = moment (expressed as functions of x and y) at any point of a column, created by externally applied forces;
N = factor of safety; subscripts c and y denote whether the factor of safety is based on P_c or P_y;
n = a number in mathematical progression;
P = an external, concentrated load:
    P' = the value of P that would hold the column in equilibrium;
    P_a = allowable load;
    P_c = critical value of P;
    P_y = value of P that produces a direct stress σ_y;

q = a substitution constant defined by Eq. 12a;

r = least radius of gyration;

s = an indefinite arc length, part of the total length L of a deflected column;

ds = a differential of s;

W = a total weight or load uniformly distributed;

w = a unit, uniformly distributed load; w_w = a unit, uniformly distributed wind load;

x = distances parallel to the longitudinal reference axis of a column;

y = distances transverse to x-distances; lateral deflection of a column at any point:

y' and y'' = first and second derivatives of y with respect to x;

y_e = "exact" values of y (see Table 5);

y_0, y'_0, and y''_0 = initial deflection, slope, and approximate curvature at any point;

Z = section modulus;

β = upper limit of the angle ϕ;

Δ = a change in any quantity in a given interval of length i; thus, Δy = a change in the deflection y;

δ = total deflection at the center of a column;

ϵ = maximum deflection of an unloaded column;

θ = slope of a deflected column at a given point;

λ = half the projection of the deflected column on the x-axis (see Fig. 1(a));

ρ = radius of curvature;

σ = normal stress:

σ_b = bending stress at any point;

σ_t = total stress;

σ_y = yield-point stress; and

ϕ = a substitution angle defined by Eq. 10a.

DISCUSSION

R. Robinson Rowe,[6] M. ASCE.—The use of the more precise expression for the elastic curve of a uniform column, instead of the usual approximation, should be accompanied by a few words of its history and analogous applications in engineering. The particular curve is known as the "arch of Yvon-Villarceau" from the name of the mathematician who according to Rankine first investigated its properties by the use of elliptic functions. It is the curve whose curvature is proportional to distance from a directrix, or simply,

$$\rho\, y = \text{constant} \qquad (53)$$

James Bernoulli discovered this shape in the seventeenth century in his study of an indefinitely broad sheet supporting a liquid mass. This is the "hydrostatic cord" assumed by sheet metal flumes; and the constant is the unit tension divided by the unit weight of the liquid. Formulas equivalent to Eq. 19*d* for seven cases have been developed by J. B. Macphail,[7] M. ASCE.

Yvon-Villarceau found the same shape inverted for his arch, designed to support a fluid-like earth without moment. This is also called the "hydrostatic arch." The constant, of course, is the unit compression, or thrust, divided by the unit weight of earth. His formulas in elliptic functions and many special properties of the curve are quoted by Rankine.[8]

Rankine noted the analogy between the hydrostatic arch, the hydrostatic cord, and the elastic curve for a spring or column under the action of a pair of equal and opposite end forces.[9] Eq. 3 by Rankine is equivalent to the author's Eq. 3 and Rankine's Eq. 7, to the author's Eq. 27*b*. In addition to the author's six cases (Table 2), Rankine showed several representative forms of undulating distortion, which can actually develop in free-ended columns with intermediate lateral support. He did not repeat the elliptic expressions for the curve; but he referred to those for the hydrostatic arch, and noted that the constant was $\frac{E\,I}{P}$ and the directrix, the neutral position of the column.

George Winter,[10] M. ASCE.—An elegant and interesting analytical treatment of some general features in the theory of elastic stability is presented in this paper. It will be of value to men interested in this field, primarily as a systematic presentation of known facts. It is unusual, in a paper covering such a vast and established field, not to find a single reference to previous work of other investigators. Indeed, the only references given relate to texts in the field of pure mathematics, concerning the analytical tools used in the derivations.

---

[6] Senior Bridge Engr., State Div. of Highways, Sacramento, Calif.

[7] "*The Engineering Journal*, December, 1934, p. 528.

[8] "Applied Mechanics," by W. J. M. Rankine, 4th Ed., pp. 190-195.

[9] *Ibid.*, p. 353.

[10] Associate Prof., Civ. Eng., Cornell Univ., Ithaca, N. Y.

The origin of the analysis of columns beyond the Euler critical load goes back to Bernoulli and Euler, in the first half of the eighteenth century. Complete treatments of this problem of the so-called "Elastica" are found in such standard texts as A. E. H. Love's "Mathematical Theory of Elasticity" and R. V. Southwell's "Introduction to Theory of Elasticity." In the latter work Professor Southwell presented material which is not only identical in substance and mathematical method with the author's treatment,[11] but he includes a chart [12] that gives all the configuration shown by Mr. Ryder in Table 2. Furthermore, the information in Table 1 is given by Professor Southwell with some data added.[13] To be sure, there are differences of emphasis and detail, but none in substance and method.

In his discussion of practical columns and beam columns by the small deflection theory (differential Eq. 33), the author offers no information that cannot be found in generally available publications. In particular, Eq. 34 merely represents a superposition of customary solutions of beam columns as presented by S. Timoshenko,[14] D. H. Young[15] and others. The derivation of the Euler load from a beam-column analysis, Eq. 36, can be found in any standard textbook.

Numerical methods of integration applicable to columns and beam columns have been developed by N. M. Newmark, M. ASCE (among others). Professor Newmark indicates[16] how his method can be extended into the large deflection range; although, in view of the purely academic interest of this problem, he omits its elaboration. These numerical procedures are identical in substance, although not necessarily in detail, with those used by the author.

It is not implied in this review of the literature that the author's material is not new in some of its details and elaborations and that it was not independent work that led him to his findings. However, the fundamental content of this paper and its methods, as well as its conclusions, have been part and parcel of established knowledge for a considerable time.

With regard to its practical implications, it should be noted that the major part of this paper deals with phenomena beyond the Euler critical load. In the first sentence of section 5, however, the author himself notes that "Columns loaded above the critical point, of course, would never be used as structural members." This substantial part of the paper, therefore, is of no practical help to the designing engineer. As far as columns loaded below the critical point are concerned, Fig. 6 is misleading; because practical columns do not behave in the way assumed for this figure, as is firmly established by numerous experimental and analytical investigations. Initial imperfections as well as nonelastic behavior for shorter columns make this graph inapplicable. On the other hand, the methods of dealing with practical columns as proposed by the author in section 4 (beginning at Eq. 32) have long been used in practical design.

---

[11] "An Introduction to the Theory of Elasticity," by R. V. Southwell, Oxford Univ. Press, 2d Ed., 1941, pp. 429-436.

[12] *Ibid.*, p. 430, Fig. 112.

[13] *Ibid.*, pp. 434-435, Tables 1 and 2.

[14] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1st Ed., 1936, Chapter I.

[15] *Transactions*, ASCE, Vol. 101, 1936, p. 422.

[16] *Ibid.*, Vol. 108, 1943, p. 1187.

The fact that in this approach "there is no critical point for an actual column" is again part and parcel of these methods.

It is generally realized that the field of column analysis and design is in an unsatisfactory state. This situation, however, is not due to lack of analytical knowledge on the behavior of isolated columns in the elastic range. Indeed, the details of this topic have been worked out in the most exhaustive manner; and the author's paper, dealing exclusively with such isolated, elastic columns, adds no new evidence of value in the practical column range.

The real and substantial problems in column design, on the solution of which further progress depends, are (*a*) the behavior of isolated columns beyond the elastic range, and (*b*) the analysis of columns as component parts of structures (trusses, frames, bents) both in and beyond the elastic range. It is hoped that future investigators will turn their attention to these topics, which are of fundamental importance for the development of more accurate and rational structural column design.

ROBERT M. MAINS,[17] ASSOC. M. ASCE.—In consideration of the voluminous literature on column behavior, a new paper on the subject should perform one of two functions: Set forth solutions for problems hitherto unsolved, or present more convenient methods for applying existing solutions to the practical column. The author's work on theoretical columns is interesting in that it restates, with some changes in notation, problem solutions which are readily available in such books as "Theory of Elastic Stability" by S. Timoshenko.[18] Indeed, many of these solutions date back to the Euler-Bernoulli correspondence concerning problems of the "elastica" about 1740.

In studying this paper, the writer hoped that it would contain forms of solution which would permit more convenient computations of practical columns. However, the methods used in the section dealing with practical columns seem to be much more cumbersome than existing methods. For example, the results of Col. 13, Table 5, which give deflections for Fig. 8, were checked with a brief half page of computation by the use of the numerical procedure of N. M. Newmark, M. ASCE, combined with the relation:[19]

$$w_a = \frac{w_i}{\frac{P_c}{P} - 1} \qquad (54)$$

in which w_a is the additional deflection due to load P; w_i is the sum of the initial deflection and the deflection due to lateral load; P_c is the critical buckling load; and P is the applied end load. The results obtained from this computation agreed with those of the author within 1%, which is certainly adequate for practical purposes. A similar saving in the labor of computation, although not so marked, may be made by the application of the same technique to the problem in Fig. 9.

---

[17] Asst. Prof., Civ. Eng., Cornell Univ., Ithaca, N. Y.

[18] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1st Ed., 1936.

[19] *Transactions*, ASCE, Vol. 108, 1943, p. 1181, Eq. 4.

Although the author's equations for the calculation of deflections are adequately tied in with computations of stress, it is not made clear just how buckling loads are to be determined for the practical column which fails by elastic buckling instead of by yielding. The relationship between deflections and buckling loads is emphasized in Professor Newmark's method, and the buckling load is obtained conveniently in the process of solution.

Since there are practical problems (although not necessarily in bridges and buildings) in which elastic buckling occurs before yielding of the column, the writer fails to see any advantage in clarity of theory or simplicity of application for the author's methods over "step-by-step" processes already available.

CHI-TEH WANG,[20] ESQ.—Although the nonlinear theory of column buckling in the elastic range was first developed by G. Kirchhoff[21] almost ninety years ago, a recapitulation for the average practical engineer still appears to be praiseworthy. Mr. Ryder's paper presents an excellent résumé of the theory, together with an enlightening discussion relating to the stability of the columns. Several points, however, are in need of further clarification.

In conclusion (3), Section 3, it was concluded that "(3) At values of P/A close to the critical, the deflection is indeterminate, and may assume any fairly small value." This was supposedly proved by showing that the curves of $\frac{M_i}{A\,L}$ and $\frac{M_e}{A\,L}$ have the same slope or are tangent to each other at $\frac{\delta}{L} = 0$. It is a simple matter to show that when two curves are tangent to each other they need not necessarily meet at more than one point. Of course, they may. To prove that there is actually only one point at which these two curves meet, reference is made to Eq. 13, in which $k^2 = \frac{P}{E\,I}$ and $q^2 = \left(\frac{k\,\delta}{2}\right)^2 = \frac{P\,\delta^2}{4\,E\,I}$. When $P = P_c = \frac{\pi^2\,E\,I}{L^2}$, $k^2 = \frac{\pi^2}{L^2}$, and $q^2 = \frac{\pi^2\,\delta^2}{4\,L^2}$. The left-hand side of Eq. 13 is then equal to $\frac{\pi}{2}$. The elliptical integral on the right-hand side will be equal to $\frac{\pi}{2}$ only if q^2 is zero. Since $q^2 = \frac{\pi^2\,\delta^2}{4\,L^2}$, it is zero only if $\delta = 0$; that is, at the critical load the deflection is zero, and at any other load close to the critical value the deflection, however small, has a definite value and can be calculated from Eq. 13. When this is proved, Eqs. 29*a* and 29*b* are equal only at the point $\delta = 0$ and not at any other point or points close to $\delta = 0$, and the original conclusion (3) is therefore incorrect.

Mr. Ryder mentions in the "Synopsis" that he is merely re-emphasizing the well-known fact that at the critical point the deflection is indeterminate. The so-called "well-known" fact is accepted as such because most writers in strength of materials deal only with the linear theory and, therefore, reach that erroneous conclusion. Actually, from the nonlinear theory, it is clear that the so-called "buckling load" is the load at which the straight form of the ideal column begins to be unstable. At any load greater than the critical load, the column has two possible forms of equilibrium, the straight form, which is unstable, and the bent form, which is stable. For any load which is less than, or equal to, the critical value, the ideal column has only one form of equilibrium, namely, the straight form. Accepting the result obtained from the linear theory, some writers consider the equilibrium at the critical load as neutral equilibrium since the deflection then seems to be indeterminate. It is evident that such terminology is not correct in the strict sense.

---

[20] Asst. Prof. of Aeronautical Eng., Daniel Guggenheim School of Aeronautics, New York Univ., New York, N. Y.

[21] "Ueber das Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes," by G. Kirchhoff, *Journal für reine und angewandte Math.*, Vol. 56, 1859.

Another objection may be raised with respect to Fig. 5, in which Euler's column formula has been extended to the yield point of the material. It should be noted that Mr. Ryder's theoretical discussion does not hold true if the stress exceeds the elastic limit of the material, namely, the proportional limit. For some alloy steels the proportional limit is approximately 90% of the yield stress. Perhaps in these cases, if a large factor of safety is used, diagrams such as Fig. 5 could be accepted for design purposes, as is the usual practice of civil engineers. For aeronautical engineers, who have to deal with aluminum alloys most of the time, this practice has long been found to be unsatisfactory because the proportional limit in some cases is only about 50% of the yield stress.

Since the author is interested in the "rational" explanation of column behavior, the writer feels that this is a point worth mentioning. The most accurate experiments on columns, up to the present time, probably were those made in the Berlin-Dahlem material testing laboratory by using a very elaborate construction for the end supports of the columns.[22] For structural steel columns with a yield stress of 45,000 lb per sq in., the test results show that there is a marked deviation from the Euler curve above $\frac{P_c}{A} = 39{,}000$ lb per sq in.[23] Theoretically, the use of Euler's column formula for stress beyond the proportional limit cannot be considered to be correct. Between the proportional limit and the yield stress one should use the tangent modulus formula, which recently has been verified theoretically by F. R. Shanley. Fig. 5 should then be modified as in Fig. 10 in which $\frac{P_t}{A} = \frac{\pi^2 E_t}{(L/r)^2}$, and E_t is the tangent modulus of the material and is defined as the slope of the compressive stress-strain curve at the stress corresponding to $\frac{P_t}{A}$. The same objection can be applied to Fig. 6.

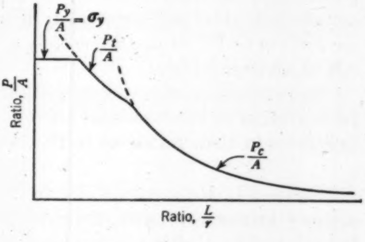


FIG. 10

---

[22] "Neuere experimentelle Beiträge zur Frage der Knickfestigkeit," by K. Memmler, *Proceedings*, 2d International Cong. for Applied Mechanics, Zürich, 1926, pp. 357–363.

[23] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1st Ed., 1936, p. 174, Fig. 109.

In the "Summary," it is emphasized that in a strict sense there is no critical point for an actual column. This statement is indeed very true. If one uses the linear theory, the load versus deflection curve of an actual column in compression is approximately a rectangular hyperbola with asymptotes $P \rightarrow P_c$ when $\delta \rightarrow \infty$. This condition, originally discussed by R. V. Southwell,[24] is interesting, because from the tests of actual columns the buckling load of a corresponding ideal column can be approximately determined. Thus, a relation between the theoretical discussion of ideal columns and the testing of actual columns can be obtained. It is obvious that when δ is not small the nonlinear equation must be used; the theoretical discussion of actual columns will yield no result concerning the buckling load.

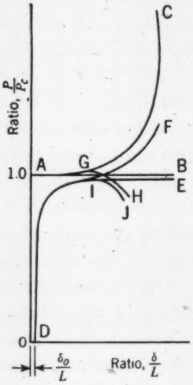


FIG. 11

The behavior of ideal and actual columns as obtained from various rational theories can be summarized in Fig. 11. Suppose that the column is slender and that the material can sustain the compressive stress $\frac{P}{A}$ without failure of elasticity. The loading of an ideal column will follow the curve OAB according to the linear theory and OAC according to the nonlinear theory. At some point G on curve OAC, the deflection δ is so large that the sum of direct compressive stress and bending stress due to the bending moment $P\,\delta$ is sufficient to produce elastic breakdown. Beyond this point the actual curve AGH will begin to drop away from AC, since the material can now sustain less stress than before. For an actual column with an initial deflection of δ_0, the loading of the column will follow the curve DE according to the linear theory, and the curve DF according to the nonlinear theory. At point I, the maximum stress in the column reaches the elastic limit and then the actual curve DIJ will drop away from DF. It should be noted that the upper part of curve DE is approximately a rectangular hyperbola which approaches AB as an asymptote.

In conclusion, the writer wishes to congratulate Mr. Ryder on his clear presentation of the nonlinear theory which will clarify for many engineers the questions in their minds as to the behavior of columns.

ERNEST L. ROBINSON,[25] Assoc. M. ASCE.—As any devotee of the sport of archery knows very well, there is nothing uncertain or mysterious about the behavior of an elastic column beyond the Euler length. In 1915, and in 1916, the writer published two analyses of the curve whose ordinate is everywhere proportional to the curvature. The first gave[26] equations for a deflecting column connecting load and deflection both with the original length and the deflected length, and presented graphic solutions of these equations, as well as charts for the required elliptic integrals. The second paper presented charts[27] for the hydraulic catenary connecting the middle depth with the tension, area, wetted perimeter, and mean hydraulic radius as well as a fairly large diagram showing the actual shape of the curve.

---

[24] "On the Analysis of Experimental Observations in Problems of Elastic Stability," by R. V. Southwell, *Proceedings*, Royal Soc. of London, Vol. 135, 1932, pp. 606–616.

[25] Structural Engr., Turbine Generator Eng. Div., Gen. Elec. Co., Schenectady, N. Y.

[26] "Curves for Strength and Deflection of Very Long Columns," *Engineering News*, June 10, 1915, pp. 1108–1109.

FREDERICK L. RYDER,[28] ESQ.—It is stated in the discussions, particularly by Professors Winter and Mains, that a good part of the material in the paper is to be found elsewhere in the literature. In view of this criticism, the writer wishes to clarify the purpose and scope of the paper, as follows:

It is well known that various aspects of the analytical treatment of the column problem have been investigated previously. Although these treatments have no doubt been adequate for those scholars who were able to integrate them into a comprehensive theory, the writer felt that in no case did they present to the average engineer a unified, cohesive theory of column behavior. As taught in the schools, for example, column behavior is discussed largely in terms of empirical results, presumably because there is no paper that presents a reasonably simple but well-rounded analytical treatment. The result is that most engineers do not understand the rationale of column mechanics.

To remedy this situation, the writer has attempted to develop, from the beginning, a logical and sequential column theory, geared to the needs of the modern engineer. Because of the analytic and self-developing nature of the work, it was unnecessary to use the methods and results of previous investigators, and in fact they were not used.

Aside from what he hopes is a more meaningful and useful organization of certain known results, the writer believes that certain parts of the paper are essentially new—for instance, the discussion of column stability in Section 3, which is essential to an understanding of column mechanics. Also, the essence of the step-by-step method appears to be entirely new, at least as applied to columns. In this connection, the writer wishes to point out a basic feature—namely, that his method is not one of successive approximations, and consequently does not require a good guess as a first approximation. This procedure is in contrast with previous methods, such as that of N. M. Newmark,[29] M. ASCE. The difference would seem to be important in the case of complicated columns, as, for example, a cargo boom, which often consists of pipes of different sizes joined at their ends, with multiple hoisting blocks applying both lateral and eccentric axial loads at different points near the end of the boom.

The writer had also hoped that Fig. 3, which presents a convenient solution of slender columns loaded above the critical point, was new. However, it turns out that curves of a similar nature have been published by Mr. Robinson. The writer extends his apologies.

With regard to specific ideas presented in certain discussions, the writer wishes to reply as follows:

To Mr. Rowe he wishes to extend his thanks for the interesting historical notes.

---

[27] "Design Charts for Open Flumes with Catenary Section," by Ernest L. Robinson, *Engineering News*, March 23, 1916, pp. 566–567.

[28] Development Engr., Shirgun Corp., New York, N. Y.

[29] "Numerical Procedure for Computing Deflections, Moments, and Buckling Loads," by N. M. Newmark, *Transactions*, ASCE, Vol. 108, 1943, p. 1161.

Professor Winter states that a discussion of columns loaded above the critical point is of no practical value. The writer believes that a study of such columns is necessary to obtain a proper insight into the behavior of the more usual columns. Furthermore, such columns are often used as machine and instrument parts. This use, to be sure, is of interest to mechanical engineers rather than to civil engineers.

As Professor Mains declares, the writer did not extend his analysis to include the determination of the buckling load for columns of other than the ideal shape. The reason is simply mathematical complexity, for it would have led to the solution of Eq. 3 with the parameter I a function of x, and not necessarily continuous. Quite often columns of nonideal shape have nonideal loading, in which case they may be completely analyzed, both as to stress and as to deflection, by the step-by-step method.

Professor Wang indicates that, at values of P/A close to the critical, the deflection of an ideal column is a determinate problem, contrary to the writer's statement in Section 3, conclusion (3), and may be evaluated from Eq. 13. The writer believes nevertheless that his original statement is correct. A numerical investigation of Eq. 13 in the vicinity of $q = 0$ will show that, as q is varied, K varies by a lesser order of magnitude. This demonstration constitutes the mathematical and physical definition of indeterminateness of q, and hence of deflection.

Professor Wang's clarification of the behavior of the column between the proportional limit and the yield point is most enlightening from a rational point of view, and would seem to be of definite value in close design.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2331

CLEANING AND GROUTING OF LIMESTONE FOUNDATIONS, TENNESSEE VALLEY AUTHORITY

A SYMPOSIUM

PAGE
Foreword 80

Fort Loudoun Dam.
By Arthur L. Pauls, M. ASCE 82

Douglas Dam.
By T. F. Taylor, Assoc. M. ASCE 102

With Discussion by Messrs. Berlen C. Moneymaker, C. E. Blee, Robert A. Laurence, James B. Ward, J. C. Nowell, John F. Partridge, and Portland P. Fox.

Note.—Published in December, 1946, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

FOREWORD

Under terms of the basic act and subsequent legislation, the Tennessee Valley Authority (TVA) has designed and constructed sixteen major hydroelectric projects to control and utilize the water of the Tennessee River and its tributaries, thus providing for navigation, flood control, and the generation of electric power. These major projects (see Fig. 1) are located at the most feasible sites to insure the unified development of this river and its tributaries consistent with the economical adaptation of structures to the geology, topography, and hydraulic conditions.

The geology of these sites is interesting. The local formations involved are the Knox dolomite, Chickamauga limestone, and some subordinate shale members. In general, both the Knox dolomite and the Chickamauga limestone are fairly soluble when considered over geologic intervals of time, and for this reason numerous seams and cavities abound in the region.

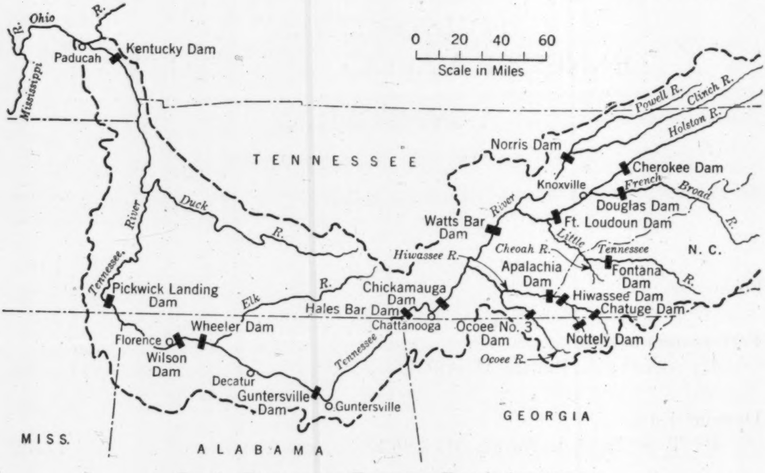

Ohio
R.
R.
Mississippi
Paducah
Kentucky Dam
0 20 40 60
Scale in Miles
Powell R.
Clinch R.
Holston R.
Norris Dam
TENNESSEE
Cherokee Dam
Knoxville
French
Broad
R.
Douglas Dam
Watts Bar Dam
River
Ft. Loudoun Dam
Little
Tennessee
Duck
R.
Hiwassee R.
Cheoah R.
Apalachia Dam
Fontana Dam
R.
N. C.
Tennessee
River
Pickwick Landing Dam
Elk
R.
Chickamauga Dam
Hales Bar Dam
Hiwassee Dam
Chatuge Dam
Wheeler Dam
Chattanooga
Ocoee No. 3 Dam
Nottely Dam
Florence
Wilson Dam
Ocoee R.
Decatur
Tennessee
Guntersville Dam
Guntersville
GEORGIA
MISS.
ALABAMA

FIG. 1.—DAMS OF THE TENNESSEE VALLEY AUTHORITY

In the construction of these major hydroelectric structures for the development of the valley, an important by-product, especially for the civil engineering profession, has been the mass of technical data made available as a result of the many problems encountered and overcome. A record of the foundation treatment required in the construction of these projects provides one type of technical data which, by its nature, is of particular value to civil engineers. Prior to this Symposium, two groups of papers published in the Society's *Transactions* have described the foundation treatment at Norris, Guntersville, Chickamauga,[1] Kentucky, and Hales Bar[2] dams. The material included in these papers concerned either the geology or the treatment required at the dam site. The two papers presented herewith serve to complete this special series.

Foundation treatment at the Douglas and Fort Loudoun dams has followed the general procedure developed since 1936 for TVA dams founded on limestone. The success of the treatment illustrates that gravity type dams may be founded on limestone satisfactorily provided that proper methods of excavation, cavity filling, and grouting are employed.

At the beginning of construction the late Theodore B. Parker, M. ASCE, was chief engineer. He was succeeded on July 1, 1943, by C. E. Blee, M. ASCE. Arthur L. Pauls, M. ASCE, was chief construction engineer. E. C. Eckel, Affiliate, ASCE, was chief geologist until his death in November, 1941, when he was succeeded by Berlen C. Moneymaker, Assoc. M. ASCE.

At Fort Loudoun Dam, J. K. Black, M. ASCE, was project manager; and J. F. Partridge, M. ASCE, construction engineer. E. C. McClenagan was construction superintendent.

On Douglas Dam, Lee G. Warren, M. ASCE, was project construction manager; T. F. Taylor, Assoc. M. ASCE, construction engineer and later acting project manager; and James S. Lewis, Jr., M. ASCE, construction superintendent.

Appreciation is expressed to A. H. Weber, civil engineer, for assistance given in the preparation of the first paper in this Symposium.

---

[1] "Foundation Experiences, Tennessee Valley Authority: A Symposium," *Transactions*, ASCE, Vol. 106, 1941, p. 685.

[2] "Unusual Cutoff Problems—Dams of the Tennessee Valley Authority: A Symposium," *ibid.*, Vol. 110, 1945, p. 947

FORT LOUDOUN DAM

BY ARTHUR L. PAULS,[8] M. ASCE

SYNOPSIS

Undoubtedly, the most distinctive feature of the foundation at Fort Loudoun Dam was the presence of solution cavities at unprecedented depths below the water table. Solution occurred at depths as great as 300 ft below the original ground surface and 260 ft below the water table. Experience with this foundation has demonstrated that structural deformational features rather than rock solubility often control the depth and extent of weathering. Established grouting procedures and practices are described only briefly to permit a more complete discussion of the deep mining operations in the large bedding cavities both for consolidation and for the construction of an effective cutoff. In the north core wall, cavities were mined to a depth of 220 ft under the most difficult working conditions.

INTRODUCTION

Fort Loudoun Dam is the ninth and final dam, on the Tennessee River, necessary for extending the navigable channel to Knoxville, Tenn.—a distance 650 miles upstream from the confluence of the Tennessee and Ohio rivers at Paducah, Ky. (see Fig. 1). The structure is located about 1 mile from Lenoir City, in Loudon County, Tenn.; 602.3 miles above the mouth of the river; 1.2 miles above the confluence of the Little Tennessee River; and 47.7 miles downstream (or 26 miles by highway) from Knoxville.

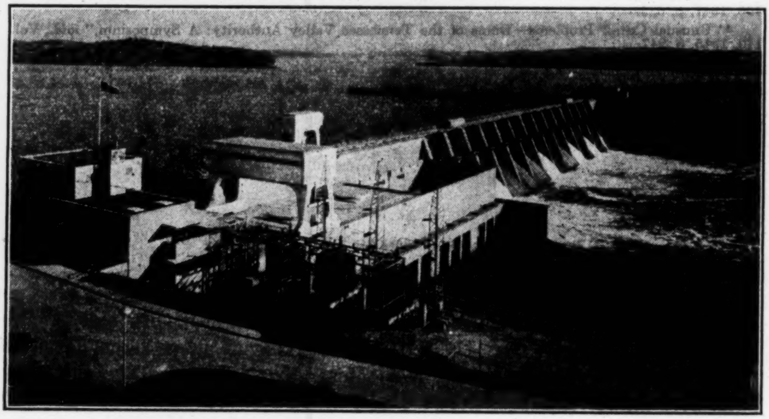


FIG. 2.—FORT LOUDOUN DAM VIEWED FROM THE NORTH BANK OF THE RIVER

---

[8] Alhambra, Calif.; formerly Chf. Constr. Engr., TVA, Knoxville, Tenn.

The over-all length of the combined earth and concrete dam (see Fig. 2) is 4,185 ft; and the structure is 95 ft higher than the original river bed or 165 ft from the lowest excavation. The south embankment is a rolled earth fill dam 2,600 ft long between the lock and the south abutment. The navigation lock between this earth embankment and the spillway has a chamber 360 ft long and 60 ft wide with a maximum single lift of 80 ft. The spillway section has fourteen bays, each 40 ft wide closed with Tainter gates 32 ft high. The powerhouse and intake span the original main river channel and have an over-all length of 280 ft in four bays. The service bay is stepped into the north bank and a core wall extends about 300 ft into the north abutment. A saddle dam on the north rim, 2,283 ft upstream from the main dam, is a rolled fill structure 525 ft long.

Selection of Site

As early as 1927, geologic studies were made by the Army Engineers in the vicinity of Fort Loudoun Dam to determine the availability of a suitable dam site. Investigations were continued by the TVA, and many sites were explored within the 10-mile to 12-mile reach of the river defined by the operating level of Watts Bar Dam downstream and by the cost of dredging the upper reaches of the Watts Bar pool as balanced against the cost of a higher dam. The occurrence of major faults and other geologic disturbances in the area limited the number of usable sites both topographically and geologically. Furthermore, the underlying rock of the entire area is more or less calcareous and therefore subject to weathering and solution.

Seven sites were studied, and four of these were explored by diamond core drilling. The Belle Canton Island site at mile 602.3 was finally selected; and, although this site was obviously not the best topographically, it did appear to be geologically preferable to any of the other sites available (with two exceptions, both of which had other disadvantages that offset the more suitable foundation). Enough drilling was completed at the Belle Canton Island site to disclose that the weaknesses could be corrected although the magnitude of the problem, as shown by the drilling, grouting, and concreting quantities in Table 1, was fully recognized when treatment was begun.

TABLE 1.—Drilling and Grouting Quantities

Feature	Drilling (Lin Ft)			Grout (cu ft)	Concrete (cu yd)
	Diamond	Wagon	Calyx		
South embankment	86,557	57,588	4,601	68,500	1,395
Lock	103,405	99,674	1,282	97,128	405
Spillway	122,837[a]	38,364	3,703	127,510	3,277
Powerhouse and intake	63,929[a]	1,600	1,377	24,839	1,196
North abutment	31,093	...	4,044	117,781[b]	3,543
North reservoir rim	34,713	...	...	408,976[b]	...
Saddle dam	3,305	...	...	222,653[b]	...

[a] Includes drain holes. [b] Includes clay content in cement grout.

GEOLOGY

The topography of the Fort Loudoun dam site was not as favorable as that at the other sites because the alluvial valley is unusually wide—more than 2,800 ft (see Fig. 3). The river channel, approximately 1,800 ft from bank to bank, was originally divided by three alluvial islands into four separate channels from 100 ft to 300 ft wide. At the north abutment of the dam, the main channel was cut into the base of the ridge that formed the right bank of the river along the apex of a broad meander beginning just upstream. In the process of cutting into the right bank, this meander receded from the higher ground leaving a clay-covered terrace slope which forms the south abutment of the dam. The topography ranges from El. 730 in the river bed to El. 900 on the crest of the flanking ridges.

The only bedrock exposed at the dam site was that in the bluff on the north abutment ridge. Elsewhere the rock was covered by overburden varying in thickness as much as 40 ft and consisting of both alluvial and residual deposits, although the alluvial material capped most of the area where outcrops were not present. Stratigraphically, the underlying rock belongs to the Ordovician period of east Tennessee. Of the belt of rocks in the vicinity, only the Tellico and Sevier formations are actually involved in the dam foundation.

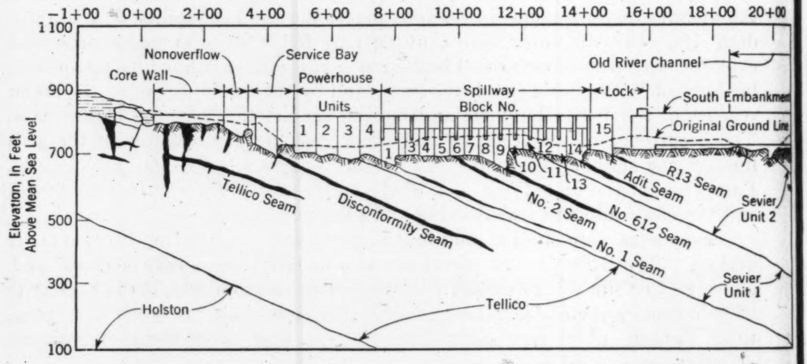
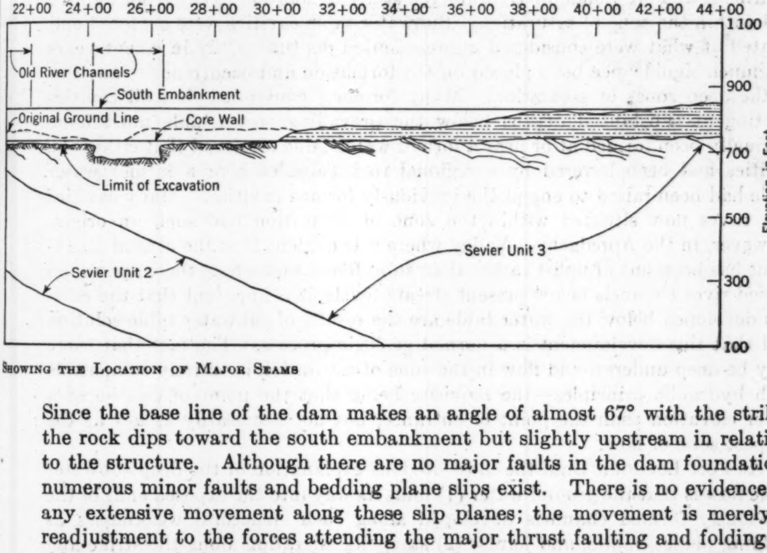

−1+00
0+00
2+00
4+00
6+00
8+00
10+00
12+00
14+00
16+00
18+00
20+00
1100
900
700
500
300
100
Elevation, In Feet Above Mean Sea Level
Nonoverflow
Service Bay
Core Wall
Powerhouse
Units
1
2
3
4
Spillway Block No.
1
3
4
5
6
7
8
9
10
11
12
13
14
15
Lock
Old River Channel
South Embankme
Original Ground Lin
Tellico Seam
Disconformity Seam
No. 2 Seam
No. 612 Seam
No. 1 Seam
Adit Seam
R13 Seam
Sevier Unit 2
Sevier Unit 1
Holston
Tellico

FIG. 3.—GENERALIZED GEOLOGIC SECTION

The Tellico formation is generally regarded as sandstone, but in this belt it is an arenaceous limestone; and, in spite of the siliceous character of the formation, it is nevertheless soluble. Because of the irregularity of the bedding and the presence of less soluble wavy stylolitic bands, cavities are not readily formed along the bedding planes. A distinctive feature of this formation at Fort Loudoun is the presence of an intraformational disconformity approximately 100 ft below the contact of the Sevier formation. There was an old erosion surface along the bedding at this position on which from 3 ft to 5 ft of shaly and fossiliferous limestone were deposited. This old erosion surface had been further attacked by solution, and a cavity was present along it.

The Sevier formation as a whole is quite diversified—ranging from a coarsely crystalline marble, through limestone and shaly limestone, to a dense blue-gray calcareous shale. It has been found practicable to divide this formation into three units (see Fig. 3), each with a predominant lithologic type. Unit 1, or the lower 225 ft, is predominantly limestone. Near the top of this unit the amount of shale becomes progressively more abundant and grades into the part designated as unit 2. As a whole, this unit is more soluble than the remaining part of the Sevier formation; hence, the cavities are usually larger and are either filled or partly filled with a clay residue. Unit 3 is mostly a gray calcareous shale; but, as in the other two units, there is a certain amount of interbedded shaly limestone varying in thickness from 20 ft to 100 ft. As this unit is less resistant structurally than the lower units, it has suffered considerable deformational effects and consequently is deeply weathered.

The geologic structure at the site is typical of the regional structure with a strike of north 40° to 45° east and an average dip of from 30° to 35° southeast.



SHOWING THE LOCATION OF MAJOR SEAMS

Since the base line of the dam makes an angle of almost 67° with the strike, the rock dips toward the south embankment but slightly upstream in relation to the structure. Although there are no major faults in the dam foundation, numerous minor faults and bedding plane slips exist. There is no evidence of any extensive movement along these slip planes; the movement is merely a readjustment to the forces attending the major thrust faulting and folding in this region. A thrust fault in the Tellico formation is unique in that it is the only thrust fault of any magnitude at the site that was not developed along the bedding. However, the movement is not considered extensive although a cavity has been formed along this weakness. Almost without exception, the many solution cavities developed in the foundation were controlled by solution along geologic structures—faults, joints, and bedding planes. At the intersection of these fractures, the larger and more continuous cavities and seams were developed. Solution followed intricate patterns with channels and cross channels predominating, separated at irregular intervals by rock pillars. The presence of these pillars greatly augmented the bearing strength of the formation, but a certain amount of consolidation grouting was considered necessary. Contrary to the normal conception that relatively more cavities are to be expected in the purer limestone, the cavities in this area were primarily controlled by structural features rather than by the purity of the rock. Also, founding a dam on impure, less soluble, limestone does not necessarily insure a better foundation than one on a pure, more soluble limestone—provided the latter is free of deformational features.

For convenience in the field but of no regional geologic significance, the master cavities or weathered bedding seams were designated from lower to upper as the Tellico, disconformity, No. 1, No. 2, No. 612, adit, and R13. These seams and their positions relative to the dam are shown in Fig. 3.

FORMATION AND CHARACTER OF CAVITIES

An excellent opportunity was presented during foundation excavation and treatment at Fort Loudoun Dam for the study of solution cavities both above and within the zone of saturation. Subwater table cavities were explored and treated at what were considered unprecedented depths. Only in recent years has much significance been placed on the formation and occurrence of cavities in the deep zones of saturation. Many formerly contended that all cavities existing at substantial depths below the prevailing water table must have originally been formed at or near a former water table level and that either the cavities had been lowered by a regional rock subsidence or a former water table had been raised to engulf the previously formed cavities. Many cavities and caves now situated within the zone of saturation had such an origin. However, in the Appalachian Valley where it is evident that the crustal movement has been one of uplift rather than subsidence, and where there are no old buried river channels below present stream levels, it is apparent that the cavities developed below the water table are the results of subwater table solution and that this development is a normal geologic process. The fact that there may be deep underground flow in the zone of saturation is entirely compatible with hydraulic principles—the requisite being that the point of exit be at a lower elevation than the point of entrance, but not necessarily as low as the deepest zone of flow.

At Fort Loudoun Dam the most obvious explanation of the deep subwater table flow is that the ground water (1) finds its way into the exposed ends of the previously formed channels developed along such structural weaknesses as bedding planes, faults, and joints; (2) starts its migration along the strike and down the dip of these conducting channels in irregular circuitous paths to a depth determined by the hydraulic head, by the solubility of the rock, and by the structural weaknesses; and (3) resurges upward to the point of discharge. The flow and resulting creation of cavities proceed downward until an upward or lateral movement is less resistant than continued downward flow. Thus, a system of both vertical siphons and low-angle inverted siphons develops, one group below the other, the depth dependent primarily on the differential head. Often, the watercourses are clogged with insoluble residue that causes the water to seek deeper levels as solution continues. The existence of an inverted siphon flow was actually observed during excavation and was further attested by the presence of cold-water springs or boils in the river bed. Dye injected into holes upstream from the cofferdam showed in the boils encountered in the upper few feet of excavation more than 1,000 ft away.

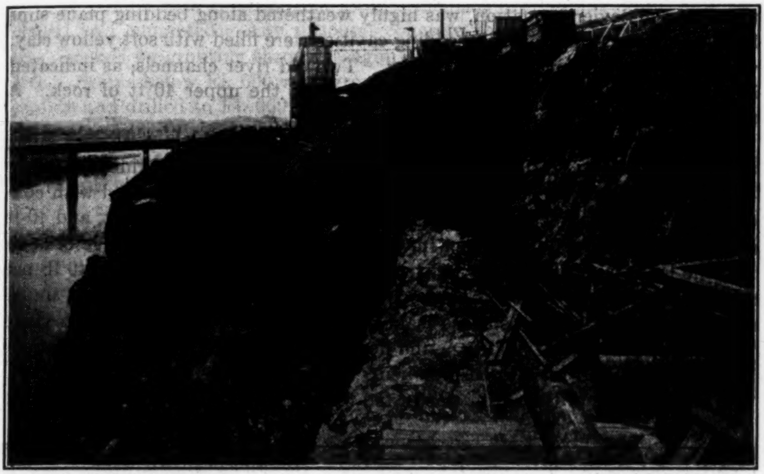


FIG. 4.—NORTH ABUTMENT, SHOWING TYPICAL ROCK PINNACLES

In the north abutment, solution has resulted in the formation of a series of deep troughs separated by intervening pinnacles (see Fig. 4). This formation, occupying a ridge with a steep bluff at the encroaching meander of the river, demonstrated an interesting point—namely, that the presence of a steep and apparently sound limestone rock bluff is not a guarantee that sound rock extends backward into the bluff; it may be nothing more than a false front with deep weathering just behind it.

FOUNDATION INVESTIGATION

For purposes of exploration the pattern of core drilling consisted of holes on lines parallel to the old preliminary drill ranges to enable the utilization of holes previously drilled. Holes were spaced on 50-ft centers with some intermediate check holes. Ordinarily, 3-in. diamond drill bits were used. The depth of holes depended on the geologic structure and ranged from 33 ft to 198 ft below the original rock surface.

For the lock location, ten ranges were drilled from upstream to downstream—a total of ninety-one holes and 8,580 lin ft. In addition, one 36-in. calyx drill hole 75 ft deep was sunk for visual inspection of the rock in place. Exploratory holes were drilled concurrently with construction, and each section was drilled in advance of excavation to determine the problems involved. A total of 58,193 lin ft of diamond drilling and 365 lin ft of calyx drilling was completed for the exploration of the dam foundation. (A general description of calyx drilling has been presented elsewhere by Mr. Lewis.[4])

SOUTH EMBANKMENT

The bedrock of the south embankment, predominantly calcareous shale of unit 3 of the Sevier formation, was highly weathered along bedding plane slips from El. 730 to El. 690. The bedding cavities were filled with soft yellow clay, which was the residuum of weathering. Two old river channels, as indicated in Fig. 3, were extremely cavernous throughout the upper 40 ft of rock. A typical weathered section is shown in Fig. 5.

A trench was excavated to fairly sound rock from the lock southward along the line of cutoff to Station 29+63. It was approximately 3 ft wide; and, after 4-in. vertical pipes were set on 2-ft centers, the trench was backfilled with concrete to a depth of 5 ft. Wagon drill holes were drilled in 20-ft, 30-ft, and 40-ft stages on 2-ft centers using the 4-in. pipe for casing. After each stage of drilling, the holes were thoroughly washed using pressures of 20, 30, and 40 lb per sq in., respectively, for each of the three stages, followed by grouting under similar pressures. Some surface leakage of grout occurred during each stage but it appeared farther upstream and downstream for each subsequent stage—a good indication that a cutoff was being obtained as the holes were deepened. The effectiveness of the cutoff was tested by grouting 60-ft diamond drill holes on 10-ft centers. Very little additional grout was accepted by these holes, and the cores recovered indicated excellent grout penetration along the weathered shale beds.

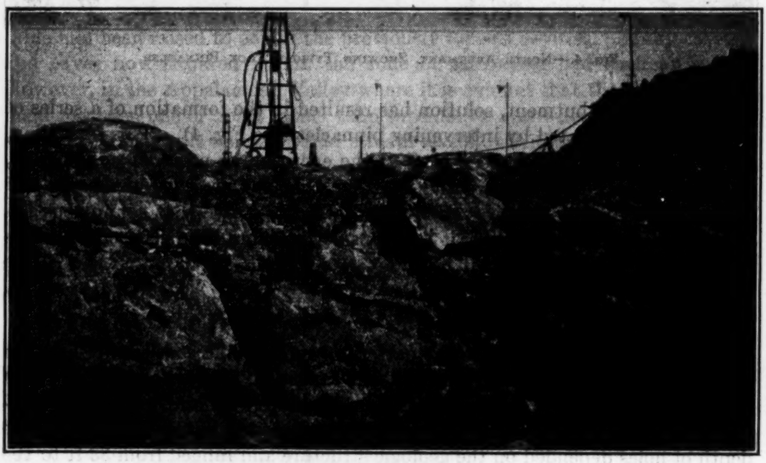


FIG. 5.—WEATHERED ROCK IN THE OLD RIVER CHANNELS; SOUTH EMBANKMENT

---

[4] "Large Core Drills Aid Construction at Chickamauga Dam," by James S. Lewis, Jr., *Transactions* ASCE, Vol. 105, 1940, p. 850.

Rather than attempt to treat the intersection of the cutoff with the old river channels in a similar manner, a positive cutoff was secured by backfilling the 38-in. calyx drill holes with concrete. Originally, these holes were drilled to El. 690 on 5-ft centers with the intervening unsound rock being removed. Because the practice proved both expensive and dangerous, the holes were backfilled with concrete and an intermediate overlapping 38-in. hole was drilled and concreted. Any unsound concrete that might have been placed was checked by grouting through wagon drill holes in the center of each 38-in. hole. Seams below El. 690 were grouted through diamond drill holes on 10-ft centers and drilled to El. 660. Where mining between the 38-in. holes on 5-ft centers was possible, wagon drill holes on 2-ft centers were grouted to form the intervening cutoff and to consolidate any unsound concrete. While this cutoff was being constructed, sufficient water entered the holes to prohibit drilling until the entire area was encircled with diamond drill holes on about 5-ft centers 20 ft upstream and downstream, and grouted using neat cement with calcium chloride added to accelerate the set.

Permeability and porosity tests indicated that the overburden south of the core wall was essentially impervious and that a satisfactory cutoff could be obtained by stage grouting. Holes were first drilled 20° from the vertical to intersect any strike joints present; but, since there was a relatively small grout acceptance, the remaining holes, for economy, were drilled vertically on 6.25-ft centers and grouted in 10-ft stages to El. 680 with deeper check holes to El. 640 every 50 ft. The first stage was grouted at a pressure of 0.5 lb per foot of overburden and each successive stage at an additional 5 lb per foot of overburden until the last stage, which was grouted at a pressure of 50 lb per sq in.

Lock

In most of the rock underlying the lock river wall and the downstream half of the land wall, closely spaced weathered seams and cavities were present in depths from 30 ft to 40 ft. Below this zone the cavities were separated by from 20 ft to 50 ft of intervening sound rock that possessed considerable bearing strength in the form of irregularly spaced rock pillars. As the major cavities in the lock foundation—those along the bedding plane slips—were inclined to the southeast with an upstream pitch, they became progressively nearer the rock surface toward the downstream end of the lock.

Obviously, with so many deep cavities and seams existing under the lock area, it was not practicable to excavate the entire zone of cavitation. If a stratum of sound rock were deep enough, although superimposed on defective material, it could support the lock structures through arch action; and, as a precautionary measure, the entire lock area was subjected to consolidation grouting prior to cutoff treatment—in most instances, before concreting operations. Even with this program of foundation treatment, excavation to depths as great as 60 ft was necessary for the river wall foundation.

The depth and spacing of the consolidation holes were contingent on rock conditions and foundation requirements. Both first-stage holes 20 ft deep and second-stage holes 40 ft deep were sunk with wagon drills on 10-ft and 5-ft centers in parallel lines from 10 ft to 12 ft apart. Stage one was usually grouted at a pressure of 20 lb per sq in.; and stage two, at a pressure of 40 lb per sq in. The program was supplemented with diamond drill check holes, and grout pipes were calked into any open seams for grouting after concrete was poured, as shown in Fig. 6. Under the river wall and lower miter sill, the grouting program was supplemented by mining through calyx drill holes wherever necessary to secure a cutoff and to augment bearing strength.

Fig. 6.—Pipes Set in Lock Foundation for Consolidation Grouting After Placing Concrete

The cutoff across the lock consisted essentially of a shallow curtain of wagon drill holes 40 ft deep grouted at low pressure and a deeper curtain of diamond drill holes from 100 ft to 150 ft deep grouted at high pressure. The low-pressure curtain was designed primarily to reduce, to a minimum, the percolation under the structure through small cracks and joints in the rock surface. Stage one of the low-pressure holes was drilled 20 ft deep on 2-ft centers and grouted at a pressure of 20 lb per sq in. The holes were redrilled to 30 ft, washed, and then grouted at a pressure of 30 lb per sq in. They were finally drilled to 40 ft, washed and grouted at a pressure of 40 lb per sq in. Diamond drill holes for the high-pressure curtain followed the same line as those for the low-pressure curtain and varied in spacing and depth to satisfy structural and geological conditions. In highly weathered areas, grouting alone was not considered adequate and 36-in. calyx drill holes were spaced on such centers that it was possible to mine out unsound material between the holes and to backfill the holes with concrete. Because of the soundness of the near-surface rock under the upper miter still, the low-pressure grout curtain was eliminated and the high-pressure curtain was utilized to grout the deep bedding seams.

Spillway and Powerhouse

Limestone of Sevier unit 2 underlies the spillway from its intersection with the lock to the 9–10 block line where it grades into unit 1 for the remainder of the spillway area to the 1–2 block line. At this line a gradational contact with the Tellico formation underlying the powerhouse takes place, see Fig. 3. Exploratory drilling had shown the rock of the spillway to be intricately dissected by weathered seams and cavities from the top of rock at El. 730 to El. 700. Below this zone of weathering, the rock was essentially sound except for the presence of several master bedding cavities that were remarkably continuous and extended to great depths.

The No. 1 seam, the No. 2 seam, the No. 612 seam, and the adit seam (Fig. 3) were the structural features that controlled the spillway excavation pattern. Other structural weaknesses were comparatively tight except at or near the intersection of bedding cavities and faults. As in the lock, some compromise had to be made between open excavation and a program of foundation treatment. Excavation was generally carried to El. 700 (El. 712 for the apron), and the additional rock was excavated down the dip of these major bedding seams to a point where mining and grouting were more economical than was the process of making a further open cut. The resulting spillway excavation had the appearance of a level rock surface modified by the removal of large wedge-shaped blocks at the outcrop of the major bedding planes.

Except for the disconformity seam which underlies practically the entire powerhouse area and several minor strike joint cavities, the foundation rock of the powerhouse was only slightly weathered and little excavation beyond design requirements was necessary. The intake section was excavated to El. 715 with additional rock removal along the weathered bedding planes in intakes 2 and 3 and with removal of weathered rock at the outcrop of the No. 1 seam in intake 4. Excavation for the draft tubes was governed by design with the exception of a wedge-shaped block along the outcrop of the disconformity seam in Sevier unit 1.

Consolidation grout holes were drilled throughout the entire spillway area to augment the bearing strength of any unsound rock. Consolidation consisted not only of grouting through wagon drill holes and diamond drill holes, but also of mining large seams and cavities through calyx drill holes and backfilling these mined zones with concrete. The most extensive mining operation for consolidation was along the adit seam under spillway block 15. The full width of the block, upstream and downstream, was excavated by undermining; and, because of the overburden, the operation was accomplished in stages. A section 10 ft wide was first excavated and a concrete pillar was placed for support. This procedure (Fig. 7) was then repeated until the entire area had been mined out and backfilled with concrete.

Consolidation grouting of the intake sections was a simple procedure compared with the extensive and intricate program necessary under most of the spillway blocks. Excavation left only the deep disconformity seam to be treated, and this was accomplished through a uniform pattern of diamond drill holes on 10-ft centers grouted at a pressure of 50 lb per sq in. Occasionally, holes were taken to the deeper Tellico seam but it was found to be essentially tight.

All areas of unsound rock which might be potential aquifers for leakage under the spillway and powerhouse were treated by constructing a cutoff curtain extending the full length of the structure along the upstream face. The method of treatment varied to meet the conditions as they were encountered. It consisted of stage grouting through diamond drill holes and wagon drill holes, and of mining and backfilling with concrete through calyx drill holes—in those areas in which grouting alone was not sufficient. Mining involved the removal of a part of the unweathered rock immediately surrounding the actual seam to provide a tunnel large enough for a man to work. Most of these cutoff tunnels were about 4 ft high and from 4 ft to 5 ft wide. On completion of the mining program, the tunnel was backfilled with concrete and the calyx drill holes were filled with concrete to the surface to secure a hydrostatic head on the tunnel concrete that greatly reduced the possibility of voids and air pockets.

FIG. 7.—MINING ADIT SEAM UNDER SPILLWAY BLOCK 15

Exploratory drilling had shown the No. 612 seam (Fig. 3) to be badly weathered throughout most of the spillway section. Eight calyx drill shafts were started in spillway blocks 10, 11, and 12, but their completion was prevented by a prohibitive amount of water that could not be controlled successfully by pumping. An auxiliary cutoff was grouted across block 10 along the strike to prevent this leakage and to form a barrier against which a cutoff could be grouted. Holes were drilled and grouted first on 16-ft centers, followed by holes on 8-ft centers and 4-ft centers, and finally on 2-ft centers throughout blocks 10 to 14. The entire procedure was repeated in three stages—drilling and grouting the same holes first on 8-ft centers, then on 4-ft centers, and finally on 2-ft centers. On completion of this work, the calyx drill holes and mining operations were completed without further water problems.

Low-pressure grouting, through wagon drill holes, was applied to spillway blocks 12, 13, and 14 only. The remainder of the spillway was subjected to closely spaced deeper diamond drill holes grouted in stages at a pressure of

100 lb per sq in. Final spacing of holes was usually 2 ft on centers; and, in the powerhouse area, holes were spaced on 4-ft centers. Fig. 8 shows the pattern of cutoff treatment across the spillway area and Fig. 9 shows similar treatment given to the powerhouse foundation.

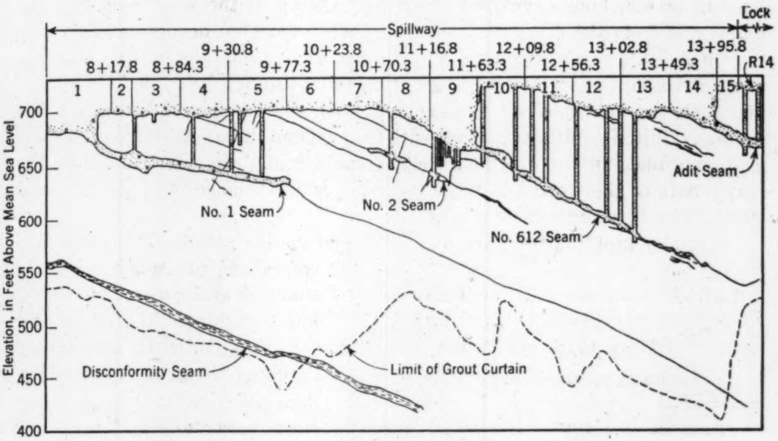

Spillway
Lock
9+30.8
10+23.8
11+16.8
12+09.8
13+02.8
13+95.8
8+17.8
8+84.3
9+77.3
10+70.3
11+63.3
12+56.3
13+49.3
R14
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
700
650
600
550
500
450
400
Elevation, in Feet Above Mean Sea Level
No. 1 Seam
No. 2 Seam
No. 612 Seam
Adit Seam
Disconformity Seam
Limit of Grout Curtain

FIG. 8.—SECTION ALONG SPILLWAY CUTOFF SHOWING TREATMENT OF THE LARGE BEDDING CAVITIES

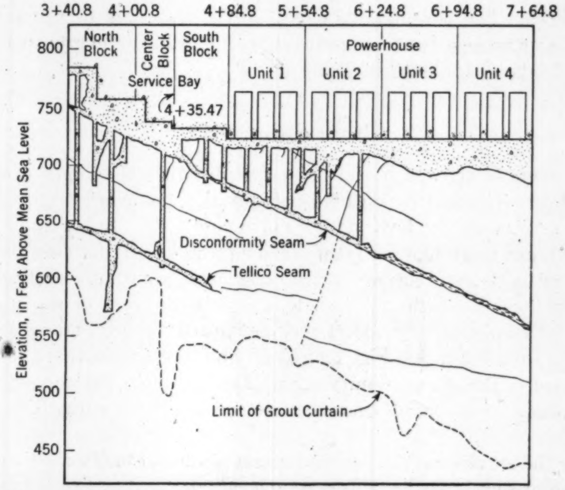

3+40.8
4+00.8
4+84.8
5+54.8
6+24.8
6+94.8
7+64.8
North Block
Center Block
South Block
Powerhouse
Service Bay
Unit 1
Unit 2
Unit 3
Unit 4
4+35.47
800
750
700
650
600
550
500
450
Elevation, in Feet Above Mean Sea Level
Disconformity Seam
Tellico Seam
Limit of Grout Curtain

FIG. 9.—SECTION ALONG POWERHOUSE AND SERVICE BAY CUTOFF SHOWING TREATMENT OF LARGE BEDDING CAVITIES AND STRIKE JOINTS

Although it was not part of foundation treatment, an extensive grouting program undertaken for sealing leakage under the cofferdam was largely responsible for controlling the flow of water into the river bed construction area.

The first-stage cofferdam enclosing the lock and the ten adjoining spillway blocks was essentially tight; but, as excavation progressed below the overburden, several boils developed along the outcrop of the weathered bedding seams and from the diamond drill holes that penetrated deeper cavities. As further leakage was anticipated with deeper excavation, the entire perimeter of the cofferdam was subjected to grouting—first through holes on 20-ft centers and then through holes on 10-ft centers with some holes on 5-ft and 2.5-ft centers. Initial grouting was done with neat cement; but, since large quantities were consumed in the first holes, clay-cement grout was substituted for the major part of the work. For the more persistent leakage, asphalt was used successfully to stop the flow of water.

A similar grouting program was developed for the second-stage cofferdam comprising the remainder of the spillway and powerhouse area, but grouting of this stage was completed prior to unwatering. Although the second-stage cofferdam was tighter than the first-stage cofferdam and presented practically no water problems, its imperviousness was not completely ascribed to the grouting of the cofferdam before unwatering. The large quantities of clay-cement and asphalt injected into the first-stage cofferdam had considerable effect on the tightness of the second-stage cofferdam as was revealed by the excavation. Drilling and grouting quantities for the cofferdams are listed in Table 2.

TABLE 2.—DRILLING AND GROUTING QUANTITIES FOR THE COFFERDAMS, FORT LOUDOUN DAM

(Spillway and Powerhouse Areas)

Cofferdam	Drilling (lin ft)	Grout (lin ft)	Asphalt (cu ft)
Stage one..........	42,850	668,940	9,408
Stage two..........	26,240	157,090	500

NORTH ABUTMENT

Exploration of the north abutment revealed that the depth of excavation required to reach sound rock would necessitate a nonoverflow gravity type of dam between the service bay and the north core wall. In a manner typical of the entire north abutment area, the Tellico formation underlying the proposed nonoverflow section and service bay was dissected by a complex series of vertical joints along which weathering had taken place to great depth, leaving high pinnacles of comparatively sound rock separated by deep troughs filled with a sandy clay residuum (see Fig. 4). This condition was further aggravated by the presence of the disconformity seam (Fig. 3) which outcropped in block 3 of the nonoverflow section and necessitated a deeper excavation than was anticipated.

Under the service bay, the disconformity seam was badly weathered and the foundation was excavated to this seam over most of the service bay area. With this depth of excavation, the only consolidation necessary under the service bay consisted of grouting through diamond drill holes on 10-ft centers along ranges 20 ft apart throughout the three upstream blocks. Consolidation grouting was supplemented by driving a 4-ft tunnel along the unexcavated part of the disconformity seam and filling it with concrete, as shown in Fig. 10.

Since it was impracticable to remove the rock to the bottom of the several deep vertical joints encountered in the nonoverflow section, excavation was

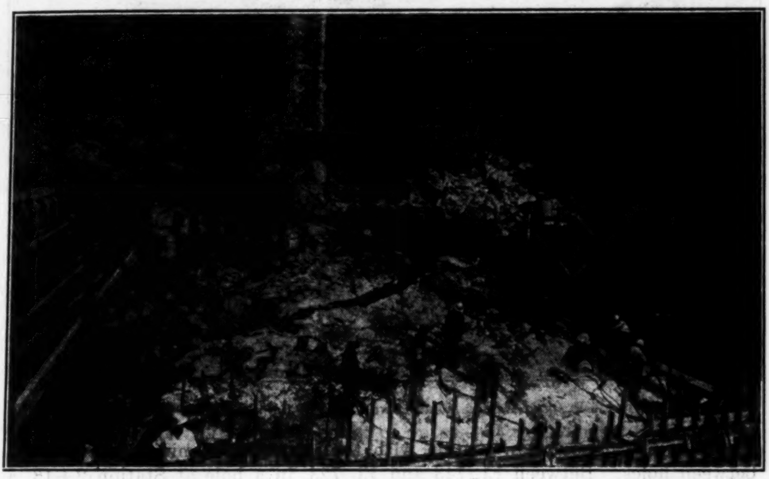


Fig. 10.—Disconformity Seam and Associated Strike Joint in the Service Bay

stopped at a depth considered adequate for bearing strength. A cutoff was constructed by mining all unsound areas through deep calyx drill holes, backfilling with concrete and subsequently grouting through closely spaced diamond drill holes. As soon as excavation in nonoverflow blocks 1 and 2 was completed, concrete was poured to El. 800 to improve working conditions and to prevent the sliding of material into the cut. Barrel forms, as shown in Fig. 11, were brought up through the pour at several points to reduce later calyx drilling.

The seams treated were of two major types—(*a*) the shallow vertical weathered joints not removed by excavation, and (*b*) the deeper continuous Tellico seam with associated strike joints. Shallow treatment in block 1 of the nonoverflow section was unnecessary as all rock below the disconformity seam was essentially sound. In block 2, several weathered joints were mined through calyx drill holes. These holes were later deepened for mining the Tellico seam. Excavation to the disconformity seam in block 3 revealed a series of shallow but closely spaced badly weathered joints; and, instead of mining between holes, 36-in. calyx drill holes were first sunk on 5-ft centers and backfilled with concrete; intermediate holes were drilled to overlap the first holes. Similar treatment was given throughout block 4 with three holes later deepened to the Tellico seam.

Treatment of the Tellico seam under the nonoverflow and service bay section saw a difficult operation. Not only did the seam occur at depths as great as 150 ft, but its treatment was complicated by the presence of weathered strike joints that required additional and deeper mining. Fig. 12 will assist the reader in the discussion to follow. Three calyx drill holes, about 5 ft apart, were drilled in the nonoverflow section with the intention of mining the cavities between holes. Between El. 743 and El. 723, in a hole at Station 2+13, a strike joint was encountered; it was thoroughly mined and a concrete plug was placed to El. 745. The plug was then drilled out and drilling was continued to the Tellico seam where excessive water entered the excavation. Concrete plugs were poured in the calyx drill holes, and a ring of six diamond drill holes grouted at high pressure, was effective in stopping the flow of water.

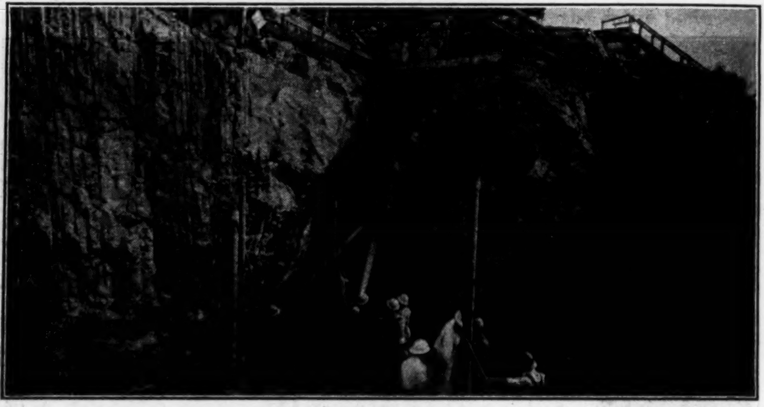


FIG. 11.—PLACING BARREL FORMS AND 4-IN. CASING FOR LATER DRILLING

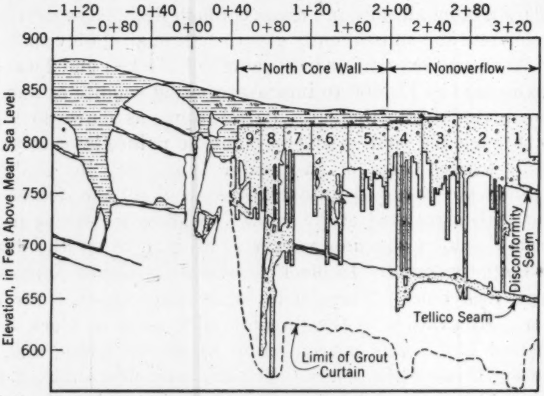

−1+20
−0+40
0+40
1+20
2+00
2+80
−0+80
0+00
0+80
1+60
2+40
3+20
900
850
800
750
700
650
600
Elevation, in Feet Above Mean Sea Level
North Core Wall
Nonoverflow
9
8
7
6
5
4
3
2
1
Disconformity Seam
Tellico Seam
Limit of Grout Curtain

FIG. 12.—SECTION ALONG THE NORTH CORE WALL AND THE NONOVERFLOW SECTION CUTOFF

Preparatory to tunneling southward along the Tellico seam, the steep wedge-shaped cut, caused by mining the strike joint, was poured to El. 665 to provide a level floor from which to work. The tunnel was driven along the Tellico seam southward until it intersected a previously drilled hole at Station 2+75. A weathered strike joint at Station 2+70 was mined below the Tellico seam to El. 650 where weathering had ceased; but, to eliminate the possibility of any deeper solution along the joint, six shallow diamond drill holes were drilled from the tunnel floor along the down-dip side of the joint and grouted.

Another calyx drill hole intersected the Tellico seam at Station 3+12, and mining was started northward to intersect the first section of the tunnel. Excellent progress was made in this section despite the presence of a weathered bedding seam intersecting the Tellico seam at Station 3+12 and pinching out at Station 2+63. By using the Tellico seam as the floor and the bedding cavity as the roof of the tunnel, a cutoff was obtained along both these structures in a single operation. After the tunnels were joined, they were cleaned and backfilled with concrete.

Concrete was placed in such a manner that only a minimum of redrilling was necessary in any hole to be used for further mining. Treatment of the Tellico seam by mining north of Station 2+08 was not required; however, the unmined area was subjected to grouting. Before tunneling the Tellico seam southward under the service bay from Station 3+12, the area was subjected to a grouting pattern to check the flow of water that was indicated by exploratory drilling. Three calyx drill holes were bored to the seam. The area was found to be more badly weathered than had been anticipated; therefore, mining was continued to Station 4+72, as indicated in Fig. 9. Several strike joints in the service bay area were also intersected and treated similarly to those encountered in the nonoverflow section.

Following the mining operation, the entire nonoverflow and service bay were supplemented with grouting through diamond drill holes on 2-ft centers throughout the service bay and blocks 1 and 2 of the nonoverflow section. Blocks 3 and 4 of the nonoverflow section were treated with holes on 4-ft centers.

North Core Wall

Rock conditions under the north core wall were similar to those under the nonoverflow section. However, the deep weathering was largely confined to two major areas separated by a ridge of comparatively sound rock. The first of these areas was under blocks 5 and 6 and the second—by far the most deeply weathered part of the entire dam foundation—existed under blocks 8 and 9 (see Fig. 12). Exploratory drilling disclosed a nearly vertical oblique joint striking normal to the dam axis and weathered to the unprecedented depth of approximately 220 ft. This joint had been subjected to some lateral movement, thought to be local, but possibly with a direct bearing on the depth and degree of the solution.

An open cut excavation was first made to El. 800. Further excavation to bare the rock for deep drilling and foundation treatment was done by trenching, and heavy timbering was required to effect this operation. A trench 4 ft wide was dug to El. 745 at the south end and to El. 758 at the north end. Below these levels, weathering was restricted to the narrow vertical joints which were mined out through calyx drill holes and backfilled with concrete.

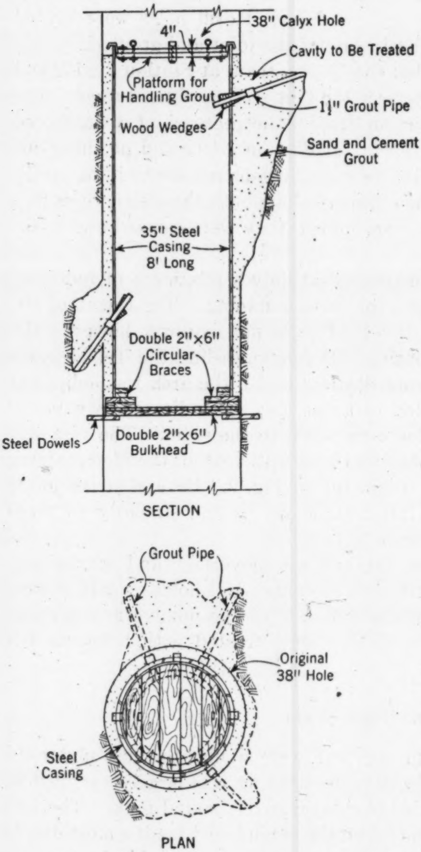

4"
38" Calyx Hole
Cavity to Be Treated
Platform for Handling Grout
1½" Grout Pipe
Wood Wedges
Sand and Cement Grout
35" Steel Casing 8' Long
Double 2"×6" Circular Braces
Steel Dowels
Double 2"×6" Bulkhead
SECTION
Grout Pipe
Original 38" Hole
Steel Casing
PLAN

FIG. 13.—METHOD OF CALKING A CAVITY THAT INTERSECTED THE CALYX DRILL HOLE

Before drilling was begun, the bottom of the trench was filled with concrete to El. 800 and El. 796 at the north end and south end, respectively. Barrel forms were again set as required to reduce the drilling footage. Treatment under blocks 5, 6, and 7 of the core wall consisted of both shallow and deep calyx drilling. In some areas a shallow cutoff was obtained with overlapping 36-in. calyx drill holes spaced on 2.5-ft centers. Although this method was possible for shallow cutoffs, it was not feasible for treating deeper cavities because of the large amount of drilling and the difficulty, at greater depths, of obtaining the alinement necessary.

The most difficult foundation problem—not only of the north core wall but also of the entire project—was the deep vertical oblique joint under block 8. As in other parts of the foundation, calyx drill holes provided shafts for the mining operation. The first of these holes was sunk at Station 0+86.8 (see Fig. 12); and, at El. 712, the hole intersected a highly weathered area at the intersection of the oblique joint and the Tellico seam. Another hole was drilled 10 ft southward at Station 0+96.8, and the cavity was cleaned between these two holes to El. 703 and filled with concrete. The concrete plugs thus formed were redrilled, and mucking was continued to El. 695 where weathering once again was confined to the vertical joint; and the cavity was again filled with concrete. The hole at Station 0+96.8 was then no longer required and was backfilled with concrete.

The mucking, backfilling, and redrilling processes continued in 7-ft stages in the remaining hole to El. 674 where the water inflow became excessive. Another hole at Station 0+76.8 was then drilled to a slightly lower elevation and connected to the cavity with jackhammer holes. A pump was installed to handle the water, and mucking was continued. One hole in the rock was kept open for a sump, by offsetting if necessary, and the other hole was used for mucking.

At El. 635 a slide forced a temporary abandonment of the treatment until the seam-filling material was consolidated by grouting. Preparatory to grouting the seam through diamond drill holes, several minor joints intersecting the walls of the hole and the oblique joint had to be plugged to avoid loss of grout. These joints were calked by inserting a 35-in. steel casing into the calyx drill hole at the point to be treated and hand packing grout into the seam between the casing and the periphery of the hole, as shown in Fig. 13. In addition, nipples for pressure grouting were calked into the seams wherever possible. Four 1.5-in. grout pipes were driven into the seam where the slide occurred and a 4-ft concrete cap was placed. After grouting through these pipes at a pressure of 100 lb per sq in., jackhammer holes were drilled from the calyx drill hole into the seam at several elevations and grouted at high pressures. Concurrently, two ranges of diamond drill holes, one from 10 ft to 40 ft upstream at Station 0+76.8 and the other a like distance downstream at Station 0+86.8 were drilled normal to the core wall for grouting the seam. Holes were drilled on 10-ft, on 5-ft, and, in some places, on 2.5-ft centers and grouted at high pressure.

When the concrete plug was drilled out of the calyx drill hole, the sandy clay residuum had been consolidated, and the process of treating the cavity was continued to El. 593 where it pinched to a fraction of an inch. Deeper treatment was accomplished by placing 1,065 cu ft of grout into two perforated pipes driven in the seam and by injecting 105 cu ft of grout through jackhammer holes drilled into the seam from the adjacent calyx drill hole. A 4-in. pipe was encased in the concrete backfill, and a later diamond drill hole, bored into the seam to El. 478.5, was grouted with 787 cu ft of cement at a surface pressure of from 100 lb to 120 lb per sq in. as the final treatment. Supplementing the calyx drill hole procedure, diamond drill grout holes were drilled on 4-ft centers for the entire length of the cutoff to consolidate any possible concrete shrinkage in the mined areas and to treat minor cracks and joints.

North Reservoir Rim and Saddle Dam

The bearing of the ridge forming the north reservoir rim parallels the strike of the geologic structure with the dip of the rock toward the reservoir. Normally, such a structure would provide a satisfactory foundation; but the presence of a complex pattern of weathered strike and dip joints made a certain amount of rim treatment necessary to prevent leakage downstream or seepage northward through the ridge separating the reservoir from Muddy Creek Valley.

The north rim was treated by placing a grout curtain extending from the end of the core wall upstream more or less along the strike of the rock a distance of about 800 ft. Diamond drill holes, inclined 45° upstream and northeast to secure a maximum intersection with weathered dip joints, were grouted on 20-ft centers. These holes were reinforced by grouting vertical holes on 10-ft centers in the vicinity of the north core wall and on 20-ft centers farther upstream. Except for the vertical holes near the core wall (which were grouted with cement) the north reservoir rim was grouted exclusively with a clay-cement mix consisting of three parts clay to one part cement.

Drilling was discontinued whenever a cavity or group of cavities was encountered, and the hole was grouted before further drilling. The processes of drilling, grouting, and redrilling were continued throughout the length of the cutoff curtain until the final drilling of each hole showed no water loss and revealed a considerable margin of sound rock below the last cavity. For the upper 440 ft of the curtain, only two to four stages of grouting were necessary, but deeper weathering (to El. 590) toward the north core wall required grouting in as many as eight stages. Although the multiple-stage grouting procedure required a larger quantity of grout than would probably have been used with the single-stage method, the method afforded positive proof of the effectiveness of the grout curtain.

Only moderate treatment of the saddle dam was necessary because of the deep layer of overburden, the dip of the rock toward the reservoir, and the low head to which the structure was subjected. Grout holes were drilled along a zone of comparatively sound rock parallel to the axis of the saddle dam and at an angle of 45° to the south to intersect the predominant vertical joints. Holes were first drilled on 25-ft centers with intermediate holes on 12.5-ft centers. Closer spacing was necessary in some areas. The depth of grout penetration varied from El. 680 to El. 720, depending on the rock structure.

Drilling and Grouting Equipment and Materials

A maximum of sixteen diamond drills and six large-diameter calyx core drills were driven for foundation treatment in addition to several pneumatic wagon drill holes. Diamond drills were powered by four-cylinder 10-hp gasoline engines except when used in tunnel or galleries where an electric motor was substituted. Calyx core drills were driven by either 20-hp electric motors or 35-hp gasoline engines. Manufacturers' standard bortz bits were used for diamond drilling. Calyx core bits were 38 in. in diameter or smaller, and casings were made of ¼-in. steel from 54-in. to 33-in. diameters.

Seven grout mixers were used on the project, two of the type developed at Boulder Dam in Arizona and Nevada and five of the type developed at Norris Dam in Tennessee.[5] All grouting was done with 7-in. by 5-in. by 10-in., air-driven, double-acting, reciprocating pumps equipped with rubber valves and iron fluid pistons with replaceable rubber sleeves. The pumps and mixers were mounted together on skids so that they could be moved easily. The grout pumps were also used extensively for pumping water from tunnels during mining operations; and, during the treatment of the north core wall, these pumps were used successfully in a vertical position suspended in the calyx drill hole from an A-frame at the surface. In this position the pump did not require priming as the discharge line to the top of the hole remained full of water when the pump was not operating and provided a continual prime. This pump also operated satisfactorily when completely submerged in water.

---

[5] "Foundation Experiences, Tennessee Valley Authority: A Symposium," *Transactions*, ASCE, Vol. 106, 1941, p. 687.

The use of packers was limited at Fort Loudoun because of the stage-grouting methods generally required. In practically all instances where packers were used, the grout found its way through open channels and into the hole above the packers, thus grouting them in. For this reason, the use of the packers, in general, was abandoned.

For foundation grouting standard Portland cement mixed with water was used entirely, with the exception of the north reservoir rim and saddle dam as previously discussed. The water-cement ratio varied between 3.0 and 0.5 by volume. Grouting in an area was usually started with a thin grout and the water-cement ratio was gradually decreased. As pressures increased indicating possible refusal, the water-cement ratio was again increased.

SUMMARY

This paper describes the treatment of solution channels and cavities in a limestone foundation at unprecedented depths below the water table. The conditions described indicate that structural deformational features rather than rock solubility often control the depth and extent of weathering. In the Fort Loudoun limestone foundation, solution occurred down to a depth of 300 ft below the original ground surface and 260 ft below the water table. The deep mining operations both for consolidation and an effective cutoff challenged the combined efforts of the designing engineers, field engineers, geologists, and special construction personnel, all of whom contributed to the satisfactory solution of this difficult and unusual foundation problem. Contrary to the normal conception that relatively more cavities are to be expected in the purer limestone, the cavities in this area were primarily controlled by solution along geologic structures—faults, joints, and bedding planes, rather than by the purity of the rock. On this project the geologists really came into their own and to them due credit is given.

DOUGLAS DAM

BY T. F. TAYLOR,[6] ASSOC. M. ASCE

SYNOPSIS

The foundation rock beneath the concrete structure of Douglas Dam and part of the adjacent rim was dolomitic limestone of the Knox formation. Large bedding cavities existed at each abutment of the concrete dam and the rock in the original river bed was highly cavernous. That part of the rim which was formed by shale was relatively tight.

The procedure used to prepare the foundation for the concrete dam consisted of excavating the highly cavernous rock, and mining, cleaning, and backfilling with concrete the large bedding cavities that were overlain with sound rock. These operations were supplemented with an intensive grouting program over the entire foundation for the concrete dam.

The reservoir rim adjacent to the dam was grouted principally where the outcropping rocks were limestones. Very little foundation treatment was needed in the shale rim. An analysis of final grout consumptions in all areas treated show that a satisfactory foundation was obtained. Observations made more than six months after the reservoir had been filled to maximum pool level indicated that seepage from the reservoir did not exist.

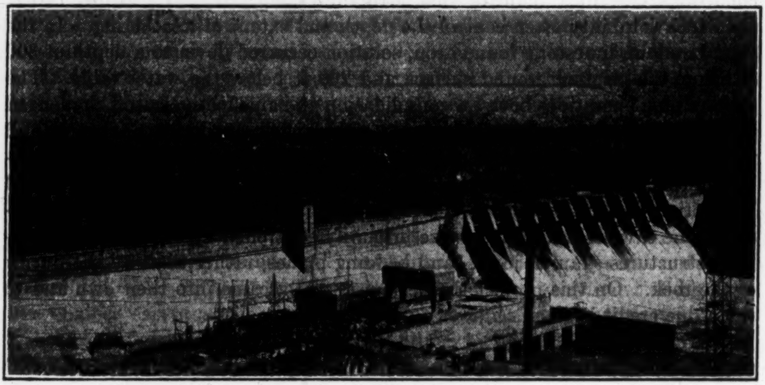


FIG. 14.—DOUGLAS DAM

INTRODUCTION

Douglas Dam is in Sevier County, Tennessee, on the French Broad River approximately 32 miles above its mouth (see Fig. 1). The site was located at a convergence in an otherwise broad valley. The original river channel at that point was about 500 ft wide at normal flow, with the bottom at approximately El. 868, and flanked on each side by a relatively narrow alluvial flood plain, on the average at El. 880. The south bank of the river was an almost vertical rock bluff. The north bank sloped away in the form of a steep sodded hill.

---

[6] Supt., Tennessee Eastman Corp., Oak Ridge Tenn.; formerly Constr. Engr., TVA, Douglas Dam, Dandridge, Tenn.

The concrete dam is approximately 200 ft high above the lowest point of the foundation and is 1,682 ft long at the crest. The main dam and powerhouse contain about 542,000 cu yd of concrete. Fig. 14 is a view of the completed dam, and Fig. 15 shows the plan of the structure and the geologic profile along the axis.

The topography of the reservoir rim, extending approximately 5 miles from the south end of the concrete dam, required the construction of eight earth dams in existing low saddles. The largest of these earth saddle dams contains 474,000 cu yd of compacted clay, and the total of all the earth saddle dams was 532,000 cu yd.

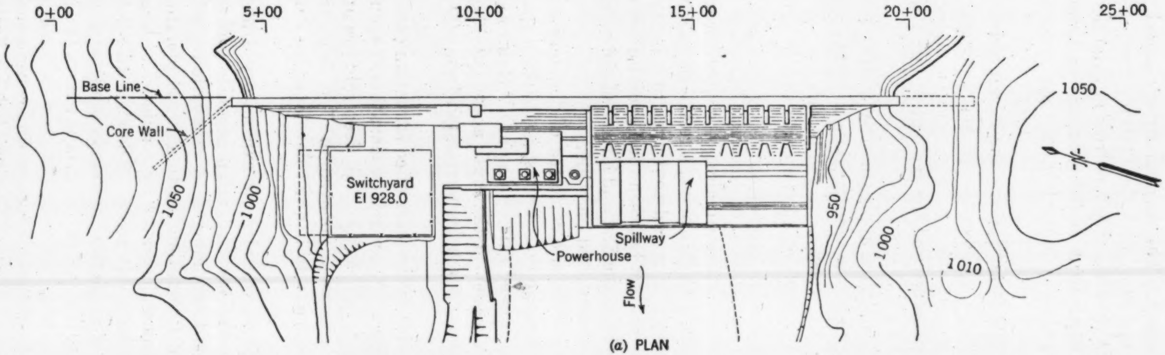
The foundation for the concrete dam is composed of dolomitic limestone of the Knox formation. The dolomite extends southward along the reservoir rim for a distance of 1,500 ft to Station 35+00. At this point, a relatively narrow band of limestone of the Lenoir formation begins and continues to a contact with the Athens shale on the south. The contact between the Lenoir limestone and the Athens shale is under the north end of the first and largest of the saddle dams. The remaining saddle dams are constructed on shale.

Preliminary exploratory borings at the dam site consisted of diamond core drill holes on three parallel ranges. The range lines were spaced 100 ft apart and were approximately parallel to the proposed axis of the dam. Holes were drilled on these ranges at 100-ft intervals, except at the south abutment where a few holes were drilled at 50-ft intervals. The holes penetrated from 100 ft to 250 ft into bedrock.

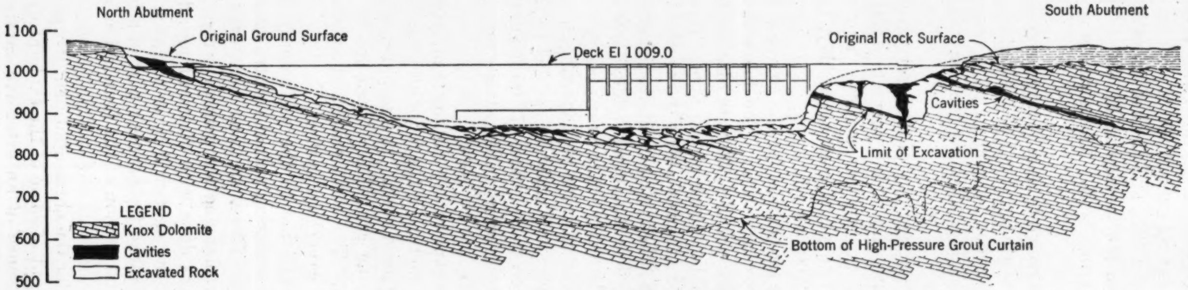
The exploratory drilling completed prior to the start of construction gave a general picture of the foundation, including the approximate elevation of sound, unweathered rock. It also indicated that, in the river bed, deep solution and weathering caused the formation of an extensive honeycomb of cavities and fissures. The weathering and solution apparently had not developed into any regular cavernous system.

In addition to the preliminary borings, considerable exploratory drilling was done during actual construction operations to develop accurate geologic sections. These sections were used for determining final excavation grades and for planning the general program of foundation treatment. Exploratory work was done with diamond core drills supplemented by a number of large-size, 42-in., shot drill holes to check questionable areas. The shot drill holes were planned to become a part of a tentative method of foundation treatment. A general section of the foundation along the axis of the dam is shown in Fig. 15(*b*).

Under the north abutment it was noticed that air was passing continuously in and out of some of the exploratory holes. These "blowing" holes indicated the presence of an extensive cavity in the foundation rock strata. It was


0+00
5+00
10+00
15+00
20+00
25+00
Base Line
Core Wall
1 050
1 000
Switchyard
El 928.0
Powerhouse
Spillway
Flow
950
1 000
1 010
1 050
N

(*a*) PLAN


North Abutment
South Abutment
1 100
1 000
900
800
700
600
500
Original Ground Surface
Deck El 1 009.0
Original Rock Surface
Cavities
Limit of Excavation
Bottom of High-Pressure Grout Curtain
LEGEND
Knox Dolomite
Cavities
Excavated Rock

(*b*) PROFILE ALONG THE AXIS

FIG. 15.—PLAN OF DAM AND GEOLOGICAL PROFILE

anticipated that this cave would have to be explored, and would receive special treatment.

At the south abutment a large bedding-joint cave was encountered which extended for some distance into the reservoir rim beyond the south end of the concrete dam. Drill holes in the rim indicated that no special foundation problem existed beyond the limits of the cave.

The anticipated program for foundation treatment consisted mainly of a general procedure for the concrete dam, plus a special treatment of the cavities existing at the north and south abutments and some curtain grouting in the reservoir rim areas.

NORTH ABUTMENT

The natural slope of the north riverbank at the dam site was nearly parallel to the dip of the foundation rock. Weathering in the upper strata of bedrock was fairly uniform and averaged from 8 ft deep to 10 ft deep over most of the area. The weathered rock was removed by open cut methods to expose good foundation rock.

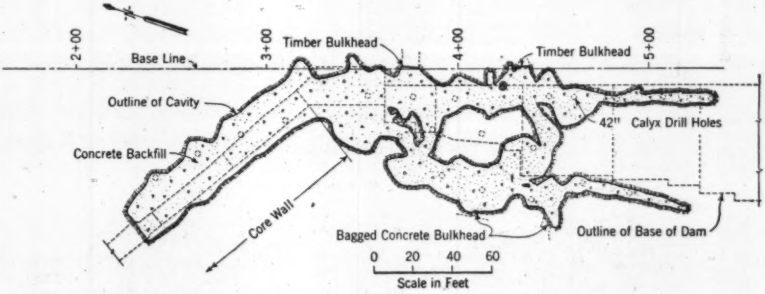


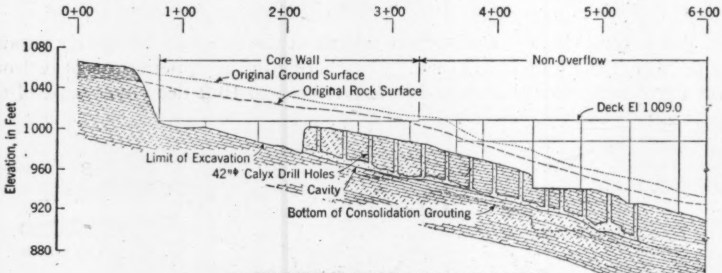
FIG. 16.—BEDDING-JOINT CAVITY UNDER NORTH NONOVERFLOW SECTION

The area in which the original exploratory drilling had indicated that an extensive cave developed along a bedding plane was further checked with 42-in. calyx drill holes. It was found that the cave extended under most of the area occupied by the north nonoverflow section of the dam and also under the north core wall. The location of the cave is given in Fig. 16, with the outline of the base of the dam shown superimposed on the plan of the cavity.

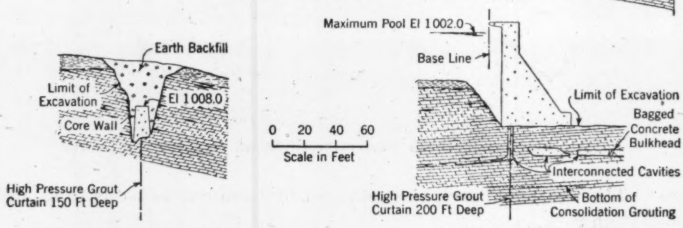
The bedding-joint cave was partly filled with clay, and it was intercepted by several strike joint cavities running normal to the axis of the dam. These strike joint cavities formed enlargements in the main cave, which made the maximum height of the cave approximately 5½ ft. About 40 ft of sound rock existed above the roof of the cave after the foundation for the dam had been excavated to final grades. The removal of this sound rock above the cave was not considered practicable or desirable. To provide adequate bearing and watertightness, it was decided to mine out and fill with concrete that part of the cavity lying under the area of the concrete dam and core wall.

To accomplish the mining and concreting operations, 42-in. calyx drill holes were drilled from rock surface to the roof of the cave on approximately 25-ft centers. Small hoists were set up over the drill holes to remove the muck and excavated material from the cave. After the cavity was mined out, it was thoroughly washed and cleaned with compressed air and water. Fig. 17 shows a section through the cave and gives the position of the large calyx drill holes.

Near the south end the height decreased so that it was necessary to enlarge the cave to provide working room for cleaning and washing. At the point where mining methods were stopped, the cave was approximately 0.6 ft thick.


0+00
1+00
2+00
3+00
4+00
5+00
6+00
1080
1040
1000
960
920
880
Elevation, in Feet
Core Wall
Non-Overflow
Original Ground Surface
Original Rock Surface
Deck El 1009.0
Limit of Excavation
42"ϕ Calyx Drill Holes
Cavity
Bottom of Consolidation Grouting

(a) TREATMENT OF CAVITY AT NORTH END OF DAM


Earth Backfill
Limit of Excavation
El 1008.0
Core Wall
High Pressure Grout Curtain 150 Ft Deep
0 20 40 60
Scale in Feet
Maximum Pool El 1002.0
Base Line
Limit of Excavation
Bagged Concrete Bulkhead
Interconnected Cavities
High Pressure Grout Curtain 200 Ft Deep
Bottom of Consolidation Grouting

(b) SECTION AT STATION 1+85

(c) SECTION AT STATION 4+48

Fig. 17.—Details of Treatment for the North Bedding Cavity

As shown in Fig. 16, there were comparatively few solid rock pillars supporting the roof of the cave.

Large quantities of water were used in the washing operations; but it was not necessary to install pumps in the cavity. The cave extended upstream and downstream far enough so that the water drained freely by gravity.

At the north end of the core wall, the rock was extensively weathered above the roof of the cave. Therefore, it was necessary to excavate by open cut methods until the core wall could be extended to the floor of the cavity. This is shown in Fig. 17. In all cases it was found that the floor of the cave was formed by a thin shaly bed which retarded the weathering below it.

Formwork was necessary in some cases to confine the limits of the concrete backfill and to establish construction joints. The original 42-in. calyx drill holes for mucking purposes were also used to place concrete. Filling was started from the calyx drill holes at the south end of the cavity and advanced northward. The floor of the cave received a thick bed of sand-cement mortar placed immediately in advance of the concrete—and as speedily as possible, to provide a hydrostatic head of fluid concrete. This method proved satisfactory; by filling the calyx drill holes with concrete, sufficient head was created to provide a relatively tight fill against the roof of the cave.

A total of 1,298 cu yd of concrete with ¾-in. (maximum size) aggregate was used to backfill the cave. The average mix had a water-cement ratio of 0.55 by weight and contained 1.83 bbl of cement, 0.79 tons of sand, and 0.74 tons of coarse aggregate per cubic yard of concrete.

After the cave had been completely backfilled, it was further treated by consolidation and curtain grouting with neat cement. The grout holes were extended through the cave from the foundation of the dam.

RIVER BED

The upper 20 ft to 60 ft of bedrock in the original river channel were extremely cavernous. This section formed the foundation for the spillway and a part of the powerhouse and intake section of the dam. Large cavities were found along many beds and were interconnected by cavities extending along both strike and dip joints. Some of the cavities were large enough to be termed caves and were continuous for long distances, principally along the strike of the rock. Solution was intensified because the strike of the rock was approximately parallel to the flow of the river. The cavities were partly filled with clay, sand, and gravel, and some of the foundation was so greatly weathered that it presented the appearance of clay and gravel in which nested huge floating boulders. Fig. 18 shows this condition.

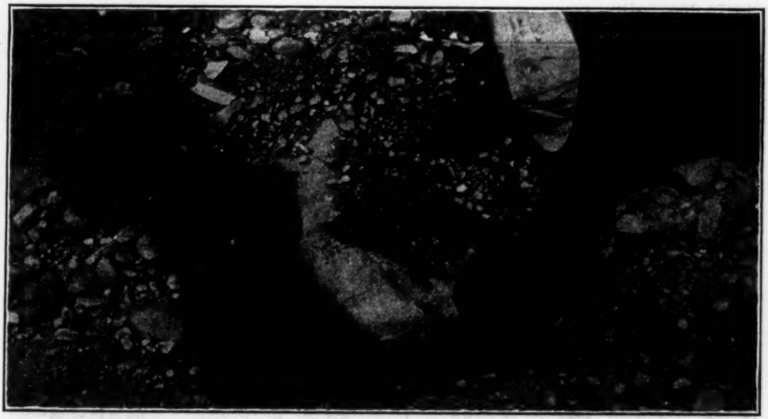


FIG. 18.—TYPICAL VIEW OF ROCK CONDITIONS IN THE RIVER BED FOUNDATION PRIOR TO FINAL EXCAVATION

The excavation of this cavernous rock presented an extremely difficult construction problem because of the large flow of water coming into the cofferdam area from the various cavities. When the first-stage cofferdam was unwatered, large leaks or boils occurred in the river bed which threatened the entire excavation program.

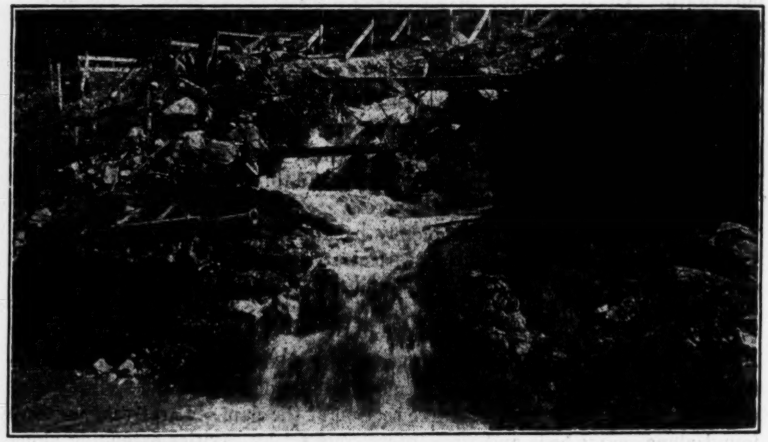


Fig. 19.—Cavernous Rock and Leakage Under the Upstream Cofferdam

Fig. 20.—Water Boil in the Excavation Area

To continue excavation, extensive grouting with asphalt, clay, and cement mixtures and with other materials was tried in an effort to cut off the flow of water coming in the work area. This was supplemented by large-scale clay blanketing in the river bed both upstream and downstream from the cofferdam.

In asphalt grouting, care was exercised to prevent asphalt from entering the foundation rock lying under the area of the dam below the final excavation grades. Although these efforts succeeded in cutting off a large part of the inflow of water, leakage was still a major problem during excavation operations. For a considerable period, thirty pumps discharging 100,000 gal per min were in constant operation to keep the cofferdam area dewatered sufficiently to permit excavation to progress. Fig. 19 shows the cavernous rock beneath the upstream arm of the first-stage cofferdam. One of the many large leaks is shown and asphalt may be noticed where it had been used in cofferdam grouting. Fig. 20 shows one of the many boils of water in the middle of the excavation area. Although the construction difficulties and the cofferdam grouting are not part of the foundation treatment program, they are mentioned to indicate the scope of the foundation problem in the river bed.

As the weathered and cavernous rock was removed and excavation progressed in depth, the leakage was confined to the sides of the excavated area. Despite construction difficulties, all foundations were excavated to sound, principally unweathered rock; and, by isolating small areas, concrete for the dam was placed on a dry foundation. A few relatively minor seams remained below the final excavation grades. These were treated with consolidation and curtain grouting.

South Abutment

The alluvial flood plain on the south side of the river at the dam site ended at a vertical rock bluff approximately 75 ft high. Beyond the top of the bluff was a gently sloping upland. From the top of the rock bluff southward for a distance of 327 ft, the rock formed the foundation for the south nonoverflow section of the concrete dam. Early exploration indicated that extensive excavation would probably be required in this abutment. Prominent deeply weathered strike joint solution channels extended through the foundation normal to the axis of the dam. In addition to these solution channels, there were less pronounced bedding cavities. Terrace clay and gravel were present in the overburden as well as in the cavities.

The program followed in the south abutment area consisted of excavating by open cut methods to a well-defined bedding cavity. Below this cavity the vertical fissures were mined out and backfilled with concrete. Fig. 21 shows the excavation in the south abutment and one of the major strike joint solution channels. The area was excavated 134 ft below the original ground surface, 115 ft being rock.

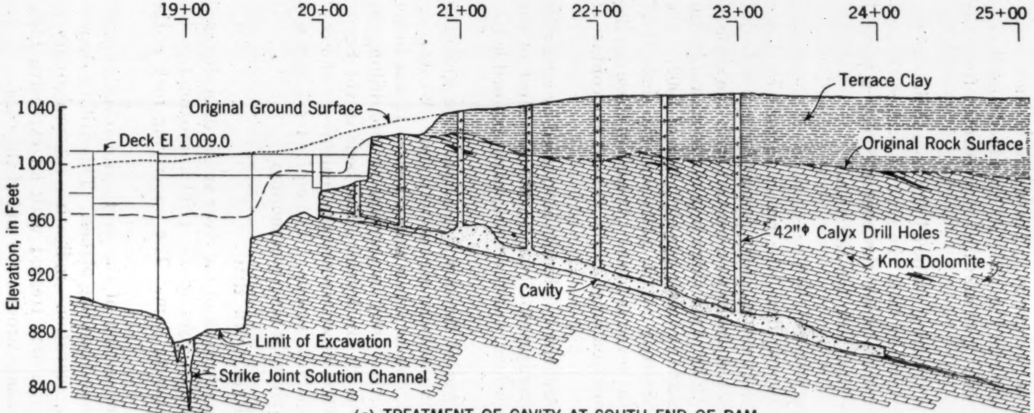
The exploratory drilling also revealed the presence of a large bedding cavity beginning at the end of the south nonoverflow section of the concrete dam and extending into the rim. The cave was partly filled, and at some points tightly filled with clay, sand, and terrace gravel, with the terrace gravel normally in lenses. Exploratory drilling indicated the presence of the cave for approximately 700 ft between Stations 20+0 (El. 963) and 27+0 (El. 790).

Enlargements of the cave existed at strike joint solution cavities. The cave was excavated by drift mining supplemented by 42-in. shot drill holes on 50-ft centers extending vertically from the ground surface to the floor of the cave. A drift was excavated following the cavity down the dip from a portal near the end of the concrete dam. Fig. 22 shows a section through the cave and gives the position of the shot drill holes. Mining operations were extended for a distance of 410 ft from El. 963 at Station 20+0 to El. 860 at Station 24+10. The height of the cavity ranged from 3 ft to 10 ft except for one section 50 ft long, which was 25 ft high. In some instances, it was necessary to excavate rock to obtain sufficient working room in the cavity. During the mining operations no difficulty was experienced with ground water until the cavity reached El. 873, which was river level. Below this elevation, periodic pumping was necessary to allow excavation to proceed.

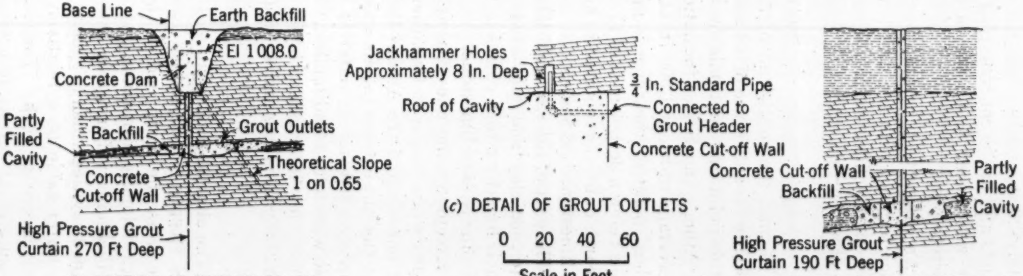
FIG. 21.—STRIKE JOINT SOLUTION CHANNEL IN SOUTH ABUTMENT

To prevent leakage, a concrete cutoff wall or plug was designed and placed in the cavity as a water stop. The concrete plug extended from the floor to the roof of the cavity. The cavity was cleaned and the concrete was placed by using the large-diameter shot drill holes. Concrete filling was begun at the low end of the cave and proceeded up the dip; and wood forms were used to maintain construction joints and the proper width of the plug. Care was exercised at the south end of the plug to obtain a solid tie into rock, at which point the plug was widened considerably. The plug contained 2,458 cu yd of concrete using a maximum size of aggregate of 1½ in.

After the concrete had contracted, the joint between the roof of the cave and the top of the concrete plug was grouted. Where necessary, openings be-


19+00
20+00
21+00
22+00
23+00
24+00
25+00
1040
1000
960
920
880
840
Elevation, in Feet
Original Ground Surface
Deck El 1009.0
Terrace Clay
Original Rock Surface
42"Φ Calyx Drill Holes
Knox Dolomite
Cavity
Limit of Excavation
Strike Joint Solution Channel

(*a*) TREATMENT OF CAVITY AT SOUTH END OF DAM


Base Line
Earth Backfill
El 1008.0
Concrete Dam
Partly Filled Cavity
Backfill
Grout Outlets
Theoretical Slope 1 on 0.65
Concrete Cut-off Wall
High Pressure Grout Curtain 270 Ft Deep
Jackhammer Holes Approximately 8 In. Deep
$\frac{3}{4}$ In. Standard Pipe
Roof of Cavity
Connected to Grout Header
Concrete Cut-off Wall
0
20
40
60
Scale in Feet
Concrete Cut-off Wall
Partly Filled Cavity
Backfill
High Pressure Grout Curtain 190 Ft Deep

(*b*) SECTION AT STATION 20+25

(*c*) DETAIL OF GROUT OUTLETS

(*d*) SECTION AT STATION 22+00

FIG 22.—DETAILS OF BEDDING CAVITY AT THE SOUTH ABUTMENT

tween the roof of the cavity and the concrete plug were calked to prevent grout leaks, and the grout was supplied through nipples previously placed on approximately 20-ft centers along the roof of the cavity. The detail of these grout outlets and of the concrete plug is shown in Fig. 22. As an added precaution, clay was placed to fill the cavity both upstream and downstream from the concrete plug.

An extensive program of cement grouting followed the completion of the concrete plug, with the drill holes extending from the ground surface through the concrete filled seam. This procedure made the concrete plug a part of the grout curtain extending from the dam into the rim area.

South Reservoir Rim

Beyond the south end of the concrete dam, the rim is formed by Knox dolomite for a distance of 0.4 mile. A limestone of the Lenoir formation begins at this point and continues for a short distance to a contact with the shale of the Athens and Sevier formations. The shale ridge extends southward with an average width of 1,000 ft, at a maximum reservoir elevation of 1002. The highest point of the ridge is approximately at El. 1240, and the shale ridge drops below a design crest of El. 1019 at seven saddles. At each of these saddles an earth dam was required in addition to a large saddle dam at the contact of the limestone and shale.

The part of the rim underlain by the Knox dolomite is covered by a blanket of terrace clay and gravel which averages 43 ft in thickness. Most of the Lenoir limestone rock comes to the surface, and there are many exposed ledges and pinnacles of limestone.

Except for the one cave at the end of the concrete dam, there was little evidence of solution and weathering in the rocks of the south rim. Weathering had not affected the Athens shale except for a few minor beds. It was found that the unweathered shale was not absolutely impervious as core drilling encountered streams of water at relatively low levels. The beds in the Sevier shale were folded and faulted. The shales were somewhat calcareous and contained a few interbedded layers of shaly limestone. The entire formation was essentially insoluble and without cavities or deep weathering.

North Reservoir Rim

The alluvial flood plain on the north bank of the river at the dam site ends in a steeply sloped sodded bank which rises to El. 1200 approximately 0.3 mile from the north end of the concrete dam. The underlying rock is the Knox dolomite formation.

Preliminary borings in the short section of the north rim revealed well-defined cavities which in general followed bedding planes. This drilling showed the water table at a relatively low elevation—indicating the probability of open cavernous rock. The north rim was treated with an extensive high-pressure grout curtain to prevent leakage around the end of the dam.

Grouting of the Concrete Dam Foundation

Consolidation Grouting.—The entire foundation beneath the concrete dam and powerhouse, with the exception of the spillway apron, was consolidated with neat cement grout for a depth of approximately 40 ft. The drilling and grouting progressed in stages with the holes on a regular pattern. The first-stage holes were placed on 20-ft centers parallel and perpendicular to the base line of the dam. Second-stage holes were placed midway between first-stage holes, third-stage holes midway between second-stage holes, etc. The drilling and grouting of a stage in an isolated area were completed before the next stage was begun. The quantity of grout consumption in a hole in one stage determined whether an additional stage hole was required. The strict theoretical pattern of holes was varied in some places. In general, wagon drills were used for all the consolidation holes. In some cases where the total depth of hole exceeded 40 ft, diamond core drills were used. All holes were vertical except that, in places where known vertical solution channels existed, some angle drilling was done to obtain most effective grouting.

The consolidation holes and the seams intercepted by them were washed with air and water under pressure through nipples grouted into the drill holes at the foundation surface. No attempt was made to wash or to grout individual seams in the consolidation holes. The holes, however, were water tested to determine the proper consistency of neat cement grout with which to start the grouting treatment.

The area that formed the foundation for the north nonoverflow section of the dam was given special treatment because of the cavernous area beneath. The consolidation holes were extended through the concrete backfilled cave, and every effort was made to secure a tight seal between the top of the concrete and the roof of the cave.

When consolidation grouting was first begun in the north abutment it was noticed that the grout traveled several hundred feet downstream from the limits of the dam. To prevent the grout from traveling into areas where it was not needed, the limits of the foundation of the dam were outlined with drill holes 40 ft deep spaced on 5-ft centers. These holes were grouted with a thick cement grout fed by gravity to form a curtain to confine the consolidation grout. This method probably eliminated a large percentage of grout waste.

For all consolidation grouting, a maximum pressure of 40 lb per sq in., measured at the surface of the hole, was used. In general, most of the grouting was completed prior to placing concrete in the base pours of the dam. Uplift gages were installed at intervals for checking any possible uplift in the foundation rock. The grout used was composed of neat cement and water and varied from 0.6 to 2.0 parts of water to one part of Portland cement by volume.

Low-Pressure Curtain Grouting.—A grout curtain to form a water cutoff in the foundation beneath the dam was extended in a line just downstream from, and parallel to, the upstream face of the concrete dam. This curtain was formed by grouting shallow holes, in general averaging about 50 ft in depth. Pressures not exceeding 40 lb per sq in. were used. The low-pressure curtain was drilled and grouted in conjunction with the consolidation grouting. The treatment given these holes was similar to the consolidation program, except that the low-pressure curtain holes were treated more intensively and were drilled mainly with diamond core drills.

Drilling and grouting were done in stages with the first-stage holes on 24-ft centers; the second-stage holes reduced the spacing to 12 ft; the third-stage holes reduced the spacing to 6 ft; etc. The resulting minimum spacing at any one place on the low-pressure curtain was 1.5 ft; and the maximum spacing, 12 ft. Water testing was used to determine the initial grout consistency and grout mixes and pressures were identical to those used in consolidation grouting.

High-Pressure Grout Curtain.—The relatively low elevation of the water table at the dam site, together with known information regarding seams and cavities, indicated the need of a deep high-pressure grout curtain along the upstream face of the dam. To determine the depth of this curtain, a number of deep check holes were drilled, water tested, and grouted. The results obtained in the check drilling and grouting of these holes indicated that the curtain should extend to a maximum depth of 200 ft. A number of small seams were encountered between 100 ft and 200 ft below the surface of the foundation.

The high-pressure curtain followed the line of the low-pressure grout curtain. Fig. 15(*b*) shows the relative position of the grout curtain. The drilling and grouting were done in stages with the first-stage holes placed on 18-ft centers. Successive stages reduced the hole spacings by one half, and the number of stages required for an area was determined by the grout consumption of the preceding holes. The minimum spacing of holes at any one point in the final curtain was 1.4 ft; and the maximum spacing, 9 ft. The depth of drilling was decreased where the grout consumption of previously drilled holes indicated a satisfactory curtain at lower levels. The minimum depth of holes was maintained at 100 ft.

The normal procedure was to wash and grout the lowest known seam or cavity. The packers were then raised and the operation repeated with each seam until all had been grouted. This procedure allowed each seam to be washed and grouted prior to the beginning of the washing of the overlying seam. Ample time was allowed between grouting and washing operations to permit the grout to harden. This process was accomplished by providing sufficient flexibility in plant equipment to allow either washing or grouting to be conducted at other locations.

Almost all the drilling and grouting were done from the lower drainage gallery after the bulk of the concrete had been placed. Thus, additional protection was afforded against uplift when using the higher grouting pressures. The maximum grouting pressure was 200 lb per sq in. measured at the top of the hole, and was in excess of the hydrostatic pressure on any seam, but less than the unit weight of the overlying rock and concrete. A neat cement grout, varying from 0.6 to 5.0 parts of water to 1 part of cement by volume, was used.

The results of the high-pressure curtain grouting as accomplished by successive stages are illustrated in Fig. 23(*a*). This curve shows the grout consumption per foot of depth of hole for each successive stage of grouting.

Rim Grouting

South Rim Grout Curtain.—A grout curtain was provided extending from the south end of the concrete dam along the reservoir rim. Tied into the end of the high-pressure grout curtain beneath the concrete dam, this curtain extended to the south end of the first saddle dam. Extensive drilling and grouting operations were employed in the area between Station 20+50 and Station 27+00 to check, tighten, and provide a water stop around the south end of the concrete dam. These operations also completely checked the cave that had been mined out and backfilled with a concrete plug.

The drilling and grouting were done in stages—as many as were necessary to check any area of high grout consumption—with the first-stage holes on 100-ft centers. In some cases, five stages were used, resulting in hole spacing of 6.25 ft. All holes were driven with diamond core drills to El. 860 or lower—slightly lower than the elevation of the original river bed.

No seam washing was attempted in the rim grouting. Neat cement grout with calcium chloride added to accelerate the setting time was used.

The neat cement grout varied from 0.6 to 2.0 parts of water to 1 part of Portland cement by volume, with 3% of calcium chloride ($CaCl_2$) by weight of cement added. The effect of the rim tightening by successive grouting stages is indicated in Fig. 23(*b*), which shows the grout consumption per foot of depth of hole for each stage.

Check drilling and grouting indicated that no other intensive grouting program was necessary in other parts of the south rim except at saddle dam No. 1, where considerable grout was injected, particularly at the contact between the Lenoir lime-

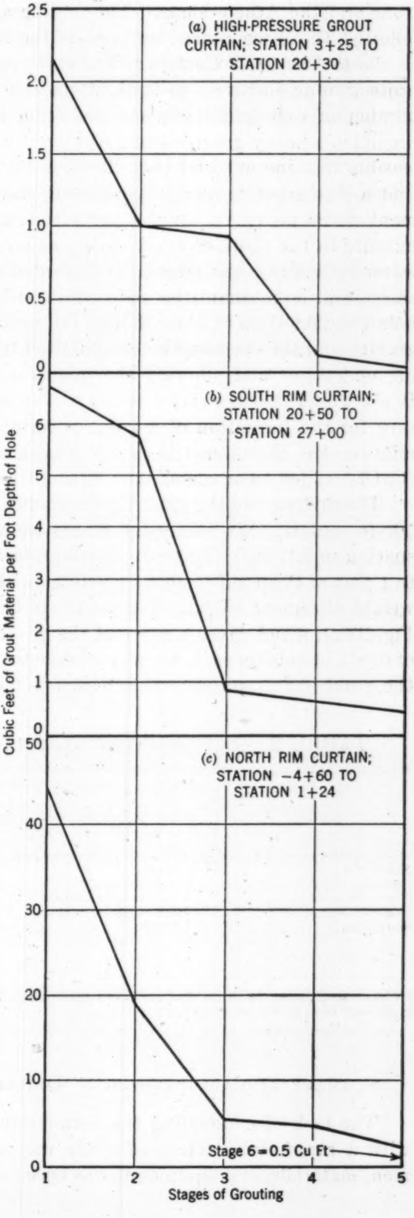


FIG. 23.—RATE OF GROUTING CONSUMPTION

stone and the Athens shale. The drilling and grouting under this saddle dam followed the same procedure used in the rim area south of the concrete dam.

North Rim Grout Curtain.—The north reservoir rim possessed some natural protection against seepage in the form of a long earth blanket. However, the number of well-defined dip cavities lying along the line of the grout curtain predicted a heavy grout consumption. It was considered neither necessary nor feasible to mine out and backfill these cavities with concrete; but it was felt that a clay grout or a grout containing clay in a prescription mixture with cement would provide a satisfactory cutoff against seepage. A clay and cement mixture in the proportion of 1 part cement to 4½ parts of clay and 6 parts of water by volume was tried. The grout was pumped into the drill holes as a slurry and large quantities were consumed. The maximum consumption per hole was 20,905 cu ft, even though the pressures used were only slightly above gravity and the consumption was limited to 500 cu ft before suspending grouting operations and allowing the mixtures to set for a period of three days. It was found that the clay mixture was spreading much farther than was necessary for the formation of a grout curtain; and, accordingly, the clay-cement mixture was abandoned in favor of neat cement grout with calcium chloride ($CaCl_2$) added to accelerate the setting time.

The drilling and the grouting were done in stages with the first-stage holes on 100-ft centers. Six successive stages were required, which reduced the hole spacing to 3.12 ft. The neat cement grout varied from 0.6 to 2.0 parts of water to 1 part of Portland cement by volume with 3% of calcium chloride ($CaCl_2$) by weight of cement added. The results of the north rim tightening are shown in Fig. 23(*c*), which gives a curve of the grout consumption in cubic feet per foot of depth of hole for each successive stage to stage five. At stage six (not shown) the grout consumption had decreased to 0.5 cu ft.

TABLE 3.—DRILLING AND GROUTING QUANTITIES, DOUGLAS DAM

Section	Structure	Drilling Quantities (Lin Ft)			Grout (cu ft)
		Wagon[a]	Diamond[a]	Total	
North rim	Curtain	...	15,758	15,758	145,919
Concrete dam[b]	Consolidation	51,700	16,714	68,414	33,806
Concrete dam[b]	Low-pressure curtain	8,160	13,164	21,324	9,702
Concrete dam[b]	High-pressure curtain	...	39,380	39,380	51,340
South rim	Curtain	...	23,991	23,991	76,669
Total	....	59,860	109,007	168,867	317,436

[a] Wagon drill holes, 2¼ in. in diameter; and diamond drill holes, 3½ in. in diameter. [b] Only neat cement grout was used in the concrete dam.

SUMMARY AND COMPARISON OF DRILLING AND GROUTING QUANTITIES

The task of conducting the foundation treatment program in conjunction with a rapid construction schedule was difficult. A judicious disposition of men, materials, and equipment was necessary to obtain satisfactory results and costs. Approximately three hundred men were utilized on the drilling and grouting program. Table 3 shows the linear feet of drilling for the various features and the quantities are separated to show wagon drilling and diamond drilling as well as the quantities of grout consumption. An interesting comparison of the results of the grouting may be obtained from the three curves in Fig. 23.

The rock conditions were different for each case as demonstrated by the rate of grout consumption per foot of depth of hole in stage one of grouting. The north rim having large cavities took 45 cu ft per foot of depth; and the concrete dam where rock was good, and also preconsolidated, took only 2.2 cu ft. On the south rim where the rock was reasonably tight and a concrete plug was constructed, the first-stage rate was 6.7 cu ft per foot of depth. A satisfactory tightening was obtained by the use of successive stages of grouting and by intensive treatment. The rate of grout consumption in the final stage was 1 cu ft or less per foot of depth of hole in all cases.

Drilling and Grouting Equipment

Standard equipment was used for drilling and grouting. Two-compartment, mechanically-agitated, mixers were used, and the grout was pumped by 7 by 5 air-driven duplex reciprocating pumps. Standard manufacturers' set bortz bits were used on the diamond core drills and the core drills were both air driven and gasoline motor driven. Two types of packers were used in the grout holes—a simple mechanical packer and an expanding rubber packer.[1]

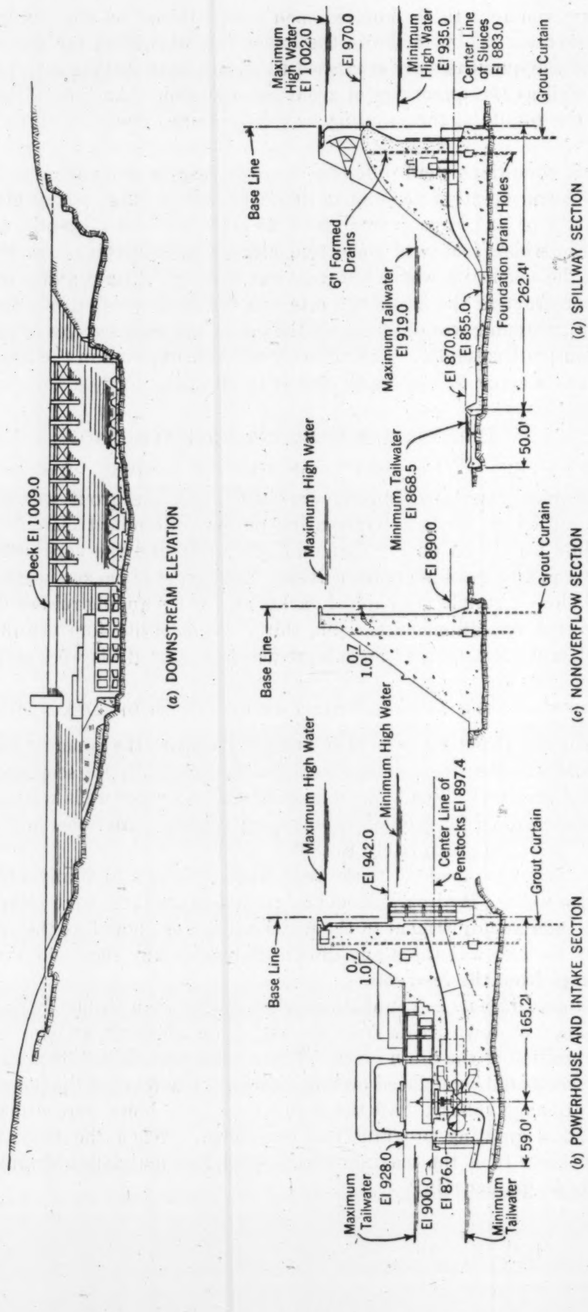
During peak operations on the job, thirty wagon drills, fourteen diamond drills, eight grout mixers, and three large-diameter shot drills were employed.

Ground-Water Investigations and Foundation Drainage

Ground-Water Investigations.—The areas adjacent to the concrete dam and strategic points in the reservoir rim were watched carefully for seepage during the period of reservoir filling. A program of ground-water wells was laid out, and wells were drilled for ground-water measurements. All holes were driven below river bed to approximately El. 868.

Observations were made of these wells regularly and also of any additional existing farm wells. Hydrographs of the ground-water table were plotted. A gradual rise and an adjustment in the water table were noted as the reservoir rose, but in no case was there any drastic change or any sign that would indicate leakage from the reservoir.

Foundation Drainage.—Foundation relief drains were drilled beneath the concrete dam just downstream from the high-pressure grout curtain. Fig. 24 shows the position of the drain holes. These holes were drilled 30 ft deep into foundation rock and were placed on 8-ft centers. The relief of the holes was in the lower drainage gallery. All the foundation drain holes were drilled after the foundation grouting program was completed. When the reservoir had filled to within 2 ft of the maximum pool level, the foundation drains had a flow of only 56 gal per min.


(a) DOWNSTREAM ELEVATION

(b) POWERHOUSE AND INTAKE SECTION

(c) NONOVERFLOW SECTION

(d) SPILLWAY SECTION

Deck El 1009.0
Maximum Tailwater
El 928.0
El 900.0
El 876.0
Minimum Tailwater
59.0'
165.2'
Base Line
0.7
1.0
El 942.0
Maximum High Water
Minimum High Water
Center Line of Penstocks El 897.4
Grout Curtain
Base Line
0.7
1.0
Maximum High Water
El 890.0
Minimum Tailwater El 868.5
Grout Curtain
50.0'
El 870.0
El 855.0
Maximum Tailwater El 919.0
6" Formed Drains
Foundation Drain Holes
262.4'
Base Line
Maximum High Water El 1002.0
El 970.0
Minimum High Water El 935.0
Center Line of Sluices El 883.0
Grout Curtain

Fig. 24.—Sections Showing Position of Relief Drains

A regular pattern of holes was also drilled over the area under the spillway apron. These holes were drilled 10 ft into foundation rock and extended through the concrete slab of the apron to provide relief from uplift pressures.

SUMMARY

Sufficient geologic information made possible advance planning and scheduling. This information was augmented by additional exploration during the construction period. Careful coordination of excavation and concreting with foundation treatment was essential, because of an extremely fast construction program. Substituting concrete for grout wherever feasible, as well as confining grout mixtures within required limits, minimized the waste of cement. Large-diameter core holes used in connection with mining and placing concrete plugs allowed visual subsurface examination and assured a positive cutoff.

DISCUSSION

Berlen C. Moneymaker,[7] Assoc. M. ASCE.—The two papers in this Symposium illustrate the magnitude of problems encountered in the construction of dams on limestone foundations in a region that has been humid for a long time, geologically. The rocks of the Tennessee Valley have been exposed to the relentless attacks of the agents of weathering for millions of years. Siliceous rocks have decayed to great depth and the carbonate rocks (limestone, marble, dolomite, and calcareous shale) have weathered by solution. Soluble rocks not only have been "worn down" to their present elevation largely by solution, but they are also "honeycombed" by caves and smaller solution channels to considerable depths. Cavities are largest and most numerous in the upper part of the bedrock, becoming smaller and less numerous with increasing depth.

Of the sixteen dams completed by the Tennessee Valley Authority (TVA) to date, nine are founded entirely or partly on limestones. Three additional dams in the area, built before the creation of the Authority, are also founded on limestone foundations. Although some of these sites were much better than others, all of them were to some degree cavernous.

Mr. Pauls' paper on Fort Loudoun Dam is a fairly complete record of difficulties met during the construction period. The first geologic problem involving this project was the selection of a dam site. A preliminary geologic investigation of that section of the Tennessee River in which Fort Loudoun was to be located, undertaken in May, 1935, soon revealed that no really good dam sites were to be found. To locate the best site available, a detailed study was made of the river from mile 576 to mile 609. The geology of this 33-mile stretch of the river was carefully mapped and fourteen potential dam sites were selected for investigation.

Some of these sites (see Table 4(*a*)) had been considered by the U. S. Army Engineers; others had not. A few holes drilled by the engineers at the Marble Bluff site showed the rock to be cavernous. The numerous sinks at both Marble Bluff and Bogart Island indicated extensive bedrock solution. At the Loudon sites, foundation conditions were thought to be fairly good but the reservoir rim was low, thin, and cavernous. The Rock Quarry Bar sites were considered the best dam sites in the entire 33-mile stretch of the river; but they were also ruled out because of the low, thin, and cavernous reservoir rim. The Bussell Shoals sites and the Coulter Shoals site were soon found to be on deeply cavernous rock.

When the operating level of Watts Bar Dam was established at El. 745, several of these sites were automatically dropped from consideration and the search for the Fort Loudoun site was thus limited to a 12-mile stretch of the river. Exploratory drilling at four sites which had not been ruled out because of geologic or engineering reasons revealed that the Belle Canton Island was the best site available. This site was therefore selected as the site for Fort Loudoun Dam.

---

[7] Chf. Geologist, TVA, Knoxville, Tenn.

The rock involved in the foundation of Fort Loudoun is nonuniform in character. It ranges from the coarsely crystalline sandy limestone of the Tellico formation, through coarsely crystalline pure limestone, finely crystalline impure and shaly limestone, to calcareous shale in the Sevier formation. Structurally, the strata of both formations have an average strike of about N 42° E and dip southeastward at angles of from 30° to 35°. The limestones of both formations are cut by numerous steep to vertical joints and the shaly members of the Sevier are somewhat folded (see Fig. 3) but less extensively jointed. In both formations, the limestones have been dissolved away along all structures sufficiently open to permit the movement of ground water. The shales of the Sevier formation are not soluble; but in many places the soluble calcium carbonate has been leached from them, leaving seams of soft mud. Although such mud seams are not cavities, they are actually little different from mud-filled solution cavities as foundation defects.

From the beginning it was recognized that the Fort Loudoun site contained serious defects of considerable magnitude. Even while the dam site was under exploration, the chief geologist, the late Edwin C. Eckel, Affiliate ASCE, appraised it in the following words: "The Belle Canton Island [Fort Loudoun] dam site is much worse than Chickamauga." The correctness of this appraisal was verified by the Authority's Board of Consultants, whose report of June 18, 1942, stated that "No site of the whole Tennessee Valley System has shown so many weaknesses."

TABLE 4.—SITES INVESTIGATED

Site	River mile
(*a*) FORT LOUDOUN DAM	
Marble Bluff[a]	577.43
Bogart Island[a]	583.75
Loudon:	
Site No. 1[b]	592.06
Site No. 2[c]	592.23
Rock Quarry:	
Bar No. 2[c]	594.72
Bar No. 1[d]	594.83
Lenoir Shoals[c]	601.05
Lenoir Ferry[c]	601.65
Belle Canton Island[f]	602.30
Bussell Shoals:[e]	
Site No. 3	603.30
Site No. 2	603.40
Site No. 1	603.54
Sister Island[a]	605.03
Coulter Shoals[a]	608.30
(*b*) DOUGLAS DAM	
Cain Island[g]	19.73
Kykers Ferry[h]	28.87
Douglas[h]	32.28
Dandridge[h]	44.81

[a] Knox dolomite limestone. [b] Tellico sandstone and shale. [c] Tellico sandstone and limestone. [d] Tellico shale. [e] Chickamauga, Holston, and Tellico limestones. [f] Tellico limestone, Sevier shale and limestone. [g] Rutledge limestone. [h] Knox dolomite.

Mr. Pauls' paper is a detailed account of the nature of these numerous defects and the methods by which they were overcome to afford excellent foundation conditions for Fort Loudoun Dam.

Mr. Taylor's paper on Douglas Dam deals with a project in which the problems presented by a limestone foundation were entirely different from those at Fort Loudoun. The only point of similarity between the two projects was that the problems at both were the result of solution of the bedrock.

The selection of a site for Douglas Dam was a comparatively simple matter. Before the final choice was made, a 30-mile stretch of the river was considered. Many localities were easily ruled out on basis of topography and the general poorness of foundation conditions, but four sites were considered in some detail. These sites are listed in Table 4(*b*).

Preliminary geologic field studies revealed that the rock at all these dam sites was cavernous, but the Douglas site appeared to be much better than the other three. It met the engineering requirements somewhat better than the other sites and was selected as the site for Douglas Dam.

Douglas Dam is located on a rather narrow belt of Knox dolomite, which is exposed along the northwest limb of a broad, relatively flat syncline. At the dam site, the strata strike from N 40° to 60° E and dip southeastward at angles of from 15° to 20°. The Knox dolomite consists mainly of cherty dolomite and limestone, although there are some sandy beds in the formation. Except for the sandy layers and the chert, the formation is of fairly uniform solubility throughout.

The preliminary drilling proved that the upper 20 ft to 60 ft of bedrock was extremely cavernous, especially in the original channel and flood plain areas, and that a sizable open cavity existed in each abutment. Subsequent excavation revealed that the cavities were developed along joints and bedding planes. The two large abutment caves and many of the smaller cavities were developed along both joints and bedding planes, but some of the smaller openings were restricted to one structure or the other.

The bedding-joint cave in the right abutment was a "blowing cave." It was intercepted by a drill hole in the early stages of exploration and the "blowing" phenomenon was noticed by the drillers. Two other holes in the same general area exhibited like behavior. In the summer, these holes became "sucking" holes. The "blowing and sucking" behavior of the drill holes suggested that the cave was extensive and proved that somewhere it had a natural opening, although such an opening was never found. The principal problem presented by the two abutment caves was that of watertightness; however, there was no definite assurance that the 40 ft of sound rock overlying the right abutment cave afforded adequate support for the end of the dam. Although these caves required a considerable amount of painstaking work under none too pleasant working conditions, their treatment did not involve any difficult or unusual problems. They were cleaned out and then plugged with concrete. The cave in the south abutment, in some places, was a rather crooked channel, and it was found to be more economical to mine out enough rock to provide for a straight cutoff plug instead of following the meander-like bends of the natural opening.

As indicated in Mr. Taylor's paper, the greatest problem presented by the cavernous rock at Douglas was that of controlling the water in the cofferdams. The strike of the strata was essentially parallel to the river channel. Solution cavities developed along bedding planes and strike joints served as conduits to pipe water into the cofferdams. All attempts to shut the water out of the cofferdams were ineffective, and excavation progressed with difficulty, block by block. Pumps in constant operation discharged 100,000 gal per min for some time, the maximum discharge being about 105,000 gal per min.

In spite of the tremendous problem of controlling water, an excellent watertight foundation was prepared. The transition between highly cavernous rock and tight, noncavernous rock was nearly everywhere sharp. The depth to which solution had progressed varied from place to place but there was no relation between the depth of solution and the lithologic character of the rock.

The two papers form a valuable record of the geologic problems, and their solution at dam sites in limestones. Mr. Taylor's paper on Douglas Dam is especially interesting as a record of a hard-fought and desperate struggle between engineer and river under the handicap of a short construction schedule. At times the outlook was dark for the engineer, who finally won by dint of his sound experience with limestone foundations in general and his very thorough knowledge of the geologic conditions at Douglas Dam in particular.

C. E. **Blee**,[8] M. ASCE.—The paper dealing with the Fort Loudoun Dam is of particular interest to the writer since he was directly in charge of the construction of this dam as project manager for the first eighteen months, being succeeded by J. K. Black, M. ASCE, who had been construction engineer on the project. It is a very comprehensive and well-presented account of the complex foundation conditions encountered and of the procedures employed to overcome these difficulties.

Perhaps the most significant feature of the foundation conditions at the Fort Loudoun site was the comparatively great depth to which the solution channels extended. This is stressed in the paper as is also the fact that structural deformation was often the determining factor which controlled the depth and extent of disintegration rather than the solubility of the rock. This latter effect was readily discernible on the ground from the crisscrossed lines of disintegration following bedding planes, joints, and faults and always having an enlarged zone of disintegration at the intersection of these features. In the case of a flowing stream where differences in head are available, there is, of course, no reason, from the standpoint of hydraulics, why water should not find its way through open cracks or fissures thus producing solution at considerable depths. At some dam sites there are geological conditions which definitely limit the depth to which solution can go. For instance, at the site of the Chickamauga Dam, there is a stratum of bentonite which effectively stopped the penetration of water and so afforded a horizon below which it was known that foundation treatment need not be carried.

At Fort Loudoun one of the chief difficulties from the construction standpoint was in cutting off the flow of water into the excavation areas. Nearly 700,000 cu ft of grout, including neat cement, clay cement, and asphalt, were required to control the flow through the rock into the first-stage cofferdam. In sinking the calyx holes within the cofferdam area, it was often necessary to encircle the hole with a line of supplementary grout holes to cut off the flow of water through seams encountered at greater depth.

Only by the utilization of the experience gained and of devices developed during comparatively recent years could a major dam and lock structure be successfully constructed on a foundation as bad as that which existed at the Fort Loudoun site. At Fort Loudoun the engineers of the Tennessee Valley Authority (TVA) had the benefit of previous experience and of the devices developed at the comparatively large number of dams, including Norris, Chickamauga, Guntersville, Kentucky, and Hales Bar, which had been built on difficult limestone foundations—as well as the experiences of other organizations recorded in the technical literature. By the publication of this Symposium together with those previously published in the *Transactions* of the Society, the experience gained by TVA is made available to the profession as a whole.

---

[8] Chf. Engr., TVA, Knoxville, Tenn.

Effectiveness of the cutoff and grouting operations at Fort Loudoun is demonstrated by the low rate of seepage past the dam site. Holes 3 in. in diameter were drilled just downstream from the cutoff on approximately 11-ft centers throughout the spillway section. These open drainage holes extend from 30 ft to 80 ft into the rock beyond the excavated surface and so can be expected largely to intercept any seepage through the foundation rock under the concrete structure. The seepage, as measured by the discharge from the total of these holes, amounts to less than 10 gal per min with a full reservoir. Careful records have been kept of the ground-water levels in both abutments of the dam and these indicate that there is very little seepage through the abutments.

As stated by Mr. Pauls in the "Summary," the operations for consolidation and for an effective cutoff challenged the combined efforts of the designing engineers, field engineers, geologists, and construction personnel. The close cooperation that existed between the engineers, geologists, and construction personnel was an outstanding feature of the Fort Loudoun job; that the challenge presented by the difficult foundation conditions was successfully met is attested by the results achieved.

A complete and explicit account of the methods and operations used in foundation treatment at Douglas Dam is given by Mr. Taylor. The information thus made available should be useful to anyone having a similar foundation problem.

Cavities encountered in the foundation rock at Douglas Dam were, in general, larger and more continuous than those at Fort Loudoun but were not as complex nor did they extend to such great depths. At Douglas, blowouts and leakage through cavities into the cofferdam areas presented particularly severe problems because of the extremely fast construction schedule for the job—calling for completion in less than fourteen months after work was started. Use of large-diameter calyx drill holes as construction devices to give working access to disintegrated seams or cavities underlying sound rock, and to afford a means of depositing concrete in these seams after being cleaned out, is well illustrated in the work described. Another and rather unusual device was the use of a grout curtain to confine, to the desired area, the grout used for consolidation.

As in the case of Fort Loudoun, the measurements of foundation drainage and the ground-water investigations at Douglas Dam demonstrate the effectiveness of the foundation treatment.

ROBERT A. LAURENCE,[9] ESQ.—Not only is the value of close, continuous cooperation between engineers and geologists emphasized in this Symposium, but also the changed and increased duties and responsibilities of the geologist since the late 1920's on such construction projects are illustrated. The changes result chiefly from the exhaustion of excellent sites and the emphasis on multipurpose projects. The discovery of ideal sites is becoming almost impossible; but, as the Douglas and Fort Loudoun examples show, detailed preliminary investigations and geologic study can result in the rejection of the most unsatisfactory sites, and a thorough knowledge of the subsurface conditions is necessary to insure proper treatment of the foundation finally selected. The Fort Loudoun site was selected after investigation of fourteen alternate sites, and the Douglas site was one of four originally considered. Furthermore, drilling for grouting the Douglas cofferdam revealed that, had the axis of the dam been located more than 100 ft upstream or downstream, an additional 10 ft to 20 ft of highly cavernous rock would have been encountered in the spillway and powerhouse foundations.

---

[9] Regional Geologist, U. S. Geological Survey, Jefferson City, Tenn.; formerly Geologist, TVA, Douglas Dam, Tennessee.

These examples also illustrate the effect of seemingly insignificant lithologic or structural features in determining the course and extent of solution and weathering. The north abutment cave at Douglas Dam was caused by the presence of a shaly layer less than 1 in. thick, which had deflected the ground water along the 20° dip of the beds, thus concentrating the solution. Although (see Fig. 17) the cave was large enough for mining and backfilling only to Station 5 + 32 (El. 891), weathering and small cavities were present along this bed as far as Station 13 + 0 (El. 679). The thin insoluble bed, which caused development of the south abutment cave for a total distance of more than 700 ft down the dip, was even less conspicuous.

The extent and continuity of cavities along bedding planes and strike joints are further shown by the fact that dye placed in the river, more than 1,000 ft upstream and downstream from the cofferdam, appeared in the same leak in the excavation area within from 30 min to 45 min.

No description of the Douglas foundation is complete without mention of the unusual filled sink on the south abutment between Stations 18 + 90 and 19 + 53. This sink was developed before the deposition of the Lenoir limestone, and was filled with 65 ft of extremely fine-grained volcanic ash and 55 ft of thin-bedded dolomite. It extended to El. 847; but, since the lower 40 ft of the volcanic ash was fresh, compact, and insoluble, excavation was "bot-

TABLE 5.—RESULTS OF PRELIMINARY CORE DRILLING AT BELLE CANTON SITE, FORT LOUDOUN DAM

(Includes All Drilling Through October 31, 1939)

Rock type	No. of holes	Total in rock (ft)	Recovery (%)	Loss (%)	Cavity (%)
Tellico[a]	14	718.8	90.70	1.06	8.24
Tellico[b]	53	4,990.3	84.42	0.59	14.99
Sevier I[a]	12	658.5	86.71	4.02	9.27
Sevier I[b]	13	662.1	78.96	5.62	15.42
Sevier II[a]	42	2,225.8	85.58	10.70	3.72
Sevier III[a]	21	1,258.2	76.95	22.42	0.63
Total	155	10,513.7	(84.00)[c]	(5.91)[c]	(10.09)[c]

[a] Vertical holes. [b] Angle holes. [c] Averages.

tomed" at El. 881. Many core drill holes were put through the bottom of this filled sink but no outlet was found. Deep solution on the strike joint channel shown in Fig. 22 was closely related to this sink hole.

Under the heading, "Geology," Mr. Pauls states that:

> "Contrary to the normal conception that relatively more cavities are to be expected in the purer limestone, the cavities in this area were primarily controlled by structural features rather than by the purity of the rock."

This statement is misleading. Although it is true that the shape, extent, and localization of the cavities were controlled by structural or sedimentary features, nevertheless all the major cavities at Fort Loudoun, as shown in Fig. 3, were in the Tellico, Sevier unit 1, and the lower part of Sevier unit 2. Although unit 3 and the upper section of unit 2 also suffered intense deformation, the rocks were much more shaly, and so suffered less from solution. The distribution of cavities in the various rock types is also demonstrated by Table 5, summarizing results of the exploratory drilling.

JAMES B. WARD,[10] JUN. ASCE.—To those concerned with the investigation, preparation, and treatment of foundations, the experiences set forth in this Symposium are invaluable. Since dams are built to serve a variety of purposes, it is no longer practicable or possible to locate the sites only in areas where good foundations are available. It is often necessary to locate these structures where they will best serve the purpose for which they were intended, and to improvise methods and procedures to meet the foundation conditions as they are. Such was the case in the selection of the Fort Loudoun Dam and Douglas Dam sites, the cleaning and foundation grouting of which are so ably presented by Mr. Pauls and Mr. Taylor. Moreover, the authors of this Symposium have demonstrated the utility of adequate geologic information in the planning and again in the construction of dams.

To appreciate the magnitude of exploration and foundation treatment required for the successful completion of the Fort Loudoun Dam, one has merely to examine the drilling, mining, and grouting quantities. The fact that more than 800,000 lin ft of drilling (including 584,000 lin ft of diamond core drilling, 16,000 lin ft of large-diameter calyx drilling, about 10,000 cu yd of concrete to backfill cavities and mined-out seams, and approximately 2,000,000 cu ft of cement and cement-admixture grouting) was necessary, testifies to the enormity of the undertaking. Seldom, if ever, have the ingenuity and combined efforts of engineers, geologists, and construction specialists been so challenged.

Undoubtedly, the most striking feature of the Fort Loudoun foundation is the fact that subwater table cavities and weaknesses were explored and treated at what are thought to be unprecedented depths, below any previous water table level. The method by which this was accomplished was well covered in the Symposium. The formation of such deep-lying cavities has been the subject of much discussion. Observations made almost daily during the construction of Fort Loudoun Dam support the explanation postulated in Mr. Pauls' paper. It may be emphasized that the clogging or damming up of subterranean channels by the insoluble residue from the solution of the impure carbonate rock appears to play an important role in the formation and rerouting of cavities and channels to deeper levels as solution continues. This may well explain the scarcity of cavities at great depths in the purer or more soluble carbonate rocks, such as those encountered at Douglas Dam. Moreover, the statement that the formation of cavities may be more dependent on structural deformation than on the relative solubility of the rock also deserves emphasis.

---

[10] Dist. Geologist, Soils and Geology Section, U. S. Engr. Office, Mobile, Ala.; formerly Resident Geologist, TVA, Fort Loudoun Dam, Lenoir City, Tenn.

Of considerable interest and importance to those engaged in the selection and investigation of dam and quarry sites in limestone and related rock formations is the condition encountered in the north abutment ridge. What appeared as a steep and apparently sound limestone bluff developed into nothing more than a false front with extensive weathering and solution just behind it, causing a series of deep troughs separated by intervening pinnacles. This condition may be found where steep bluffs have been formed by encroaching meanders and may be due to the fact that overburden is not able to form on a steep bluff that has been or is being cut back or undercut and, hence, rain and surface water drain away immediately. In the case of ridges behind bluffs, it is possible for the overburden to accumulate, forming a catchment for the rain water. The water, in turn, seeps downward to, and finds access along, joints or other areas of structural weakness in the rock. The ground water is thus recharged in a manner that permits continuous solution. In the exploration and treatment of a formation of this nature, horizontal and low-angle core drill holes may be used most advantageously.

It was apparent at the outset of foundation exploration that detailed and reliable geologic information, properly portrayed and prepared well in advance of construction, was mandatory for the effective and economical preparation and treatment of this highly defective dam foundation. Thus, a staff of geologists consisting of a resident geologist, with from two to three assistants (an organization larger than is usually required on a project of this nature), was in residence during most of the construction period; but, to re-emphasize the magnitude of the geologic and foundation work involved—at the peak of construction as many as sixteen diamond core drills and six calyx shot drills were in continuous operation—three shifts, seven days each week. During the course of the project several hundred geologic drawings, including sections, sketches, plans, and models were prepared which portrayed the foundation condition and aided in the determination of the required excavation, as well as the type and probable extent of foundation treatment. The exploratory drill pattern was such that geologic sections could be prepared not only along ranges parallel and normal to the axis of the dam structures, but also, by interpolation, along any other direction desired. With a myriad of such sections, they were difficult to use unless they were studied in their relationship with one another. For this reason, what are thought to be novel methods of displaying subsurface conditions were devised. After considering peg, glass, and cardboard models, it was evident that some type of transparent or sectional model would be best suited for the visualization or interpretation between the various related sections. Hence, after experimentation, it was found that a transparent model could be constructed most economically by using rigid celluloid film positives, developed from photographic negatives of the geologic sections. These film positives were assembled in an upright position on a rectangular wood base, on which an outline of the particular dam structure had been superimposed. The transparent sections were properly oriented by dovetailing, and were correctly arranged on, and attached to, the wood base by fastening them through metal eyelets to piano wire that was suspended along the top and bottom of four upright steel rods securely anchored at each corner of the rectangular base. The completed model enabled at a glance the visualization of the foundation conditions in an accurate three-dimensional manner.

Another type of model used to advantage may best be described as a hollow, sectional model of plywood construction. It consists simply of mounting a front and reverse print of a geologic section on the front and back of a piece of thin plywood of similar dimensions, and arranging a series of related sections so mounted on a wooden base as to correspond with their location to each other and to the dam structure being represented. The sections are arranged and held securely on the base in an upright position by the slit grooves provided in the sides of a 3-in. upright board wall built around the base. Both longitudinal and transverse sections were dovetailed together in the manner of the transparent model described herein; and the sections were cut with a band saw along the line of the proposed excavation indicated thereon, thus enabling one, simply by removing the upper part of a section, to study the foundation conditions that existed before, and those that would exist after, excavation. This model was invaluable both in the planning for the removal of a maximum of the required excavation with mechanical equipment and in the determination of the method and probable extent of foundation treatment required.

That the difficult foundation conditions at Fort Loudoun were successfully overcome is attested by the fact that seepage through the foundation at full reservoir is inconsequential. Thus, it is indeed gratifying that the contributions of the geologists on this project have been so generously acknowledged in Mr. Pauls' paper. Assisting the writer at intervals during the construction of this project were the following geologists: Earl C. Van Horn, William V. Conn, Coy L. Watson, and Chilton E. Prouty.

J. C. NOWELL,[11] M. ASCE.—In considering Mr. Taylor's able paper on the extensive cleaning and grouting operations required at Douglas Dam, it may be of interest to connect it with the study of the original, preliminary drilling records and with the selection of the position of the axis of the dam. Early drilling had disclosed numerous cavities, but restrictions on the authorized scope of the preliminary program had prevented drilling sufficient intermediate holes to determine the extent or the interrelation of the numerous major cavern areas which were encountered later. The Douglas project was suddenly authorized for immediate construction about two months after the Japanese attack on Pearl Harbor, Hawaii, with the unprecedented schedule of closure set at fourteen months, to provide additional power for expanding aluminum production.

---

[11] Asst. Mgr., Eng. Dept., Gen. Electric Realty Corp., Schenectady, N. Y.; formerly Project Engr., Design Dept., TVA, Knoxville, Tenn.

It was essential to locate the dam axis immediately and to begin construction without awaiting the results of additional exploratory drilling. In the interests of speed it was perhaps fortunate that the topography closely limited the location of the axis, especially at the bluff on the left abutment. Brief study was given to swinging the right end of the axis slightly to the west in an effort to place the north abutment farther downstream from the creek just above the dam. This creek had cut somewhat of a gash into the sloping north bank, indicating a possible area of fractured rock with increased subsurface weathering.

On the other hand, such a shift in the north abutment would have resulted in obvious disadvantages in the layout of the structures. It was decided, therefore, to proceed full speed on the preferred basis of a straight alinement at right angles to the stream bed. Subsequent drilling indicated such lack of improvement in the condition of the rock downstream as to have resulted in no major reduction in the cleaning and grouting operations for foundation treatment. However, in limestone and dolomite areas especially, it seems advisable to emphasize that a preliminary drilling pattern on 100-ft squares (as mentioned in Mr. Taylor's paper) is insufficient to assure any definite opinions as to the expense or time required for the construction of a major dam. It was fortunate that the foundation conditions actually encountered did not materially delay the completion of the project as a whole. However, this should be credited to the resourceful methods of the construction personnel, rather than to any adequate exploratory program, which had been impossible due to the lack of authorization for extensive investigations.

Mr. Taylor has well described the rather wide variety of methods used in the foundation treatment for the Douglas project. It is interesting to review the principal factors considered in determining the specific treatment to be applied to any particular part of the foundation. In addition to the usual requirements, such as adequacy, permanency, and cost of treatment, special emphasis was placed on speed to meet the unusually fast schedule for closure of the dam in the spring of 1943. With wartime demands on the aluminum company's operations and on the atom bomb project at near-by Oak Ridge, Tenn., the question of speed, rather than that of cost, was given primary consideration. In general, therefore, mass excavation and other mechanical methods were selected, rather than alternatives requiring more handwork and extended periods of time for the foundation treatment. For example, when a difficult cavernous condition was disclosed under the north half of the spillway apron, a redesign was determined on dropping the apron grade in this area. Thus, a heavy general shovel cut, excavated to an elevation below the weathered rock, was decided upon rather than the filling of the cavities below the cap rock. The latter measure would have been especially slow because of cofferdam leakage conditions described in Mr. Taylor's paper.

Similar need for the speed of mass excavation and concreting was one of the prime considerations in deciding to carry the left abutment blocks in the vicinity of Station 19 down to the level of a bedding cavity which was followed down the dip to about El. 890 (see Fig. 15). Brief consideration was given to an alternative scheme based on treating and leaving in place the partly fractured and somewhat weathered rock above the bedding cavity, with the load on the foundation then reduced by the adoption of a concrete core wall embedded in a rolled fill section, rather than the standard concrete gravity dam section. Such a scheme indicated a possible saving in expense, but would have affected completion dates adversely, especially as it would have been necessary to place the rolled fill in winter. It was again obvious, therefore, that the removal of the questionable rock not only would result in a more conventional and conservative design, but would be essential to meet required completion dates since mass excavation and concreting would eliminate much tedious drilling, grouting, and other "dental" work in treating cavities, which would have delayed the project.

As Mr. Taylor has stated, the heights of the caves on both the right and left abutments were increased by drift mining to permit better access for men and equipment in placing the concrete plugs. This also assisted in the bonding and keying of the concrete to the sound rock. It was particularly required for the left abutment where heavy sound rock, both above and below the cave, permitted the use of a narrow concrete plug designed to take the upstream headwater load by shear into the roof and floor of the cave.

It would be of interest to have available even an approximate estimate of the total extra cost of mass methods used in the interests of speed to reduce cleaning and grouting operations. In considering such problems, the large quantities of extra excavation and concrete involved are likely to appear inordinately expensive as compared with the estimate usually prepared for cavern filling, grouting, and other similar measures. On the other hand, the large quantities in themselves tend to reduce unit cost. Also, such mass measures eliminate the uncertainty of difficult "dental" work and questionable cavern grouting; and they provide a conservative design with little chance for the development of trouble from future leakage. Especially with wartime and postwar shortages of labor, there are obvious advantages in substituting machine operations for handwork. It seems probable, therefore, that little if any actual extra expense was encountered on the Douglas project, in lowering mass excavation grades, as compared with the expense of the extra and especially difficult foundation treatment that would have been necessary if the original tentative excavation grades had been held.

In spite of continuous wartime pressure to keep the job on schedule, regardless of the foundation difficulties encountered, it would appear that the project manager (Lee G. Warren, M. ASCE), Mr. Taylor, and others not only maintained a conservative point of view, which assured a thoroughly safe foundation, but also simultaneously used a wide variety of resourceful methods in the cleaning and grouting of the foundations for the Douglas project.

John F. Partridge,[12] M. ASCE.—The profession is indebted to Mr. Pauls for his fine exposition on the treatment of the foundation for the Fort Loudoun Dam on the Tennessee River. This paper, together with previous papers convering other Tennessee Valley Authority (TVA) projects,[1,2,13,14] provides a strong background for a technique for handling foundation problems, which problems a few years ago would have been the cause for rejection of otherwise highly advantageous dam sites. This is of great import to the national economy, since before the rivers of the United States are completely controlled there will be an increasing scarcity of ideal dam sites.

---

[12] Constr. Engr., TVA, Fort Loudoun Dam, Lenoir City, Tenn.

Great strides have been made by the construction industry, in the development of methods and machinery for applying principles of modern foundation treatment, but much remains to be done in this field. The continued cooperation of geolgists, design engineers, construction personnel, and the supervisory field engineers is essential to progress in this work.

Standard TVA practice dictated that geologic data of an exploratory nature be furnished continuously to the design section, and preliminary designs be referred back to the construction forces for criticism on their suitability to current construction procedure. Both final designs and construction schedules incorporated the suggestions and recommendations of the field forces. In certain instances the field geologists were granted the supervision of construction activities, under the general supervision of the field engineers. This procedure was facilitated by the organizational setup which placed all construction activities pertaining to grouting and foundation treatment under an associate engineer, reporting directly to the construction superintendent, and nominally under the technical direction of the construction engineer.

The human equation, involving the vigilance of the staff of geologists, the core drillers and core drill inspectors, and the supervisory engineering forces, is the largest single factor in the successful treatment of foundation rock which has been weathered to the degree encountered at Fort Loudoun Dam. Each linear foot of the cutoff curtain is a potential hazard. The repeated thinning and swelling of the area of decomposition, as well as the weathering along the bedding seams and along the intersecting planes of fractures, require an alert and constantly repeated inspection, to insure the continuity of the plane of the cutoff curtain. Seams and intersecting joints in the plane of the curtain must be mined out, cleaned, and washed, until a "live-rock" contact, a "concrete-to-live-rock" contact, or a "grout-to-live-rock" contact is secured. Every step in the work (including both drilling and grouting) must be described accurately and conscientiously. Each step must be recorded and evaluated—including the logging of the drill holes, the plotting of drill holes to show all joints and fractures in the rock formation, and the recording, by stages, of all grout quantities. Effectiveness is often a matter of relative values. When there is doubt, intermediate check holes should be drilled. A liberal drilling program in the exploratory stages of the work, will result in economies in the entire program.

In all phases of the work at Fort Loudoun Dam, the resident geologist and and the grouting superintendent had "their hands full." The great footage of

---

[13] "Masonry Dams: A Symposium," *Transactions*, ASCE, Vol. 106, 1941, p. 1154.

[14] "Large Core Drills Aid Construction at Chickamauga Dam," by James S. Lewis, Jr., *ibid.*, Vol. 105, 1940, p. 850.

cores from all core drilling operations meant many hours of examination and logging. Drilling and grouting were continuous operations. It was from the records of these operations that the picture of the weaknesses in the foundation rock was built up. Actual core losses, where present, painted a clear picture. In other instances it was only possible to piece together a clear view of the situation, by observing the "relative density" of lines of fracture (dip, strike, and oblique joints), revealed by plotting the logs of drill holes on cross-section paper, as profiles of each respective range. Except for this method of detection, much time would have been lost and much grout wasted, because to stop persistent leakage it would have been necessary to make a general coverage of an area, instead of proceeding first with a selective coverage of suspicious areas, revealed by the multiplicity of minor fractures detected in the recovered cores.

The drilling and grouting program at Fort Loudoun Dam involved the following total quantities (see Table 1):

Description	Quantity
Diamond drilling, in linear feet	445,839
Wagon drilling, in linear feet	197,226
Calyx drilling, in linear feet	15,007
Grout, in cubic feet	1,067,387
Concrete, in cubic yards	9,816

Construction was started in July, 1940, and power was delivered over the transmission lines in the autumn of 1943. It is obvious that the drilling and grouting program had to be pressed to completion throughout all stages of the construction work.

The general scheme of foundation treatment was formulated during the exploratory stages of the work with a pattern of holes on 50-ft, 60-ft, and 100-ft centers. The detail was developed and expanded as intermediate and supplementary drill holes supplied additional information. The logging of all holes drilled required the presence of a large force of geologists and inspectors. All cores were preserved until the information could be plotted and coordinated, formations were proved, and the data and findings were checked by the local staff and the board of consultants. Exploratory holes were subsequently augmented by intermediate holes, reducing the spacing at each drilling by one half. Each set of holes was grouted before redrilling and grouting, unless washing between holes was done to increase the efficacy of the grouting or to trace any interconnection of cavities and seams. Packers in various holes, and a liberal use of dye introduced into the wash water, aided in the latter operation.

Drilling and grouting were continued through all stages of evacuation. When concreting was started, pipe casing was carried up through the concrete, to facilitate subsequent drilling operations. There were some exceptions to this procedure when drilling through the concrete proved necessary. In the case of the grout curtain for the spillway, powerhouse intake, and the service-bay sections, the heel of the dam at the upstream face, at ground level, was poured as an integral part of the dam structure, in a projection upstream. This provided a base for setting up the diamond and the calyx drills, and permitted the drills to be in continuous operation without interference with concreting operations as the various blocks of the dam were raised above. All grout holes and calyx drill shafts in these areas were drilled through this projecting lip, and thus afforded a positive connection between the grout curtain and the monolithic structure of the dam proper. The concrete backfill, in the mined-out sections and the large cavities along the line of the cutoff, was placed under pressure, induced by pouring upgrade, through successive vertical shafts (36-in. calyx drill holes). These areas were drained of trapped air and subsequently grouted through grout pipes placed before the concrete was poured. Further consolidation was insured by the regular pattern of curtain drilling and grouting, done after the completion of mining and concreting operations.

Reference is made by Mr. Pauls to the formation and the character of the cavities, particularly in the zone of saturation. This deep weathering and the formation of cavities below the water table caused no small annoyance, before the significance of the process of their formation was recognized. Water was fed into the subwater table strata, over a wide area outside the cofferdam, wherever bedrock (with or without an overburden of gravel) was in contact with the waters of the river channel. Percolating to great depth, by direct or indirect paths, the waters emerged within the cofferdam area, either as seepage over a wide area, or as flow in quantity at points of weakness in the rock formation. Emergence usually occurred along dip joints or weathered bedding seams, particularly at the trace of their intersection. Leakage along the strike joints usually was more direct and in greater quantity, but at lesser depth.

The resultant characteristics of the area inconvenienced the work, through the difficulty of tracing the source of the leakage, (*a*) in the control of the water for construction purposes, and (*b*) in the determination of the foundation treatment, and the probable extent of such treatment. Once the principle of deep percolation was recognized, it was evident that control of the leakage was possible only by interception, since control of the rock surface outside the cofferdam was obviously impractical.

Interception was possible by stage drilling and grouting, (1) on the riverbank, immediately outside the cofferdam, (2) through the base of the cofferdam, (3) at any point between the cofferdam and the area of excavation for the dam, (4) along the upstream or downstream face of the foundation excavation, or (5) within the area of the excavation proper. Grouting with asphalt, a mixture of cement and clay, and neat cement grout were all used successfully. Asphalt proved to be much better than coal-tar pitch. The latter is slower to melt, more likely to chill and plug the hole, and it emits fumes of brownish hue which create dangerous as well as disagreeable working conditions. An asphalt with melting point of 165° F to 175° F, proved to give the best results. Experiments with clay-cement mixes showed best results with 9 cu ft of clay (absolute volume) to 3 ft of water. Lesser amounts of cement were unsatisfactory and showed considerable shrinkage. In holes showing very large "takes," a 9:2 mix was used, completing with a 9:3 mix at refusal. In the 9:3 mix actually 12 cu ft of clay were used instead of 9 cu ft, because of the loose measure. Although the water content never exceeded 7.5.cu ft per batch, it was varied below this amount, in accordance with the desired fluidity. The clay was obtained from borrow pits in the vicinity of the dam. It was red in color, nonplastic, free of sand, but given to crumbling on drying. It was screened through half-inch hardware cloth, to remove large lumps, vegetation, or other large objects that might cause damage to the pumps or clog the lines. Central station grout mixing was used successfully throughout all grouting operations. Mixes were pumped with facility, and at times the varying distances exceeded 1,000 ft.

In general, asphalt grouting was used in the more open formations, particularly along the outcropping of badly weathered seams, or where the joints being treated carried water in quantity sufficient to carry away the cement and cement-clay mixes as fast as pumped in. Sandbags were placed over surface leaks as they developed, thus allowing the asphalt to cool and harden sufficiently to prevent further loss of asphalt. Cement-clay mixes were used when large takes were experienced, particularly in cofferdam grouting, in the treatment of the north reservoir rim, and in the grouting of the saddle dam. For the cutoff and curtain grouting under the lock and the concrete sections of the dam, neat cement grout was used. Cement grout was used similarly in the treatment of the south embankment and the north core wall (including the north abutment rim immediately upstream from the core wall).

Seepage from deep penetration was stopped by consolidation grouting in a pattern over the entire area of excavation. Interception of heavy leakage required a concentration of effort to provide grout in quantity at a point a sufficient distance down the dip of the rock to permit hardening and so to offset excessive leakage. A range of grout holes, parallel to the strike of the rock, gave a curtain at right angles to the intersecting dip or oblique joints and tapped the weathered bedding seams. Such a condition in block 10 of the spillway section is described by Mr. Pauls. There the asphalt grouting proved most effective in the open weathered formation along the outcrop of the bedding seams, and was secured by subsequent grouting with cement below (down the dip) and in back of the asphalt curtain.

The drilling and grouting program for the south embankment started in the early stages of construction, and was pushed constantly in order to complete the work before beginning to place the earth fill embankment. The magnitude of the task was not understimated and the job was completed with a minimum of interference with construction operations. Interference resulted chiefly from a lack of proper facilities for the disposal of wash water from washing operations preceding the grouting. In preparing for the cutoff trench (which was excavated southward from the lock to a point well beyond the old river channels), the first step was the excavation of the overburden of silt, clay, sand, and gravel. This was done for a sufficient width upstream and downstream from the line of the proposed concrete core wall, to give space for the use of power scrapers and rollers during the placing and compaction of the earth fill in the embankment. The removal of the overburden probably increased the lateral leakage during the grouting, but an even greater advantage resulted from then being able to observe directly the results of the stage drilling and grouting.

In describing the treatment of the foundation rock where the core wall of the south embankment crossed the old river channels, it probably would have been more informative to have stated that sufficient water to require control thereof by grouting entered the area from upstream along the strike of the rock formation. The leakage was traced by dye, placed in seepage ponds which formed in borrow pits at the inner toe of the upstream cofferdam embankment, several hundred feet distant. A line of diamond drill holes, on 5-ft centers, drilled parallel to, and some 20 ft upstream from, the cutoff trench and a similar line with fewer holes 20 ft downstream virtually encircled the area. The grouting consolidated the enclosed area, intercepting from upstream the direct seepage or flow into the area and from downstream the backflow of subsurface water from within the cofferdam enclosure.

During the planning stage preceding construction, it had been suggested in anticipation of leakage under the cofferdam, that a line of grout holes be drilled and grouted, at intervals, along the entire length of the cofferdam. The alternative was to begin stripping and excavation, for the dam, and, as leakage developed within the area of excavation, to drill and grout to the extent necessary. The first would have greatly increased the scope of what was estimated to be a large and expensive drilling and grouting program. The second presented the disadvantage that if grouting of the cofferdam later proved necessary, in order to reduce the leakage within the area of the excavation, it might be difficult to prevent excessive loss of grout where the overburden had been removed by stripping and excavation. The decision was to proceed with the excavation and to follow with the grouting to control the leakage as it developed. No question was ever raised as to the soundness of this decision. Leakage under the cofferdam was serious in a few areas only. Some of these areas required intensive treatment; others required no treatment.

Permeability and porosity tests, in the area of the south abutment of the dam, indicated that the overburden to the south of the zone in which a core wall was considered necessary was essentially impervious, and that a satisfactory cutoff could be obtained through stage grouting without resorting to an open trench and concrete core wall. However, the provision in the scope of the project for a future canal and flood channel from the Little Tennessee River resulted in the necessity of reinforcing the overburden on the upstream face of the south embankment at the extreme south end. At that point the excavation for the proposed channel would be flooded by the impounded waters. The cut exposed certain lenses of gravel extending toward the axis of the dam and well under the body of the embankment. The reinforcement of the overburden consisted of removing the exposed gravel, to a depth depending upon its texture and its position, and its replacement with embankment material rolled in place.

With respect to the lock, the spillway, and the powerhouse sections, Mr. Pauls' paper, complete with illustrations, appropriate sections, and photographs, gives an excellent picture of the problem, and little can be added without needless repetition.

At the north abutment it should be visualized that the vertical oblique joints, transverse of the line of the north core wall, were continuous for an unknown distance upstream and downstream from the line of the cutoff curtain; and that they existed as open cavities from the rock surface to depths exceeding 200 ft, although no evidence of their presence appeared on the surface of the hillside. They were irregular in outline, with branching arms funneling into the main trunk; at times they exceeded 5 ft in width; for the most part they were filled with a residuum of arenaceous material saturated with water below the water table, to form a quicksand.

The presence of some such cavities was partly expected, because large sinks and open cavities were observed immediately following the first rains, and subsequent to stripping the overburden along the ridge above the north abutment for the construction of access roads and a storage area for concrete aggregate. At a later date, when work was started on the north core wall, the significance of these earlier sinks was recognized; and further disturbance of the natural ground surface along the north rim in the vicinity of the north abutment was avoided.

In the construction of a watertight cutoff in the area, it was appreciated that any core wall, or curtain, constructed by backfilling with concrete after all weathered material and residuum had been removed by mining operations, must be functional in design to withstand the development of a pressure head against the wall with reservoir full. To prevent rupture of the wall, and any movement of the ruptured section in a downstream direction, the concrete backfill was wedged against such movement, by cutting into the side walls of the joint cavity. Steel reinforcing bars were also placed to insure a monolithic structure with adequate support by the rock formation in place. This reinforcement was kept clear of known locations of intended calyx drill holes; although, if an occasional bar became displaced, there was no complication because the calyx drill would cut both steel and concrete with equal facility. In the treatment through the deep holes at Stations 0+86.8 and 0+96.8 (Fig. 12), the 7-ft stages mentioned could not always be maintained because of the varying degree of consolidation of the residual material filling the cavity. The safety of the men was the primary consideration fixing the depth of material removed at any one time. Stages of excavation actually varied from 4 ft to 10 ft.

Because of the nature of the material filling the cavity along these vertical joints, consolidation by grouting at the prevailing depths was variable in its effectiveness. Apparently the external pressure balanced the grouting pressure (external, relative to the grouting system, and internal, relative to the quicksand or filling medium), and the displacement, by grout, of the ground-table water in the quicksand was extremely local in its effect. The writer suggests that the grouting of such a medium, be selected as a problem for research in the field of soil mechanics. The displacement of the water of saturation, in a quicksand or in a sandy clay residuum under pressure, with slow or quick setting chemical grouts or neat cement grout, has not been reported, to the writer's knowledge, in current engineering literature.

A mechanical difficulty encountered was the difficulty in maintaining the intended spacing of the grout holes for their full depth. The drill holes were deflected in the drilling, or were divergent because of inaccuracies in the leveling up of the drills themselves. This resulted in both unequal spacing at depth and in divergence from the plane of the drill curtain. Often the holes failed to strike the cavity, resulting in a loss of the hole and of the time consumed in drilling the hole. In drilling to depths of 200 ft this time loss was serious.

The time consumed in stage drilling and grouting, compared with the time of step procedure in drilling, mining, concreting, and redrilling with calyx drill holes, was a continually disputed point. In the writer's opinion, a more efficient operation would have resulted from a more extended grouting program.

Reference to Fig. 12 reveals a badly weathered area just beyond the end (block 9) of the north core wall. This region was thoroughly explored, and was deemed to be a critical area. Because of its depth, laterally, from the impounded waters in the reservoir, it was believed to be safe against the action of percolating waters. To insure against such lateral percolation, the rim treatment upstream from the north abutment grouting was definitely tied into the north core wall.

During the entire treatment of the north core wall area, and the north rim as far as the saddle dam, a check was maintained of the general effectiveness of the grouting program. This was done through a pattern of drilled wellholes, covering the area of the hillside extending from the future water line of the reservoir, up over the crest of the ridge and down the opposite slope away from the reservoir, to the base of the hill at Muddy Creek. Water-level readings were made twice daily and plotted as water-level contours. Changes in water-table elevations were thus detected and, with due allowance for the effects of wash water, were used in the grouting operations, as a means of detecting the efficiency of the grouting operations.

Portland P. Fox,[15] Assoc. M. ASCE.—Few, if any, dams in the world have received such thorough and extensive foundation exploration and treatment as those described by Messrs. Pauls and Taylor. The authors are to be commended for an excellent presentation of much data of interest to engineering geologists.

Of the nine dams constructed on the main stem of the Tennessee River, only one was constructed on noncarbonate rocks. Watts Bar Dam is constructed on interbedded, fissile shale and sandstones of variable hardness. At the other eight main dams founded on carbonate rocks, open and filled cavities and channels, of various sizes and shapes, were found along bedding planes, faults, unconformities, and joints. The conditions at each site are quite different from the others because of the geology of the area. When confronted with numerous and different problems at each site, the geologist found the particular site at a given time a little worse than any others.

It is difficult to state that any one site is worse than another, but the foundation at Fort Loudoun Dam offered about the most varied types of cavities. The persistent steep dips and the variable physical character of the limestones and shales were chiefly responsible. Kentucky Dam was noted for both small and large persistent cavities parallel to bedding planes from El. 300 above sea level to El. 30 below sea level. Large, and nearly vertical, solution channels developed along joints to a depth of 200 ft below the top of rock. The foundations of Chickamauga and Guntersville dams ably demonstrated the effectiveness and the influence on solution in limestone by thin interbedded layers of shale or bentonite. Hales Bar Dam foundation demonstrated concentrated honeycomb cavities developed along, and adjacent to, two minor thrust faults in a thick-bedded, relatively pure limestone. Fort Loudoun Dam foundation demonstrated the great depth to which solution takes place in steeply inclined limestones in a warm humid climate and the control of the geologic structures on solution of the limestones.

---

[15] Regional Geologist, U. S. Bureau of Reclamation, Billings, Mont.

The papers form valuable records for other engineers, but they would be even more valuable if the authors would give the unit cost of the excavation, the average unit cost of the diamond, wagon, and large-diameter calyx drill holes, and the unit cost of the two types of groutings used.

The success in the rapid and efficient treatment of these most difficult foundation problems during World War II was due to the availability of highly trained construction personnel.

The writer has had the opportunity to explore a number of dam sites on the upper Missouri River and its tributaries. Several of these sites are located on limestones as soluble as any in the Tennessee Valley Authority (TVA) region. There is a marked contrast in the degree of cavitation in limestones in this cold dry climate as compared with conditions in the TVA region. Relatively pure limestones comprise the foundation rock at the Yellowtail Dam in south central Montana on the Big Horn River. Recent core drilling has proved that cavitation is limited.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852

TRANSACTIONS

Paper No. 2332

STABILITY OF SOIL SLOPES

By Ek-Khoo Tan,[1] Jun. ASCE

With Discussion by Messrs. Paul Baumann, Gregory P. Tschebotarioff, Jacob Feld, Earl M. Buckingham, K. B. Hirashima, Donald M. Burmister, and Ek-Khoo Tan.

Synopsis

All cohesionless soils assume an angle of repose. On the other hand, when the soil possesses some cohesion the natural slope can be made steeper. A study of the phenomena and mechanics involved in the failure of earth slopes is reported in this paper. In the model studies of sand slopes used to investigate the slide phenomenon, the angle of repose of cohesionless soil was found to be an entirely superficial phenomenon and independent of the height of the slope. However, the inclination and the height of a bank of cohesive soil were found to depend upon cohesion. A technique to create an equivalent cohesion in a cohesionless soil was developed to reproduce actual sliding failures on a small-scale model. These laboratory failures were found to be remarkably similar to those observed in nature; and they proved ideal for studying the mechanics of failure.

Photoelastic studies were made on gelatin models to investigate the distribution of stresses, and their relative order of magnitude. There was a remarkable resemblance between the pattern of shearing stress and the pattern of shearing strain obtained by means of the sand model. It appears that failure is initiated in a plastic region near the top of the slope.

An approximate mathematical solution of the slope problem based on the theory of plasticity was developed to determine the possible position of the sliding curve and the stress conditions leading to failure. The theory shows the slip lines to be arcs of circles. A sliding failure is imminent whenever the height and inclination of slope, and the shearing properties of soil, bear such a relation to each other that a plastic region develops and tends to enlarge progressively.

Introduction

One of the most important, and at the same time most interesting, problems in the field of soil mechanics is the investigation and analysis of the stability of

---

Note.—Published in January, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Asst. Chf., Aerodromes. Air Routes and Ground Aids Section, Provisional International Civ. Aviation Organization, Montreal, Que., Canada.

soil slopes. Whenever a natural bank of cohesive soil is exposed and left unsupported laterally, experience has shown that there are certain critical height-to-slope relations necessary for stability. When these conditions are exceeded, a failure by sliding is inevitable. This slide phenomenon has been observed extensively by engineers, and has been found to follow typical forms of failure.

The most complete analysis of landslide phenomena was conducted by the Swedish Geotechnic Commission, whose investigations were prompted by the failure of a quay wall in Göteborg in 1915. Subsequently, the commission made a complete study of slides in railway cuts, and found that the shape of the sliding curve in cohesive materials showed a decided curvature that could be approximated by an arc of a circle. The expression of the fundamental principles of the "Swedish circular arc methods" are credited to K. E. Petterson[2] and S. Hultin.[2] In 1927 W. Fellenius[3] and H. Krey[4] perfected the method for practical application. Subsequently, contributions were added by Karl Terzaghi,[5] M. ASCE, J. Jaky,[6] and D. W. Taylor,[7] Assoc. M. ASCE.

Part I of this paper concerns the phenomenon of slides and the mechanics of failure, which were investigated by means of small-scale model slopes of soil. Part II concerns the photoelastic investigations made on gelatin models in order to obtain a clearer concept of the distribution, and of the relative order of magnitude, of the shearing stresses within a slope as the failure condition is approached. Part III is devoted to the development of an approximate mathematical solution of the slope problem based on the theory of plasticity, which permits the determination of the possible position and form of the sliding curve and of the approximate stress conditions leading to failure.

I. Investigation of Soil Slopes by Models

Model tests of soil slopes were made to study the mechanics of failure in cases where the slide phenomenon is controlled by gravitational influences only. In cohesionless sand the angle of repose represents a limiting condition when the soil assumes a natural stable slope. When the soil possesses some cohesion, on the other hand, it is capable of standing at a considerably steeper slope than that of a cohesionless material. The maximum inclination and height to which a stable earth slope can be constructed depend primarily on the character and physical properties of the earth in the slope, and on the moisture conditions.

The apparatus used for the model investigation on sand (see Fig. 1) consisted of a galvanized iron box with tight joints and an open end. This box, placed on a firm support, was provided with a device for tilting it to any desired angle. The angle of slope of soil was measured by a clinometer. A "coarse to fine" sand was used throughout the investigation.

---

[2] Statens Jarnvagers, Geotekinson Comm., Slutbetankande, May, 1922.

[3] "Erdstatische Berechnungen mit Reibung und Kohäsion," by W. Fellenius, Ernst, Berlin, 1927.

[4] "Erddruck, Erdwiderstand und Tragfähigkeit des Baugrundes," by H. Krey, Ernst, Berlin, 1932.

[5] "The Mechanics of Shear Failure on Clay Slopes and Creep of Retaining Walls," by Karl Terzaghi, *Public Roads*, December, 1929.

[6] "Stability of Earth Slopes," by J. Jaky, *Proceedings*, 1st International Conference on Soil Mechanics and Foundation Eng., Harvard Univ., Cambridge, Mass., Vol. II, 1936, Paper G-9, p. 200.

[7] "Stability of Earth Slopes," by D. W. Taylor, *Journal*, Boston Soc. of Civ. Engrs., July, 1937.

1. Angle of Repose Phenomenon for Dry Cohesionless Materials.—A series of tests was made with the sand deposited in two density states, loose and dense. In each case the slope was formed to its natural angle of repose of 33°, which was determined by a clinometer. A very loose state was obtained by spreading sand in thin layers with a large funnel, the thickness of the layer being controlled by the height of a funnel tip above the sand. Quite a dense state was obtained by using a vibrating device which compacted the sand to the desired density. The technique of placing the sand was developed so as to obtain reproducible consistent results. The apparatus was then tilted slowly until the first visible movement of the sand was observed. The angle of repose of dry sand used in these tests was found to vary between two limiting values—a maximum of 36° just before a slide, and a minimum of 33° just after a slide. The lower angle, which occurs after a slide, is created by the momentum of the material as it slides down the slope in an effort to seek a stable position. Failure was observed to begin at the top of the slope and was confined entirely to the surface layers of the slope. Because of the superficial nature of the slide phenomenon, the angle of repose of dry material is independent of the height of the slope; the slope always tending to flatten itself by superficial slides to the angle of repose. The results in Table 1 for a series of tests on the given material, compacted at different densities, show that the angle of repose is practically independent of the density of the material.

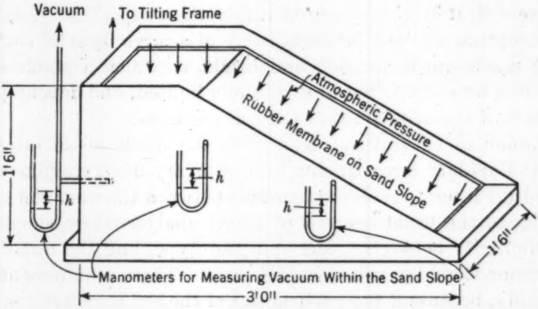

Vacuum
To Tilting Frame
Atmospheric Pressure
Rubber Membrane on Sand Slope
1'6"
h
h
h
1'6"
Manometers for Measuring Vacuum Within the Sand Slope
3'0"

FIG. 1.—MODEL APPARATUS FOR INVESTIGATING THE STABILITY OF SAND SLOPES

TABLE 1.—ANGLE OF REPOSE ϕ, IN DEGREES; AVERAGE OF A SERIES OF TESTS FOR EACH MATERIAL

Material	Void ratio e	Maximum ϕ	Minimum ϕ
Sand:			
Loose	0.73	36	33
Compacted	0.65	36	33.5
Loose Gravel:			
⅜-in.	...	36.5	31
¾-in.	...	42.5	27

To determine the effect of the coarseness of the material on the angle of repose, tests were made on gravels separated into ⅜-in. and ¾-in. sizes. The maximum angle of repose for a ⅜-in. material (36.5°) is only slightly larger than sand, whereas for the ¾-in. material the angle of repose (42.5°) is noticeably larger. This is due to the greater interlocking effect as the particle size increases. Angles of repose of 45° are a frequent occurrence in talus slopes of coarse fragments. The angle of repose of 31° after a slide, for the ⅜-in. material, is noticeably smaller than that for the sand; and for the ¾-in. material the angle of repose (27°) is still smaller, the momentum effect being more pronounced for the larger particles in flattening the slope.

2. Phenomenon of Slides in Slopes of Cohesive Soils.—When a slide occurs in a cohesive soil, it is a deep-seated rather than superficial phenomenon involving the rupture and the breaking away of a large mass of earth from the slope. The maximum inclination and height to which a stable slope of cohesive soils can be constructed are definitely limited, and depend primarily on the magnitude of the cohesion that the soil possesses.

The cohesion in natural soil deposits is the result of an internal system of forces: (1) Capillary forces arising from capillary moisture films at the grain contact; and (2) intermolecular attractions between the finer soil grains.

According to the usual concept of model analysis there should first be a geometric similarity between model and prototype, and the forces controlling the phenomenon should have a constant ratio. These requirements cannot be satisfied readily, because if the particle size of the soil is reduced in accordance with the scale ratio, the properties of the soil are altered materially. Gravitational and molecular forces cannot be altered. Capillary forces are difficult to control in the laboratory in a small-scale test. This means that every model test is really full scale. However, in a small-scale model, if a slide phenomenon can be produced similar to slides observed in nature, then it can be stated that the performance of the model is similar to nature.

In order to reproduce this phenomenon in the small-scale model it was necessary to develop a special technique which, because cohesion is the important factor, involved creating an equivalent cohesion in a cohesionless material that could be controlled so as to satisfy model requirements and reproduce a slide failure in the laboratory. However, it must be remembered that the influence of the mass of the soil involved in a failure cannot be learned from a small-scale model; but the model does afford a good insight into the mechanics of failure.

Cohesion is essentially an intergranular stress phenomenon created in one case by capillary forces arising from the capillary moisture film at the grain contact or by molecular forces between the finer soil grains, which resist shearing displacements of the grains. Thus, natural cohesion is the product of an internal force times the tangent of the angle of friction between grains. In the technique described herein an identical situation can be created by substituting a uniform system of external force for these natural internal systems of forces which are the source of natural cohesion.

The equivalent cohesion in the model soil slope was obtained by covering the cohesionless sand slope by a rubber membrane. This was stretched loosely over the slope but was tightly clamped to the sides of the apparatus, as shown in Fig. 1, so that the air in the soil mass could be evacuated. The soil was thus subjected to a uniform external pressure which could be maintained constant and be easily controlled or varied at will. This uniform external pressure produced an essentially uniform intergranular stress throughout the soil mass. The most important advantage was that the cohesion could be made sufficiently small so that a failure by a slide could actually take place.

Model slopes were formed: (1) With sand placed in a very loose state, and (2) with sand placed in a medium dense state. The sand was then transformed to a soil possessing an equivalent cohesion. The applied uniform system of external forces was determined by measuring the vacuum within the soil mass by means of three water manometers in Fig. 1, placed at intervals along the slope. The fact that they gave equal readings showed that the pressure was practically uniform over the slope. The apparatus was then tilted until a failure occurred. The angle of slope was measured; the form of failure was observed after the rubber membrane was removed. A new slope was built for each test. Fig. 2 shows the relations between the equivalent cohesion and the resulting angle of the slope at failure for the loose and dense states. It is important to note that: (*a*) The angle of slope at failure increases linearly with the equivalent cohesion except for very small value where the restraining influence of the rubber membrane itself is appreciable compared with the air pressure; and (*b*) the density of the sand has a very marked effect on the angle of slope at failure, because of the important increase in the angle of friction with increase in density.

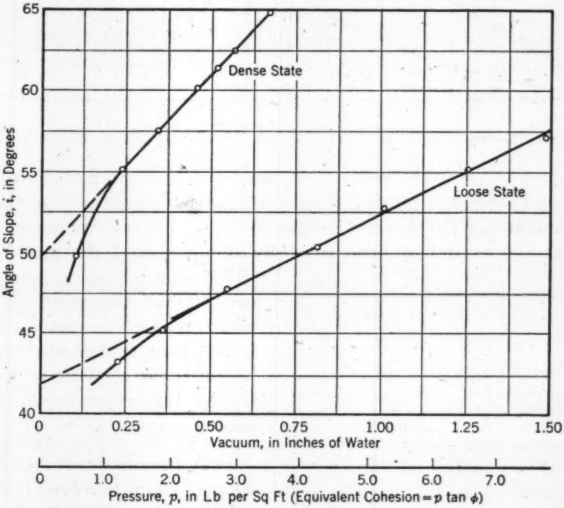


Fig. 2.—Relation Between the Angle of Slope at Failure and the Magnitude of the Equivalent Cohesion

3. Model Investigations of the Deformations Within the Soil Slope and the Location and Form of the Rupture Surface.—A procedure was devised to obtain a record of the deformation within the soil slopes, and the locations and forms of the rupture surface. A piece of glass, which was cut to the shape of the sand slopes and carefully ruled with vertical lines of lampblack and oil, about 1/16 in. wide (Fig. 3), was placed within the soil mass. After the glass had been placed against the side of the apparatus, the sand was carefully deposited against it,

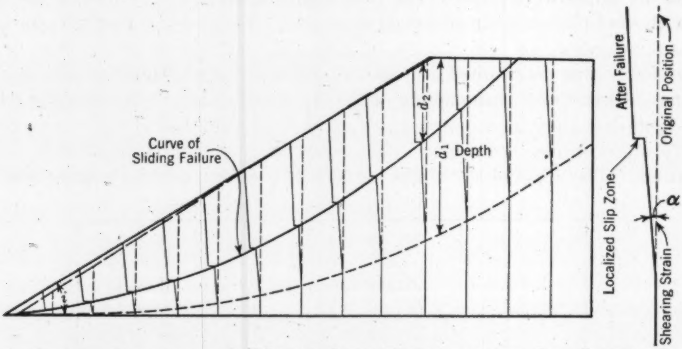


FIG. 3.—SHEARING DEFORMATIONS AND SLIDING CURVE DEVELOPED IN THE SAND SLOPE AT FAILURE

and the slope was formed as before. When any deformation or sliding occurred within the soil mass, the movement of the sand was registered on the glass by the displacements of the vertical lampblack lines. These movements were observed after the sand had been carefully removed and the glass plate was withdrawn. Displacements of about one fifth of the width of the line could readily be detected. Fig. 3 illustrates a typical curve of failure. These investigations reveal the following important facts, which are shown graphically in Fig. 4:

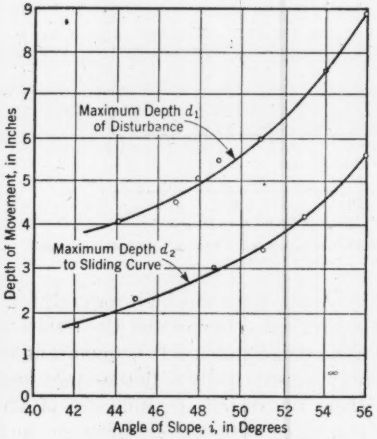


FIG. 4.—RELATION OF THE DEPTH OF MOVEMENT TO THE SLOPE ANGLE i AT FAILURE

(1) As cohesion increases, a larger mass of soil is involved in the failure and the principal failure curve recedes deeper from the surface of the slope;

(2) In all cases the zone of disturbance extends a considerable distance below the final sliding curve; and

(3) An appreciable volume change was observed in the soil slope as evidenced by the vertical movements of the surface of the slope. A subsidence occurred in the case of loose sand, and an expansion in the case of dense sand.

The angular displacements of the lines, or shearing strains, as shown in Fig. 3, reached a maximum value of about 7°. This represents a shearing strain of 0.1225 in. per in.

A study of the displacements within the soil slope was made at different stages prior to failure. These measurements establish the fact quite clearly that failure occurs in a soil slope after the strain exceeds a certain rather well-defined maximum value.

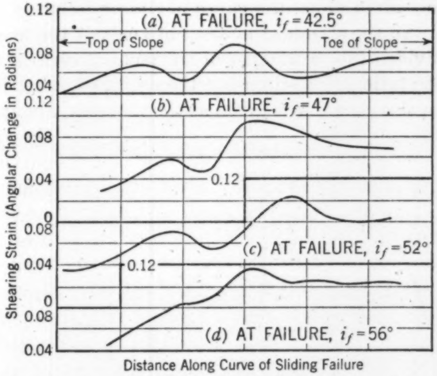


FIG. 5.—DISTRIBUTION PATTERN, SHEARING STRAINS ALONG THE ADJUSTED CURVE OF SLIDING FAILURE, FOR VARIOUS ANGLES OF THE SLOPE i_f AT FAILURE

This maximum strain, which is associated with first evidence of an incipient failure by a slide, seems independent of the angle of slope and occurs at or near the middle of the sliding curve. The curves of Fig. 5 show the distribution of shearing strains along the sliding curve at impending failure. The wave pattern is quite consistent in all tests, with a maximum at or near the middle of the sliding curve. This peculiar wave propagation, or development of shearing strains, in the soil slope at impending failure bears a close resemblance to the variation of maximum shearing stress obtained in the photoelastic studies on gelatin models shown subsequently in Fig. 9.

4. Investigation of the Failure of Model Vertical Banks.—It is a well-known phenomenon in nature that cohesive soils can stand up in a laterally unsupported vertical bank. In these tests the initial equivalent cohesion was made much greater than required for the stability of the bank of the given height. The cohesion was then decreased slowly until the bank failed.

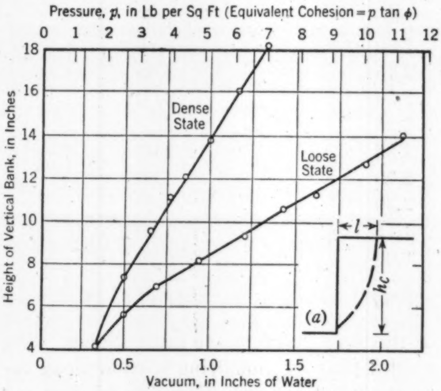


FIG. 6.—RELATION BETWEEN THE CRITICAL HEIGHT OF A VERTICAL BANK AND THE MAGNITUDE OF EQUIVALENT COHESION

The results of the investigation (see Fig. 6) show that the critical height at which the laterally unsupported embankment will stand vertically is a linear function of the cohesion. When failure occurred, the break at the surface of the embankment was found to be approximately 37.3% of the critical height of the embankment. This agrees quite closely with actual failures observed in nature where breaks occur at 0.4 h_c to 0.5 h_c (see critical height h_c of bank in Fig. 6(*a*)).

5. Analysis of the Curves of Failure by the Swedish Circular Arc Method.—A series of the observed curves of failure was analyzed by the usual Swedish circular arc method. The principles of this method are based on the following assumptions: (1) The problem is a two-dimensional one; (2) failure takes place along an arc of a circle; (3) the mass of soil above the failure curve tends to rotate about some center of rotation; and (4) the shearing resistance is fully mobilized along the curve of sliding failure.

In the standard method of analysis the mass of soil above a sliding curve is divided into a number of elementary vertical sections and the usual principles of static equilibrium are then applied; that is, $\Sigma M=0$; $\Sigma H=0$; and $\Sigma V=0$. In this case ΣM is taken as the moment, about the center of the circular arc, of the resisting forces along the sliding curve which moment, for stability, must balance the moment of the forces causing sliding. The lateral earth-pressure forces acting against the vertical side of each element are considered as conjugate stresses and the resultant is zero. The weight of the soil element acts through the centroid of the element and is resolved into its tangential and normal components, W_t and W_n, respectively, on the sliding curve. In Fig. 7 let the material above the critical slide plane be divided into sections 1 to 9, exerting forces or weights W_1, W_2, etc. Denote the respective components of weight by W_n (normal) and W_t (tangential), with numerical values as follows:

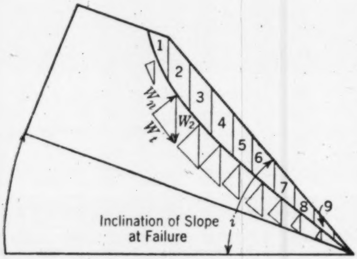


FIG. 7.—SWEDISH CIRCULAR ARC ANALYSIS

Section	W_n	W_t
1	1.00	1.75
2	2.75	3.00
3	3.50	3.00
4	3.25	2.20
5	2.80	2.00
6	2.70	1.60
7	2.20	1.50
8	1.75	1.20
9	1.00	0.70
	20.95	16.95

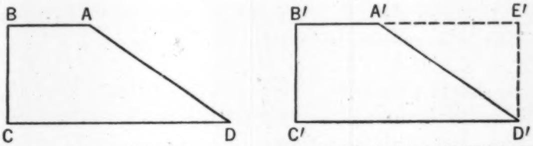
The tangential component W_t is the force that tends to cause sliding. The normal component W_n tends to resist sliding by mobilizing friction along the sliding surface, which, according to the usual concepts of sliding friction, can become equal to $W_n \tan \phi$ as a limiting value. The ratio of the forces resisting sliding to those causing the sliding is usually defined as the factor of safety. According to this concept this factor of safety should decrease to unity at impending failure.

The observed curves of failure, obtained from the tests on cohesive soil slopes, were analyzed and the results showed that, the angle of internal friction

ϕ being 33°, the factor of safety at failure was $\frac{\Sigma W_n}{\Sigma W_t}\tan\phi = \frac{20.95 \times 0.649}{16.95} = 0.80$; that is, somewhat less than unity. The significance of a factor of safety less than unity at failure is not fully understood. The real mechanics of failure are evidently not fully taken into account by the simplified Swedish circular arc method. The circular arc method assumes an average condition along the entire length of arc. The actual stresses set up in the soil slopes probably bear very little resemblance to those assumed in the circular arc method.

II. Photoelastic Studies of Gelatin Models

Photoelastic studies were made on gelatin models of slopes to obtain a clearer picture of the distribution, and the relative order of magnitude, of the shearing stresses set up within a slope. Such studies offer great possibilities for investigating the distribution of shear stresses and the stability conditions of soil embankments. Since the problem was to determine the stresses, in the soil slope, created by gravity forces, it was necessary to use a photoelastic material sensitive enough to show the stresses set up by its own weight. Gelatin seems to be quite satisfactory material for this purpose. The technique of making gelatin models has been fairly well developed.[8,9,10] Gelatin, glycerin, and water were mixed in the proportion 12:14:76. Glycerin was added to produce a tougher and more elastic gel. The higher the concentration of the gelatin, the more sensitive the gel becomes. The properties one should look for in a gel are a high degree of translucency, optical sensitivity, and elasticity. It was found that a few drops of hydrochloric acid added to the mixture would yield a more translucent model.



(a) MODEL SLOPE (b) BEFORE REMOVAL OF TEMPLATE

Fig. 8.—Similitude Relations in a Gelatin Model

A glass flume, 17 in. by 10 in. by 3⅝ in., with 3/16-in., removable, plate-glass front walls and back walls (see Fig. 8) served as a container for the gelatin model. A perforated templet A′ D′, made of galvanized iron and shaped to simulate the desired slope, was placed in the container. The perforations in the templet permitted the gelatin to be poured so that a continuous medium with a level upper surface was formed while the gelatin was cooling and setting. The gelatin model slope could then readily be cut to shape after the gelatin had set.

---

[8] "Gelatin Models for Photoelastic Analysis of Stress in Earth Masses," by F. B. Farquharson and R. G. Hennes, *Civil Engineering*, April, 1940, p. 211.

[9] "Gelatin Models," by T. R. Cuykendall, Eastern Photoelasticity Conference, May, 1939.

[10] "The Application of Photoelasticity to Soil Mechanics," by R. G. Hennes, 1st Northwestern Photoelasticity Conference, March, 1940.

With this perforated templet in place, and before the upper part A′ E′ D′ above the templet had been removed, the stress distribution in the gelatin solution was evidently hydrostatic. The distribution remains practically unchanged when the gelatin sets, as shown by the fact that there was no double refraction when the gelatin block was put into the polariscope.

To prevent the gelatin model from sticking to the glass side of the flume, and to eliminate friction, cellophane was attached to the glass side by a thin layer of vaseline. It was found necessary to allow the gelatin to cool to the same temperature as the flume before it was poured, to prevent the cellophane from wrinkling and to avoid disfiguring the surface of the model.

After the solution was poured into the flume, it was then allowed to set for a period of at least 30 hr, but not more than 48 hr, the period in which the shrinkage of the gel becomes excessive. The flume was placed on its side, the glass plates were removed carefully, and the cellophane was peeled off. The material above the templet was discarded and the templet itself was severed carefully from the model. The gelatin and the inner face of each glass plate were lubricated with thick mineral oil and the glass plate was replaced. This procedure was followed meticulously on both sides of the model. When the model, now shaped to the desired slope, was returned to its vertical position, the body forces came into play, causing the gelatin to deform.

1. Similitude Relations of Gelatin Models.—In making gelatin models certain conditions should be satisfied: (1) Surfaces AB and AD in Fig. 8 must be free of stress for the natural slope; and (2) the stresses in the slope should be confined to those caused by gravitational forces only.

In the gelatin model the surfaces BC and CD, while simulating conditions in the model sand slope, introduce a rigid boundary which does not exist in natural slopes. However, at failure, when a plastic state exists within the region of failure, the effect of the rigid boundaries may become less important. To reproduce the desired boundary conditions in the gelatin an ideal procedure would require that the gravitational force be applied and released alternately. However, since this is not possible, the aforementioned experimental procedure, involving the templet, was used. Before the perforated templet (A′ D′ in Fig. 8) was removed, the block of gelatin A′B′C′D′ was under a hydrostatic stress distribution. When the templet and wedge A′E′D′ are removed, the free boundary condition along plane A′D′ is satisfied, the free boundary condition along A′B′ is unchanged, and the gravitational body force in the volume A′B′C′D′ is brought into play.

2. Distribution of Maximum Shearing Stresses; Isochromatics.—If a transparent model of gelatin is stressed in its plane, and, if linearly polarized light is sent through the stressed body by means of Nicol prism, interference phenomena and colored fringes called isochromatics are produced which can be observed on a screen by the use of a second Nicol prism. On the screen these isochromatic lines will appear in repetition, each color corresponding to a certain principal stress difference:

$$(\sigma_1 - \sigma_2) = \text{constant} = 2\,\tau_{max} \dots\dots\dots\dots\dots\dots (1)$$

in which σ is the unit principal stress; and τ is the shear stress. The isochromatics gave a clear picture of the distribution and the order of magnitude of the maximum shearing stress throughout the slope in the gelatin model. The outside surface of one glass plate was divided into 3-in. squares, because the lenses of the polariscope allowed only a field of 5 in. in diameter to be observed. When the entire field for one position of the model had been mapped, the angle of tilt was increased and the isochromatic pattern was again mapped. The tilting caused a slight variation in the configuration and intensity of the maximum shearing stresses.

Two gelatin models were made and showed identical patterns of isochromatics. A calibration of the gelatin was made by cutting off a right prism 3 in. by 3 in. by 6 in., and the fringes resulting from its own weight were measured. It was found that the calibration constant for the fringe order corresponded to 84% of height, *h*, of the prism. This calibration permits an approximate quantitative study of the maximum shearing stresses. The values of the maximum shearing stresses, as determined on the gelatin model, were transformed to the corresponding values in the sand model by multiplying the calibration constant for the maximum shearing stress values obtained on the gelatin model by a scale ratio:

$$\frac{\gamma_s}{\gamma_g} \times \frac{L_p}{L_m} = \gamma_r \, L_r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which γ_r is the scale ratio of the unit weight of soil (γ_s) to the unit weight of gelatin (γ_g) and L_r is the linear scale ratio of the sand phototype (L_p) to the gelatin model (L_m).

To study the possible conditions with respect to the boundary BC and CD, Fig. 8, which are likely to exist in the sand model or in the prototype slope, photoelastic model studies were made for two conditions—with an adhering and a lubricated base. By separating the gelatin along boundary BC the tension in the material was eliminated at this boundary. When the model adheres to the base, there is a concentration of stress at the toe of the slope; but, when the base is separated and lubricated, the stress concentration at the toe disappears and there is also a general decrease in stress throughout the model as shown in Fig. 9(*a*). The separation of the gelatin from the boundary BC decreased the stresses in the vicinity of this boundary, but increased the stresses near the toe of the slope. These represent possible limiting conditions. Judging from the stress and strain patterns, where one fringe order $= 0.84\,h = (\sigma_1 - \sigma_2) = 2\,\tau$, it is probable that the actual stress condition in the sand lies somewhere between these two limiting conditions, since the shearing stresses along the rigid boundary at the toe of the sand slope, where the height is small, must also be relatively small.

The gelatin model was formed to an initial slope of 29°. The distribution of shearing stresses formed a regular pattern within the slope and increased in intensity as the angle of slope was increased by tilting successively to 34°, 39°, 44°, and 49°, as shown in Fig. 9. To interpret these stress conditions in terms of mechanics of failure in the sand slope, contours of equal shearing strength have been superimposed on the isochromatic patterns in Fig. 9. The positions of these lines were determined by means of the scale ratio of Eq. 2. The calibration constant (0.84% of height, *h*, of gelatin for one fringe order) is substituted in Eq. 2. The lines are always parallel to the surfaces of the slope.

It is important to note that the wave pattern of shearing stress bears a close resemblance to the pattern of shearing strains in Fig. 5 obtained in the model investigations on slopes of sand possessing equivalent cohesion. The significance of the wave pattern in Fig. 5 is now made clear. In the same general region where the shear-stress contour lies above the corresponding contour of shearing strength, the shearing strains are a maximum, and the material has reached a plastic state. Physically the shearing stress cannot exceed the shearing strength; but the excess stress must be transferred to the adjacent regions where there is a reserve of strength. Thus the plastic state encroaches on the elastic region and spreads progressively until, finally, a rupture surface develops and the slope fails by a slide.

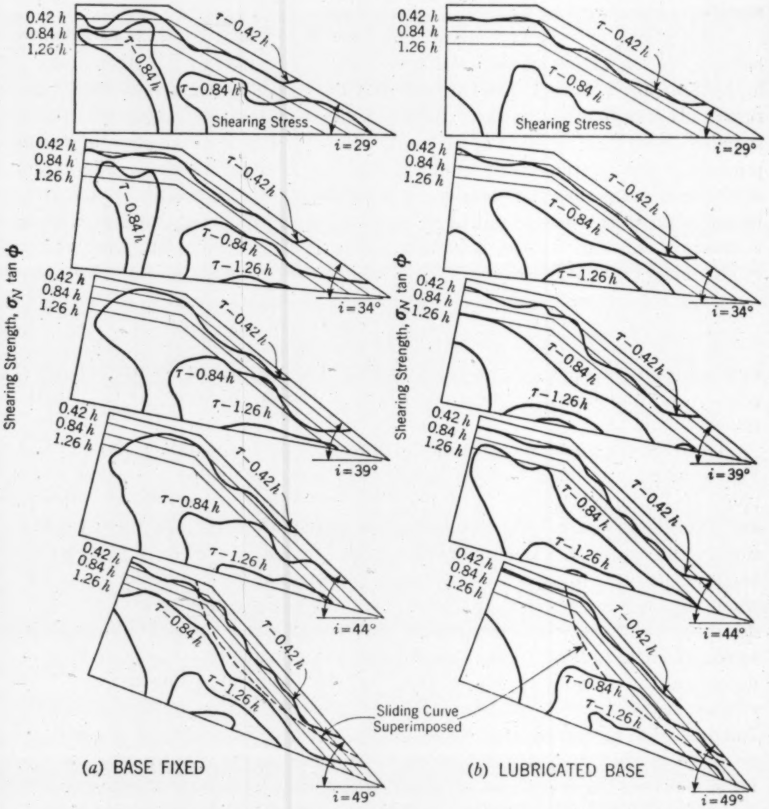

0.42 h
0.84 h
1.26 h
τ−0.42 h
τ−0.84 h
τ−0.84 h
Shearing Stress
i = 29°
τ−1.26 h
i = 34°
i = 39°
i = 44°
i = 49°
Shearing Strength, σ_N tan φ
Sliding Curve Superimposed
(a) BASE FIXED
(b) LUBRICATED BASE

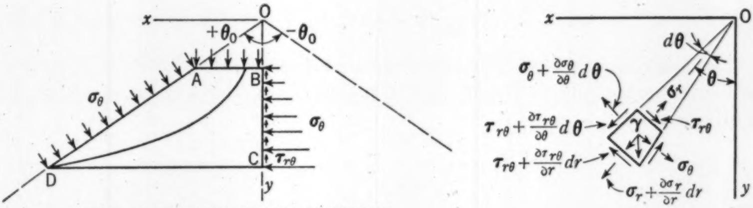
Fig. 9.—Isochromatic Patterns for Difference Values of the Slope *i*

It is now evident that, at 29°, a sand slope is stable. As the slope is tilted to the angle of repose or slightly greater (34°) a plastic region develops only near the surface of the slope, which accounts for the superficial nature of the angle-of-repose phenomenon.

When the angle of slope is to be increased, the equivalent cohesion must be increased, accordingly, to some higher value in order to maintain a stable slope. The uniform cohesion, in effect, raises the line of constant shearing strength with respect to the line of constant shearing stress so that the regions of reserve of strength slightly exceed the plastic regions. For this given value of cohesion, when the slope is tilted slightly above the critical value, the plastic region encroaches on the region of reserve of strength and the sliding curve finally develops.

III. Theoretical Plasticity Analysis

Part III concerns the development of an approximate mathematical theory for the plastic failure of a soil slope due to its own mass, which would permit the determination of the possible position and form of the sliding curve and of the approximate stress conditions leading to failure. A theoretical investigation was made based on the assumption that a wedge is formed by two intersecting planes (OD and Oy in Fig. 10), within which the state of stress has progressed from an elastic to a plastic state.


x
O
$+\theta_0$
$-\theta_0$
A
B
σ_θ
σ_θ
C
$\tau_{r\theta}$
D
y
x
O
$d\theta$
$\sigma_\theta + \frac{\partial\sigma_\theta}{\partial\theta} d\theta$
θ
σ_r
$\tau_{r\theta} + \frac{\partial\tau_{r\theta}}{\partial\theta} d\theta$
γ
$\tau_{r\theta}$
$\tau_{r\theta} + \frac{\partial\tau_{r\theta}}{\partial r} dr$
σ_θ
$\sigma_r + \frac{\partial\sigma_r}{\partial r} dr$
y

(a) EXTERNAL BOUNDARY LOADING CONDITIONS. SLOPE OF SAND. (b) EQUILIBRIUM CONDITIONS

Fig. 10.—Hypothetical Wedge of Plastic Material

The general principles of the plasticity theory developed by A. Nádai[11] were used to study the phenomenon of failure of slopes. Fig. 10(*a*) represents the model of a sand slope governed by the following conditions:

(1) A uniform normal external pressure is applied on surfaces AB and AD equal to that creating the equivalent cohesion in the sand model;

(2) Surfaces AB and AD are free from shearing stresses; and

(3) Since the wedge is symmetrical about Oy, the lateral displacement of surface BC is zero.

According to the theory of plasticity the stress in the plastic region is independent of the plastic deformation. The solution of the problem is approximate because it is necessary to assume the entire wedge to be in plastic state, whereas in nature only the region above the rupture surface is in a plastic state.

---

[11] *Zeitschrift für Physik*, Berlin, 1924, p. 106.

In nature, no rigid boundaries exist which would influence somewhat the stress condition within the plastic zone. In the laboratory, this problem is complicated by the fact that body forces are acting, that the boundary conditions cannot be satisfied readily, and that the solution of nonlinear differential equations is required. Therefore, it was decided to investigate only a certain class of solutions to develop one that would represent the physical phenomenon reasonably well. The theoretical investigation is concerned with determining the plasticity equations, which would define approximately the stress conditions for the plastic state and would permit drawing the potential slip planes in the slope.

In his presentation of the theory of elasticity, S. Timoshenko[12] states the two differential equations of equilibrium required for defining the stresses in the elastic region. For the two-dimensional case, using polar coordinates and the body forces shown in Fig. 10(*b*):

$$\sigma_r + r\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{r\theta}}{\partial \theta} - \sigma_\theta + r\,\gamma\cos\theta = 0 \ldots\ldots\ldots\ldots (3a)$$

and

$$\frac{\partial \sigma_\theta}{\partial \theta} + r\frac{\partial \tau_{r\theta}}{\partial r} + 2\,\tau_{r\theta} - r\,\gamma\sin\theta = 0 \ldots\ldots\ldots\ldots (3b)$$

In the plastic region it is necessary to introduce an additional equation, which expresses the fact that, in the plastic state, the stresses are in equilibrium; thus:

$$\left(\frac{\sigma_r + \sigma_\theta}{2}\right)\sin\phi + c\cos\phi = \sqrt{\frac{(\sigma_r - \sigma_\theta)^2}{2} + \tau^2_{r\theta}} \ldots\ldots (4a)$$

or

$$\tau_{\max} = \sqrt{\frac{(\sigma_r - \sigma_\theta)^2}{2} + \tau^2_{r\theta}} \ldots\ldots\ldots\ldots (4b)$$

The condition of plasticity may be expressed by either Eq. 4*a* or Eq. 4*b*. Eq. 4*b* represents the common concept of the maximum shearing stress condition of failure; whereas Eq. 4*a* represents the failure condition in a cohesive soil having both an angle of friction, ϕ, and cohesion, c. Eq. 4*b* is used to obtain a solution, and to satisfy the requirement of a constant maximum shear stress throughout the plastic region. It is directly applicable to the problem represented by the gelatin model but only approximately so for a cohesive soil.

A solution of the problem consists in finding a set of functions for the stresses σ_r, σ_θ, and $\tau_{r\theta}$ which satisfies Eqs. 3 and 4*b*, and of selecting values at the boundaries reasonably consistent with the boundary conditions.

The first step is to derive a fundamental equation governing the shearing stress by eliminating σ_r and σ_θ from Eqs. 3 and 4*b*. Writing Eq. 4*b* in the form—

$$(\sigma_r - \sigma_\theta) = \pm\, 2\sqrt{\tau^2_{\max} - \tau^2_{r\theta}} \ldots\ldots\ldots\ldots (5a)$$

let the right-hand side (for simplicity) be represented by $\frac{A}{r}$, or:

$$r\,(\sigma_r - \sigma_\theta) = \pm\, A \ldots\ldots\ldots\ldots (5b)$$

---

[12] "Theory of Elasticity," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1934.

Differentiating with respect to r and θ,

$$\pm \frac{\partial^2 A}{\partial r\, \partial \theta} = \frac{\partial^2}{\partial r\, \partial \theta} [r\, (\sigma_r - \sigma_\theta)] = \frac{\partial}{\partial \theta} \left[(\sigma_r - \sigma_\theta) + r \left(\frac{\partial \sigma_r}{\partial r} - \frac{\partial \sigma_\theta}{\partial r} \right) \right] \, ..(6a)$$

and

$$\pm \frac{\partial^2 A}{\partial r\, \partial \theta} = \frac{\partial \sigma_r}{\partial \theta} - \frac{\partial \sigma_\theta}{\partial \theta} + r \frac{\partial^2 \sigma_r}{\partial r\, \partial \theta} - r \frac{\partial^2 \sigma_\theta}{\partial r\, \partial \theta} \, \ldots\ldots\ldots\ldots (6b)$$

Differentiating Eq. 3*a* with respect to θ and substituting the value of $\frac{\partial \sigma_\theta}{\partial \theta}$ into Eq. 6*b*:

$$\pm \frac{\partial^2 A}{\partial r\, \partial \theta} = - \frac{\partial^2 \tau_{r\theta}}{\partial \theta^2} + r\, \gamma \sin \theta - r \frac{\partial^2 \sigma_\theta}{\partial r\, \partial \theta} \, \ldots\ldots\ldots\ldots (7a)$$

Differentiating Eq. 3*b* with respect to r, multiplying through by r, and substituting in Eq. 7*a*:

$$\pm \frac{\partial^2 A}{\partial r\, \partial \theta} = - \frac{\partial^2 \tau_{r\theta}}{\partial \theta^2} + r^2 \frac{\partial^2 \tau_{r\theta}}{\partial r^2} + 3\, r \frac{\partial\, \tau_{r\theta}}{\partial r} + r\, \gamma \sin \theta + 0 - r\, \gamma \sin \theta$$

$$= - \frac{\partial^2 \tau_{r\theta}}{\partial \theta^2} + r^2 \frac{\partial^2 \tau_{r\theta}}{\partial r^2} + 3\, r \frac{\partial \tau_{r\theta}}{\partial r} \, \ldots\ldots\ldots\ldots\ldots\ldots (7b)$$

The equation in its final form is obtained by substituting Eq. 3*a* in Eq. 7*b*, which then proves to be independent of body forces:

$$\pm \frac{\partial^2}{\partial r\, \partial \theta} \left(2\, r \sqrt{\tau^2_{\max} - \tau^2_{r\theta}}\right) = - \frac{\partial^2 \tau_{r\theta}}{\partial \theta^2} + r^2 \frac{\partial^2 \tau_{r\theta}}{\partial r^2} + 3\, r \frac{\partial \tau_{r\theta}}{\partial r} \, \ldots\ldots (8)$$

Eq. 8 governs the shearing stress at every point in a material in a plastic state at incipient failure. Eq. 8 turns out to be independent of body forces and therefore becomes identical with that obtained by Mr. Nádai[13] in an investigation, which did not involve body forces. Following the general procedure, adopted by Mr. Nádai, solutions of Eqs. 3 and 4*b*, for the plastic state, are investigated herein because they are independent of the coordinate r.

The determination of the stresses in the wedge requires first the solution of the fundamental differential for the stresses σ_θ and σ_r. For this purpose the assumption is made that the shearing stress is a function of θ only. Integrating $\tau_{r\theta}$ with respect to θ:

$$\frac{\partial \tau_{r\theta}}{\partial \theta} = \mp 2 \sqrt{\tau^2_{\max} - \tau^2_{r\theta}} + 2\, C\, \tau_{\max} \, \ldots\ldots\ldots\ldots\ldots (9)$$

in which the constant of integration is designated by $2\, C\, \tau_{\max}$. Integrating Eq. 9, in turn,

$$2\, \theta + \alpha = \int \frac{\partial \tau_{r\theta}}{\tau_{\max}\, C \mp \sqrt{\tau^2_{\max} - \tau^2_{r\theta}}} \, \ldots\ldots\ldots\ldots (10)$$

---

[13] "Plasticity," by A. Nádai. Eng. Societies Monographs, McGraw-Hill Book Co., Inc., New York, N. Y., 1931.

An auxiliary function ψ is introduced such that

$$\tau_{\max} \sin \psi = \tau_{r\theta} \qquad (11)$$

Eq. 10 then becomes

$$2\,\theta + \alpha = \int \frac{\cos \psi}{C \mp \cos \psi}\, \partial\, \psi \qquad (12)$$

Using the lower plus sign of Eq. 12 only:

$$\partial\theta = \frac{\cos \psi}{2\,(C + \cos \psi)}\, \partial\psi \qquad (13)$$

Eq. 13 represents the conditions in which the stresses are brought into play in resisting the boundary loading, as distinguished from the case, in which the boundary loads are induced as reactions of the stresses.[12] From the equilibrium Eqs. 3 and 4*b*, by substituting the value of $\frac{\partial \tau_{r\theta}}{\partial \theta}$ and $\tau_{r\theta}$ from Eqs. 9 and 11 and introducing now the body forces:

$$r\,\frac{\partial \sigma_r}{\partial r} = -\,2\,\tau_{\max}\, C - r\,\gamma\, \cos\theta \qquad (14a)$$

and

$$\frac{\partial \sigma_\theta}{\partial \theta} = -\,2\,\tau_{\max} \sin \psi + r\,\gamma\, \sin\theta \qquad (14b)$$

Having determined the values of $\partial\theta$ and $\partial\psi$, the fundamental differential stress Eqs. 14 for σ_r and σ_θ can now be solved by substituting for $\partial\theta$ from Eq. 13. Eq. 14*b* then becomes

$$\partial\sigma_\theta = -\int 2\,\tau_{\max}\, \frac{\sin\psi \cos\psi}{2\,(C + \cos\psi)}\, \partial\psi + \int r\,\gamma\,\sin\theta\, d\theta \qquad (15)$$

Integrating Eq. 15, the expression for σ_θ is:

$$\begin{aligned}\sigma_\theta = {} & +\,\tau_{\max}\,(C + \cos\psi) - \tau_{\max}\, C \log\,(C + \cos\psi) \\ & -\, r\,\gamma\,\cos\theta + f_2(r) \qquad (16)\end{aligned}$$

in which the constant of integration is a function of r only. Integrating Eq. 14*a*:

$$\sigma_r = -\,2\,\tau_{\max}\, C \log r - r\,\gamma\,\cos\theta + f_1(\psi) \qquad (17)$$

in which the constant of integration is a function of ψ only. In order to satisfy the condition of plasticity (Eqs. 4*b* and 11), the variables in r and θ must vanish identically. Since—

$$\sigma_r - \sigma_\theta \approx 2\,\tau_{\max} \cos\psi \qquad (18)$$

—it follows that the functions of $f_2(r)$ and $f_1(\psi)$ must equal, respectively,

$$f_2(r) = -\,2\,\tau_{\max}\, C \log r + C_1 \qquad (19a)$$

and

$$f_1(\psi) = \tau_{\max}\,[\cos\psi - C \log\,(C + \cos\psi)] + \tau_{\max}\, C + C_1 \qquad (19b)$$

Let the constant

$$C_1 = \tau_{\max} C \log a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

in which a is a length measured radially from the apex of the wedge. Substituting these values of the functions into Eqs. 16 and 17 the equations for the stresses σ_r and σ_θ in their final form are:

$$\sigma_r = -\tau_{\max} C \log \frac{r^2}{a^2} (C + \cos \psi) - \tau_{\max} \cos \psi - r \gamma \cos \theta + \tau_{\max} C \quad (21a)$$

and

$$\sigma_\theta = -\tau_{\max} C \log \frac{r^2}{a^2} (C + \cos \psi) + \tau_{\max} \cos \psi - r \gamma \cos \theta + \tau_{\max} C \ldots (21b)$$

To obtain the values of the function ψ Eq. 12 is integrated to yield the values of ψ implicitly as a function of θ and C:

$$2\theta + \alpha = \pm \frac{2C}{\sqrt{C^2 - 1}} \tan^{-1} \left(\frac{\sqrt{C + 1}}{\sqrt{C - 1}} \tan \psi/2 \right) \mp \psi \ldots\ldots\ldots (22)$$

The curves in Fig. 11 were obtained from Eq. 22 (with $C > 1$ and $0 < \theta < \theta_0$) for determining the numerical values of the stress Eqs. 21.

The general expressions for the stresses σ_r, σ_θ, and $\tau_{r\theta}$ (see Eqs. 11 and 21) should satisfy the boundary conditions of the soil slope. These conditions require that the shearing stress $\tau_{r\theta}$ be equal to zero along line OD. (θ equals θ_0, the wedge angle in Fig. 10(a)) and that the normal stress σ_θ be approximately uniform as originally assumed along [the boundary OD.

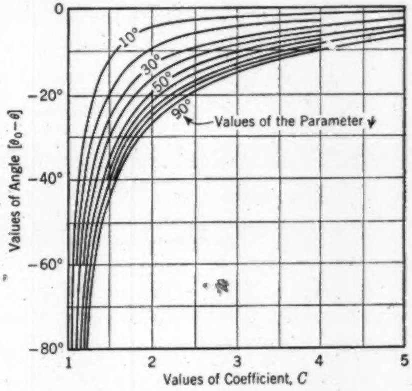
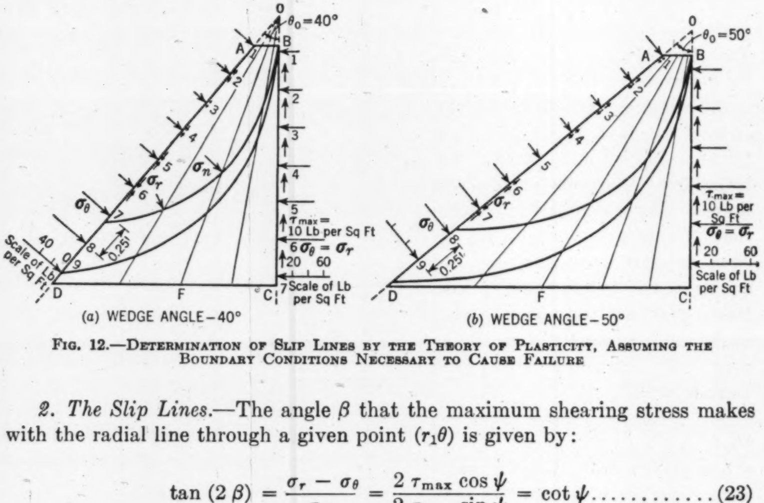


FIG. 11.—VALUES OF THE PARAMETER ψ FOR DETERMINING THE SHEARING STRESS τ IN A WEDGE OF PLASTIC MATERIAL

1. Boundary Conditions.—A study of the curves of Fig. 11 shows that, in order that the shearing stress given by Eq. 11 may have real values throughout the entire wedge, it must have its maximum value $\tau_{r\theta} = \tau_{\max}$ along boundary OC, Fig. 10, where ψ is equal to 90° by Eq. 11. Along plane OD where θ equals θ_0 the shear stress must be equal to zero. Hence, by Eq. 11, ψ must also equal zero along plane OD. A plastic region exists where $\tau_{\max}$ has become equal to the maximum shearing strength that can be mobilized.

For a wedge angle θ_0 of 40°, the value of coefficient C is determined from Fig. 11 for $\theta = \theta_0$ and $\psi = 90°$, where $\tau_{r\theta}$ is a maximum, and C is found to be 1.6. The value of $\tau_{\max} =$ (10 lb per sq ft) was obtained from Fig. 2 for a slope of 50° (θ equals 40°).

A comparison of the magnitude of σ_θ for the boundary OD in Fig. 12 for wedge angles of 40° and 50°, respectively, shows that lower boundary stresses are required for a wedge angle of 50° in a plastic condition (that is, less equivalent cohesion is required for stability) than for a wedge angle of 40°. On the boundary OC the normal stress should increase linearly in the y-direction in accordance with the usual concepts of earth-pressure phenomenon. This condition is satisfied reasonably well except near the top of the slope. The normal stress loading on boundary OD, on the other hand, does not satisfy quite so well the required assumed condition of a uniform loading, except possibly in the upper part of the slope. It is believed that the distribution of σ_θ is reasonably uniform, as assumed, and that it represents a good approximation for the upper sliding curves. The boundary condition for the wedge can therefore be assumed to be satisfied reasonably well. Fig. 12 shows also that, in the upper part of the slope, tension can exist (σ_r is positive); and, therefore, tension cracks may develop in cohesive soil.



(a) WEDGE ANGLE-40°

(b) WEDGE ANGLE-50°

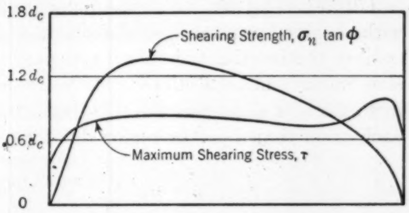
FIG. 12.—DETERMINATION OF SLIP LINES BY THE THEORY OF PLASTICITY, ASSUMING THE BOUNDARY CONDITIONS NECESSARY TO CAUSE FAILURE

2. The Slip Lines.—The angle β that the maximum shearing stress makes with the radial line through a given point $(r_1\theta)$ is given by:

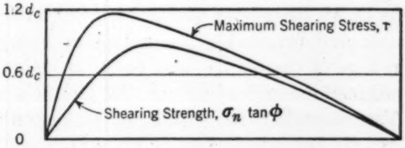
$$\tan(2\beta) = \frac{\sigma_r - \sigma_\theta}{\tau_{r\theta}} = \frac{2\,\tau_{\max}\cos\psi}{2\,\tau_{\max}\sin\psi} = \cot\psi \ldots\ldots\ldots\ldots (23)$$

When the directions of maximum shearing stress are found by Eq. 23 and Fig. 11, for a large number of points in the plastic region, the slip lines can be drawn as shown in Fig. 12 for wedge angles of 40° and 50°. These slip lines are arcs of circles and have the same general forms as the sliding curve actually observed in nature and in the model tests.

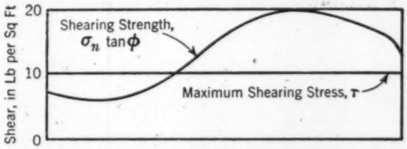
The stress conditions at failure were investigated along the sliding curve by the photoelastic method (inclination of gelatin slope, 49°), Swedish circular arc method (inclination of sand slope, 49°), and by the theoretical plasticity method (analysis of the potential slip line, wedge angle, and slope inclination, 45°). The distribution of the shearing stress in relation to the shearing strength along the sliding curve is shown in Fig. 13. The stress and strength relation obtained from the Swedish circular arc method is quite different from that determined by the photoelastic and plasticity analyses, which indicate that failure is initiated in a plastic region near the top of the slope which encroaches on the region having a reserve of strength, until a failure curve develops.



(*a*) PHOTOELASTIC ANALYSIS



(*b*) SWEDISH CIRCULAR ARC METHOD



(*c*) PLASTICITY ANALYSIS

Fig. 13.—Distribution of Maximum Shear Stresses and of the Shearing Strength Along the Slip Lines

CONCLUSION

The angle of repose of cohesionless soil was found to be entirely a superficial phenomenon. Therefore, it is independent of the height of the slope. The angle of repose for any given material lies between two limiting values—a maximum just before a slide, and a minimum just after a slide. It appears to be independent of the density but is affected by the coarseness of the material. On the other hand, the inclination and the height of a stable slope of a cohesive soil were found to be direct functions of the cohesion and were greatly influenced by the density of the material. By creating an equivalent cohesion in a cohesionless soil, it was possible to reproduce slides in the laboratory which were remarkably similar to those observed in nature, the failure curve receding from the surface deeper into the slope with increasing cohesion. It was also possible to observe the mechanics of failure, and the deformations and strains within the slope. The distribution of strain along the failure curve was found to form a consistent pattern with a maximum near the top of the slope where a failure appeared to be initiated after the strain exceeded a rather well-defined value.

The photoelastic studies in gelatin models afforded a clear picture of the distribution and relative order of magnitude of the maximum shearing stress in the slope. There was a close resemblance between the pattern of the shearing stresses and the pattern of the shearing strains obtained by means of the sand model, the plastic regions coinciding with the regions of maximum strain. The plastic regions develop gradually near the top of the slope and encroach upon the regions of reserve of strength, until a rupture curve is formed, followed by a slide.

The plasticity theory permitted the determination of the possible position and form of the sliding curves and of the approximate stress conditions leading to failure. The slip lines were found to be arcs of circles which agree closely with observed natural slide phenomena. The analysis shows that it is possible to have tensile stresses in the upper part of the slopes which might lead to tension cracks and final failure. A failure is imminent whenever the height and inclination of slope and the shearing properties of soil bear such a relation to each other that a plastic region develops.

ACKNOWLEDGMENTS

This paper is based on a thesis entitled "Stability of Soil Slopes."[14] The writer wishes to express his appreciation to the following members of the faculty of Columbia University in New York, N. Y.: J. K. Finch, M. ASCE, chairman of the Civil Engineering Department and now Dean of the Engineering School; to D. M. Burmister, Assoc. M. ASCE, for advice, encouragement, and innumerable and invaluable discussions on every aspect of the research and critical review of this paper; and to R. D. Mindlin, Assoc. M. ASCE, for his helpful suggestions and advice on the photoelasticity and plasticity analysis. G. P. Tschebotarioff, M. ASCE, of Princeton University at Princeton, N. J., and M. G. Salvadori, Assoc. M. ASCE, of Columbia University, reviewed the paper critically before presentation.

---

[14] "Stability of Soil Slopes," by Ek-Khoo Tan, thesis submitted to Columbia Univ., New York, N.Y., n 1945, in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Engineering.

DISCUSSION

Paul Baumann,[15] M. ASCE.—Research in the mechanics of slides in soil slopes has been aided materially by this paper. This is particularly true from an academic point of view as the sand models were necessarily confined to a definite boundary at the base. The models were not influenced by part or full saturation and therefore could not be expected to incorporate the complexities which seem to be typical of slides in prototypes such as dams, canal banks, and mountain slopes. An ingenious device was used to simulate "cohesion" in the materials used in the soil tests.

Sand Model Tests.—The application of atmospheric pressure on the exposed surfaces of the sand models does not truly represent cohesive action. It represents the compressive effect of these outer forces. Hence, it lends support to these surfaces and causes consolidation of the soil and, thereby, changes the stress distribution that would be established in the presence of true cohesive forces in the prism. The latter would be mutually balanced within the mass without the aid of outer forces.

To avoid confusion, the term "outside pressure" will hereafter be used in referring to the simulated cohesion. It must also be kept in mind that the lateral pressures in the prism are affected, to some extent, by the sloping top which is established in the course of tilting the model, and that the shear distribution along the inclined base is likewise affected. In the course of this discussion the effect of outside pressure and of tilting prisms will be further studied and compared with the conditions that would establish themselves without tilting.

It would have been helpful if the author had shown a screen analysis and had furnished information on the density and moisture content of the material in question. It is hoped that this will be done in the closing discussion to complete the paper.

A most interesting and valuable feature was brought out in the course of the tests on the sand models—the relation of depth to the limit of disturbance and to the surface of rupture. The insight into the mechanics of sliding failure and its division between a plastic and an elastic zone is thereby enhanced.

Prototype slides, however, are usually related to soil moisture, and to the pore water stress that results from it. This is particularly true of slopes that are exposed to water subject to fluctuation. With few exceptions, landslides observed in nature have likewise been closely linked with the moisture content in the soil and the stresses caused by this moisture, either static or dynamic. The tests described in the paper naturally do not shed any light on these phenomena; they are limited entirely to gravitational influences on the soil itself, and to the pressure acting on the outer surface.

The problem of analyzing the initial stress distribution in a soil prism, such as that used in the tests, has never been uniquely solved. This solution may be approached in the following manner:

---

[15] Asst. Chf. Engr., Los Angeles Flood Control Dist., Los Angeles, Calif.

Let line CB in Fig. 8 be the axis of symmetry of an embankment, the dimensions of which are equal in feet to the values specified in inches by the author. Then, distance CD is equal to one half of its base. The shear distribution along the base may be approximated[16] in distributing the later pressure E_o as shown in Fig. 14. Actually, the shear is distributed as indicated by the dashed line. The hatched area then represents the total lateral pressure per unit width of prism acting on a vertical plane through point A (see Fig. 14). Its value is ascertained by the Coulomb formula for cohesionless granular soil in:

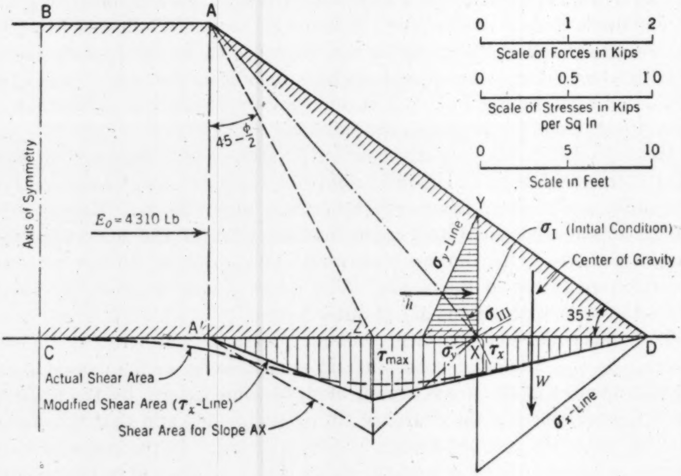

B
A
0
1
2
Scale of Forces in Kips
0
0.5
1.0
Scale of Stresses in Kips per Sq In
0
5
10
Scale in Feet
Axis of Symmetry
$45-\frac{\phi}{2}$
Y
σ_y-Line
σ_I (Initial Condition)
Center of Gravity
$E_0 = 4310$ Lb
h
σ_{III}
35±
A'
Z
C
D
τ_{max}
σ_y
X
τ_x
W
σ_x-Line
Actual Shear Area
Modified Shear Area (τ_x-Line)
Shear Area for Slope AX

FIG. 14

$$E_o = \frac{\gamma\, h^2}{2} \tan^2 \left(45 - \frac{\phi}{2} \right) \dots\dots\dots\dots\dots\dots (24)$$

in which γ, the unit weight of soil, equals 90 lb per cu ft (assumed); ϕ, the angle of friction, equals 33° (according to Mr. Tan); and $h = \overline{AA'}$.

It is significant to note that maximum, modified shear occurs along line AZ, which means that it also occurs near the top of the slope. This was confirmed by the test. The rapid increase in the shear intensity with the steepening of the slope is also significant (see dashed and dotted line in Fig. 14). Tilting the prism causes an increase in the shear intensity which is even more pronounced in that E_o increases with the tilt. Thus, tilting the base mobilizes forces which are different from those created by the steepening of the slope alone. Next, consider a small triangular prism near the toe of the slope which, in the initial state, is held in equilibrium by the stresses σ_x, σ_y, τ_x, τ_y, and the body force W.

---

[16] "Soil Mechanics," by D. P. Krynine, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1941, p. 272.

The triangle formed with σ_y as the base represents the horizontal force H which is equal and opposite to the shear S_x between points X and D in Fig. 14. The vertical (downward) shearing stress τ_y at point X is equal to τ_x. Hence, the vertical shear S_y may be determined. The weight of the prism is W. The prism is drawn to a large scale in Fig. 15 and the stresses and forces applied. The bearing pressure σ_x at point X must be determined before the principal stresses can be ascertained. This is done by combining H and S_y, and W and S_x, into resultants and by combining the two. The final resultant R falls closely in line with W and permits the σ_x-stress at point X to be determined. This stress is not equal to the pressure caused by the height XY of the prism. Fig. 16 shows the determination of the principal stresses and their directions by Mohr's circle. These directions are transferred to point X in Fig. 14. As a boundary condition for the initial state all stresses must be zero along line AD.

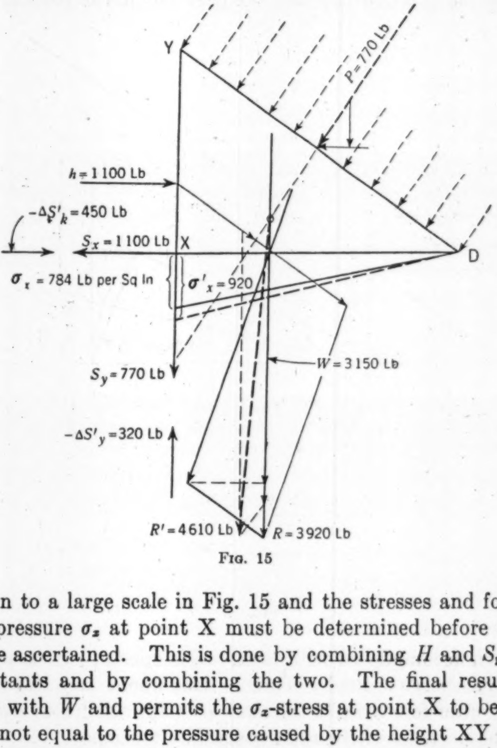


Fig. 15

Now, let the outside pressure P be applied uniformly so as to correspond to 1 ft of water column, equal to 1 in. in the model test as shown in Fig. 15 by dashed lines. Clearly, the stress condition along the surface has changed; the normal stress is no longer zero, nor is the longitudinal stress necessarily along the outside fiber. Combining P with R gives a new resultant R' which reduces τ_x and τ_y and increases σ_x. The corresponding principal stresses and their directions are shown (by the dashed line) in Fig. 16.

If the outside pressure P were increased indefinitely, the soil mass would eventually become a solid, and the section could be treated as a monolith. The stress condition would then again undergo a complete change. Hence, the application of outside pressure creates stress conditions different from those resulting from cohesion. It primarily creates a pre-stressed condition, as discussed by the writer in connection with the San Gabriel Dam No. 1 of the Los Angeles County (Calif.) Flood Control District.[17]

A better simulation of cohesion than that achieved by using the rubber membrane over the outer surface and the part vacuum in the voids of the soil could have been accomplished with mass forces created by the rotation of the model at various speeds around a center. This would have had the advantage of eliminating outer forces, since the mass (centrifugal) forces would naturally apply to all the particles of the sand mass simultaneously.

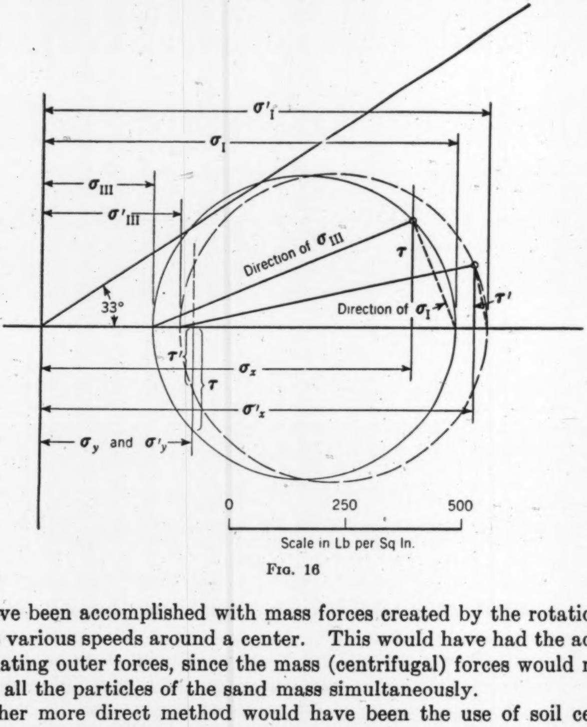


FIG. 16

Another more direct method would have been the use of soil cement of various mixes. Actual cohesion and internal friction could then be determined by tests.

Swedish Circular Arc Analysis.—It appears as if the discrepancy in the safety factor at failure in the Swedish circular arc method of analysis may be due to two principal factors: (1) The angle of internal friction evidently was considered to be equal to the minimum angle of repose of 33°, whereas, in effect, it probably was greater; and (2) the analysis apparently was made on the sand models that had been subjected to outside pressure on the rubber membrane. These forces should be superposed on the gravitational and lateral forces and the stresses along the rupture surface determined in similar manner, as outlined by the writer in connection with the soil prism (see Fig. 15). Apparently, the outside pressure was simply assumed to increase normal components along the rupture surface without simultaneously decreasing the tangential components. Thus, although factor (1) would cause an increase in the numerator, factor (2) would have the opposite effect on the denominator, and the two combined might well result in a factor of safety approaching unity.

---

[17] "Design and Construction of San Gabriel Dam No. 1," by Paul Baumann, *Transactions*, ASCE, Vol. 107, 1942, pp. 1648–1651.

Photoelastic Tests.—It is somewhat difficult to agree as to the "remarkable resemblances" between the results of the photoelastic tests and those of the sand tests, particularly in view of the fact that the photoeleastic tests do not show the formation of the surface of rupture. The shear distribution in the gelatin models is quite at variance with the one in the sand models except for the fact that both show a wave pattern.

Theoretical Plasticity Analysis.—Like most researchers in the field of soil mechanics, the author had to make considerable concessions, particularly in the way of approximations, to avoid terms of hopeless complexities for analytical solution. This circumstance, no doubt had much to do with the fact that Karl Terzaghi, M. ASCE, has defined soil mechanics as a descriptive science. The mathematical analysis is nevertheless remarkable, although the results, as shown in Fig. 13, bear little resemblance to either the Swedish circular arc or the photoelastic analyses.

Conclusion.—There is no question in the writer's mind but that the author's work will lend itself advantageously for further advance in the knowledge of the idiosyncracies of soil slopes, either natural or artificial.

Gregory P. Tschebotarioff,[18] M. ASCE.—A noteworthy feature of this paper is the attempt to apply the mathematical analysis of stress conditions within a soil mass by means of the theory of plasticity. The results of the several methods of investigation, as presented in the paper, do not appear to lead to exactly the same conclusions. However, such differences as have been registered are of value since they serve to reveal the limitations of some methods of investigation.

Fig. 3 presents, in clear graphical form, the fact that a surface of failure can develop only at some distance from a rigid boundary after the shearing strains have reached a certain value. This is a very significant finding. These shearing strains develop within the soil mass beneath the surface of failure in a soil zone which is frequently assumed as not affecting the location of the surface of failure.

Fig. 5 shows that the maximum strains occur close to the toe of the slope as determined from the displacements of lampblack lines caused by motions of the sand grains. In this respect, these curves appear to be in agreement with the general trend of stress distribution represented in Fig. 9, which summarizes the results of the photoelastic studies. There again the greatest stress concentration is shown to occur at the toe of the slope. It seems to the writer that this is a direct consequence of the presence of the rigid boundary along the base of the model slopes, and of the fact that the inclination of this boundary varied as a function of the slope i and increased with it. Judging from the isochromatic patterns shown in Figs. 9(*a*) and 9(*b*), the lubrication of the base did not noticeably reduce the stress concentration at the toe of the slope, and even appears to have increased it somewhat. In both cases (as stated by the author in Part II, section 2), the stress concentration at the toe of the slope must have been strongly influenced by the separation of the gelatin from the rear vertical face BC of the flume (see Fig. 8) which prevented the development of tension at that boundary.

---

[18] Associate Prof., Civ. Eng., Princeton Univ., Princeton, N. J.

When dealing with natural slopes in the field, one usually encounters conditions in which the cohesive soil mass extends well beyond the toe of the slope both in the vertical and in the horizontal directions. It is doubtful that stress concentrations of an intensity similar to the one shown in Fig. 9 could occur then, irrespective of whether tension cracks had, or had not, developed near the top of the slope. Accordingly, in the writer's opinion, one of the most interesting and significant contributions of the paper lies in the fact that the considerable importance of boundary conditions is made apparent, as well as the influence of these boundary conditions on the redistribution of stresses within the soil mass, and on the location of the surface of failure.

JACOB FELD,[19] M. ASCE.—The thesis of the paper is to prove—by experiment, by experience, by deduction from the measured stress distribution in gelatin models, and from mathematical analysis of stress distribution in idealized soil masses—that failure within a soil body results from plastic strain. Plastic failure methods applied to cohesive materials are consistent with present thought; but, when the failure of pure granular materials is likewise explained, a new concept has been added.

The basic differences in shear deformations of different states of materials have been explained by E. C. Bingham.[20] When a perfectly elastic solid is subjected to a shearing force, a strain develops and disappears entirely when the stress is removed. In this case of elastic deformation, there is no permanent displacement and, therefore, no flow. If the solid is not perfectly elastic, a part of the deformation remains as a permanent displacement and work has been performed in overcoming the internal resistance of the material. The flow resulting in the displacement may be of three types:

1. Viscous or linear flow in which the amount of the deformation is proportional to the acting force, although such relationship may not become apparent until the force has acted over a measurable time period;
2. Hydraulic or turbulent flow; and
3. Plastic flow in which the slightest shearing force starts a deformation which continues indefinitely.

The phenomenon, plasticity, contains the concepts of internal friction and mobility, acting simultaneously. True plasticity can only be exhibited by a material in the fluid state, for, when the material changes to a solid state, deformation ceases.

The laws of friction in solids and the laws of friction during viscous flow are quite different.[21] For solids, frictional resistance is: (1) Directly proportional to the component of the exterior forces acting normal to the surface of rupture; (2) greater before the beginning of motion than during motion; and (3) independent of the area of contact. For viscous flow, frictional resistance is: (1) Independent of the load; (2) equal to zero if the velocity is zero and is directly proportional to the velocity; and (3) proportional to the area of contact.

---

[19] Cons. Engr., New York, N. Y.

[20] "Fluidity and Plasticity," by E. C. Bingham, McGraw-Hill Book Co., Inc., New York, N. Y., 1922, p. 4 et seq.

[21] *Ibid.*, Chapter VIII.

A number of these concepts were enunciated by C. A. Coulomb[22,23] as "preliminary propositions" to his studies of the rupture of bodies, pressure of earth, and equilibrium of arches. He postulates that:

> *a.* Frictional shear failure resistance is proportional to the pressure acting normal to the failure surface; and
>
> *b.* Cohesion is a measure of the resistance which a body presents to a disunion into two parts, and in a homogeneous mass the cohesion is proportional to the area of the section of separation.

The author's statement that failure in earth slopes is a plastic phenomenon requires the assumption that the materials immediately adjacent to the fractures are in a fluid condition; otherwise failure would be localized and not progressive. Examination of slide failures in soil masses indicates that such an assumption is possible. It explains the existence of void spaces along the rupture surfaces. Discrete particles of solid matter in motion are fluid.

To the list of references proving the existence of curved rupture surfaces, the author should add Coulomb's derivation of the same type of surface. It is somewhat involved mathematically, but, nevertheless, it is the earliest derivation of the rupture curve of a soil having both frictional and cohesive resistances.[24]

The apparatus used by the author to induce variable cohesive properties to granular soils is ingenious. For materials with high angles of slope, the fixed bottom surface will affect the properties of the internal mass. The diagram in Fig. 1 does not give the angle of the side walls, but the gelatin models in Fig. 9 show an angle of 29°. The apparatus tests should be repeated with a bottom sloping downward from the toe, so that the angle of soil is at least as large as the largest angle of natural slope to be measured.

The artificially induced "cohesion" by partial evacuation of granular materials brings to mind an interchange of discussions between the late H. de B. Parsons, M. ASCE, and the writer about 1932. In an attempt to explain why readings in lateral pressure tests were lower than expected, Mr. Parsons[25] postulated the existence of a partial vacuum in the soils tested. The writer summarized his arguments by showing how the existence of cohesion within a soil mass gave the necessary explanation of the pressure readings and the internal stress relationship was the same as when a partial vacuum was assumed.[26]

---

[22] "Essai sur une application des règles de maximis et minimis à quelques problèmes des Statique, relatifs a l'Architecture," by C. A. Coulomb, *1773 Mémoires de mathematique présentés à l'Académie Royale des Sciences par divers savants*, Vol. 7, 1776.

[23] "Theorie des Machines Simples," by C. A. Coulomb, Paris, 1821, pp. 318–363.

[24] *Ibid*, p. 345, Propositions 15 and 12.

[25] "Vacuum in Soil Mechanics," by H. de B. Parsons, *Civil Engineering*, December, 1933, p. 690.

[26] "Vacuum or Cohesion in Soils," by Jacob Feld, *ibid.*, April, 1934, p. 219.

In sections 1 and 2 under the heading, "I. Investigation of Soil Slopes by Models," the author distinguishes between the angle of repose phenomena in "dry cohesionless materials" as purely a surface readjustment and in "cohesive soils" as a deep-seated internal rupture. There must be a zone of gradation in action, and some relationship between the minimum cohesion (probably as a ratio of internal frictional resistance) in a soil to place it into the "cohesion class." All soils that show the superficial phenomenon are not totally lacking in cohesion.

The author explains the existence of cohesion in natural soil deposits (section 2) as the result of capillary forces in the moisture between the finer soil grains. This explanation is not probable. The writer has experimented with pulverized bentonite (a colloidal natural clay) which when perfectly dry shows no cohesive properties and acts exactly the same as a very fine sand. However, a very slight moisture content makes the bentonite a slimy gel. Dry particles cannot be packed close enough (in soils) to permit intermolecular attraction. Powders can be made to stick together without adding a fluid medium, but only under very high pressures. On the other hand, thin moisture films will draw particles closely together; but, even in this case, the force of capillarity of the moisture films is much larger than the gravitational attraction between the molecules of the separated particles.

Cohesion is more likely to be generally explained as the interplay of dissimilar materials in intimate contact. For instance, the introduction of a liquid into a granular mass results in cohesion only if the solid aggregate is wetted by the liquid. If the liquid does not wet (adhere to) the solid grains, the physical resistance of the granular mass is not increased. Actually, for some lubricating types of liquids, the internal friction may be decreased. If a gas and a liquid are added to a granular mass, with conditions such that the gas is occluded on the surface of the grains and the liquid tends to dissolve the gas, a very plastic mix is obtained. Similarly, large cohesive properties result from the addition of colloidal ingredients or the mixing of materials which form colloidal covering of the soil grains. Any of these conditions increases the "resistance to disunion" between grain particles, the increase being many times the natural gravitational attraction of like grains to each other. The author's experimental method permits a study of varying equivalent cohesive strengths in any type of soil to be easily made.

The results given in Fig. 2, at zero vacuum, for loose and dense sand are not consistent with the values in Table 1. Possibly, the dashed line extrapolations in Fig. 2 should be eliminated, and the graphs for loose and dense state extended to meet at a value of 36° for the angle of slope with no vacuum acting.

In section 4 of the test results, the author reports a linear relation between the critical height of an unsupported embankment and the cohesion of the soil. In 1773 Coulomb developed a formula for the height h_f of a bank which will stand vertically unsupported as

$$h_f = c \frac{l}{m} \qquad \text{(25)}$$

in which c is the coefficient of cohesion; and m and l are coefficients depending on the weight of the soil and value of the internal friction, such that the earth pressure,[24] p, equals:

$$p = m\,h^2 - c\,l\,h \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

Coulomb's result also gives a linear relation between the free standing height h_f and the cohesion c.

All the author's methods, as well as the reports by H. G. Moulton[27] and the late J. C. Meem,[28] the late David Molitor,[29] Lazarus White,[30,31] D. P. Krynine,[32] Members, ASCE, and the references listed by the author under the heading, "Introduction," tend toward a solution defining the surfaces of rupture and of stable unsupported slope as circular, conchoidal (Mr. Moulton),hyperbolic (Mr. Molitor), and sometimes spiral. The unanimity of determination is the form of the curve—namely, concave; yet, the problem cannot be declared solved as long as the largest experimental report, from the Panama Canal slide correction work, gives just the opposite type of curve, definitely convex.[33]

Earl M. Buckingham,[34] Assoc. M. ASCE.—The preparation of this paper has required a considerable amount of painstaking work. To the writer, however, it seems that the planning and interpretation were not commensurate with the effort expended in the laboratory. Tests as simple as working a lump of clay between the fingers will demonstrate the fallacy of trying to simulate the action of impervious cohesive soil by increasing the friction in sand. The forces of capillarity, molecular attraction, and pore pressure are much too complicated to be replaced by a simple external pressure. Part I of the paper must therefore be considered applicable only to clean dry sand.

Although the internal structure of clay has been known for years,[35,36] the profession has been slow to accept the application of this knowledge. Soils that have a high percentage of sand possess a true internal friction, and some of the more permeable silt clays may consolidate rapidly enough to develop an increase in strength when loaded. However, the application of the Coulomb conception of cohesion plus friction to an impermeable clay may be highly misleading. This is true even though values of "cohesion" and "friction" may have been determined in the laboratory.

Consolidation is frequently difficult to prevent in the laboratory; but it might occur very slowly in the natural deposit from which the sample was taken. Instead of developing more frictional resistance, an increase in loading might cause sufficient deflection to develop dangerous pore pressures and materially reduce the shearing strength. Figs. 17 and 18 show typical results of some shear tests made recently by the writer with equipment of the three-ring type. Fig. 17 represents an unconsolidated estuary mud composed of particles of the silt and clay sizes. Each sample had a constant shearing strength for all normal pressures below a critical value at which the granular structure broke down and the mud began to liquefy and lose its strength.

---

[27] *Transactions*, A.I.M.M.E., 1920, Section 13, pp. 1–20.

[28] *Transactions*, ASCE, Vol. LX, June, 1908, p. 1.

[29] *Journal*, Assn. of Eng. Societies, Vol. 13, 1894, p. 12.

[30] *Engineering News-Record*, Vol. 85, 1920, p. 1268.

[31] *Ibid.*, Vol. 86, 1921, p. 439.

[32] *Ibid.*, July 16, 1931, p. 93.

[33] *Proceedings*, ASCE, February, 1947, p. 179 and p. 185, Fig. 2.

[34] Draftsman, City of Oakland, Oakland, Calif.

[35] "The Structure of Clay and Its Importance in Foundation Engineering," by Arthur Casagrande, *Journal*, Boston Soc. of Civ. Engrs., April, 1932.

[36] "Progress Report of the Special Committee on Earths and Foundations," *Proceedings*, ASCE, May, 1933, p. 797.

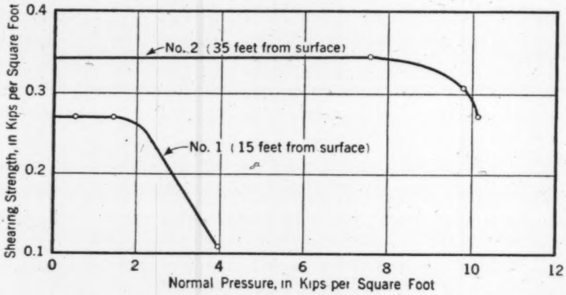


FIG. 17.—SHEAR TESTS, ESTUARY MUD

After the principle had been demonstrated with mud, an attempt was made to obtain the same effect with a stiff and fairly dry marly clay, as shown in Fig. 18. The sample was first tested at zero normal load, and failed at a shearing load of 3,030 lb per sq ft. After shifting the sample to expose new shear faces, a normal load of 12,700 lb per sq ft was applied. When the shearing stress reached 3,030 lb per sq ft, water appeared on the surface of the sample between the rings. Failure did not occur until the shear oad reached 5,760 lb per sq ft. In the natural deposit, the escape of even a small quantity of water might not be possible, and slow failure might begin at the stress under which water began to extrude from the test sample. A casually run test would have indicated an "angle of friction" of 12° 10′.

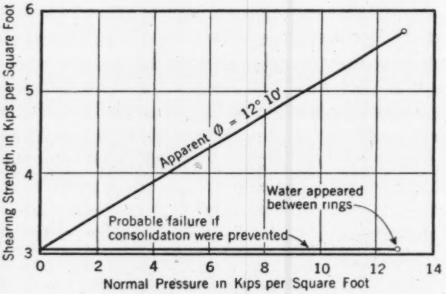


FIG. 18.—SHEAR TESTS, MARLY CLAY

Because a saturated clay ordinarily loses its strength when it is deformed, complete failure may result from a local concentration of stress even if the average stress is below the safe strength of the soil. Progressive failure of this type is common in natural slopes; but it is not considered in any analysis based on "cohesion" plus "friction."

In Part I, section 5, Mr. Tan states: "The significance of a factor of safety of less than unity at failure is not fully understood." Lack of understanding in this respect is due to the failure to distinguish between the angle of internal friction and the angle of repose. For the arc analyzed, the angle of internal friction, $\phi_i = \tan^{-1} \frac{16.95}{20.95} = 39°$. Failure along either a straight line or a circular arc does not involve internal deformation, and the only resisting forces are those acting on the failure surface. Hence, if the condition that causes failure comes into being slowly, the factor of safety at failure cannot be less than unity. Other failure sections, such as a logarithmic spiral or a broken line, involve internal deformations which are not ordinarily considered in the computation. For this condition, a computed factor of safety of less than unity is possible. Assumed failure sections of all types are arbitrary and probably differ from the section of least resistance along which failure will occur. Therefore, a factor of safety greater than unity at failure is at least theoretically possible for all assumed sections. Uneven distribution of stress should not be of major importance when dealing with a material that can yield without losing its shearing strength. If it were a factor, it would increase rather than decrease the apparent factor of safety at failure.

The author does not state how he computed the distribution of shearing stress by the Swedish arc method. Although the interfacial pressures between slices are neglected when dealing with the arc as a whole, they are quite important when considering an individual slice. The distribution of both stress and frictional resistance depends on these pressures, and a method of solution would be of considerable interest.

The straight-line relationship between critical height of a vertical bank and vacuum is not surprising. Considering unit thickness, volumes and weights vary as the square of the linear dimensions, whereas the areas to which the vacuum is applied vary as the first power. Therefore, it is necessary for the vacuum to vary as the dimensions to maintain similitude. The relation between vacuum and angle of slope is not so simple geometrically, but could probably be reduced to a constant coefficient of friction if the shapes of the failure sections were known. The writer recommends that Mr. Tan include, in his closing discussion, detailed measurements of representative failure sections which he produced. These may well prove to be the most valuable part of the investigation. Unless an actual slip has occurred and has been located, current methods of analysis depend on an assumed section. Although the various methods yield fairly comparable results, experimental data might make possible a more intelligent choice between the several conventional sections.

K. B. HIRASHIMA,[37] ESQ.—This paper, which is a valuable contribution to the literature on slope stability in that it provides an insight into the mechanics of slope failure, should be of interest to all engineers dealing with earth structures, such as highway cuts and embankments, earth dams, etc. In spite of considerable literature on the subject, cited by the author in the "Introduction," it is a phase of engineering science to which most engineers pay altogether too little attention. The subject of slope stability is one of great complexity involving many factors that are indeterminate and not fully understood. One needs only

---

[7] Testing Engr., Territorial Highway Dept., Honolulu, Hawaii.

to observe the number of slope failures that actually occur to realize the truly complex nature of slope stability.

The stability of slopes in cohesionless soil is easy to understand. Slope failure in such cases is, as the author concludes, purely a superficial phenomenon. Hence, for a mass of cohesionless soil to be stable, it is only necessary that the side slopes be less than the angle of repose. It is noteworthy that Table 1 gives two values of the angle of repose, a maximum and a minimum, whereas most textbooks give but a single value. In regions subject to earthquakes and other such disturbances, important earth structures should perhaps have slopes not exceeding the minimum angle of repose, thus decreasing the tendency to slide.

In a cohesive soil, the cohesion is sometimes defined[38] by

$$c = P_i \tan \phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

in which P_i is the hypothetical internal pressure and $\tan \phi$ is the coefficient of internal friction. The internal pressure P_i acts uniformly in all directions.

In his model test, the author uses an ingenious arrangement to impart an "equivalent cohesion" to a sandy mass by applying a uniform atmospheric pressure to the side slopes and top. This externally applied force gives rise to a stress system within the mass which can be resolved, at any given point, into a uniform direct stress, the same in all directions, and a shear stress. Theoretically, if this shear stress is of any consequence, it should affect the stress distribution due to the gravitational forces.

In the present model study it is not possible to reproduce all the conditions that exist in the prototype. An actual earth structure, such as a highway embankment, is bonded to the foundation soil—that is, there is a certain continuity to the soil mass. In a model, the soil mass is necessarily discontinous because of the model's rigid base. The change in boundary conditions affects the stress distribution; for instance, it is possible for the circle of sliding failure to undercut the toe of the slope, the so-called "Swedish break." For an embankment built on solid rock, that is, a rigid base, the failure line must necessarily pass through or above the toe of the slope.

In Fig. 6 is given the maximum height at which an unsupported bank of cohesive soil will stand vertically. It would have been interesting if the author had given additional data so as to make possible a comparison with the usual theoretical formula:[39]

$$H = \frac{2\,c}{w} \tan \left(45 + \frac{\phi}{2} \right) \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

in which H is the maximum height of unsupported bank of cohesive soil; c is the cohesion; w is the unit weight of soil; and ϕ is the angle of internal friction. The paper presents a comparison of three methods of slope analysis: The photoelastic analysis, the Swedish circular arc method, and the theoretical plasticity analysis. For routine use, the Swedish circular arc method is the only practical one. This method deals essentially with averages. Thus, if the shearing stress tending to cause sliding failure is less than the shearing strength of the soil along the assumed arc of failure, the soil mass is assumed to be safe according to the usual rule. However, a careful study of the author's analysis will show that a dangerous condition may, nevertheless, exist—that is, certain regions within the soil mass may be so highly stressed that a condition of plasticity prevails. The gradual spreading of the plastic zone may lead to eventual failure, and it is well to remember in this connection that the spreading of a plastic zone within a large mass of earth is an extremely slow process that may require weeks, months, or even years.

---

[38] "Soil Mechanics," by D. P. Krynine, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1941.

[39] "Present Status of Soils Investigations," by C. A. Hogentogler and E. S. Barber, *Proceedings*, 19th Annual Meeting, Highway Research Board, National Research Council, Washington, D. C., 1939, p. 377.

In actual engineering earth structures, the pore water pressure[40, 41] must be known. The author does not give any data as to moisture content in any of his experiments. Apparently, it was necessary to use the sandy material in a dry state in order to control the "equivalent cohesion" within narrow limits.

The writer is familiar with a soil which has a natural moisture content of more than 200%, based on the dry weight of the soil, in its undisturbed state. A vertical bank from 8 ft to 10 ft high can be cut through this material. The slope will be stable and there will be no drainage of water from the sides, in spite of the tremendous water content. On the other hand, building a highway embankment of any great height with such a soil is quite a problem, in view of the fact that rainfall conditions in the region where this particular soil is found are such that drying out the soil is impractical. Plastic zones will readily form within such a mass because of the high water content. The progressive creep of these plastic zones which may take years, has given rise to sudden settlements. However, it is the pore water pressure that is the basic cause of these difficulties, and the designer must have some idea of how this pressure acts in order to arrive at a safe design. Incidentally, in this particular soil a type of failure occurs that has not been mentioned thus far. If for any reason the moisture content of this soil drops below a certain critical value, as, for example, in times of drought, it turns irreversibly into a granular sandy mass, and the side slopes will fail in chunks along the surfaces separating the dry mass from the wet mass.

DONALD M. BURMISTER,[42] Assoc. M. ASCE.—A distinctive contribution to the subject of stability of soil slopes has been made by the author—first, by presenting a new technique which may be used for investigating the broad field of stress displacement and failure phenomena in soils by the use of an "equivalent cohesion"; second, by providing an insight into the mechanics of failure of soil slopes and a better understanding of the conditions leading finally to failure; and, third, by interpreting such phenomena by photoelastic studies and by a mathematical treatment of the subject of failure in the plastic state. The writer wishes to consider in detail certain fundamental concepts involved in this study.

The first is the concept of cohesion. The author states in Part I (under the heading, "2. Phenomenon of Slides in Slopes of Cohesive Soils"):

---

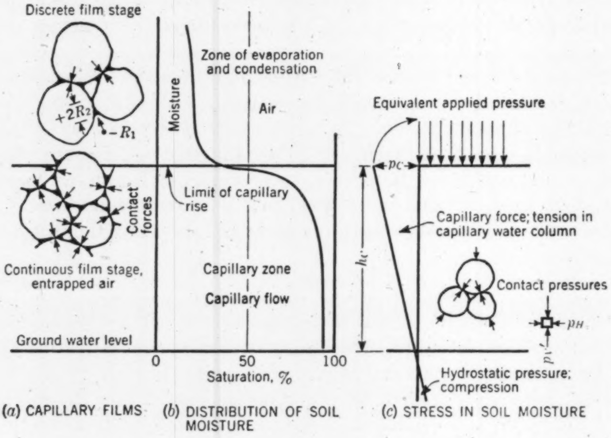
[40] "Factors of Safety to Be Applied in the Utilization of Shear Test Data," by D. M. Burmister, *Proceedings*, 19th Annual Meeting, Highway Research Board, National Research Council, Washington, D. C., 1939, p. 439.

[41] "Theoretical Soil Mechanics," by Karl Terzaghi, John Wiley & Sons, Inc., New York, N. Y., 1943.

[42] Associate Prof. of Civ. Eng., Columbia Univ., New York, N. Y.

> "Cohesion is essentially an intergranular stress phenomenon created in one case by capillary forces arising from the capillary moisture film at the grain contact * * *."

The truth of this statement should be obvious for sands and silts of cohesionless character. For example, the moisture films left in the fine sand at Daytona Beach, Fla., when the tide recedes converts the sand (most unstable when completely dry) to a firm hard surface, which makes an excellent race track. The importance of capillary cohesion can be judged from the fact that stable slopes in sands and silts of cohesionless character, but containing moisture films at the grain contacts, rarely have to be cut to angles flatter than about 45°. This cohesion is relatively permanent and can be relied on by engineers, otherwise all slopes in such materials would have to be cut to the angle of repose. The water in the contact moisture films does not evaporate entirely, or as readily as does free water, particularly if fine silt has been deposited at the grain contacts by rain water percolating downward through the deposit. The stress conditions induced by capillary forces acting in the contact films are shown in Fig. 19. Three zones are involved: A saturated zone below the ground-water level; a capillary zone extending to the upper limit of the capillary rise, in which interconnected capillary films form a continuous system through which capillary flow can take place; and a zone of condensation and evaporation above the upper limit of the capillary rise, in which discrete water films occur at the grain contacts, but are not interconnected. The film relations are shown in Fig. 19(*a*); the state of stress in the contact water films is given by La Place's equation:


Discrete film stage
$+2R_2$
$-R_1$
Contact forces
Continuous film stage, entrapped air
Ground water level
Moisture
Zone of evaporation and condensation
Air
Limit of capillary rise
Capillary zone
Capillary flow
0
50
100
Saturation, %
Equivalent applied pressure
p_C
h_c
Capillary force; tension in capillary water column
Contact pressures
p_H
p_V
Hydrostatic pressure; compression

(*a*) CAPILLARY FILMS (*b*) DISTRIBUTION OF SOIL MOISTURE (*c*) STRESS IN SOIL MOISTURE

FIG. 19.—CAPILLARY FORCES AND INITIAL STRESSES IN COHESIONLESS SOILS

$$p_c = g \rho h_c = T_s \left(\frac{1}{R_1} = \frac{1}{R_2}\right) \qquad (29)$$

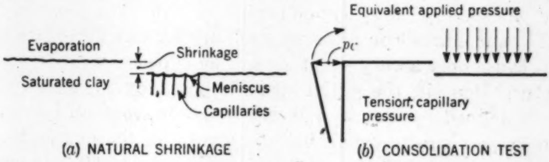
in which g is the acceleration of gravity; ρ is the mass density; h_e is the equivalent height of capillary rise; T_s is the surface tension of water; R_1 is the radius of the meniscus whose center is outside the film and is negative; and R_2 is the radius of the contact film. Eq. 29 does not limit the magnitude of either p_e or h_e, which in fine soils may considerably exceed the value for water in the normal state (30 ft of suction head).

Distribution of the moisture in the soil column, in percentages of that required for saturation, is shown in Fig. 19(*b*). The state of stress in the interconnected water films below the limit of capillary rise is shown in Fig. 19(*c*), the maximum tension in the water films occuring at the limit of capillary rise. This maximum stress condition in the water films must be exactly balanced by a stress condition in the grain structure of the soil—that is, by compressive stresses at the grain contacts. In the capillary region the stress conditions in the grain structure of the soil are identically the same as those that would be imposed either by applying a uniformly distributed vertical pressure at the limit of capillary rise equal to the maximum tension, as shown, or by evacuating the region beneath a rubber membrane, which takes the place of the water films at the same level. The stress condition in the grain structure of the soil would be uniform with depth, but would not be isotropic, since the relation between the vertical and horizontal stresses is given by

$$p_h = p_v \left(\frac{\mu}{1 - \mu}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

in which p_h and p_v are, respectively, that part of the horizontal stress and the vertical stress due only to capillary forces at a point, and μ is Poisson's ratio, varying from 0.2 to 0.3 for sands. Above the limit of capillary rise, in the region of evaporation and condensation where the films are not interconnected, the stress conditions should be isotropic.

The most important fact is that under such circumstances an initial stress condition exists in the grain structure of the soil due to forces other than gravity. These stresses greatly increase the resistance of particles to relative displacements, particularly near the surface, where the gravitational forces may be small as compared with these initial stresses due to capillary forces. Therefore, the concept of an "equivalent cohesion" is applicable to such problems, because, as stated by the author, capillary cohesion is equal to the capillary force (or tension) times the tangent of the angle of friction.



(*a*) NATURAL SHRINKAGE (*b*) CONSOLIDATION TEST

FIG. 20.—CAPILLARY FORCES AND INITIAL STRESS IN CLAY

It may not be so obvious that the concept is also applicable to clay-soils and clays. First, it is necessary clearly to separate two concepts: Cohesion, which is essentially an initial stress condition, whatever its source; and cohesive action, which involves plasticity and is quite another matter. The second concept will be deferred to the subsequent discussion on plasticity.

Conditions existing in a natural deposit of clay, exposed for the first time to evaporation forces by an excavation, are shown in Fig. 20. Evaporation of water from the surface of the clay results in an equal volume change in the region affected. Shrinkage of a mass of clay represents a strain, and hence the existence of a corresponding stress causing it must be recognized. This stress originates in the capillary meniscii formed at the surface of the shrinking clay by the evaporation processes, which bring into play active capillary forces at the surface of the clay, as shown in Fig. 20(*a*). The clay mass is subjected to a uniform stress. An identical situation is produced by applying an external pressure in a consolidation test, and the magnitude of the capillary pressure acting at any stage can be determined, inasmuch as a certain stress must be associated with a certain strain for the given material. Shrinkage is really a compression of the clay under capillary forces. At the shrinkage limit, the resistance to further compression exceeds the maximum value of the capillary pressure that can be developed. Further evaporation merely causes the soil to dry out and causes air to enter the voids. Sufficient water (which cannot be completely evaporated at ordinary temperatures) remains in the soil as very fine capillary films to maintain this final stress condition, which may exceed 200 kg per sq cm.

If a series of clay specimens is prepared for unconfined compression tests—the first being tested immediately and the remainder tested at increasing intervals of time so that the water lost by slow evaporation produces approximately equal changes in water content and therefore in density—the strength of the specimens will show a regular increase with the increase in density. It may be properly reasoned, therefore, that cohesion is not an inherent, single-valued property of clays, as frequently thought, but is a function of density or water content. The really significant fact, however, is that the strength, and hence the cohesion, is a function of the initial stress conditions imposed on the specimen, whatever the source. Identical results would be obtained by preconsolidating each specimen under a series of increasing external pressures and permitting equilibrium to be reached by free drainage of the excess pore water. As soon as the specimen is removed from the consolidating device to be tested in unconfined compression, capillary forces will replace the external pressures, because only a very slight expansion is necessary.

In a natural deposit, the initial stress imposed by the action of capillary forces will be defined by Eq. 30 and will range between an isotropic state of stress when Poisson's ratio equals 0.5 and a ratio of the horizontal stresses to the vertical stresses of 0.667 when Possion's ratio equals 0.40. In some deposits of clays the material has been consolidated under much greater pressures than those imposed by the present overburden. In nature the initial state of stress is maintained under such conditions because of the impermeable nature of the clay, which does not readily permit absorption of water and hence volume increase, and because of the relatively small vertical expansion associated with the reduction of the vertical stress to that imposed by the present overburden.

Therefore, the most significant fact about natural clay deposits is their initial stress condition, regardless of the source. It should be clear that, from the standpoint of the initial stress conditions, the new technique of imposing an "equivalent cohesion," or rather an initial stress condition, is a most valuable contribution for investigating and studying the field of stress, displacement, and failure phenomena in soils. The influence of the size of the mass of soil involved in the displacements or in failure cannot be learned from a small-scale model, but the model does give an insight into the mechanics of the phenomena and a better understanding of the stress displacement relationships and of the conditions leading to failure.

The second concept is that of plasticity, as a distinctive clay quality, which may be defined as the capacity to be deformed continuously under the action of a constant shearing stress without appreciable volume change. This quality is due primarily to the colloidal scale-like character of the true clay minerals, and to the very impervious nature of clay. On the other hand, fine sand and silt are not plastic, because if a ball of the moist soil is shaken in the hand, the surface becomes wet and shiny, indicating a densification and a much more pervious character. If the ball is squeezed between the thumb and forefinger, the surface dries and water is sucked into the ball by the expansion of the soil under such action. In such a material the initial stress condition can change during deformations, because the stress in the pore water changes, becoming a positive hydrostatic stress with a compression and a negative stress or tension with a tendency for expansion. In clays, with their characteristic qualities and impervious nature, the initial stress condition is maintained unless the rate of deformation is exceedingly slow so that volume changes may occur. The dry sand model test with an imposed initial stress condition conforms more closely to the situation for a saturated clay than for a saturated fine sand or silt, particularly if Coulomb's equation for the shearing strength of clay—

$$S = c + N \tan \phi \ldots\ldots\ldots\ldots (31)$$

—can be considered to apply reasonably well. The chief difference is the magnitude of the angle of friction, ϕ, of clay, if such a factor exists for clays. The ratio of the horizontal to vertical stresses will be somewhat different, being of the order of from 0.667 to 0.875 for clay and of from 0.25 to 0.50 for sand.

As noted previously, plasticity is a distinctive characteristic of clay. Viscous or fluid flow is characteristic of liquids and suspensions and depends on only one property of the liquid, as expressed by

$$v \approx \frac{F}{\eta} \ldots\ldots\ldots\ldots (32)$$

in which v is the velocity of flow; F is the force or head causing the flow; and η is the viscosity of the liquid or suspension. Plastic flow is basically different in one important respect. Bingham's[21] law states that plastic flow depends on two properties of the plastic material, such that

$$v \approx \frac{F - F_o}{\lambda} \ldots\ldots\ldots\ldots (33)$$

in which F_o is the yield value of the material, and λ is its consistency.

In viscous flow no static equilibrium is possible even under the slightest pressure or head. The significant fact about plastic flow is that, for stresses below the yield value, static equilibrium is possible and plastic flow or yielding ceases in time. If it were not for this important fact, it would be impossible to found any structure on clay. Below the yield value the flow is in the nature of a very slow plastic creep, which in most cases ceases after a time; above the yield value plastic flow can take place indefinitely at a constant rate under a constant shearing force and without appreciable volume change. The magnitude of the yield value is determined by the initial stress condition imposed on the clay at some time in its geologic history.

The third concept is that of the plastic state. It is necessary to differentiate clearly between plasticity as a clay-like quality (as previously defined), and the plastic state defined as a definite state of stress in the theory of plasticity without regard to the nature or the kind of material involved. A region can be said to have reached the plastic state when everywhere within the region the shearing stresses have become equal to the shearing strength of the material. This condition of plasticity has been expressed mathematically in Eq. 4, where the maximum shearing stress has become equal to the maximum shearing strength or cohesion, *c*. The maximum shearing stress can become equal to but cannot exceed the shearing strength of the material. This means that the plastic zone must spread to areas having a reserve of shearing strength until the total shearing load is carried—if such a condition is possible. In clays there will be a zone in which Eq. 4, the law of plastic flow, applies, and the failure will proceed essentially without volume change. In sands it is conceivable that flow failure may occur if a volume decrease occurs during the failure and the failure can take place rapidly.

Certain significant facts are shown in Figs. 3 and 4, and three regions may be defined. Above the final slip line is a plastic region in which the shearing stresses have become equal to the shearing strength. Below this is a transition zone, in which relatively large displacements have occurred, but below values required to mobilize the full shearing strength. Below this transition zone is an elastic zone, in which deformations are insignificant. Fig. 4 shows that, as the initial stress is increased, the angle of slope at failure being related to this initial stress through Fig. 2, the phenomenon changes from an essentially superficial one typical of cohesionless sand under very low values of the initial stress to one which is typical of the deep-seated character associated with slides in clays. The final sliding curve in each case appears to be localized at a definite depth, depending on the magnitude of the initial stress. The phenomenon shown in Fig. 3 is typical of the "toe failure slides," wherein the horizontal bottom of the excavation is at the level of the toe, or where a slide occurs in a weaker material in a bank. Usually this weaker soil is underlain by a higher strength material which limits the failure curve to the interface between the two materials. The phenomenon associated with a lubricated base (Fig. 9) is of interest because a number of rolled earth dams, built on foundations containing a plastic clay layer relatively near the surface, have shown considerable lateral movement during construction.

It is also significant to note that in the plasticity theory developed by the author the existence of an initial stress condition is an essential condition for deep-seated sliding failure curves, as shown in Fig. 12, and that the surface of the upper part of the slope is in tension (above point 3 in Fig. 12(*a*) and above point 5 in Fig. 12(*b*)). A fact of great practical importance is that failure is imminent when the strain in the embankment exceeds a certain rather definite limiting value for the given material. This is a consequence not only of the stress deformation characteristics of the soil, but also of the progressive failure phenomena due to the actual distribution of shearing stresses within an embankment and to the spreading of the plastic zone. These factors, together with the concentration of shearing stresses in the middle area of the rigid base (see Fig. 9), may account for the wave pattern in the shearing strain (Fig. 5) and the corresponding stress pattern (Fig. 9), which are associated phenomena. In clays the phenomenon of progressive failure is likely to be more important than in sands (except in the dense state), because the stress-strain curves of clays usually show a characteristic large decrease in shearing strength after the maximum strength has been reached, when the test is made by strain control methods. This decrease in shearing strength is the result of disturbance effects resembling strain softening.

The fourth concept is that strains in soils and the stress conditions for failure depend exclusively on effective stresses, as stated by K. Terzaghi,[41] M. ASCE. The total stresses in a saturated soil consist of two parts. One part is the effective stress, which acts in the grain structure of the soil and causes compression; the other is the pressure in the pore water, which may be greater (or less) than the natural hydrostatic pressure, and even negative, depending on conditions. The shearing resistance that can be mobilized in a soil mass under any given circumstances depends only on the effective stresses. A change in stress in a saturated soil is always associated with a tendency toward change in its water content or volume of voids. If the soil tends to compress during shearing and the loading is rapid enough so that the water cannot escape within that short period, positive hydrostatic stresses are induced in the pore water in excess of the natural hydrostatic pressure. As a consequence, part of the load on a potential plane of shearing is carried by positive pore water pressures. The shearing resistance of the soil is then expressed by—

$$S = (N - u) \tan \phi \qquad (34)$$

in which N is the total normal pressure acting on the potential surface or rupture or sliding; u is the pore water pressure (positive or negative excess over the natural hydrostatic pressure); $(N - u)$ is the effective stress (less than N, if u is positive and greater than N, if u is negative); and ϕ is the angle of friction.

The value of the pore water pressure, whether positive or negative, depends not only on the nature of the soil and its initial state with respect to density and to stress, but also on the rate of stress application, the permeability of the soil, and the size of the soil mass involved. The influence of the mass of soil involved in the displacement, or in the failure, cannot be learned from a small-scale model because the effect of mass in such phenomena primarily concerns the effective stresses acting throughout the mass. These stresses are determined as to distribution and magnitude by the drainage conditions and by the pore water pressures. The magnitude and distribution of the pore water pressures, whether positive or negative, depend not only on the initial density of the soil with respect to the critical density (the original concept stated by A. Casagrande,[43] M. ASCE), but also on the pressure on the potential sliding surface with respect to the pressure which is critical for the soil in the given density state (an extension of the original concept). Wherever the soil stresses are less than this critical pressure, the soil will tend to expand during shearing and the effective stress in Eq. 34, since u is negative, will be greater than the applied normal stresses. Thus, the soil will have greater shearing resistance if failure by a rapid slide is a possibility. Otherwise the failure will occur slowly, if at all, after the pore water pressures are relieved by drainage. On the other hand, whenever the stresses are greater than the critical value, failure is imminent, because the soil will tend to compress during shearing, and thus will induce positive pore water pressures. The effective stress in Eq. 34 will be considerably less than the applied normal pore water pressure stresses. It is probable that in a large structure certain regions may be in the critical state whereas adjacent regions are not, because in these adjacent regions the pressures are less than the critical. It is evident that a small-scale model can give no information on this phenomenon, because the stress conditions for the critical state cannot be satisfied, either relatively or as to distribution. These stability conditions are primarily a matter of laboratory determination of the shearing strength characteristics of the soils involved and of analytical analyses of the stability conditions. The only check is the behavior of full-scale structures. The small-scale model test on dry sand with controlled initial stress conditions does, however, give a valuable insight into the mechanics of the phenomenon and a better understanding of the stress displacement relationships and of the conditions leading to failure.

EK-KHOO TAN,[44] JUN. ASCE.—The writer is grateful for the discussions of his paper and for the interest shown in this subject. Mr. Feld's remarks on the concept of plasticity and failure of soil slopes are appreciated as they lead to a better understanding of the basic phenomenon. The writer has attempted to show by a small-scale sand model the possibility of sliding failure. The purpose of the tests is to acquire an insight into the actual mechanics of failure of large soil masses by slides. The present method of analysis is to assume a failure curve and then to calculate the factor of safety. If a plastic condition is initiated at any region in a soil slope that is overstressed locally and the region of overstress is allowed to spread, a progressive failure will eventually take place. Therefore, the factor to be considered in the design of slope, it seems, is to avoid regions of stress concentration.

Mr. Feld states that not all soils showing the superficial phenomenon are totally lacking in cohesion. The tests show that for cohesionless materials the failure could not be a deep-seated one, but is essentially a surface one. It does not preclude the case of surface movement of cohesive material where no deep-seated mass movement results in a slide.

---

[43] "Characteristics of Cohesionless Soils Affecting the Stability of Slopes and Earth Fills," by A. Casagrande, *Journal*, Boston Soc. of Civ. Engrs., Vol. 23, 1936, p. 13.

[44] Shanghai, China.

In commenting on Fig. 2, Mr. Feld states that it would be more logical to extend the graphs for the loose and dense states to meet at 36°. At very low values of the vacuum, however, the restraining effect of the rubber membrane becomes rather important. It shifts the lines of failure for the loose and dense states parallel to themselves from an angle of repose of 36° maximum at actual zero equivalent cohesion to their positions shown in Fig. 2. The restraint effects are thus of the order of a vacuum of 0.55 in. of water, or 3 lb per sq ft, which represents an equivalent cohesion of 1.95 lb per sq ft.

Both Messrs. Baumann and Hirashima state that soil slope failure basically is due to pore water pressure. The writer agrees that this is an important factor contributing to failure of soil slopes. The pore water pressures developed in any given case will depend on the initial density and on the magnitude of the minor principal stress in relation to the critical values, which may vary from point to point in the mass of soil undergoing shearing deformations. The pore water pressure, conceivably, can be either positive or negative, depending on whether the soil tends to compress or to expand during shearing. Such relationships can only be studied in carefully controlled triaxial compression tests, in which the phenomena can be assumed to be sensibly uniform and homogeneous throughout the soil specimen. Introducing moisture conditions in the model studies would unduly complicate the situation. It was apparent that any attempt to simulate such phenomena in a model was impractical. Many more uncontrollable conditions also would be introduced, such as a lack of homogeneity, uniform density, and uniform moisture content; unknown failure conditions; and difficulty in obtaining actual failures in such models.

Mr. Baumann suggests the use of soil cement mixtures. Such a model would have to be a full-scale model, because the forces involved in the model used are relatively small, a vacuum of 1 in. of water representing an equivalent cohesion of only 3.4 lb per sq ft. Mr. Baumann also suggests that centrifuge methods would simulate cohesion better. It is obvious, however, that if the gravity forces are increased in a certain ratio the shearing resistance that can be mobilized is also increased in exactly the same ratio. Hence, if the slope were stable before centrifuging, it would remain stable during such a test at any increase in the mass or gravitational forces, excepting only if the model were tilted during the operation.

To overcome all the various difficulties, and to limit the number of variables, this new technique was developed by introducing an equivalent cohesion. Dry sand was used throughout all the tests. The sieve analysis of the sand used is:

Sieve No.	Percentage finer by weight
4	100
8	90
16	76
30	54
50	28
100	9
200	0

The maximum density was 116 lb per cu ft and the minimum, 99 lb per cu ft. The loose state used in the tests had a density of about 100 lb per cu ft and the dense state, one of about 110 lb per cu ft. The angle of friction in the loose state by shearing tests was 33°; in the dense state this angle was 41°.

As Mr. Feld points out, cohesion is a very complicated phenomenon taking into consideration the size of soil particles, intermolecular attraction, and capillarity of moisture films, as well as the interplay of dissimilar materials in intimate contact. The experiments on the soil slopes were set up to permit a study of the influence of varying, equivalent cohesive strength in an ideal soil and to avoid all uncontrollable factors.

Messrs. Baumann and Buckingham suggest that the discrepancy in the analysis by the use of the Swedish circular arc method may be due to the fact that the angle of repose (33°) was used as the angle of internal friction. The angles of internal friction of some sands may vary between 27° for a loose state and 45° for a dense state. The 33° angle used in the computation was taken to be the average consistent with the density of the model sand, which was slightly above the loosest possible state. Tests on gelatin models indicate that the stress pattern in a soil slope is rather different from that assumed in the Swedish circular arc method of analysis. This conventional method is essentially one dealing with averages along the whole failure curve. The experiments have shown that failure is initiated through a local concentration of stresses. Spread of this concentration thus produces a slide.

Professor Tschebotarioff points out the importance of boundary conditions in the distribution of stresses within a soil mass. This is of particular importance when the soil mass is subjected to external applied pressure. The writer agrees with Professor Tschebotarioff that the rigid boundary as used in the test on gelatin models has an influence on the stress distribution of the model.

Mr. Buckingham stresses an important concept—that for clay deposits whose moisture content is prevented from escaping, thereby making consolidation of the deposit impossible, failure may take place when the deposit is subjected to a superimposed load. It is true that in such cases the laboratory shear test employed should approximate conditions in the field, and the quick unconsolidated shear test results should be employed.

Mr. Hirashima asks that additional data be supplied to make possible a comparison between the experimental and theoretical determination of the maximum height of the unsupported bank of cohesive soil that will stand vertically. The formula for the theoretical determination should read

$$H = \frac{4\,c}{w} \tan\left(45 + \frac{\phi}{2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (35)$$

The restraining effect of the rubber membrane was found to be rather large. The curve for the loose state in Fig. 6 should be a straight line through the origin, approximately parallel to the experimental line. By producing the experimental line to the vertical axis, the effect of the restraint can be estimated to be approximately equivalent to 4 in. of height. Introducing the values of the unit weight and of the angle of friction into Eq. 35, and deducting the 4 in. gives values within the order of magnitude to be expected.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2333

EXPERIMENTAL OBSERVATIONS ON GROUTING SANDS AND GRAVELS

By Alfred Machis,[1] Jun. ASCE

With Discussion by Messrs. James B. Hays, William S. Foster, Frederick J. Smith, and Alfred Machis.

Synopsis

The laboratory research reported in this paper was conducted to provide data on the use of cement slurries to prevent the contamination of water wells. The initial impetus for this investigation was the acute situation resulting from the infiltration of salt water into wells that threatened, severely, the quality of the Baltimore, Md., underground waters.

A water-bearing formation may be contaminated by salt water in one of two ways. If the formation has an outcrop under the sea, the sea may move into the formation when excessive pumping reduces the internal pressure. Such a situation may exist in some parts of the upper aquifers of the Baltimore industrial area. However, there is no way in which this condition may be alleviated by the use of cement. Many industries which originally had wells in these formations abandoned them and drove new wells tapping deeper fresh-water strata.

In many instances the new wells also became contaminated by salt water and the question then arose as to how the salt water entered the new formations. Investigation revealed that in general the deeper formations were not receiving salt water directly from the sea but were being contaminated chiefly by salt water leaking down between the well casing and the drilled hole from the upper strata. This leakage may at first seem impossible since almost all wells are drilled through clay beds. However, it has been shown by the use of caliper logs that the actual diameter of a drilled well usually is considerably larger than the diameter of the bit. Therefore, the reason why leakage does occur down the outside of a casing set into such a hole should be obvious. In some instances this leakage occurs in the same well that is producing the water; in other cases, it may occur in some distant well, often an abandoned one, and

---

Note.—Published in November, 1946, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Formerly 1st. Lt., Corps of Engrs., U. S. Army, Baltimore, Md.

may then pass into the producing formation. For cases of leakage around the outside of the well casing, grouting may be used to prevent further deterioration of water quality.

The seriousness of the situation was emphasized in 1944 by John C. Geyer,[2] Assoc. M. ASCE. In some instances the wells were producing water with a salt content as high as 3,000 ppm whereas other wells in the same formation contained only from 8 ppm to 12 ppm of salt.

The problem cannot be solved by abandoning (without sealing) the contaminated wells and drilling new ones. The capital investment in the average well of this region is about $15,000. On the other hand, a well may be repaired successfully in most cases for less than 10% of $15,000. In this light it becomes difficult to justify the extra expenditure of drilling a new well; and, in addition, an abandoned well that has not been properly sealed is a greater hazard to the area as a whole than is the same well when it is being pumped. In the latter case the well may serve to contaminate all the lower fresh-water formations by acting as an open conduit between the strata. In fact such cases as this do exist and are increasing materially the difficulty of solving the salt-water contamination problem.

Grouting New Casing

Many methods may be used to grout new casing into a well. Essentially, however, all these methods fall into one of the three main classes known as bailer, tubing, or casing methods. In each method the cement mixture commonly employed is a "neat" proportion of water and cement without any addition of sand. This mixture is also variously referred to as "cement slurry" or "cement grout." In all cases, the cement is mixed with the desired proportion of water before it is allowed to pass into the well.

Dump Bailer Method of Cementing.—In the dump bailer method, the slurry is lowered to the bottom of the well in a cylindrical container and is emptied through a foot valve. This method of cementing is too limited in its scope and, therefore, has been superseded by improved methods.

Tubing Method of Cementing.—In the tubing method of cementing, the slurry is pumped into the bottom of the well through a tube down the center of the well. Since the cement must be forced out around the outside of the casing, it is obvious that there must be a free passage for the entire length of the casing. This condition is usually insured by establishing the free circulation of water down around the casing shoe and up to the surface.[3]

Casing Method of Cementing.—In this method of cementing, the slurry is pumped directly through the casing down to the bottom of the well. The slurry is usually enclosed between two moving plugs to keep it from becoming diluted or contaminated.[4]

---

[2] "Ground Water in the Baltimore Industrial Area," by J. C. Geyer, Maryland State Planning Comm., Baltimore, May, 1945.

[3] "Methods of Shutting Off Water in Oil and Gas Wells," by F. B. Tough, *Bulletin No. 163*, Bureau of Mines, U. S. Dept. of Commerce, Washington, D. C., 1918, plate facing p. 38

[4] *Ibid.*, plate facing p. 48.

Repair of Leaking Wells

The repair of a leaking well is a much more difficult problem than the grouting of a new casing. Where possible, the casing of a leaking well should be pulled and the well should be grouted in the same manner as a new well. This is seldom possible, however, because the casing will usually be tightly bound by cavings of the earth wall. For this same reason, it may also be impossible to establish a circulation of water or mud around the outside of the casing. For such a circumstance the method of "squeeze cementing" developed in the oil industry must be applied. "Squeeze cementing" is the name applied to a process whereby the cement slurry is forced into, or against, a formation through perforations in the casing. The method is very similar to the tubing method of cementing except that the slurry is forced directly against the salt-water formation instead of down around the casing shoe. The highly successful utilization of this process in the oil well industry warrants close study for possible adaptation to the water well industry.

Sealing Abandoned Wells

The sealing of wells which are to be abandoned is highly essential to prevent them from acting as open conductors between fresh and contaminated water-bearing formations. It is first necessary to pull all casing that is not too securely bound in the formation. The remaining sections must then be completely slotted or perforated so that the slurry can enter behind them at many points, thus insuring the complete filling of all cavities and channels behind the casing. The previous practice of simply filling the inside of the casing has shown itself to be insufficient, for leakage may still follow a path down the outside.

After the casing has been removed or perforated, a process similar to squeeze cementing may be used to seal the well. The hole should be grouted by progressive stages from bottom to top. An attempt made to seal the entire well in one operation without raising the outlet of the grout tube may cause the grout pipe to stick and thus make a failure of the job.

Cement Penetration Through Sands

In determining what happens to the grout that is pumped into a well, the first question that must be answered pertains to the penetration of the cement grout through the sand strata. Thus, with a given sand formation, may the grout be expected to remain in the hole or will it pass into the pores of the sand and away from the well?

In addition to solving the problem of the water well industry, such data should also be valuable in shedding light on the problems of shutting off water leakage through earth dams, levees, railroad embankments, foundations, etc. In the water well industry the problem is to prevent the excessive loss of slurry into the formation; but, in these other cases, the problem is to find the best method of forcing the grout into the pores of the sand so as to form an impermeable wall.

Materials.—The type of cement used may be a factor in determining the quantity of slurry that will pass into the formation. There are three general grades from which a cement may be chosen—high early strength, regular Portland, or special oil well cement. For this work, a Portland cement was chosen since it is the type most often used.

The characteristics of the sand used will also have an effect upon the penetration of the slurry into the sand. It is easy to understand that a sand with large pores will readily admit the passage of cement slurry whereas a sand with pores smaller than the cement grains will not pass any of the slurry. By considering the pore size and shape together as a single quality of the sand, a measure of this quality may be obtained in the expression of the sand permeability. To secure a wide range of permeabilities, sands of uniform size were used in this work. The permeabilities of all the samples are given in a subsequent section of this paper. Since an expression of the permeability completely defines the hydraulic characteristics of the sand, natural formations with the same permeabilities should react to cement penetration in the same manner as the laboratory samples.

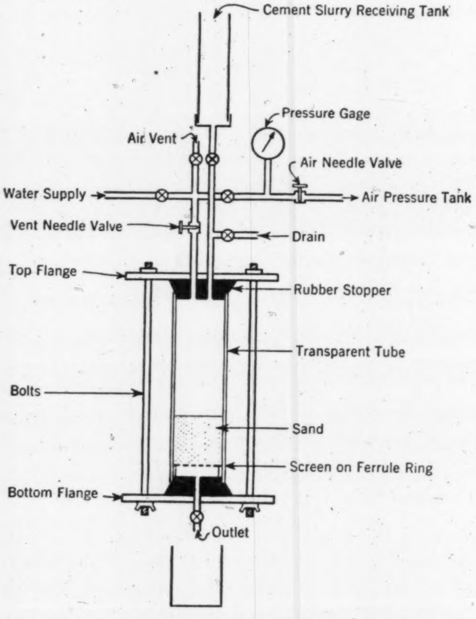


FIG. 1.—SCHEMATIC DIAGRAM OF CEMENT SLURRY PENETRATION APPARATUS

Other Factors Affecting Penetration.—In addition to the type of sand and cement used, the factors of temperature and pressure may change the properties of the cement slurry and, therefore, also may affect the penetration of the slurry into the sand. In work done by R. Floyd Farris,[6] a number of cements were tested at various temperatures and pressures to determine the effect of these factors on the change of consistency with time. From an examination of the study it appears that, in cementing water wells, the temperature and pressure effects probably will not be of any significance.

---

[6] "Effects of Temperature and Pressure on Rheological Properties of Cement Slurries," by R. F. Farris, *Transactions*, A.I.M.M.E., Vol. 142, 1941, p. 117.

Apparatus.—The apparatus, shown in Fig. 1, used in this investigation, consisted of a transparent tube of glass or plastic which contained the sand sample being tested. The tubes used were 1½ in. in diameter and from 7 in. to 11 in. long. The sand was supported within the tube by a wire cloth soldered to a ferrule ring. Rubber stoppers, used as gaskets in the ends of the tubing, were squeezed tightly between two steel flanges by four long bolts. The flanges were drilled and tapped at the three points shown, to receive ¼-in. steel pipe.

Procedure.—The sand sample to be tested was weighed and poured into the transparent tube. The best procedure for removing all air completely from the sand was to immerse the tube in a beaker of water and to allow the water to rise slowly from the bottom of the sand. The transparent tube was placed in the apparatus and the excess water was allowed to flow out until the meniscus was just at the top of the sand. The sand was vibrated to compact it by tapping the bottom flange with a rubber hammer.

The porosity of each sample was determined by measuring the height of the sample and computing the porosity from the volume and specific gravity of the sand. The cement and water were proportioned to give the desired consistency and mixed. The slurry was poured into the receiving tank from which it could be added to the transparent tube. A fresh slurry was used for each run.

TABLE 1.—PENETRATION OF CEMENT SLURRIES THROUGH SANDS

Slurry[a] (gal per sack) (1)	Porosity n (2)	Air pressure[b] (3)	Dry cement[c] (%) (4)	Volume reduction[d] (5)
(*a*) TYLER SIEVE NOS. 4 TO 8 (4.699 MM TO 2.362 MM)				
6	42.8	30	...	...
6	40.9	30	19.7	0.41
7	40.1	15	85.6	0.03
8	39.4	5	>76[e]	0.13
(*b*) TYLER SIEVE NOS. 8 TO 10 (2.362 MM TO 1.651 MM)				
6	41.0	30	...	...
6	42.3	30[f]	24.8	0.42
7	39.2	20	82.8	0.08
8	39.3	30[f]	31.3	0.44
8	39.2	15	95.4	0.06
8	39.8	5	...	...
9	39.6	30[f]	39.6	0.37
9	39.2	15	92.2	0.06
10	39.8	15	95.0	0.00
10	39.2	25[f]	17.8	0.52
(*c*) TYLER SIEVE NOS. 10 TO 14 (1.651 MM TO 1.168 MM)				
6	39.6	30	...	...
7	40.4	30	...	...
7	39.6	30	26.5	0.38
8	41.3	30	...	...
8	41.5	30	29.0	0.38
9	42.2	20	88.8	0.04
10	41.5	20	86.4	0.04
(*d*) TYLER SIEVE NOS. 14 TO 16 (1.168 MM TO 0.991 MM)				
8	40.8	30	...	...
8	40.7	30	0.6	0.68
8	40.9	30	1.5	0.75
9	41.6	30	...	...
9	41.0	20	79.8	0.06
10	41.0	15	...	...
12.5	41.4	25	...	...
14.5	39.2	30	...	...
14.5	39.9	30	100.	...
(*e*) TYLER SIEVE NOS. 16 TO 20 (0.991 MM TO 0.833 MM)				
7	44.6	30	...	...
8	43.6	30	...	...
8	40.9	30	6.8	0.65
8	40.1	25	2.8	0.68
9	42.2	30	...	...
9	41.2	25	1.2	0.63[g]
10	40.9	30	46.6	0.27
12.5	44.7	30	2.5	...
12.5	41.2	30	79.0	0.07
12.5	40.6	30	...	...
13.5	41.4	30	55.3	0.15
14.5	40.0	10 to 30	...	...
14.5	41.8	20	77.3	0.04
(*f*) TYLER SIEVE NOS. 20 TO 28 (0.833 MM TO 0.589 MM)				
14.5	41.4	30	...	...
14.5	40.8	30	4.1	0.39
20	41.8	20	47.4	0.15
(*g*) TYLER SIEVE NOS. 28 TO 30 (0.589 MM TO 0.503 MM)				
14.5	42.8	30	0	...
(*h*) TYLER SIEVE NOS. 30 TO 40 (0.503 MM TO 0.381 MM)				
14.5	40.9	30	0	...
8	39.9	20	0	0.67
14.5	41.2	30	0	...

[a] Gallons of water per sack of cement; in the text referred to as "6-gal slurry," etc. [b] Pounds per square inch. [c] Percentage of dry cement (by weight) that passed the testing apparatus. [d] Ratio of final volume to initial volume of slurry. [e] Test was stopped before the slurry passed through. [f] Pressure was applied slowly. [g] Apparatus clogged; test incomplete.

After the slurry had been placed, the desired pressure was admitted from an air pressure tank, and the outlet valve was opened. The slurry that passed through the sand was caught and dewatered by boiling. In this manner the weight of dry cement that passed could be determined. The initial weight of dry cement added to the sand was computed from the volume and specific weight of the slurry.

To study the effect of the foregoing method of applying the pressure suddenly, some data were also taken, using a slow application of pressure—first by opening the outlet valve and then by slowly applying the pressure. A distinct difference in results was obtained.

Data and Discussion of Results.—The data on the penetration of a cement slurry through a sand are presented in Table 1. The sand size is given as the mesh of the screen through which the sand will pass and the mesh upon which it will be retained. Mesh sizes, in millimeters, for the standard Tyler sieves are given in parentheses in Table 1.

Effect of Squeeze Pressure.—When a sand sample of a grain size smaller than 28 mesh (0.589 mm) is used, no amount of pressure will make a slurry pass through the sand. Slurries as dilute as 14.5 gal of water per sack of cement (hereinafter called "14.5-gal slurry," etc) were tested, and the results showed that the cement particles would penetrate into the sand only about three or four grain diameters before they lodged in a restricted pore. The pressure then acted only to squeeze the water from the slurry so that in a few minutes it was possible to reduce a 14.5-gal slurry to a 4-gal or 5-gal slurry. A pressure of 1,000 lb per sq in. was used to test this principle and was found to give exactly the same results as a pressure of 10 lb per sq in. The only difference was that the higher pressures squeezed the water from the slurry more rapidly.

When grain sizes greater than 28 mesh are used, the pressure becomes a more important factor. In this case some slurry can be expected to pass through the sand, and the amount passed will depend on how the pressure is applied. With a given slurry and a given sand, a large quantity of slurry may be made to pass through the apparatus by applying a high pressure very rapidly. This procedure was followed in all the tests in which the percentage of dry cement passed was measured, except the four tests given in lines 2, 4, 7, and 10 of Table 1(*b*) (see footnote[f]). In the first three cases mentioned, a pressure up to 30 lb per sq in. was applied at a rate of about 3 lb per sec. In each case the cement particles ceased to pass through before the full 30 lb per sq in. was reached and any additional pressure simply caused the water in the slurry to filter out more rapidly. In the last case about half of this rate of pressure application was used, and a smaller percentage of slurry was passed than was in the other three tests. This condition held in spite of the fact the slurry was more dilute—a factor more conducive to the passage of cement particles.

A comparison of these four tests with the other tests on the same sand where the full pressure application was instantaneous (less than 1 sec) shows that in the latter case the percentage of cement passed is considerably greater. For example, a comparison of lines 4 and 5, Table 1(*b*), each on an 8-gal slurry, shows that a rapid pressure application will cause the weight of dry cement passed to increase from 31% to 95%.

The main reason why this difference in penetration occurs with increased rate of pressure application is probably that, when a cement slurry is allowed to stand quiescent for a few moments, it will thicken. If violently stirred, the slurry will return to its initial thin concentration. It is very probable that the sudden application of pressure causes a violent surge of the lower layer of slurry which destroys the gelling properties. With the fluidity thus increased, the slurry does not bridge across the sand grains and easily passes into the sand.

Effect of Porosity.—A change in porosity of any given sand will serve to change the size of the pore in the sand and, of course, the amount of slurry which will pass through; but a better method of measuring the effect of a change in pore size can be obtained by changing the sand size. For this reason, an attempt was made to compact each sand into as stable a condition as possible and to maintain the same porosity in each test.

Effect of Core Length.—An examination of Col. 5, Table 1 (which gives the ratio between the final volume of the slurry after all the water has been squeezed from it and the initial volume), will show that the cement may be made to filter even on the coarse (4-mesh to 8-mesh) sand. The plug left on the sand is not the result of an inability of the cement particles to move through the pores of the sand, for observation through the glass sample tube shows that the particles in the pores are continually being washed out by the water filtering from the plug above. Instead, this cement plug is caused by an inability of the cement particles to enter the pores as a result of a strong tendency of the cement to bridge across the sand grains. This bridging has sufficient strength to support the cement particles so that the pressure on the slurry simply forces the water out and leaves a solid cement core. Since this filtering depends only on the action at the sand-cement interface, the length of the core has no effect in the results of the penetration tests.

By comparing the results of Col. 5, Table 1, the amount of slurry that will pass into the sand may be appraised. However, in comparing the results of tests using different consistencies of slurries, allowance must be made for shrinkage that would take place because of the loss of the excess water. To facilitate this comparison, tests on shrinkage of the cement slurry are reported in a following section. In addition to this information, the weight of dry cement that passed through the apparatus, computed as a percentage of the cement in the initial slurry volume, is also given in Col. 4, Table 1. These values would be affected to some degree by the void volume of the sample. With the more dilute slurries, the proportion retained in the pores after the sand was blown out with compressed air would be negligible. With the thick slurries, more would be retained in the pores of the sand; but, even then, the values of Col. 4 would be affected only by a slight percentage.

Effect of Sand Size and Slurry Concentration.—The sand size and the slurry concentration are two factors which have a very marked influence on the percentage of slurry passing into the sand. The large sand sizes give large pore sizes into which the grout may pass readily. When the cement is mixed with a large amount of water to give a thin slurry, the cement particles are too dispersed to bridge across the pore and so are carried into the sand. This bridging of cement particles is increased if the slurry is allowed to remain quiescent above the sand for several minutes.

To aid in giving a better description of the appearance of a grouted sand formation, photographs of a number of the samples were taken. In each instance the squeezing was stopped just before the air blew the cement out of the pores of the sand. The reason for this was to show the appearance of the sand as it would look if it were in a well that had been grouted by one of the methods previously described. The samples shown were removed from the tube and allowed to set. Part of the core was then cut away to show the interior of the sand and slurry. The bar above the letter $\overline{\text{C}}$ in Fig. 2 is used to indicate the sand-cement interface so that the part of the core below the letter $\overline{\text{C}}$ is the grout used to fill the section between the casing and the sand formation. The core section above the letter $\overline{\text{C}}$ is the sand formation penetrated by the grout. Each division on the scale beside the core is 1 mm so that the size of the core is apparent.

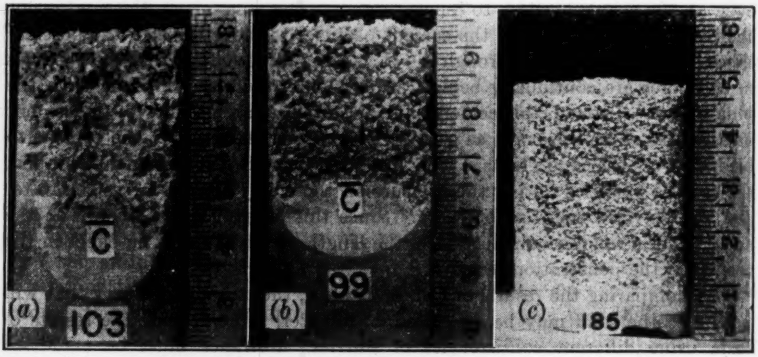


FIG. 2.—RELATIVE TEXTURE OF CORE SPECIMENS GROUTED BY SLURRY

(a) No. 8 to No. 10 Sand and 6-Gal Slurry
(b) No. 14 to No. 16 Sand and 8-Gal Slurry
(c) No. 20 to No. 28 Sand and 20-Gal Slurry

Fig. 2(*a*) shows an 8-mesh to 10-mesh sand grounted with a 6-gal slurry. The slurry has penetrated all the pores, forming a strong solid concrete. The slurry column that has not penetrated the sand has had all the excess water squeezed from it so that it is in its densest and most compact condition. The slurry columns that do not penetrate the sand can always be made to compact into this dense condition regardless of the initial water content.

Fig. 2(*b*) shows a 14-mesh to 16-mesh sand core grouted by an 8-gal slurry The pores of the sand have become too small to allow the slurry to pass a sufficient percentage of cement to fill the pores completely. In contrast to this case, a 9-gal slurry will pass about 80% of the initial dry weight of cement, and a 14.5-gal slurry will pass 100% of the cement—emphasizing the importance of the slurry concentration on penetration as the pore size decreases.

Fig. 2(*c*) shows a 20-mesh to 28-mesh sand grouted with a 20-gal slurry. About 50% of dry cement passed through the sand but the concentration within the pores of the sand was rather low. This sand was the finest into which any measurable quantity of slurry could be forced when slurries as dilute as 14.5 gal of water per sack of cement were used.

WEIGHT, VOLUME, AND SHRINKAGE OF CEMENT SLURRY

To make a comparison of the volume of slurry passing into the sand from the data of Col. 5, Table 1, it was necessary to know how much each slurry will decrease in volume as a result of the excess water in the mix. In addition to this information, it was also necessary to know the volume of the slurry per unit weight of cement in the mix to compute the amount of dry cement in the slurry above the sand prior to squeezing. To secure these data, a weighed quantity of cement and a measured quantity of water were mixed in a chemical graduate and the resultant volume was found. The results were then converted to cubic feet of slurry per 94-lb sack of cement; and these values were plotted as ordinates with the values of gallons of water per sack of cement as abscissas in Fig. 3. This curve may be considered sufficiently accurate for use on any brand of ordinary Portland cement. The values of the weight of a slurry in pounds per gallon of slurry would be of interest in field work to control the consistency of the slurry. Since a curve showing these values could be computed easily from the same data, it, too, is given in Fig. 3.

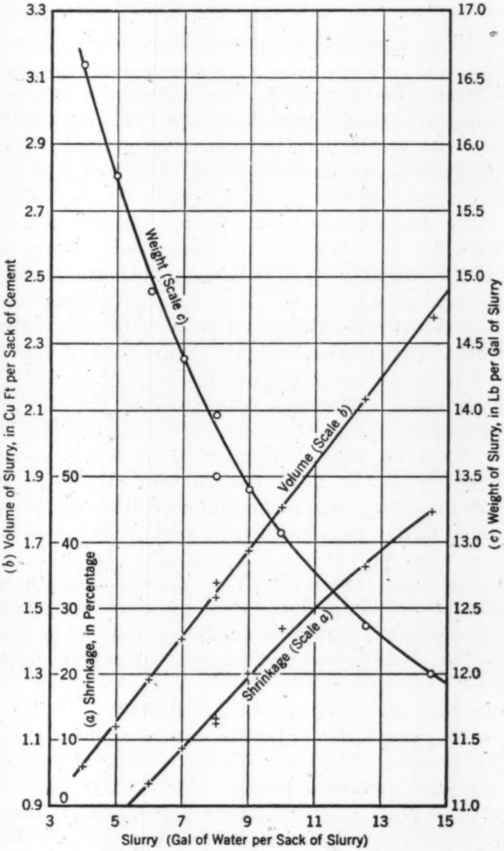


FIG. 3.—WEIGHT, VOLUME, AND SHRINKAGE OF CEMENT SLURRY

These slurries were then poured into test tubes about 5 in. high and allowed to stand for two days. The amount of shrinkage of each tube was measured and computed as a percentage of the initial height. These values also were plotted in Fig. 3. Extrapolation of this curve to zero shrinkage would give a 5.4-gal slurry—apparently containing just the amount of water necessary to hydrolyze the cement.

Sand Permeabilities

Under the heading, "Cement Penetration Through Sands: Materials," the reason why graded sands could be used to represent natural sand formations is given. For laboratory investigations this method of securing a wide range of characteristics is very convenient; but, to correlate these data on penetration with penetrations to be expected in natural sands, a better method than mesh size is necessary to describe the sand. Since the flow of slurry into a sand would depend on the hydraulic characteristics of the sand, the permeability would be the best property with which to correlate penetration. In truth, the penetration of a cement would depend on the ratio of the cement particle size to the sand pore size. However, the permeability is a measure of the sand pore size; and, since it is easy to determine, it is the most practical characteristic on which to base a correlation. From an academic point of view, the ratio of cement particle size to sand pore size is of interest; it is the key to further developments in the art of increasing or decreasing the cement penetration. Therefore, both qualities, the permeability and the sand pore size, were determined.

For most water-bearing sands, the simple variable-head apparatus of the type described by L. K. Wenzel, Assoc. M. ASCE, in 1942[6] will be found suitable for determining permeabilities. However, on this work sands were to be tested having permeabilities beyond the range of this instrument; and, therefore, a constant-head apparatus had to be used. A second important reason necessitating the use of a constant-head apparatus was that it furnished, with absolute certainty, data in the region of streamlined flow. To calculate the average size of pore from the flow of water through sand, Poiseuille's law may be used; but this law applies only to streamlined flow of fluids—not to turbulent flow. Consequently, it was necessary to find a way of determining whether the flow was streamlined or turbulent.

An examination of the literature shows that the well-known curve obtained by plotting the friction factor, calculated from Darcy's law, against the Reynolds number applies to flow through porous media as well as to flow through pipes. Several groups of investigators have worked on this same problem, Joseph Chalmers, D. B. Taliaferro, Jr., and E. L. Rawlins[7] obtained such a curve by computing the Reynolds number from the mean effective pore diameter. G. H. Fancher and J. A. Lewis[8] derived another curve of a similar shape by using the diameter of the average grain. With a knowledge of the mechanical analysis of the sand, the permeability and pore size, supposedly, could be computed from the latter curve.[8] Before this was done, however, the curve was transposed to the units used by Messrs. Chalmers, Taliaferro, and Rawlins. It was found that the two curves did not agree, and neither of the two curves could be used safely for the sand in these experiments. Nevertheless, the basic principle used by these investigators is sound; and, by obtaining such curves for the sands used in this work, the permeability and pore size of each sand could be computed.

---

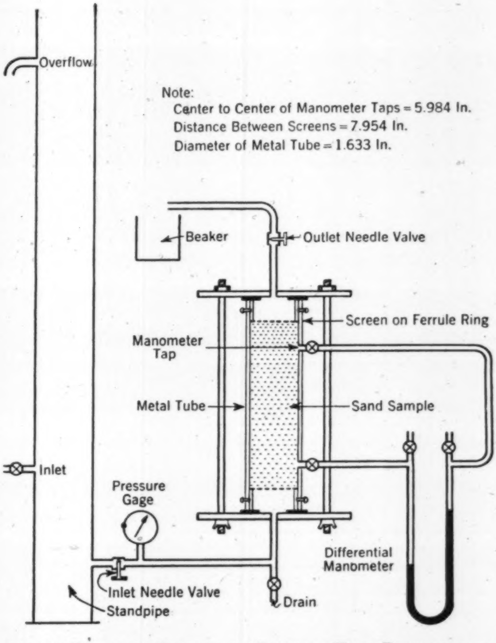
[6] "Methods for Determining Permeability of Water-Bearing Materials," by L. K. Wenzel, *Water Supply Paper No. 887*, U.S.G.S., U. S. Govt. Printing Office, Washington, D. C., 1942, p. 59.

[7] "Flow of Air and Gas Through Porous Media," by Joseph Chalmers, D. B. Taliaferro, Jr., and E. L. Rawlins, *Transactions*, A.I.M.M.E., Vol. 98, 1932, p. 388.

[8] "Flow of Simple Fluids Through Porous Materials," by G. H. Fancher and J. A. Lewis ,*Industria and Engineering Chemistry*, Vol. 25, 1933, p. 1145.

Apparatus.—To secure the data, the constant-head apparatus, shown in Fig. 4, consisting of a metal tube 12 in. long and 1.633 in. in diameter, was used. The sand sample was packed into this tube and was held in place by 65-mesh screens on ferrule rings placed at each end of the tube. Each ring was held in position by three sharp-pointed screws fitting into sharp-recessed holes. In this manner a constant volume could be secured easily for each sample. The tube was drilled and tapped for $\frac{1}{4}$-in. cone valves for manometer taps. The inner end of each tap was machined to the curvature of the tube, and wire cloth was soldered on to prevent the passage of sand. The final assembly of the taps was made carefully so that the wire cloth was flush with the tubing.

Procedure.—A weighed quantity of sand was placed in the tube and compacted by striking the tube with a hard rubber hammer. The quantity of sand added had to be just sufficient to support the top screen in its proper position. The screen could then be tightened in place. From the weight of sand, the specific gravity of the sand, and the volume of the container between the screens, the porosity of each sample could be computed.



Note:
Center to Center of Manometer Taps = 5.984 In.
Distance Between Screens = 7.954 In.
Diameter of Metal Tube = 1.633 In.

FIG. 4.—SCHEMATIC DIAGRAM OF CONSTANT-HEAD PERMEAMETER

After the standpipe was overflowing at a steady rate, water was admitted to the sand container very slowly so as not to trap any air bubbles.

An adjustment of the outlet needle valve allowed the desired rate of flow to be obtained and, at the same time, permitted the maximum possible pressure to be maintained within the tube. This maintenance of pressure within the tube reduced the possibility that the dissolved gases would separate from the water with a consequent air binding of the sand. The water passing out of the sand was timed and weighed to determine the rate of flow. The temperature of each run was measured to give the viscosity correction.

Notation.—The letter symbols in this paper defined where they first appear in the text, conform essentially with "Soil Mechanics Nomenclature" (ASCE *Manual of Engineering Practice No. 22*) and with American Standard Letter Symbols for Hydraulics (ASA—Z10.2—1942) prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942.

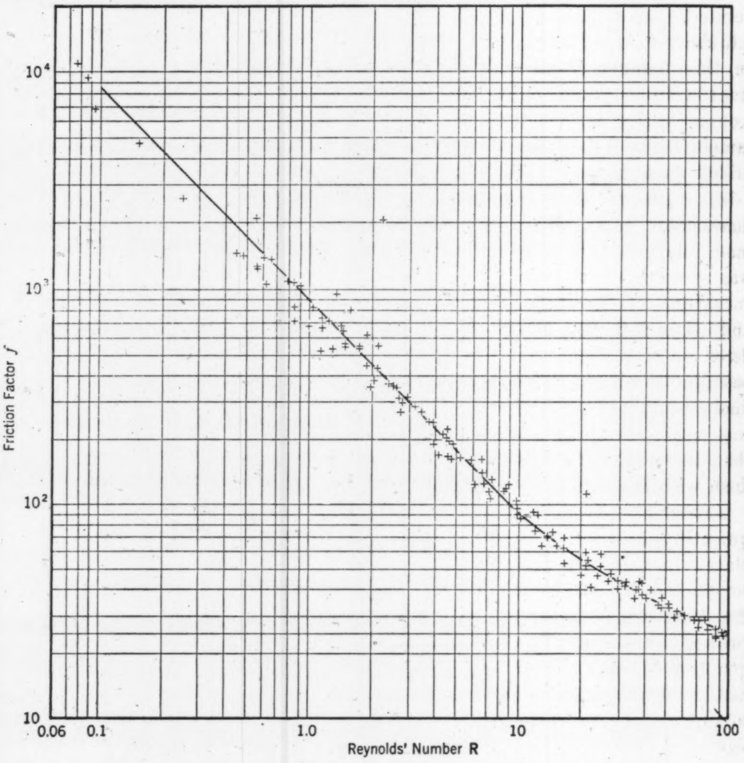


FIG. 5.—THE REYNOLDS NUMBER VERSUS FRICTION FACTOR

Data and Discussion of Results.—Six different sand sizes were tested and the value of the Reynolds number and the friction factor for each reading were computed. The plot of the two values on log-log paper gave a curve for each of the sands tested. However, all these curves were so close together that the average curve might well define them all. The plot of these two values is shown in Fig. 5.

The Reynolds number is defined as—

$$\mathbf{R} = \frac{D_g\, V\, \gamma}{\mu} \qquad \text{(1)}$$

—which is a pure number, so that any consistent set of units might well be used. If the English system is used, the values would be defined as: D_g is the diameter of the average sand grain, in feet; γ is the density of the fluid, in pounds per cubic foot; μ is the absolute viscosity, in pounds per foot per second; and V is the velocity of flow over the full area, in feet per second; that is:

$$V = \frac{B}{A_t\, t} \qquad \text{(2)}$$

in which B is the quantity of flow, in cubic feet; A_t is the cross-section area of the tube, in square feet; and t is time, in seconds. The friction factor, f, may be computed from the converted form of Darcy's law known as Fanning's formula, thus:

$$f = \frac{g\, D_g\, (p_1 - p_2)}{2\, L\, \gamma\, V^2} \qquad \text{(3)}$$

in which g is the acceleration of gravity, in feet per second per second;[9] $p_1 - p_2$ is the pressure loss due to friction, in pounds per square foot; and L is the length of core, in feet.

To determine the average grain diameter D_g, it was assumed that there was an even gradation of sizes between the two sieves used in the separation of the sand. The mean of the openings was found and used as the average grain diameter. In a case where a composite sand is used, the average grain diameter may be found by the method of Messrs. Fancher and Lewis using the formula:

$$D_g = \sqrt[3]{\frac{\sum (N\, D^3_s)}{\sum N}} = \sqrt{\frac{N_1\, D^3_{s1} + N_2\, D^3_{s2} + N_3\, D^3_{s3} + \cdots}{N_1 + N_2 + N_3 + \cdots}} \qquad \text{(4)}$$

In Eq. 4 D_s is the average diameter of the grain retained between two sieves and N is the number of such grains per gram. The values of N are given in Table 2 and were calculated from a sand density equal to 2.65 by assuming that the grains were spherical. Messrs. Fancher and Lewis proved the validity of this expression by actual microscopic count.

TABLE 2.—NUMBER OF GRAINS FOR ANY STANDARD TYLER SCREEN

Sand size (meshs per in.)	D_s, average diameter (cm)	Number (N) of grains per g
30 to 40	0.0442	8,230
40 to 50	0.0330	20,200
50 to 80	0.0227	61,900
80 to 100	0.0161	173,500
100 to 200	0.0111	530,000

Although it may at first appear incongruous to neglect the porosity completely in computing the velocity of flow (see Eq. 2), the method is found to give as good a correlation as the more complicated expression in which a porosity term is included in the velocity computation; thus:

$$V = \frac{B}{A_t\, t} \times \frac{1}{n} \qquad \text{(5)}$$

---

[9] "American Civil Engineers' Handbook," by Thaddeus Merriman and Thomas H. Wiggin, John Wiley & Sons, Inc., New York, N. Y., 5th Ed., 1941, p. 1323.

Furthermore, even the inclusion of the porosity in the calculation of the velocity would scarcely be more than an approximation of the true average velocity, for no exact method has yet been devised to measure the effect of the tortuous channel of flow or the effect of the many expansions and contractions.

When it became obvious that the curves for all sands could be represented by one line, it was no longer necessary to run tests on each sand. The permeability and pore size of any sand used in the tests on cementing could be computed from this curve.

In the region of streamlined flow the curve in Fig. 5 is a straight line with a slope of unity. When turbulent flow begins to occur, the curve gradually breaks away from the straight line. Since both the equation for permeability and the equation for pore size depend on measurements in the region of streamlined flow, only this part of the curve is of interest. A comparison between this curve and the one found by Messrs. Fancher and Lewis will show that the latter is somewhat higher in position. However, the curve found by the writer coincides (1) with the curve of T. H. Chilton and A. P. Colburn;[10] and (2) (when the data are converted to the proper units) with the curve of Messrs. Chalmers, Taliaferro, and Rawlins. An examination of the curve used by Messrs. Fancher and Lewis will show that there is a consistent difference in the friction factor which gives results exactly twice as large as those of the other workers. Thus, it appears that the difference may reflect an error in the computation of the friction factors by these writers.

TABLE 3.—SAND PERMEABILITIES AND PORE DIAMETERS

Sand size (meshes per in.) (1)	Average diameter (cm) (2)	Permeability (gal per day per sq ft) (3)	DIAMETER OF PORES (Cm)	
			Eq. 7 (4)	Corrected (5)
4 to 8.......	0.3531	127,200	0.0731	0.2535
8 to 10......	0.2007	41,400	0.0418	0.1452
10 to 14......	0.1410	18,800	0.02805	0.0972
14 to 16......	0.1080	12,100	0.02245	0.0778
16 to 20......	0.0912	8,500	0.0188	0.0652
20 to 28......	0.0711	5,200	0.01482	0.0514
28 to 30......	0.0546	3,050	0.01132	0.0392
30 to 40......	0.0442	2,015	0.00918	0.0318
40 to 50......	0.0330	1,125	0.00686	0.0238

The excellent agreement of results of data gathered by a number of different workers using widely varying granular materials and different types of apparatus indicates that this curve may be used to compute the permeability of various unconsolidated sands with a high degree of accuracy, merely from a mechanical analysis.

The permeabilities for the various mesh sizes of sand given in Table 3 were computed from velocity and pressure data taken to conform with the curve of Fig. 5. The coefficient of permeability k is given in units of gallons per day per square foot of area for a 100% gradient. All readings were adjusted to water at 60° F. To compute the coefficient of permeability, Darcy's law is used in the form:

$$k = \frac{\mu_t}{\mu_{60}} \frac{B}{A\,t\,(h/L)} \qquad \text{(6)}$$

---

[10] "Pressure Drop in Packed Tubes," by T. H. Chilton and A. P. Colburn, *Industrial and Engineering Chemistry*, Vol. 23, 1931, p. 913

in which μ_t is the absolute viscosity at the test temperature; μ_{60} is the absolute viscosity at 60° F; A is the total cross-sectional area of flow, in square feet; B is the volume of flow, in gallons; t is the time, in days; and h/L is the hydraulic gradient, in feet of water per foot of core length.

A comparison of the permeabilities of sand formations, secured from well logs with the permeabilities of Table 3, will give the equivalent sand size tested for cement penetration. On referring back to Table 1 for the data on the percentage of cement passed at various concentrations into this sand, the percentage of cement lost into the formation will be found. If the quantity lost is excessive, the slurry will have to be treated chemically to change the properties so that the loss may be reduced. On the other hand, if it is desired that the slurry penetrate into the sand as in the several cases previously described, the data will be of value in predicting the quantity of cement that will pass.

An idea of the range of permeabilities that may be expected in natural sands may be gathered from the paper by Mr. Wenzel:[6] Examples of sands tested show a range from 0.0002 gal to 90,000 gal per day.[11] The sand having a permeability of 90,000 was secured from a well on Long Island, New York, at a depth of about 200 ft. This high permeability is unusual, of course, but it is cited to show that such conditions may be encountered at times. Sands from wells at Salisbury, Md., with permeabilities from 252 to 1,190 are also reported by Mr. Wenzel. The author has tested several samples secured from outcrops in the vicinity of the Baltimore industrial area and has found sands with permeabilities of 1,000. Some samples taken from wells in the industrial area would have permeabilities of several thousand. One such sample was secured and tested to measure the passage of a 10-gal slurry. All the slurry passed through the sand readily. Whether this sample was representative of the sand in the well or whether its fines were washed off in the process of removal is unknown.

If highly permeable sands are encountered that might cause an excessive loss of cement, it would be necessary to find some method of plugging the pores. To attack the problem intelligently, it is first necessary to gain some idea of the magnitude of the pore opening. If the work of L. C. Graton and H. J. Fraser[12] is examined, it can be noted that the shape of a pore formed by spherical particles is far from a tubular opening. By systematic packing this shape may be varied to give a void ratio of from 26% to 48%. With natural sands, even more complex shapes may exist. Therefore, the only way to indicate the size of the pore is to compare it with the diameter of a capillary tube that has the same hydraulic characteristics as the sand pore—that is, with a straight tube which will pass the same amount of fluid with the same pressure loss as the tortuous path of the pore. Using the same data previously used to compute the permeabilities, the mean pore diameter may be calculated from Poiseuille's law; thus:

$$D_p = \sqrt{\frac{32\,\mu\,V\,L}{g\,e\,(p_1 - p_2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

in which D_p is the mean pore diameter, in feet; and e is the void ratio.

---

[11] "Methods for Determining Permeability of Water-Bearing Materials," by L. K. Wenzel, *Water-Supply Paper No. 887*, U.S.G.S., U. S. Govt. Printing Office, Washington, D. C., 1942, p. 13.

[12] "Systematic Packing of Spheres—With Particular Relation to Porosity and Permeability," by L. C. Graton and H. J. Fraser, *Journal of Geology*, Vol. 43, No. 8, 1935, Pt. 1, p. 785.

The values obtained by using Eq. 7 are presented in Col. 4, Table 3. However, for the correct application of Poiseuille's law to flow through porous media, it is essential to make a number of corrections in the data. The law applies only to the pressure lost in the viscous flow of a fluid through a straight circular section. In a porous medium the cross section of the flow channels is not circular but is irregular. The path is not straight but is sinuous with many alternate enlargements and contractions. C. S. Slichter,[13] using spherical particles, showed that the path of fluid flow was from 1.2 to 1.5 times the length of the core and that the average velocity of the fluid was about 1.8 times the velocity calculated from the rate of flow through the area of the voids. Messrs. Chilton and Colburn[10] claim that only about 20% of the pressure loss is caused by the friction of viscous flow and that the loss caused by the rapid expansion and contraction of the fluid is responsible for the major part of the remainder. Making these corrections on the length of path, the velocity, and the pressure would increase the value under the radical in Eq. 7 by about twelve times and the pore diameter by the square root of 12. The pore diameters corrected by this amount are given in Col. 5, Table 3. These corrected values are much closer to the size of openings that can be observed microscopically in core cross sections than are the uncorrected values.

The cement used in this work, when tested for fineness, gave 96% passing the 200-mesh sieve (0.0074-cm opening). The 28-mesh to 30-mesh sand was the largest size which did not pass a 14.5-gal slurry. Comparing the 96% cement size with the pore size of the sand shows the average pore to be five times the cement size. Taking into account the fact that the average diameter of the pore is being compared, and not the minimum constricted diameter, it appears that the cement ceases to pass into the pore when the pore becomes small enough for two or three particles to bridge across it.

From this evidence it appears that the passage of cement into a sand is limited by the ratio of the cement particle size to the sand pore size. A sand having a permeability of less than 3,000 gal per day per sq ft will not allow any measurable quantity of cement to pass into it regardless of the squeeze pressure or of any other factors.

SLURRY FILTRATION ON FINE SANDS

Thus far in the paper, the description has dealt only with factors affecting the penetration of cement slurry into sands. The next feature that requires attention is the problem of placing a cement slurry against a fine sand. The greatest percentage of wells in the Baltimore area probably penetrates sands with permeabilities less than 3,000. Hence, the cement particles will not penetrate into the sand but will build up a filter cake as the water is forced from the slurry. If the pressure gradient between the slurry and sand is high, it is quite possible that the water may filter from the slurry more rapidly than the slurry is being pumped into the well. In such a case, the slurry cake may clog the inlet tube, stop the pump, and leave an incomplete job.

---

[13] "Theoretical Investigation of the Motion of Ground Water," by C. S. Slichter, 19th Annual Rept., U.S.G.S., U. S. Govt. Printing Office, Washington, D. C., Vol. 2, 1898, p. 305.

To solve this problem, the effect of the pressure, the rate of pumping, the slurry concentration, and the sand permeability on the rate of formation of the slurry cake must be known. By correlating all these factors it may be possible to choose the correct slurry concentration, pumping rate, and squeeze pressure to insure a satisfactory completion of a grouting job for any well. When a cement slurry is placed in a well and the pressure in the slurry is greater than the pressure in the sand formation, the water will be squeezed out of the slurry leaving a filter cake on the wall of the well. In effect, this is a giant filter press and it is possible to simulate this press with a laboratory model. By starting with the simple concept of considering the filtration rate as proportional to the pressure divided by the resistance and considering the resistance as being made of two parts, the resistance of the filter cake and the resistance of the filtering medium, it is possible to arrive at a filtration equation. This equation will include all the known factors that will affect the filtration process. By measuring the constants that depend on the slurry and the filtering media, a solution of the equation will yield the result of a change in any one of the variable factors previously discussed. In this manner a precise measure of each of the variables may be obtained.

A good treatment of filtration equations and their derivations has been published by W. H. Walker, W. K. Lewis, W. H. McAdams, and E. R. Gilliland.[14] For the cement grouting of a well, a constant-pressure filtration equation will best fit the conditions usually existing in the field; and, therefore, the following equation for constant-pressure filtration[15] will be chosen for investigation:

$$\frac{p_f\, t}{W_t/A_f} = \frac{r''\, B_c}{2\, a}\, \mu\, a\, p^{k}_{f} \times \frac{W_t}{A_f} + r'\, \mu\, p^{m}_{f} \ldots\ldots\ldots\ldots (8)$$

in which p_f is the total filtering pressure on the cake, in pounds per square inch; W_t is the total weight of filtrate from the slurry, in pounds to time t; t is the total time of operation, in seconds; A_f is the total area of the filtering surface, in square inches; r' is a constant coefficient depending on the resistance of the filter bed; r'' is a constant coefficient depending on the specific resistance of the cake; B_c is the volume of the cake as it collects on the filter, in cubic inches per pound of filtrate; a is the concentration of the slurry, in pounds of solids, per pound of filtrate greater than that necessary to make a 5.5-gal slurry; m is the coefficient of plugging for the filter media; μ is the absolute viscosity of the filtrate, in centipoises; k is the coefficient of compressibility of the filter cake; and m is the coefficient of plugging of the filter media.

The value of the concentration factor a was computed quite empirically. Ordinarily, the factor a is defined as the pounds of solids per pound of filtrate; but, when this definition was applied to the data obtained in the tests, it was impossible to secure the same results with different slurry concentrations. In the experiments on the shrinkage of slurries, it was found that 5.4 gal of water per sack of cement was the quantity of water required to just hydrolyze the cement. If the value of a is based on the amount of water in excess of this quantity, which is the quantity causing the fluidity of the slurry, the a-value secured would be comparable to that which is ordinarily used for solids that do not react chemically with the filtrate. Instead of using 5.4 as the basis, the value was rounded to 5.5. This hypothesis applied to the data was found acceptable. Since the filtration equation itself is solved empirically, this definition of a is satisfactory.

---

[14] "Principles of Chemical Engineering," by W. H. Walker, W. K. Lewis, W. H. McAdams, and E. R. Gilliland, McGraw-Hill Book Co., Inc., New York, N. Y., 1937, p. 342.

[15] *Ibid.*, p. 347.

In the laboratory, only the values of $\left(\frac{r'' B_c}{2\,a}\right)$, k, r', and m (which will be constants in Eq. 8) must be measured to reduce the equation to a relatively simple form. It is then an easy matter to measure the effect of any one variable on the filtration process by changing the variable and solving the equation for the amount of filtrate lost. When the amount of filtrate lost is known, it is necessary only to subtract this quantity from the amount pumped in to determine the concentration of the remaining slurry.

In the filtration formula (Eq. 8), the term $\left(\frac{r'' B_c}{2\,a}\,\mu\,a\,p^k{}_f \times \frac{W_t}{A_f}\right)$ is a measure of the resistance of the filter cake. For a given slurry the values of $\left(\frac{r'' B_c}{a}\right)$ and of k should be constant. In practice, however, experiments show that, even with an incompressible material such as a "kieselguhr" slurry, the values vary slightly with the pressure and with the concentration of the slurry. With compressible materials the variation is often considerable.

The last term of Eq. 8 $(r'\mu\,p^m{}_f)$ is a measure of the resistance of the filter medium. Again, the terms r' and m should be constant for a given filter medium; but in practice they usually vary somewhat with the pressure and with the slurry concentration. The resistance of the filtering medium is not the resistance of the medium when no slurry is present, but rather the resistance after the solid particles have plugged the pores of the medium. Thus, the constant terms change in value with changes in the slurry and in the pressure.

Materials.—The cement and sand used in this work were the same as those described previously under the heading, "Cement Penetration Through Sands: Materials." The coarsest sand used for a filter medium was the 30-mesh to 40-mesh material, with a permeability of 2,015 gal per day per sq ft, and the finest material tested was a blended sand of from 30 mesh to smaller than 200 mesh. This sand had a permeability of 11.6 gal per day per sq ft. The other two sands used had permeabilities of 529 and 132.

The permeability of each of the sands was computed from the Reynolds number versus friction factor diagram, by the method previously outlined, and checked by a variable-head permeameter. The Reynolds number versus friction factor diagram was found to give excellent results.

Apparatus.—The apparatus used for these tests was the same as that used for the tests of cement penetration. The schematic diagram of the apparatus is shown in Fig. 1.

Procedure.—The procedure used in testing followed lines similar to those used in the tests on cement penetration. The pressure forced the filtrate from the slurry into the sand, replacing the water already there. The excess water passed out of the sand and was caught in a chemical graduate. A knowledge of the total proportion of filtrate that could be expected from the slurry was gained from experience, and an attempt was made to catch about one sixth of the filtrate in each of six graduates. The filling of each graduate was timed by a stop watch. In this manner it was possible to secure six points for each run so that a curve could be drawn. At the end of each run the apparatus was backwashed with water to remove all the slurry in the tubing.

Data and Discussion of Results.—Eq. 8 is a formula of the form $y = m x + b$ in which W_t varies with t as the pressure is held constant. If the data fit this equation, a plot of the values of $\frac{p_f\, t}{W_t/A_f}$ against the values of W_t/A_f should give a straight line of the type shown in Fig. 6. The slope of this straight line will be $(r''\, B_c/a)\ (\mu\, a/2)p^k{}_f$ and the intercept on the ordinate at $W_t/A_f = 0$ will be $r'\, \mu\, p^m{}_f$. When the value of the slope of the line is divided by $\mu\, a/2$, the remainder will be $(r''\, B_c/a)p^k{}_f$. A plot of the values of $(r''\, B_c/a)p^k{}_f$ against the values of p_f on log-log paper will give a straight line with a slope of k. Such a plot is shown in Fig. 7. The intercept of this line on the ordinate at $p_f = 1$ will be the value of $(r''\, B_c/a)$. This method of analysis furnishes the values of the two constants $(r''\, B_c/a)$ and k for the filtration equation.

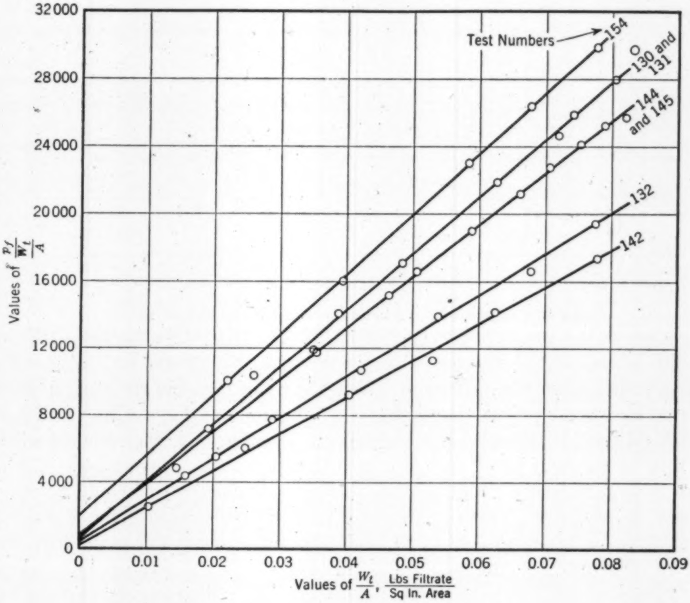


FIG. 6.—GENERAL FILTRATION CURVES

By a similar procedure it is possible to evaluate the constants r' and m. The values of $r' \mu p_f^m$, secured from the first plot of the type shown in Fig. 6, when divided by the viscosity, will give $r' p_f^m$. A plot on log-log paper of the values of $r' p_f^m$ against the values of p_f will give a straight line with a slope of m. A plot of this type is shown in Fig. 8. The intercept of the line on the ordinate at $p_f = 1$ will be the value of the constant r'.

When the constants for Eq. 8 have been found in this manner, their substitution will reduce the equation to a form which is relatively simple to handle.

The first experiments were conducted with a 30-mesh to 40-mesh sand and a 14.5-gal slurry. This very dilute slurry was chosen for the first tests because it was very fluid and easy to handle. Pressures from 5 lb per sq in. to 45 lb per sq in. were used in making the series of tests. For each run the pressure was kept constant and six readings on the quantity of filtrate with the corresponding time were taken. From these data, the series of curves of the type shown in Fig. 6 were drawn.

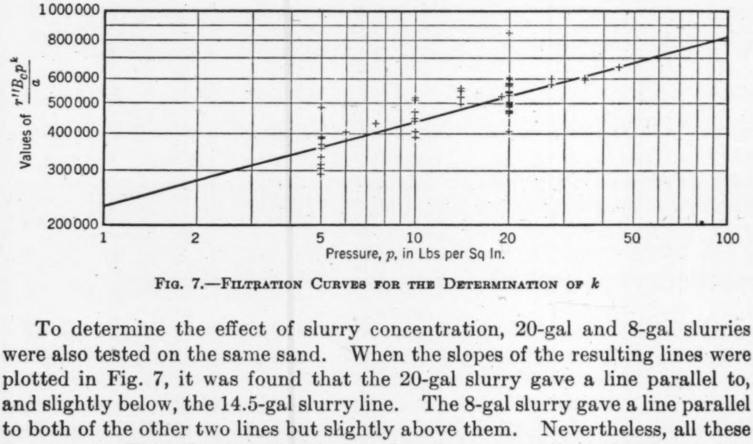


Fig. 7.—Filtration Curves for the Determination of k

To determine the effect of slurry concentration, 20-gal and 8-gal slurries were also tested on the same sand. When the slopes of the resulting lines were plotted in Fig. 7, it was found that the 20-gal slurry gave a line parallel to, and slightly below, the 14.5-gal slurry line. The 8-gal slurry gave a line parallel to both of the other two lines but slightly above them. Nevertheless, all these lines were sufficiently close together so that the average line could be used to represent any one. Furthermore, the plugging resistance given by the last term of Eq. 8 tends to counterbalance this difference. Thus, the final result will be reasonably accurate in any of the cases.

To secure the constants for the term involving the resistance of the sand, the intercepts on the ordinate at $W_t/A_f = 0$ of the original plot (Fig. 6) were divided by the value of the viscosity and plotted against the pressure readings, giving a series of curves of the type shown in Fig. 8. The intercept on the ordinate at $p_f = 1$ is the value of r' and the slope of the lines is the value of m.

The line for the 20-gal slurry is very nearly parallel to the line for the 14.5-gal slurry but is slightly above it. The value of m, therefore, is the same in each case; but the value of the intercept, r', is higher in the case of the 20-gal slurry. The 14.5-gal slurry line may be taken as representative of each case because the slight change in the constants of the slurry resistance term and the sand resistance term balance each other.

The next important step is the determination of the change in the resistance of the sand as the permeability is decreased. For this work three other sands with permeabilities of 529, 132, and 11.6 gal per sq ft were used. A 14.5-gal slurry was squeezed against each one and the filtration curves were drawn. The slopes of the "General Filtration Curves," as shown in Fig. 6, were corrected by dividing by $\mu/2$ and multiplying by a to obtain the term $\frac{r'' B_c p^{k}{}_f}{a}$. These corrected slopes are also included in Fig. 7 for the determination of $r'' B_c/a$ and k. The points define the same quantities as those previously obtained so that the cake resistance term is not affected. The filter media resistance, on the other hand, is changed considerably. The value of m is unchanged, but the value of r' increases with a decrease in the permeability. This change in r' was plotted on various types of graph paper, and it was found that, on semilogarithmetic paper (Fig. 9), it produced very nearly a straight line between the permeabilities of 2,000 gal per day per sq ft and 100 gal per day per sq ft. Somewhere between the permeabilities of 100 and 0 the straight-line relationship ceased to hold and the curve became asymptotic to the line of 0 permeability. Only one point in this region was found—the point at a permeability of 11.6. Further experience may indicate that a closer definition of this line is desirable, but for the present this paper will limit its scope to the part of the curve above a permeability of 100.

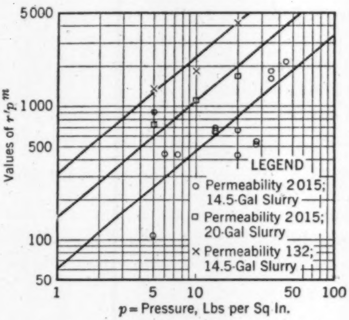


Fig. 8.—Filtration Curves for the Determination of m

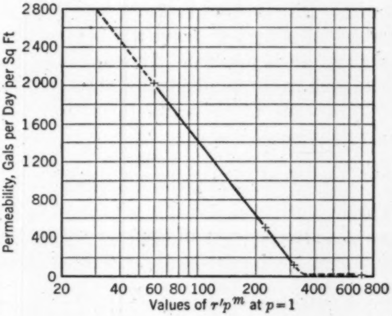


Fig. 9.—Effect of Permeability on the Value of r'

The straight-line section of the curve probably can be extended to a permeability of about 3,000. Two runs on a sand with a permeability of 5,200 were made, but about 4% of the cement passed through the sand. The filtration equation (Eq. 8) was not applicable, and no further tests in this region were made.

The slope of the curve of Fig. 7 gives a value of 0.278 for k and the intercept on the ordinate gives a value of 230,000 for $r'' B_c/a$. From Fig. 8, the slope of the line gives a value for m of 0.88. The intercept on the ordinate which is r' varies with the permeability and may be found from Fig. 9. For a sand with a permeability of 2,015, the value of r' is 60 and Eq. 8 becomes:

$$\frac{p_f t}{W_t/A_f} = 230{,}000 \left(\frac{\mu a}{2}\right) p^{0.278} (W_t/A_f) + 60 \mu p^{0.88} \quad \ldots\ldots\ldots (9)$$

To determine the importance of the various factors of Eq. 9, a typical hypothetical case of a cementing job may be assumed and the values determined. For example, assume that a 14.5-gal slurry is to be squeezed against a 10-ft salt-water formation. The hole is 18 in. in diameter and is cased with a 14-in. casing. The slurry temperature is 68° F, and the sand has a permeability of 2,015. It is desired to know how long it would take to squeeze enough water out of the slurry to leave a 5.5-gal slurry behind the casing if 30 lb per sq in. of pressure is applied. It was found that 50.8 sec would be required. The following example illustrates the solution of the problem:

The area of the sand formation is: $A_f = \pi \times 18 \times 10 \times 12 = 6{,}780$ sq in.; the viscosity of the water, $\mu = 1.005$ centipoises; and the slurry concentration factor, $a = \frac{94}{(14.5 - 5.5)\ 8.33} = 1.252$ lb of solids per pound of filtrate. The definition of a is the pounds of solid per pound of filtrate in excess of 5.5 gal of water per sack of cement. For this calculation a is the ratio of the weight of a sack of cement to the weight of 9 gal of excess water. The volume to be filled outside the casing $= \frac{\pi}{4}\left[\left(\frac{18}{12}\right)^2 - \left(\frac{14}{12}\right)^2\right] 10 = 7$ cu ft.

From Fig. 3, a 5.5-gal slurry occupies 1.22 cu ft per sack of cement. To fill 7 cu ft will require $7/1.22 = 5.74$ sacks of cement. A 14.5-gal slurry occupies 2.40 cu ft per sack of cement; and, therefore, the volume of filtrate which must be squeezed out is $(2.40 - 1.22)\ 5.74 = 6.76$ cu ft, or 423 lb. Therefore, $W_t/A_f = 423/6{,}780 = 0.0624$ lb per sq in.

The value of r' from Fig. 9 is 60. Consequently, $\frac{p_f}{W_t/A_f} = 230{,}000 \left(\frac{\mu a}{2}\right) p^{0.278}{}_f \left(\frac{W_t}{A_f}\right) + 60 \mu p^{0.88}{}_f$; $\frac{30 t}{0.0624} = 230{,}000 \frac{(1.005 \times 1.252)}{2} (30)^{0.278} (0.0624) + 60\ (1.005)\ (30)^{0.88} = 23{,}200 + 1{,}208 = 24{,}408$; and $t = 50.8$ sec.

If the sand permeability were assumed to be 11.6 instead of 2,015, the time required would be 77.6 sec. All values shown in the solution of the foregoing problem would remain the same except the value of r'. From Fig. 9, the value of r' would be 700. A sand with the permeability of 11 would be considered as a low permeability material. On the other hand, a permeability of 2,000 is very high. For comparison with these values, it may be stated that, to be considered a good aquifer, a sand should have a permeability of 250 or higher.

From these calculations it is evident that, within this range, the permeability of a sand is a factor of small importance in the filtration of a cement slurry.

Clay beds would probably absorb the filtrate so slowly that the area of clay surfaces may be neglected in the computations. Even if this were done, the time of filtration in most cases still would be so short that it would make little difference in the type of equipment necessary to place the slurry.

The concentration of the slurry is a factor of great importance when using either the tubing method or the casing method of cementing. If a slurry with a low-water content is used, the water in the slurry may filter out more rapidly than the pump can place the slurry. In this case, only a few feet of slurry may be placed behind the casing before the concentration increases to the point where it is no longer a fluid but a solid. The outlet of the grout pipe will then clog and the filter cake will build up within the pipe stopping the pump.

It becomes obvious that the pump used to place the grout must have a capacity sufficiently high to place the required volume of slurry before the concentration increases to the point where the slurry will clog the grout tube. In the previous example, a pump capacity of at least 124 gal per min would be necessary. The pump must place 13.8 cu ft of slurry, but the final volume is 7 cu ft. It should also be noted that doubling the height of hole to be cemented will double the pump capacity required, because the time computed by the filtration equation will still be the same.

The oil well industry learned long ago that high-capacity pumps were essential for the full completion of a grouting operation. Companies were formed which specialized solely in the cement grouting of wells. Operators learned that special equipment was necessary to grout a well successfully and called upon these concerns to furnish the service. In much of their advertising, great emphasis has been placed on the high-capacity, high-pressure pumps.

High pressures are neither necessary nor desirable for squeezing the excess water from the slurry. Even though high pressures are used in the oil well industry for squeeze cementing, it must be remembered that the total pressure in itself does not force the water from the slurry, but rather the pressure differential between the slurry and the sand formation. In the oil well industry, formation pressures of more than 1,000 lb per sq in. are fairly common. Before the slurry can be pumped into the well, the formation pressure as well as the friction loss of flow through the grout line must be overcome. For this reason, the pump pressure must always be higher than the effective squeeze pressure. From the example cited, it may be noted that a pressure differential of 30 lb per sq in. is sufficient to force the water from the slurry very rapidly. A low pressure differential, which increases the allowable time for pumping, would be more economical.

For an intelligent use of the filtration equation, the limits within which it applies must be understood. The equation is derived for a condition in which the slurry has immediate access to all parts of the filter medium, and the area of the medium does not change with time. Therefore, the equation can be expected to give accurate results when grouting is done by the principles of squeeze cementing. In this case, the salt-water formation is located, and the casing opposite the formation is bullet perforated. The slurry has immediate access to all parts of the formation through the many holes and is prevented from passing too far beyond the formation by cavings around the casing. Under these circumstances, it would be desirable to use a high concentration of slurry to be certain that the slurry cake forms rapidly and that the slurry does not pass down the hole away from the formation.

For the grouting of a long section of casing, Eq. 8 cannot be used. With this operation, the slurry will rapidly form a thick cake around the foot of the casing. If the pumping rate is low, the grout pipe will clog; but, if the rate is high, the velocity of flow may be sufficient to scour the cake around the casing and to keep the grout flowing up the hole. However, in its path upward the slurry will lose water to the adjacent cake so that, when it arrives at the top of the column, it will be thicker than the initial concentration. The value of a in the equation will then be different, as will also the value of W_t/A_f, because of the change in area with time.

SUMMARY

The contamination of underground fresh-water aquifers by salt-water infiltration has become a serious problem in the Baltimore industrial area and other areas. The salt water is leaking into both producing and abandoned wells from overlying formations into the fresh-water strata. This leakage generally occurs through the space between the well casing and the borehole and may be stopped by filling this space with cement slurry.

When a cement slurry is placed in a well in contact with a porous formation, the water in the slurry will be filtered from it when the pressure differential is toward the sand. If the coefficient of permeability of the sand is less than 3,000 gal per day per sq ft, the pores of the sand will be too small to allow the cement grains to enter. If the permeability is greater than this quantity, some of the cement may pass; but, since the water in the slurry may filter at the same time, the entire quantity of cement will not always move into the formation.

When a sand has a permeability of less than 3,000, the pressure will force the water out of the slurry leaving a filter cake upon the sand surface. To give an exact description of the conditions at any instant during the filtration process, the filtration equation for cement slurries was developed.

A study of the sands having permeabilities greater than 3,000 showed that three factors mainly controlled the amount of cement penetrating into the formation:

(1) The ratio of the pore size to the cement particle size;
(2) The slurry concentration; and
(3) The manner in which the pressure was applied.

Calculation of the pore diameters showed that the pore of the largest sand which would not permit the passage of a cement slurry was small enough so that two or three cement particles could form a stable bridge across the opening. No amount of pressure could be expected to force the cement into this sand. When the pores are larger, some of the cement will pass into the sand. If the slurry is thin, all the cement may penetrate; but, if the slurry is thick, the water may filter from it too rapidly to carry along all the cement particles and a filter cake will result. By using this principle, a slurry cake may be formed on a sand with a permeability as high as 127,000.

The speed with which the pressure is applied will determine the amount of cement that will penetrate. If the pressure is applied instantaneously, a greater quantity will pass than would pass if the same pressure were applied slowly. With a slow rate of pressure application, the slurry may cease to penetrate long before the maximum pressure is reached, so that high-pressure differentials are of small importance in such a case.

Under no circumstance is there any justification for using a blanket rule for the selection of the slurry concentration as has been attempted by many writers. In some cases the slurry may be very dilute, the excess water serving only as an agent to carry the cement into position, after which it is squeezed out and the cement particles are compacted. The initial slurry concentration will have no bearing on the volume, strength, or permeability of the final product. In other cases, it may be desirable to grout off a given formation without allowing the slurry to spread over other strata. In these circumstances, it would be desirable to use a thick slurry and a high squeeze pressure.

Acknowledgments

The investigations reported in this paper were made in the laboratories of The Johns Hopkins University at Baltimore. The writer is indebted to Abel Wolman, M. ASCE, professor of sanitary engineering, The Johns Hopkins University, for advice and general supervision in carrying out the investigation. The writer is also indebted to Robert R. Bennett and R. R. Meyer, Geologic Survey, United States Department of the Interior, for information on cementing practices in the Baltimore area and in the western oil fields. Special appreciation is due to John C. Geyer, associate in civil engineering, The Johns Hopkins University, for suggesting this problem.

DISCUSSION

James B. Hays,[16] M. ASCE.—Some valuable information on the gradually increasing knowledge of grouting is contributed by this paper. The forcing of cement grout into sand has generally been considered as impossible. The author has thrown light on the subject and has indicated the approximate limiting point. The principal difficulty is to adapt the laboratory results to actual conditions as found in the field. Other considerations that must generally be taken into account are variations in gradation of the material in the layer to be grouted.

One of the difficulties encountered in using thin cement grouts is the fact that sedimentation is quite rapid in such mixes as compared with thick slurries. This is due to the settling out of the coarser particles where the slurry is not moving rapidly or is not otherwise being agitated. Therefore, a natural suggestion would be to try the use of cement screened through a fine mesh—say, the 200-mesh screen. This was done at the Owyhee (Oregon) and Boulder (Arizona-Nevada) dams for grouting contraction joints. Another suggestion along the same line would be to add a dispersion agent to the cement and note the effect in grouting. Any small increase in penetration under conditions described by the author would be highly beneficial.

The writer has always found it desirable to conduct such tests under conditions as nearly normal as possible. As described by Mr. Machis, there was no water in the column above the surface of the sand when the grout was introduced. The model might have worked differently if it had been filled with water. With a column of water 100 ft long (a common condition in practice) the coarse particles of cement would have dropped to the bottom ahead of the main stream of grout. If it is possible, the sand bed should be set vertically for the test. This is a difficult matter for the laboratory.

In the field, one of the most difficult problems would be to isolate the salt-water bearing zone so that the necessary grout pressure could be applied without its escape to other zones. Grout has a tendency to follow planes or zones of weak bond such as exists between steel and clay or stone and clay.

If these zones of salt water are located during the drilling of the well, there should be better means of cementing off the unwanted section than by grouting. Such methods should involve the use of certain processes well known to the driller. Deep beds of fine sand under dams could be grouted through well holes if cost were a secondary consideration; but the process would require that the grout penetrate at least several feet—rather than inches—from the hole.

If the author has the opportunity to apply his laboratory results to a field condition, the results would be interesting.

William S. Foster,[17] Assoc. M. ASCE.—While a sanitary engineer may have been searching diligently for cross connections with an unapproved water supply in some remote corner of his distribution system, a more serious "cross connection" may have escaped notice under his very nose. This is the challenging observation presented in the Machis' paper. In well drilling the customary practice is to seal off the undesirable aquifers by producing a barrier at the casing. Nevertheless, as Mr. Machis states, and as the experience at Baltimore (Md.) shows, this undesirable water can find its way down the outside of the casing and thereby pollute the new aquifer. At Baltimore the pollution consisted of salt water which renders the supply unfit for further use.

---

[16] Chf. Engr., Comm. on Palestine Surveys, New York, N. Y.

[17] Eng. Editor, *The American City Magazine*, New York, N. Y.

Mr. Machis' system, borrowed from the petroleum industry, consists of pumping a cement slurry around the outside of the well casing and forcing it under pressure against the wall of the well. In coarse sands and gravels the cement will penetrate until the internal friction of the sand grains halts the intrusion of the slurry and allows it to block and seal off the aquifer. In fine sands the slurry will merely collect like a filter cake, and the pressure applied will squeeze out the excess water leaving a firm, strong cement barrier with a consistency of from 4 gal to 5 gal per sack. In its underground location, this barrier will have an even temperature; it will be constantly in contact with moisture; and, probably, will not be subjected to variations in pressure. Consequently, for practical purposes, it should enjoy virtually an indefinite life. As Mr. Machis states, by varying the consistency of the slurry, one should be able to seal long lengths of the well casing or to apply local applications of grout and seal off a limited area.

The fear that this application of a thin grout (often preceded by water flushed up and outside the well casing in order to establish circulation) may disturb the sand and gravel strata and make them slough against the side of the well casing, thus blocking the operation, seems unfounded. This is particularly true for wells drilled by the rotary process in which the wall of the wells is held firm by the driller's mud. In most cases the pressure of the overburden and the precipitated minerals will consolidate the sands and gravels sufficiently to make them hold their positions firmly. Actual drilling logs show that clays and shales with less internal strength slough away more readily than do sands and gravels.

One question that arises in the mind of the uninitiated (and most engineers who are not petroleum specialists fall in that class) is whether some materials other than cement might give better results. Of course, a beam or briquette made of neat cement will give an excellent account of itself on a strength basis; but it does insist on shrinking, even under favorable conditions. The writer recalls at one time helping to waterproof a curing room in a laboratory in which both temperature and humidity were to be maintained within close limits. A neat-cement layer of grout applied to the brickwork in the room shrank and cracked sufficiently to lose its effectiveness. It may be that shrinkage would not be a factor in the case of well grouting; but water has an admirable trait of persistency and an ability to enter any small opening and develop it to amazing proportions. It would seem that some other product like asphalt or a dense impervious clay, which does not shrink when temperature and humidity are constant, would do the same sealing operation equally well and would eliminate this cracking danger.

Another point that immediately comes to the attention of the practicing engineer is how he can apply this useful grouting operation in his own installations. Evidently this sealing work is not a task for amateurs. Since it is being performed in the petroleum field, there must be concerns that would be willing to do the same work in the water works field. Consequently, the water works engineer would like to know what specifications he should call for in order to encourage these concerns to bid, and at the same time protect the interests of his client.

On the basis of studies by Mr. Machis, the painstaking engineer may very well insist that, when sealing operations are required in well construction, they be accompanied by grouting the well casing in that particular area.

FREDERICK J. SMITH,[18] JUN. ASCE.—Petroleum engineers have long been concerned with the problem of leakage of fluids behind the casing of oil wells. At times their problem has been the same as that described at Baltimore, Md., in Mr. Machis' paper—namely, downward percolation of water between the casing and the well bore. At other times the problem has been the leakage of high-pressure gas or oil from the producing formation, upward between the casing and the borehole to shallower sands or to the surface. When such a leak reaches the surface around the casing there is danger of a "blowout," which results in the development of a crater around the well as the soil and subsoil is blown away by the escaping oil or gas. There is also the possibility of of the ultimate loss of the well by rupturing of the surface connections as their support is removed, as well as the danger that the escaping oil or gas may catch fire, thus adding to the general chaos.

Various methods have been developed by petroleum engineers to combat this leakage between the casing and the wall of the borehole. Some of these methods are applicable in the water works field, and others may be adapted for such use. The writer would like to submit the following suggestions, based on current petroleum engineering practices, as a solution of the leakage problem outlined in Mr. Machis' paper. It is assumed that drilling will be done by the rotary process, although these suggestions can be adapted for use in cable tool holes.

New Wells.—When a water well is drilled in an area where contaminated waters are found at shallower depths than the potable water to be exploited, the following program is suggested:

The well should be drilled to the water sand, but should not penetrate more than 3 ft or 4 ft into the sand before the protective casing is set. As shales and clays are prone to slough or "cave," a caliper survey of the borehole can be made before placing the casing in the borehole; the number and size of the "caves" can be established; and an estimate of the amount of cement needed to fill the annular space between the casing and the wall of the borehole can be prepared. Also, in light of the "caving" nature of most shales, it would be wise to have the bottom of the casing at least 1 ft below the top of the sand to prevent subsequent caving, and the development of leakage around the casing shoe, after completion of the well. The casing should be run to bottom and cemented using the "casing method" quoted by Mr. Machis (see heading, "Grouting New Casing: Casing Method of Cementing"). Enough cement grout can be pumped, by normal oil-field cementing practices, to establish circulation of the cement and to get "returns" of the cement at the surface outside the casing if so desired. During placement of the cement, the casing should be setting on the bottom of the hole. As soon as all cement has been pumped into the hole, the casing should be picked up and suspended with the bottom of the casing 1 ft below the top of the water sand.

---

[18] Geologist, Sinclair Prairie Oil Co., Seminole, Okla.

After sufficient time has elapsed for the grout to set (from 24 hours to 72 hours, depending on local conditions), the cement remaining in the casing is drilled out, and the hole is opened 1 ft below the casing shoe (which still leaves from 1 ft to 2 ft of cement in the hole). The next step is to determine whether or not any leakage exists around the casing shoe, due to a faulty cementing job. This determination can be made by using any one of several available formation testers[19] with the packer set in the bottom joint of the casing. The testers consist of a packer and a series of valves which are run into the hole on the bottom of the drill pipe or tubing. This is the standard "drill stem test" familiar to petroleum engineers. A valve, below the packer, excludes fluid from the drill pipe, and maintains atmospheric pressure inside the pipe as the testing tool is lowered into the hole. This valve is opened after the packer is set, and the resulting pressure differential between the formation pressure of the zone below the packer and the atmospheric pressure in the drill pipe allows the formation fluid (if any is present) to flow into the drill pipe above the packer. The valve automatically closes when the packer is unseated, trapping the fluid in the drill pipe; the pipe is pulled from the hole, and the amount and the nature of the fluid recovered are checked by the engineer supervising the test. A pressure recording device can be used with this testing tool to record the shut-in rock pressure, and the flowing pressure of the zone can be tested.

If the drill stem test does not recover water, the cementing job can be considered a success, and the well can then be completed using the standard practices for the area. If water is recovered on the test, however, it should be analyzed to determine if it came from the fresh-water sand below the casing shoe, or if it is contaminated water that is leaking between the casing and the wall of the borehole—indicating a poor cement job. in the latter case, the bottom 10 ft to 20 ft of the casing should be gun perforated (or bullet perforated) and cement grout should be squeezed through these perforations to seal the leakage behind the pipe. The casing shoe can again be tested following the squeeze cementing. This process of squeezing and drill stem testing should continue until a shutoff is achieved. When no further leakage is apparent the well can be completed.

Old Wells.—Three typical cases applying to old wells can be discussed.

Case 1. Well Completed in a Potable Water Sand and Later Contaminated by Leakage from Behind the Casing.—The first step in the elimination of this type of leakage should be to fill the open hole below the casing with fine sand and gravel in order to protect the face of the producing sand from any possible contamination by the cement grout. To eliminate the leakage, the bottom joint of the casing should be gun perforated, and cement grout squeezed behind the casing through the perforations. When the grout has set, the cement remaining in the casing can be drilled out, and the cement job tested as previously described. This process may have to be repeated one or more times until the leakage is completely sealed off. The sand and gravel can be removed from the hole after the shutoff is obtained, and the well then put back in production.

---

[19] "Formation Testing and Testing Equipment," by J. C. Padgett, *The Oil Weekly*, March 24, 1947, pp. 18-21.

Case 2. Leak in the Casing.—If a leak in the casing is suspected, it can be located by running a "water survey" or "electric pilot." The writer is not familiar with the details of this type of survey, although the service has been available to the petroleum industry for some time.

When the leak in the casing is located it can be eliminated in the following manner: A drillable cement retainer or plug should be set in the casing below the leak; the casing is gun perforated in the zone of leakage, and then the perforated section is squeeze cemented. When the cement has set, the cement remaining in the casing should be drilled out to a point below the leak, but above the cement retainer, and the zone of leakage drill stem tested as described herein under the heading, "New Wells." Perforating and cementing should be repeated, if necessary, until no further leakage is noted. After all leakage is sealed off, the cement retainer can be drilled out, the well cleaned out to bottom, and then put back in production.

Case 3. Squeeze Cementing of Shallow Sands Behind the Casing.—At times it may seem advisable to squeeze some cement into the shallow contaminated sands covered by the casing. This would be especially true in plugging an old well where it is impossible to reclaim the casing. These sands can easily be located if a geologist's sample log or an accurate driller's log of the hole is available. Where such logs are not available, or where the records are very poor, the sands behind the casing can be located by a "gamma-ray" survey, a relatively new process that is being used in the petroleum industry. These surveys are based on the natural radioactivity of the sedimentary rocks, the survey instrument recording the intensity of the "gamma-ray" radiation of the formations as the instrument is withdrawn from the borehole. This type of survey is advantageous because it can be run in any type of well fluid, and it will locate the various formations accurately in both cased (one or more strings of cemented casing) or uncased holes. The disadvantage of these surveys lies in the fact that some idea of the normal sedimentary sequence of the area is needed to interpret these logs properly, as sands and limestones give similar reactions. This disadvantage can be overcome (at extra cost) by making a "neutron" survey in connection with the "gamma-ray" survey, and properly correlating the two surveys (done by the service company). The "neutron" survey, in contrast to the "gamma-ray" survey, is based on the intensity of radiation from the formation as a result of bombardment by neutrons from a radioactive source suspended below the survey instrument.

It has been found that the best results are obtained, in petroleum geological studies, when the radioactivity logs are used in conjunction with any geologist's or driller's logs that are available from the same area. The aforementioned inability to differentiate between a sand and a limestone should not be too big a disadvantage in the water works field, since a porous limestone is potentially as dangerous as a sandstone as a source of contaminated water.

After the sand bodies (or limestones) have been located, they can be grouted by squeeze cementing following the procedure outlined for case 2.

The Use of Plastics to Seal Contaminated Sands.—Many oil companies have been experimenting with plastics; they have been trying to "shut off" bottom hole water by introducing liquid plastics of various types into the borehole and allowing them to harden. Some good results have been reported from such experiments, and these materials might prove very useful in the water works field. Each contaminated sand encountered in drilling a new well could be impregnated with plastic as soon as it is drilled, thereby eliminating the source of future pollution.

Service Companies.—Communities located in the areas of active petroleum exploitation are familiar with most of the service companies engaged in the activities described herein. Their characteristically colored cars and trucks are being found throughout a wider and wider area as the search for oil expands.

Locating Cement Squeezed Behind Casing.—By laboratory experiments Mr. Machis has determined the effects of such physical constants as porosity, permeability, pressure, and water-cement ratio on the problem of forcing cement grout into a sand formation. Where the porosity and permeability of a sand have been determined from cores, it should be possible, using Mr. Machis' results, to determine the theoretical quantity of cement squeezed into a subsurface sand body under set conditions of pressure, type of slurry, and time and rate of pressure application. No accurate method has been developed to check, below ground, the theoretical results to be expected from a subsurface squeeze job.

It is possible, however, to determine the vertical extent and the mass distribution of the cement following squeeze cementing. This determination is made possible by adding a radioactive mineral to the dry cement to be used, and by "gamma-ray" surveys of the borehole before and after the squeeze job. The radioactive mineral is usually "carnotite," a potassium-uranium-vanadium oxide, and is added to the cement in the ratio of from ¼ lb to 1 lb of carnotite per sack of dry cement. Carnotite, because of its nearly colloidal size and specific gravity, remains in a fairly stable mixture with the cement slurry (usually 15.5 lb per gal), and does not tend to settle out of the slurry.

In practice a "gamma-ray" survey of the borehole is made prior to the squeeze job, and is used as the reference line for all future surveys. The perforated zone is squeeze cemented, using the radioactive cement, and a second "gamma-ray" survey is run after the hole has been cleaned out. By superimposing the second survey upon the original, the vertical distribution of the radioactive cement behind the pipe can be determined by the sharp changes in the "gamma-ray" intensities between the two surveys at the upper and lower limits of cement travel.[20] Some idea of the mass distribution of the cement can be obtained from the comparison of the two surveys when it is understood that the greater the mass of treated cement at any point, the greater will be the "gamma-ray" intensity. In all such comparisons of "gamma-ray" surveys, in connection with radioactive squeeze cementing of a sand section through perforations, that the writer has seen, the "gamma-ray" intensities were highest opposite the sand section, and tapered off to the original values a short distance up or down the hole. The concentration of the "gamma-ray" intensities is due to (*a*) a filter cake mass being formed in the annular area between the casing and the wall of the borehole, or (*b*) to penetration of some of the solid particles of cement and carnotite into the sand, or (*c*) to a combination of both.

---

[20] "Some Practical Aspects of Radioactivity Well Logging," by W. J. Jackson and J. L. P. Campbell, *Technical Publication No. 1923*, A.I.M.M.E., pp. 14–16.

In some extreme cases two or more "radioactive" squeeze jobs have been performed opposite the same zone. A comparison of the "gamma-ray" surveys made after each squeeze job showed a progressive increase in "gamma-ray" intensities for each succeeding survey, indicating the probability that at least a part of the radioactive cement slurry had penetrated the sand body during each squeeze job.

This combination of radioactive cement and "gamma-ray" surveys does not give definite values to the amount of cement forced into a formation; nor does it supply the depth of penetration of the cement into the formation. The method does give an answer to the problem of the vertical travel of the cement behind the casing, and an approximate location of the main mass of the cement forced into the annular area between the casing and the wall of the hole.

Summary.—Mr. Machis' paper is an excellent addition to the literature on grouting of sands and gravels. His experiments have supplied data of much value for the solution of grouting problems. Mr. Machis has noted (see heading, "Repair of Leaking Wells"):

> "The highly sucessful utilization of this process in the oil well industry warrants close study for possible adaptation to the water well industry."

Therefore, the writer has tried to outline these oil field practices in the hope that they will aid in the practical application of the results of Mr. Machis' experiments. A positive method of locating the cement squeezed into a formation through perforations is also included.

ALFRED MACHIS,[21] JUN. ASCE.—The author is indebted to those who have taken the time to critically analyze and discuss this paper. He believes that the contribution made by Frederick J. Smith, Jun. ASCE, is especially appropriate, since it serves to focus the attention of the civil engineer on advances made in an allied field relating to a problem very similar to his own.

Conservation of a natural resource of such immense value as ground waters should be of primary importance to the engineer. Large investments are made annually in the construction of water wells and plants to treat the waters obtained from these wells. Experiences in the oil industry indicate that the leakage of an undesirable fluid down the outside of a well casing may be more economically controlled by grouting than by depending upon the separation of the fluids in refining. Similarly, it may be more economical to prevent the leakage of mineral-bearing waters into fresh-water aquifers by grouting than it would be to sustain the economic losses necessitated through the use of mineral-bearing waters.

The large number of successful results secured in the oil well industry indicates that the problem of grouting outside a well casing is not impractical. The author's attention was first directed to this problem of water contamination by an unsuccessful attempt to prevent the leakage of salt-bearing waters down the outside of a well casing into a fresh-water aquifer. Grouting was performed without a thorough understanding of the factors involved. The experimental data of this study were gathered to help provide the engineer with a basis upon which to form a logical solution to the problem.

---

[21] Research Associate in Eng., The Johns Hopkins Univ., Baltimore, Md.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2334

THEORY OF INELASTIC BENDING WITH REFERENCE TO LIMIT DESIGN

By ALEXANDER HRENNIKOFF,[1] ASSOC. M. ASCE

WITH DISCUSSION BY MESSRS. GEORGE WINTER, D. C. DRUCKER, MICHAEL R. HORNE, E. P. POPOV, BERNARD A. VALLERGA, AND ALEXANDER HRENNIKOFF.

SYNOPSIS

For the most part this paper contains the description of a method for analyzing statically indeterminate flexural structures loaded beyond the elastic limit or structures comprising material that does not obey Hooke's law. The relations between the unit strains and the unit stresses (both normal and tangential), bending moments, angle changes, and deflections, produced under these circumstances, are derived. The effects of removing a load and of moving a load along the span are also considered. Furthermore, the method is applied to mild steel I-beams, and is illustrated by a number of examples bearing on the "Theory of Limit Design" presented by J. A. Van den Broek,[2] M. ASCE. Brief discussion of this theory together with observations concerning its validity, based on the results obtained, is included in Section 18.

1. INTRODUCTION

The method proposed in this paper entails the construction of several curves derived from the stress-strain curve of the material of the structure, making proper allowance for the shape of the cross section of its members. Once the requisite curves are constructed, the indeterminate moments are found by trial. The labor involved in such an analysis is not prohibitive, although it is greater than that involved in the usual elastic analysis. A new set of curves is needed if the material of the structure or the cross section of its members is different.

For the most part, this method, with its illustrative numerical solutions, is novel, although a simpler case of the same problem, involving a rectangular beam, has been treated, in a somewhat different way, by S. Timoshenko.[3]

NOTE.—Published in March, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Associate Prof., Civ. Eng., Dept. of Civ. Eng., Univ. of British Columbia, Vancouver, B.C., Canada.

[2] "Theory of Limit Design," by J. A. Van den Broek, *Transactions*, ASCE, Vol. 105, 1940, p. 638.

[3] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1936, p. 45.

The method is based on the assumption that the cross sections of the members remain plane throughout the entire range of flexure. The same assumption has also been used by Professor Timoshenko[3] and by A. Nádai;[4] and it has some experimental corroboration.[5] In certain restricted regions the sections undoubtedly become warped by shear, but on the whole the assumption is believed to be substantially true.

In addition to this assumption the theory is restricted by the following limitations:

1. The structure consists of prismatic members with I-beam, channel, or rectangular sections, symmetrical about their neutral axes;
2. The stress-strain curve is the same in tension and compression (the only limitation of the shape of this curve);
3. The bending moment diagram of the loaded structure is assumed to be bounded by straight lines (a curved diagram being approximated by a polygonal shape);
4. The effect of normal forces is ignored;
5. Deformations produced by shearing forces are disregarded;
6. Instability is not considered to be a factor; and
7. Deformations of the structure are assumed small.

Some additional assumptions, useful in expediting the work but otherwise avoidable, will be stated subsequently.

Notation.—The letter symbols in this paper are defined where they first appear, in the text or by illustration, and are assembled for convenience of reference, in the Appendix.

2. Unit Strain—Bending Moment Relation in a Rectangular Beam

The functional relationship between the unit stress, σ, and the unit strain, ϵ, for the material of the beam must be stated explicitly in one of the following forms: As an equation, $\sigma = f(\epsilon)$; as a graph, such as Fig. 1; or as a table of corresponding values of the two variables.

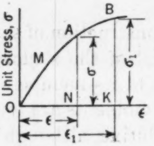


Fig. 1

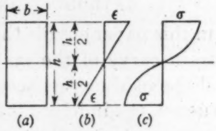


Fig. 2

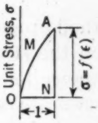


Fig. 3

Let a rectangular beam with the cross section shown in Fig. 2(*a*) be subjected to flexure. According to the stated assumptions, the unit strain diagram is linear (Fig. 2(*b*)), with zero strain at the center of the section and with equal strains ϵ at the outside fibers. It follows that the diagram of unit stress (Fig. 2(*c*)) consists of two equal parts resembling the stress-strain curve (Fig. 1) to point A, with the abscissa ϵ. The base ON = ϵ in Fig. 1 is reduced in Fig. 2(*c*) to the length $\frac{1}{2}h$. When the strain in the outside fibers of the beam is increased to a value ϵ_1, in response to a greater bending moment, the stress curve in Fig. 2(*c*) will resemble a greater length of the stress-strain curve OMBK (Fig. 1), whose base ϵ_1 is again foreshortened to the same length, $\frac{1}{2}h$.

---

[4] "Plasticity," by A. Nádai, McGraw-Hill Book Co., Inc., New York, N. Y., 1931, p. 164.

[5] "Mechanics of Creep," by J. Marin, *Transactions*, ASCE, Vol. 108, 1943, p. 459.

To find a relation between the bending moment M and the corresponding unit strain ϵ in the outer fibers of the beam, it is necessary to introduce a variable m_1, defined[3,4] as the statical moment, about the σ-axis, of an area under the stress-strain curve, taken to a variable point A on the curve, provided however that the variable base ϵ is reduced to unity (Fig. 3). The moment about the σ-axis of the area under the curve OA in Fig. 1 is evidently equal to $\int_0^\epsilon \epsilon\,\sigma\,d\epsilon$. Reducing the base ϵ to a unit length changes both the elements of area and their lever arms ϵ times; therefore, the mathematical expression defining m_1 is as follows:

$$m_1 = \frac{1}{\epsilon^2}\int_0^\epsilon \epsilon\,\sigma\,d\epsilon \qquad (1)$$

The variable m_1 may be considered as a function of ϵ. In cases when σ and ϵ are not related by a mathematical equation the value of the integral in Eq. 1 must be determined by summation, dividing the area under the stress-strain curve into a large number of sufficiently narrow strips parallel to the σ-axis.

The requisite relation between the bending moment M and the unit strain ϵ in the outer fibers of the beam is found by equating the internal moment of stresses and the external bending moment. It may be observed that for the same unit strain, ϵ, the internal moment is proportional to the breadth b and to the square of the depth h of the section, the coefficient of proportionality being simply related to m_1 in the following manner:

$$M = 2\,m_1\,b\,(\tfrac{1}{2}h)^2 \qquad \text{or} \qquad m_1 = \frac{2\,M}{b\,h^2} \qquad (2)$$

In the analysis of statically determinate rectangular beams, when the bending moment M and the dimensions of the cross section are known, Eq. 2 determines the value of the variable m_1. The unit strain ϵ can then be found from a table or graph representing the relation between m_1 and ϵ expressed by Eq. 1, and the unit stress σ in the outer fibers of the beam, by the stress-strain relation.

3. UNIT STRAIN—BENDING MOMENT RELATION IN AN I-BEAM OR A CHANNEL BEAM

It is possible to derive a rigorous expression for a function m, applicable to I-beams and channel beam sections, which can be used in the same manner as the function m_1, defined in Section 2. This function would contain two independent parameters, a feature restricting the field of its usefulness. For this reason, it has been considered wise to widen the scope of applicability of the function m by ignoring the variation of unit stress over the areas of flanges. An analogous approximation is often made in the elastic range, which is essentially the so-called flange-area method of design applied to plate girders. The error involved in this approximation in the region above the elastic limit is smaller than that involved in the elastic range. When this approximation is made, the web of the beam is assumed equal to the distance between the centers of flanges, and the areas of flanges are assumed concentrated at the extremities of the web (Fig. 4(*a*)).

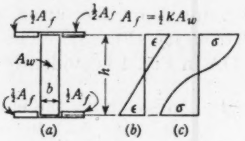


Fig. 4

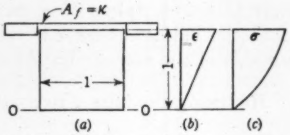


Fig. 5

To introduce the function m, imagine a section as in Fig. 5(*a*), strained and stressed as in Figs. 5(*b*) and 5(*c*). The width and depth of the web are both equal to unity, and the flange area A_f is equal to κ. The moment about the axis OO of stresses produced in this section, when it is strained as shown, is defined as the m-function. Evidently—

$$m = m_1 + \kappa\,\sigma \quad \text{(3)}$$

In Eq. 3, m_1 is the function of ϵ defined by Eq. 1, and σ is the unit stress corresponding to the same value of ϵ, thus making m also a function of ϵ.

Combining Eqs. 1 and 3:

$$m = \frac{1}{\epsilon^2}\int_0^{\epsilon} \epsilon\,\sigma\,d\epsilon + \kappa\,\sigma \quad \text{(4)}$$

The advantage of this variety of the function m over the one referred to at the beginning of this section is that it contains only one parameter κ instead of two, which widens its scope.

The function m can be easily related to the bending moment carried by the beam of Fig. 4. The web area of this beam is A_w, the depth is h, the area of one flange is A_f, and the ratio $\dfrac{A_f}{\frac{1}{2}A_w} = \kappa$.

Imagine a number of beams possessing the same flange-web ratio κ and strained to the same unit strain ϵ in the outside fibers. The following relations exist between the dimensions of the beams and the bending moments carried by them: When h remains constant and A_w varies, the internal moment varies in proportion to A_w. On the other hand, as long as A_w retains its value, the moment varies in proportion to h. Thus, the internal moment is proportional to the product $A_w\,h$. Therefore, the external bending moment M can be expressed, in terms of the function m, as follows:

$$M = 2\,m\,(\tfrac{1}{2}A_w)\,(\tfrac{1}{2}h) = \tfrac{1}{2}\,m\,A_w\,h \quad \text{(5)}$$

Eq. 5 can be applied to statically determinate I-beams and channels just as Eq. 2 can be applied to rectangular beams. The function m in Eq. 5 should be computed for the value of the parameter κ appropriate to the shape of the section.

The analysis of shear stresses requires the introduction of the function q, defined as the sum of the normal stresses developed by the section shown in Fig. 5. When the flange area κ is equal to zero, the corresponding q-function is referred to as q_1. Functions q and q_1, representing the internal normal forces, are thus analogous to the functions m and m_1, representing the internal moments, developed in the half-beam section of Fig. 5. Evidently,

$$q_1 = \frac{1}{\epsilon}\int_0^\epsilon \sigma\, d\epsilon \qquad (6a)$$

and

$$q = \frac{1}{\epsilon}\int_0^\epsilon \sigma\, d\epsilon + \kappa\,\sigma \qquad (6b)$$

4. Angle Changes

The analysis of statically indeterminate beams requires the derivation of relations between the angle changes and the unit strains and relations between the deflections and the unit strains.

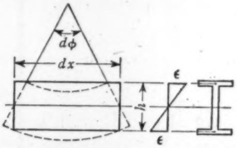


Fig. 6

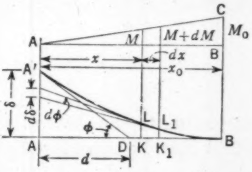


Fig. 7

Fig. 6 represents an element of length dx of a straight prismatic I-beam or channel beam with a symmetrical cross section, whose effective depth is h. It is strained to a unit strain ϵ at the centers of flanges, and the end sections rotate and form an angle $d\phi$ between them; thus:

$$d\phi = \frac{\epsilon\, dx}{\frac{1}{2} h} = \frac{2\,\epsilon\, dx}{h} \qquad (7a)$$

The angle change ϕ on a finite length of the beam is:

$$\phi = \frac{2}{h}\int \epsilon\, dx \qquad (7b)$$

to integrate Eq. 7b, ϵ and dx should be expressed in terms of the same variable.

In Fig. 7 the bending moment in the beam varies as a straight line from zero at one end to M_0 at the other end. If the shearing force in the beam is V, then, by a well-known relation:

$$dx = \frac{dM}{V} \qquad (8a)$$

Differentiating Eq. 5,

$$dM = \tfrac{1}{2} A_w h \, dm \ldots\ldots\ldots\ldots (8b)$$

Substituting Eq. 8*b* in Eq. 8*a*,

$$dx = \tfrac{1}{2} \frac{A_w h}{V} dm \ldots\ldots\ldots\ldots (8c)$$

Placing this value for dx (Eq. 8*c*) in Eq. 7*b*, the following expression for ϕ is obtained:

$$\phi = \frac{A_w}{V} \int_0^{\epsilon_0} \epsilon \, dm \ldots\ldots\ldots\ldots (9)$$

The integral in Eq. 9 represents the area between the (m-ϵ)-curve and the m-axis (Fig. 8), taken up to point A on the curve with the coordinates m_0 and ϵ_0, corresponding to the state of strain in the beam at end B with moment M_0. The integral may be considered as a function of ϵ_0. It can be computed from the (m-ϵ)-curve for different values of ϵ_0 by summing up the terms referring to a large number of narrow horizontal strips. In some special cases the integration can also be performed by calculus. The function expressed by the integral is given a symbol n, so that

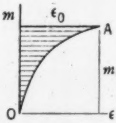


FIG. 8

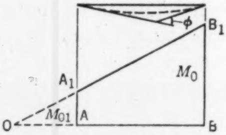


FIG. 9

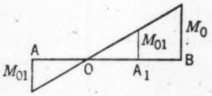


FIG. 10

$$\int_0^{\epsilon_0} \epsilon \, dm = n_0 \ldots\ldots\ldots\ldots (10)$$

Eq. 9 then becomes

$$\phi = \frac{A_w}{V} n_0 \ldots\ldots\ldots\ldots (11)$$

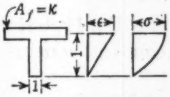
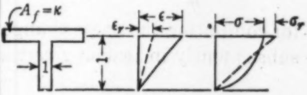
The loading conditions of the beam, to which Eq. 11 applies, are presented in Fig. 7. To solve for ϕ, compute m_0 from M_0 by Eq. 5. For the resulting value of m_0 select the value of the unit strain ϵ_0 from the (m-ϵ)-curve. For this value of ϵ_0 find the value of n_0 in the (n-ϵ)-curve, and then substitute it into Eq. 11. If the values of the m-functions and n-functions are plotted on the same sheet, or listed as in Table 1(*a*), the necessary n_0-value can be read directly against the corresponding m_0-value without referring first to ϵ_0.

Eq. 11 can be easily extended to a trapezoidal shape of the moment diagram by adding, algebraically, angle changes contributed by different parts of the beam. Thus, in Fig. 9, the angle change caused by the moments on the length AB is equal to the difference of the angle changes contributed by the triangular parts OBB_1 and OAA_1; and, since A_w and V are the same in both these parts,

$$\phi = \frac{A_w}{V}(n_0 - n_{01}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

The functions n_0 and n_{01} in Eq. 12 correspond to the strains present in the outer

TABLE 1.—STRAINS, STRESSES, AND DERIVED FUNCTIONS FOR MILD STEEL BEAMS[a] ($\kappa = 1$)

1,000 ϵ	σ $\left(\frac{\text{kips}}{\text{in.}^2}\right)$	m $\left(\frac{\text{kips}}{\text{in.}^2}\right)$				 $A_f = \kappa$ ϵ_r ϵ σ σ_r 1 1			
			(a) PRIMARY VALUES			(b) RESIDUAL VALUES			
			$10^3 n$	$10^3 u$	dq/dm	$10^3 n_r$	$10^3 u_r$	1,000 ϵ_r	σ_r
1.1	33.0	44.00	24.20	709.9	1.5	0	0	0	0
1.2	33.0	44.88	25.22	754.7	1.64	0.043	1.463	0.078	−0.66
1.3	33.0	45.56	26.06	793.3	1.77	0.113	5.084	0.161	−1.17
1.4	33.0	46.10	26.79	826.8	1.91	0.222	10.09	0.247	−1.58
1.5	33.0	46.54	27.43	856.2	2.05	0.352	16.12	0.336	−1.91
1.75	33.0	47.33	28.69	915.7	2.39	0.696	32.28	0.567	−2.50
2	33.0	47.84	29.64	960.9	2.73	1.037	48.48	0.804	−2.88
3	33.0	48.76	31.86	1,067.9	4.09	2.144	101.79	1.781	−3.57
4	33.0	49.08	32.97	1,122.4	5.45	2.861	137.16	2.77	−3.81
5	33.0	49.23	33.64	1,155.1	6.82	3.343	160.82	3.77	−3.91
6	33.0	49.31	34.08	1,177.0	8.18	3.689	177.83	4.77	−3.98
7	33.0	49.36	34.40	1,192.6	9.54	3.947	190.43	5.77	−4.02
8	33.0	49.40	34.63	1,204.3	10.91	4.132	199.71	6.76	−4.05
9	33.0	49.42	34.82	1,213.5	12.27	4.292	207.6	7.76	−4.06
10	33.0	49.43	34.97	1,220.8	13.63	4.427	214.3	8.76	−4.07
11	33.0	49.44	35.09	1,226.8	15.00	4.536	219.7	9.76	−4.08
12	33.0	49.45	35.19	1,231.7	16.36	4.625	224.1	10.76	−4.09
13	33.0	49.46	35.28	1,236.0	17.72	4.697	227.7	11.76	−4.10
14	33.0	49.47	35.35	1,239.6	19.09	4.758	230.7	12.76	−4.10
15	33.0	49.47	35.41	1,242.7	20.45	4.822	233.8	13.76	−4.10
16	33.0	49.47	35.47	1,245.5	21.81	4.877	236.6	14.76	−4.10
17	33.0	49.48	35.52	1,247.9	23.18	4.914	238.4	15.76	−4.11
18	33.0	49.48	35.56	1,250	24.54	4.958	240.5	16.76	−4.11
20	34.25	50.79	60.46	2,499	1.10	28.22	1,407	18.73	−3.84
22	35.35	52.06	87.13	3,870	1.08	53.25	2,694	20.70	−3.70
24	36.45	53.33	116.34	5,409	1.07	80.79	4,145	22.67	−3.55
26	37.5	54.61	148.34	7.135	1.05	111.06	5,778	24.63	−3.46
28	38.5	55.89	182.90	9,045	1.05	143.85	7,590	26.60	−3.42
30	39.5	57.17	220.0	11,144	1.07	179.17	9,587	28.57	−3.38
32	40.4	58.33	256.0	13,222	1.09	213.4	11,568	30.54	−3.35
34	41.2	59.43	292.3	15,361	1.10	248.1	13,612	32.51	−3.37
36	42.0	60.52	330.4	17,650	1.12	284.7	15.803	34.49	−3.39
38	42.7	61.47	365.6	19.794	1.16	318.4	17,858	36.46	−3.40
40	43.3	62.35	399.9	21,920	1.16	351.3	19,896	38.44	−3.46
45	45.0	64.68	498.9	28,200	1.09	446.6	25.950	43.38	−3.51
50	46.8	67.11	614.4	35,810	1.10	558.1	33,290	48.32	−3.53
55	48.3	69.21	724.6	43,330	1.19	664.7	40,570	53.27	−3.61
60	49.6	71.06	831.0	50,790	1.24	767.8	47,800	58.22	−3.70
65	50.75	72.75	936.6	58,390	1.20	870.4	55,180	63.18	−3.81

[a] Units are as follows: m, $10^3 n$, $10^3 n_r$, σ, and σ_r are in kips (thousand pounds) per square inch; and $10^3 u$ and $10^3 u_r$ are in $\frac{\text{kips}^3}{\text{in}^4}$.

fibers at points B and A, respectively. Eq. 12 is also applicable to the setup of Fig. 10, in which the bending moment changes its sign on the length AB, because the resultant angle change over the length AA_1 is zero.

Eq. 11 can be given a different form by substituting $\frac{M_0}{x_0}$ for the shear V and using Eq. 5:

$$V = \frac{M_0}{x_0} = \frac{A_w\, h\, m_0}{2\, x_0\,.} \qquad (13a)$$

Substituting Eq. 13a into Eq. 11:

$$\phi = \frac{2}{h}\, x_0 \left(\frac{n_0}{m_0} \right) \qquad (13b)$$

The ratio $\frac{n_0}{m_0}$ can be calculated as a new function. Eq. 13b is less convenient for computing the angle changes than is Eq. 11, but it will be used to advantage subsequently in reaching certain theoretical conclusions.

5. DEFLECTIONS

Referring again to Fig. 7, end B, at which the moment is M_0, is assumed to be fixed; and it is necessary to find an expression for the deflection of the other end, A, where the moment is zero. Consider an element dx of this beam, a distance x from the left end. The angular change $d\phi$ occurring on the length of this element causes a deflection $d\delta$ of the left end, so that

$$d\delta = x\, d\phi \qquad (14a)$$

The total deflection of the left end is then

$$\delta = \int_0^{x_0} x\, d\phi \qquad (14b)$$

which is the same expression as in the elastic theory. By Eq. 7a, $d\phi = \frac{2\, \epsilon\, dx}{h}$. Consequently,

$$\delta = \frac{2}{h} \int_0^{x_0} x\, \epsilon\, dx \qquad (15)$$

From Eq. 5 and Fig. 7,

$$x = \frac{M}{V} = \tfrac{1}{2}\, A_w\, h\, \frac{m}{V} \qquad (16)$$

Substituting Eqs. 8c and 16 in Eq. 15:

$$\delta = \frac{h}{2} \left(\frac{A_w}{V} \right)^2 \int_0^{\epsilon_0} m\, \epsilon\, dm = \frac{h}{2} \left(\frac{A_w}{V} \right)^2 u_0 \qquad (17a)$$

in which the integral—

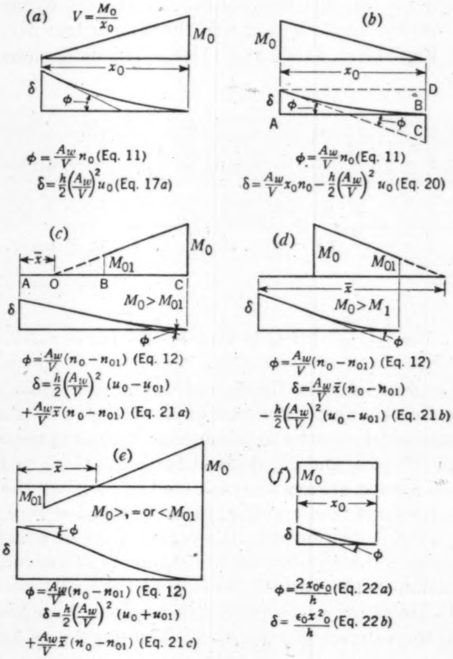
$$u_0 = \int_0^{\epsilon_0} m\, \epsilon\, dm \qquad (17b)$$

—represents a new function which signifies the statical moment about the ϵ-axis of the area between the (m-ϵ)-curve and the m-axis taken to point A on the curve with coordinates ϵ_0 and m_0 (Fig. 8). The function u_0 can be evaluated by summation. The function u_0 in Eq. 17*b* pertains to the strain condition existing at point B in the beam, where the bending moment is M_0.

The other version of this equation analogous to Eq. 13*b* is:

$$\delta = \frac{2\,x^2_0}{h}\,\frac{u_0}{m^2_0}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

In the following discussion and tables the subscript 0 in the symbols of n_0 and u_0 will be omitted, as a rule, when no specific point of the beam is under consideration.



(a) $\phi=\frac{A_w}{V}n_0$ (Eq. 11)

$\delta=\frac{h}{2}\left(\frac{A_w}{V}\right)^2 u_0$ (Eq. 17*a*)

(b) $\phi=\frac{A_w}{V}n_0$ (Eq. 11)

$\delta=\frac{A_w}{V}x_0 n_0-\frac{h}{2}\left(\frac{A_w}{V}\right)^2 u_0$ (Eq. 20)

(c) $\phi=\frac{A_w}{V}(n_0-n_{01})$ (Eq. 12)

$\delta=\frac{h}{2}\left(\frac{A_w}{V}\right)^2(u_0-u_{01})$

$+\frac{A_w}{V}\bar{x}(n_0-n_{01})$ (Eq. 21*a*)

(d) $\phi=\frac{A_w}{V}(n_0-n_{01})$ (Eq. 12)

$\delta=\frac{A_w}{V}\bar{x}(n_0-n_{01})$

$-\frac{h}{2}\left(\frac{A_w}{V}\right)^2(u_0-u_{01})$ (Eq. 21*b*)

(e) $\phi=\frac{A_w}{V}(n_0-n_{01})$ (Eq. 12)

$\delta=\frac{h}{2}\left(\frac{A_w}{V}\right)^2(u_0+u_{01})$

$+\frac{A_w}{V}\bar{x}(n_0-n_{01})$ (Eq. 21*c*)

(ʃ) $\phi=\frac{2\,x_0\epsilon_0}{h}$ (Eq. 22*a*)

$\delta=\frac{\epsilon_0 x^2_0}{h}$ (Eq. 22*b*)

FIG. 11

Formulas for the deflections, corresponding to various types of trapezoidal moment diagrams, are summarized in Fig. 11. They can be easily developed either by the method used to derive Eq. 17*a*, or from simple geometrical considerations, on the basis of Eqs. 11 and 17*a* which are fundamental. This condition is illustrated by Fig. 11(*b*). Let it be required to find the deflection at point A of a beam with a moment M_0, in relation to the tangent at point B, where the moment is zero. The deflection, of course, is assumed to be small. By Eq. 11:

$$\overline{\text{CD}} = x_0\,\phi = \frac{A_w}{V}\,x_0\,n_0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19a)$$

By Eq. 17*a*

$$\overline{BC} = \frac{h}{2}\left(\frac{A_w}{V}\right)^2 u_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (19b)$$

Finally:

$$\delta = \overline{CD} - \overline{BC} = \frac{A_w}{V} x_0\, n_0 - \frac{h}{2}\left(\frac{A_w}{V}\right)^2 u_0 \ldots\ldots\ldots\ldots (20)$$

It is necessary to remember that the two cases dealing with the triangular moment diagrams (Figs. 11(*a*) and 11(*b*)) are quite distinct. Therefore, the corresponding formulas for deflections, Eqs. 17*a* and 20, cannot be converted from one to the other by direct algebra; and the same is true regarding the three trapezoidal cases, Figs. 11(*c*), 11(*d*), and 11(*e*). The deflections for these cases are, respectively:

For case (*c*)—

$$\delta = \frac{h}{2}\left(\frac{A_w}{V}\right)^2 (u_0 - u_{01}) + \frac{A_w \bar{x}}{V}(n_0 - n_{01}) \ldots\ldots\ldots\ldots (21a)$$

for case (*d*)—

$$\delta = \frac{A_w \bar{x}}{V}(n_0 - n_{01}) - \frac{h}{2}\left(\frac{A_w}{V}\right)^2 (u_0 - u_{01}) \ldots\ldots\ldots\ldots (21b)$$

and, for case (*e*)—

$$\delta = \frac{h}{2}\left(\frac{A_w}{V}\right)^2 (u_0 + u_{01}) + \frac{A_w \bar{x}}{V}(n_0 - n_{01}) \ldots\ldots\ldots\ldots (21c)$$

In the first two trapezoidal cases the sign of the moment does not change, but in the last case it does change. The triangular cases can be obtained from the corresponding trapezoidal cases by assuming one of the end moments to be zero.

In Eqs. 11, 12, 17*a*, 20, and 21, derived for Figs. 11(*a*) to 11(*e*), the quantities A_w, V, n, and u must always be considered as positive; the same condition is true for x_0 and $\bar{x}$, except that $\bar{x}$ in Fig. 11(*c*) may either represent a positive quantity (when point A lies outside the region OB) or a negative quantity (when point A is within the distance OB). Moments M are considered positive when there is tension in the bottom fibers, and the angle changes caused by positive moments are taken as positive. Deflections δ are assumed positive, if they are above the tangent of reference. The quantities n_0 and u_0 refer to the points with the moments M_0, and the quantities n_{01} and u_{01} to the points with the moments M_{01}.

Curvilinear moment diagrams can be treated by dividing the curves into a number of sufficiently narrow trapezoids and applying appropriate equations. When a large number of trapezoids is necessary, the computation is quite laborious.

A special case, which cannot be solved by the foregoing equations, Eqs. 11, 12, 17*a*, 20, and 21, involves the rectangular moment curve shown as Fig. 11(*f*). In this case the flange strain ϵ_0 is constant all along the beam and, by Eq. 7*b*:

$$\phi = \frac{2\, x_0\, \epsilon_0}{h} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22a)$$

and

$$\delta = \tfrac{1}{3}\, x_0\, \phi = \frac{\epsilon_0\, x^2{}_0}{h} \quad \text{(22}b\text{)}$$

Angle changes occurring in the elastic range are proportional to bending moments, which follows from the well-known formula:

$$d\phi = \frac{M}{E\,I}\, dx \quad \text{(23}a\text{)}$$

and the moment diagram may thus be considered as a scale picture of the angle changes. At the same time:

$$d\phi = \frac{2\,\epsilon}{h}\, dx \quad \text{(23}b\text{)}$$

These relations will be utilized in Section 8, in computations pertaining to the elastic range.

Beyond the elastic range the angle changes increase much faster than the moments, and the moment diagram does not represent the angle changes to scale. Referring to Fig. 7, point D, where the tangents at the ends of the deflected beam meet, is a distance $\bar{d}$ from the point of zero moment, point A. By geometry and by Eqs. 13*b* and 18:

$$\bar{d} = \frac{\delta}{\phi} = x_0\, \frac{u_0}{m_0\, n_0} \quad \text{(24)}$$

In the elastic range, $\bar{d} = \frac{2}{3}\, x_0$, but beyond it $\bar{d}$ increases and may even approach x_0.

Once the expressions for the deflections and angle changes in beams are known, the indeterminate stress analysis outside the elastic range is in no way different from conventional analysis, and it merely consists in setting up the equations between the deflections and the angle changes prescribed by the conditions of restraint.

6. SHEARING STRESSES IN BEAMS

The greatest shearing stress existing on a normal or a longitudinal plane of an I-beam is present in the web of the beam at the neutral axis, and an expression will be derived for the value of this stress τ_0, following the method used in the elastic analysis.

Fig. 12 represents one half of the cross section of an I-beam and the free body diagram of the length *dx* of this section. The greatest strains at the left and right sides of this section are ϵ and $(\epsilon + d\epsilon)$, respectively. Recalling the definition of the function *q* in Eq. 6*b*, the following expression is obtained from the condition of equilibrium of horizontal forces:

$$\tau_0\, b\, dx = \tfrac{1}{2}\, A_w \left[q\,(\epsilon + d\epsilon) - q(\epsilon)\right] = \tfrac{1}{2}\, A_w\, dq \quad \text{(25}a\text{)}$$

and

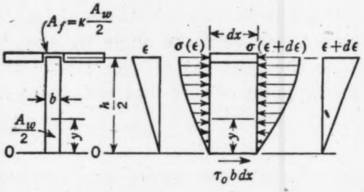
$$\tau_0 = \frac{h}{2}\, \frac{dq}{dx} \quad \text{(25}b\text{)}$$

If Eq. 8c is substituted for dx in Eq. 25b:

$$\tau_0 = \frac{V}{A_w}\frac{dq}{dm} \qquad (25c)$$

The function $\frac{dq}{dm}$ must be computed point by point, by taking increments of the functions q and m corresponding to the same increments in ϵ.

Thus, shearing stresses in inelastic range are dependent not only on the shearing force V, but also on the bending moment, since the latter determines the strain ϵ, and the derivative $\frac{dq}{dm}$ is a function of ϵ. The ratio $\frac{V}{A_w}$ represents uniform shearing stress per square unit of area of the web. The function $\frac{dq}{dm}$ therefore signifies a coefficient allowing for nonuniformity of shear distribution over the web.



(a) (b)

FIG. 12

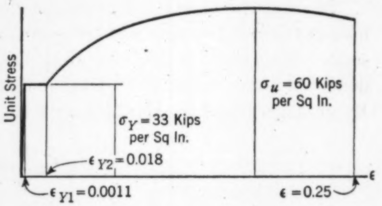


FIG. 13

A similar procedure leads to a more general formula for the value of the horizontal or vertical shearing stress at a distance y from the neutral axis (Fig. 12):

$$\tau = \frac{V}{A_w}\frac{d\left[\,q(\epsilon) - \frac{y}{\frac{1}{2}h}q_1\left(\frac{y}{\frac{1}{2}h}\epsilon\right)\right]}{dm(\epsilon)} \qquad (25d)$$

In Eqs. 25a and 25d, $q(\epsilon)$ and $m(\epsilon)$ signify the values of the functions q and m corresponding to the value ϵ of the strain in the outer fibers of the beam, whereas $q_1\left(\frac{y}{\frac{1}{2}h}\epsilon\right)$ represents the value of the function q_1 corresponding to the value of the strain $\frac{y}{\frac{1}{2}h}\epsilon$.

7. FUNCTIONS m, n, u, AND $\frac{dq}{dm}$ FOR MILD STEEL BEAMS

The foregoing bending theory will be applied to mild steel beams, whose stress-strain curve is presented in Fig. 13 and in Table 1(a). The main mechanical properties of this steel satisfy the material requirements of the 1940 Specifications of the American Railway Engineering Association for fixed-span railway bridges. They are as follows:

Unit Stresses, in Kips Per Square Inch—	
Yield point	33
Ultimate strength	60
Modulus of elasticity	30,000
Unit strain (%) at the beginning of yielding	0.11
Percentage elongation in 8 in.	25

In comparison with the actual curve, the stress-strain curve in Fig. 13 is idealized in two respects: First, the elastic section of the curve is assumed to continue straight to the yield point without any curved transition; and, second, the yield section is assumed to extend horizontally to the point where the strain hardening begins—at $\epsilon = 1.8\%$. These approximations do not distort the mechanical behavior of beams materially, and they are introduced merely for the purpose of reducing the labor of computation and not because they are inherent in the theory presented.

The necessary derived curves are computed for four values of the parameter κ—0, ½, 1, and 1½. The first of these values corresponds to a rectangular beam, whereas the remaining three cover the range of standard I-beams. Formulas pertaining to the elastic and plastic parts of the derived curves (that is, the parts within the limits of ϵ from zero to 0.11% and from 0.11% to 1.8%, respectively) can be developed by algebra as demonstrated in Sections 8 and 9.

8. Elastic Part

In the elastic range, $\sigma = E\,\epsilon$. Substituting this value in Eq. 4 and integrating:

$$m = (\kappa + \tfrac{1}{3})\,\sigma = E\,(\kappa + \tfrac{1}{3})\,\epsilon \qquad (26a)$$

Differentiating Eq. 26*a* and substituting *m* and *dm* in Eqs. 10 and 17*b* for *n* and *u*:

$$n = \tfrac{1}{2}\,E\,(\kappa + \tfrac{1}{3})\,\epsilon^2 = \tfrac{1}{2}\,(\kappa + \tfrac{1}{3})\,\sigma\,\epsilon \qquad (26b)$$

and

$$u = \tfrac{1}{3}\,E^2\,(\kappa + \tfrac{1}{3})^2\,\epsilon^3 = \tfrac{1}{3}\,(\kappa + \tfrac{1}{3})^2\,\sigma^2\,\epsilon \qquad (26c)$$

Similarly, by Eq. 6*b*,

$$q = (\kappa + \tfrac{1}{2})\,E\,\epsilon \qquad (27a)$$

and

$$\frac{dq}{dm} = \frac{\kappa + \frac{1}{2}}{\kappa + \frac{1}{3}} \qquad (27b)$$

Eqs. 26 and 27 are valid to the beginning of yielding, where the coordinates of the stress-strain curve are $\epsilon = \epsilon_{Y1}$ and $\sigma = \sigma_y$.

The specific values of these functions at the yield point under the conditions of the assumed stress-strain curve are as follows:

$$m = 33\,(\kappa + \tfrac{1}{3})\left(\frac{\text{kips}}{\text{in.}^2}\right) \qquad (28a)$$

$$n = 0.01815\,(\kappa + \tfrac{1}{3})\left(\frac{\text{kips}}{\text{in.}^2}\right) \qquad (28b)$$

and

$$u = 0.3993\ (\kappa + \tfrac{1}{3})^2 \left(\frac{\text{kips}^2}{\text{in.}^4}\right) \ldots\ldots\ldots\ldots\ldots\ldots (28c)$$

Expressions for the angle change and deflection in the elastic range, obtained by substituting Eqs. 26*b* and 26*c* in Eqs. 11 and 17*a*, and simplifying, are:

$$\phi = \frac{x\,\epsilon}{h} \ldots\ldots\ldots\ldots\ldots\ldots (29a)$$

and

$$\delta = \tfrac{2}{3}\,\frac{x^2\,\epsilon}{h} \ldots\ldots\ldots\ldots\ldots\ldots (29b)$$

9. Plastic Part

In the plastic range, $\sigma = \sigma_y$ = constant, as ϵ varies between the limits of ϵ_{Y1} and ϵ_{Y2}. Although the expression for the m-function can be obtained from the basic formula, Eq. 4, it is derived more conveniently from the first principles (Fig. 14) by taking moments about the axis OO of the normal stresses distributed over the section as shown:

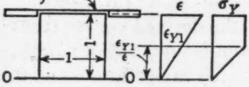


Fig. 14

$$m = \kappa\,\sigma_Y(1) + \sigma_Y(1)\tfrac{1}{2} - \frac{\sigma_Y}{2}\,\frac{\epsilon_{Y1}}{\epsilon}\,\frac{1}{3}\,\frac{\epsilon_{Y1}}{\epsilon} = (\kappa + \tfrac{1}{2})\,\sigma_Y - \frac{\sigma_Y}{6}\left(\frac{\epsilon_{Y1}}{\epsilon}\right)^2 \ldots (30a)$$

By Eq. 10,

$$n = \int_0^{\epsilon_{Y1}} \epsilon\, dm + \int_{\epsilon_{Y1}}^{\epsilon} \epsilon\, dm \ldots\ldots\ldots\ldots\ldots\ldots (30b)$$

By Eq. 26*b*,

$$\int_0^{\epsilon_{Y1}} \epsilon\, dm = \frac{E}{2}\,(\kappa + \tfrac{1}{3})\,\epsilon^2_{Y1} \ldots\ldots\ldots\ldots\ldots\ldots (30c)$$

From Eq. 30*a*,

$$dm = \frac{\sigma_Y}{3}\,\frac{\epsilon^2_{Y1}}{\epsilon^3}\,d\epsilon \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

Eq. 31 makes it possible to evaluate the second integral in the expression for n. Then:

$$n = E\,\epsilon^2_{Y1}\left[\tfrac{1}{2}\,(1 + \kappa) - \tfrac{1}{3}\,\frac{\epsilon_{Y1}}{\epsilon}\right] \ldots\ldots\ldots\ldots\ldots\ldots (32a)$$

By applying the same procedure to u,

$$u = \tfrac{1}{3}\,E^2\,\epsilon^3_{Y1}\left[\left(\kappa^2 + \frac{5}{3}\,\kappa + \frac{5}{9}\right) - (\kappa + 0.5)\,\frac{\epsilon_{Y1}}{\epsilon} + \frac{1}{18}\left(\frac{\epsilon_{Y1}}{\epsilon}\right)^3\right] \ldots (32b)$$

From Fig. 14,

$$q = (\kappa + 1)\,\sigma_Y - \frac{\sigma_Y}{2}\,\frac{\epsilon_{Y1}}{\epsilon} \ldots\ldots\ldots\ldots\ldots\ldots (32c)$$

Then, by differentiating Eqs. 30*a* and 32*c* and simplifying:

$$\frac{dq}{dm} = \frac{3}{2}\,\frac{\epsilon}{\epsilon_{Y1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (33)$$

At the beginning of yielding, where $\epsilon = \epsilon_{Y1}$, $\dfrac{dq}{dm} = \frac{3}{2}$, as in a rectangular beam.

At the end of yielding, where $\epsilon = \epsilon_{Y2} = \dfrac{1.8}{0.11}$, $\dfrac{dq}{dm} = 24.54$.

10. Strain-Hardening Part

Values of the derived functions above the plastic range can be computed only by successive summation of the increments of integrals, with all the necessary values properly tabulated in a systematic manner. The method is illustrated by an example of quantities referring to $\kappa = 1$ and $\epsilon = 0.020$. First, the necessary values of functions corresponding to the value of $\epsilon = 0.018$, immediately below $\epsilon = 0.020$, are found either by integration, or by step-by-step summation.

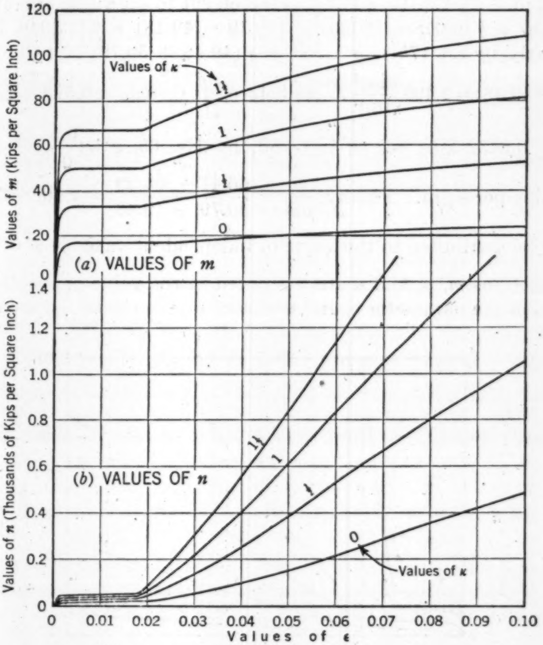


Fig. 15.—Constants for the Design of Mild Steel Beams

For $\epsilon = 0.018$.—

Solving Eqs. 30*a* and 32: $m = 49.48$ kips per sq in.; $n = 0.03556$ kips per sq in.; $u = 1.250 \frac{\text{kips}^2}{\text{in.}^4}$; and $q = 64.99$ kips per sq in. Furthermore, $\int_0^{0.018} \sigma \, d\epsilon = 0.576$ kips per sq in.; and $\int_0^{0.018} \epsilon \, \sigma \, d\epsilon = 0.005340$ kips per sq in.

For $\epsilon = 0.020$.—

The stress-strain curve gives $\sigma = 34.25$ kips per sq in. for $\epsilon = 0.020$, from which $\int_0^{0.020} \epsilon \, \sigma \, d\epsilon = 0.005340 + \frac{1}{2}(0.018 + 0.020) \times \frac{1}{2}(33 + 34.25) \times 0.002 = 0.006618$ kips per sq in. By Eq. 4, $m = \frac{0.006618}{0.020^2} + 1 \times 34.25 = 50.79$ kips per sq in. By Eq. 10, adding an increment to the value of n at $\epsilon = 0.018$, $n = 0.03556 + \frac{1}{2}(0.018 + 0.020) \times (50.79 - 49.48) = 0.006046$ kips per sq in. Similarly, by Eq. 17*b*, $u = 1.250 + \frac{1}{2}(49.48 + 50.79) \times \frac{1}{2}(0.018 + 0.020) \times (50.79 - 49.48) = 2.499 \frac{\text{kips}^2}{\text{in.}^4}$. As before, $\int_0^{0.020} \sigma \, d\epsilon = 0.576 + \frac{1}{2}(33 + 34.25) \times 0.002 = 0.6432$ kips per sq in.; and, by Eq. 6*b*, $q = \frac{0.6432}{0.020} + 1 \times 34.25 = 66.41$ kips per sq in. Finally, $\frac{dq}{dm} = \frac{66.41 - 64.99}{50.79 - 49.48} = 1.10$. This value of $\frac{dq}{dm}$ should be attributed to the value of independent variable $\epsilon = 0.019$.

The functions m, n, and u are computed to the value of $\epsilon = 0.100$, for the four values of the parameter κ, and the results of computation are presented in Figs. 15 and 16 and, in a greatly expanded form, in Table 1(*a*). The graphs, not being very accurate, serve merely as illustrations, whereas the more comprehensive table, of which Table 1(*a*) is a part, is intended for use in the analysis.

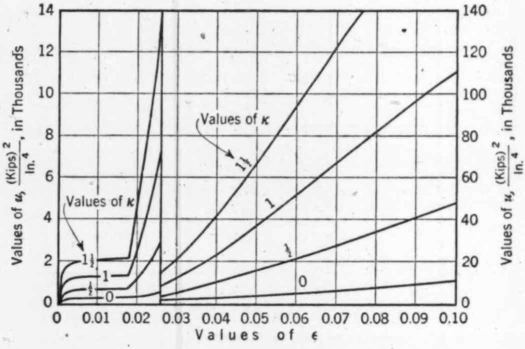


Fig. 16.—Curves for the Constant u, in Mild Steel Beams

11. EFFECT OF SHEAR STRESSES

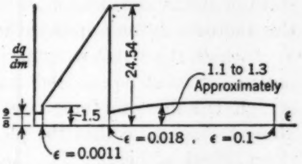
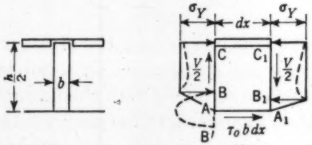


FIG. 17 (Not to Scale)

Only one set of values of the shear coefficient $\frac{dq}{dm}$, corresponding to $\kappa = 1$, is given in Fig. 17 and in Table 1(*a*), in view of the comparative unimportance of this coefficient in design. Some comments are needed on the numerical values of this coefficient, however. Everywhere outside the plastic range (that is, outside the limits of ϵ from 0.0011 to 0.018) this coefficient has a magnitude of from approximately 1.1 to 1.3, which signifies that the greatest shearing stress in the web of the beam is only from about 10% to 30% higher than the average stress per unit area of the web. In the plastic range, however, this coefficient varies independently of the value of κ, and in a straight line from the value 1.5 at $\epsilon = \epsilon_{Y1}$ to a very high value of 24.54 at $\epsilon = \epsilon_{Y2}$.

The correctness of this rather startling phenomenon is made plain by reference to Fig. 18 representing the free body diagram of a small element of an I-beam. Since normal stresses on the parts of the sections BC and B_1C_1 are constant and equal, no shearing stresses acting either on horizontal or on vertical planes are possible in these outer parts of the web BC, all shear on the cross section thus being localized on a small central part of the web BB′. The intensity of stress on BB′, indicated by a dotted parabola, may rise very high, because the resultant shearing stress over the cross section must equal the shearing force.



(*a*) (*b*)

FIG. 18

Of course, this deduction is only as authentic as the underlying assumption of plane cross sections. If the cross sections warp (which seems inevitable in the presence of high shearing stresses), some redistribution of stresses and strains, both normal and tangential, must follow—a condition which raises some doubts as to the validity of the present theory of bending in the plastic range.

In appraising the effect of this stress irregularity, several aspects of the situation will be distinguished and discussed: (1) The high value of shear stresses (2) the redistribution of normal stresses; and (3) the additional angle changes caused by warping.

(*1*) *High Value of the Shearing Stress.*—Shear stresses reach an intensity of 24.5 times the average stress per unit area of the web. From general considerations it appears certain that, if the absolute value of this stress is high, it is relieved by warping, perhaps to the level of yielding, because a high concentration of stress, sharply localized over a small area, is inconceivable in the absence of concentrated loads.

(2) *Redistribution of Normal Stresses in the Planes of Cross Sections Produced by Warping.*—Although this effect is undoubtedly present, it is likely to be small since, in the plastic and strain-hardening regions, only a slight change in the normal stresses is possible, even after an appreciable change in the strains.

(3) *Additional Angle Changes Caused by Warping.*—Such angle changes influence the values of static unknowns and, through them, affect the general state of stress and deformation over the entire structure. The significance of this factor may be appraised indirectly by estimating the length of the beam, over which the stress irregularity considered may possibly occur. In this connection, it must be realized that, if the average unit shearing stress in the web is high, the shearing force is great, and the bending moment builds up quickly along the span, so that the plastic range of normal stresses is passed over in a short length of beam. On the other hand, if the plastic state occupies a great length of the beam, the shearing force (and with it the average shearing stress in the web) must be small, and even a high coefficient of nonuniformity would not raise the maximum shearing stress beyond the yield point.

By using appropriate values for Δq and τ_0 in Eq. 25*b* the greatest length, on which the warping is possible in the most extreme case is of the order of $\Delta x = \frac{1}{4} h$. This length appears too small to produce an appreciable effect on the static unknowns.

The foregoing material seems to warrant the conclusion that the stress irregularity occurring in the plastic region of normal stresses is too small to affect the strength of beams noticeably or to interfere seriously with the flexural theory based on linearity of the cross sections. Therefore, in Sections 14, 15, and 16 no allowance will be made for the warping of cross sections in the plastic region.

12. STRESS RECESSION

In many important construction materials the stress-strain curve of unloading, in straight tension or compression, may be closely approximated by a straight line YO_1Y_1 parallel to the elastic part of the curve OA (Fig. 19). The

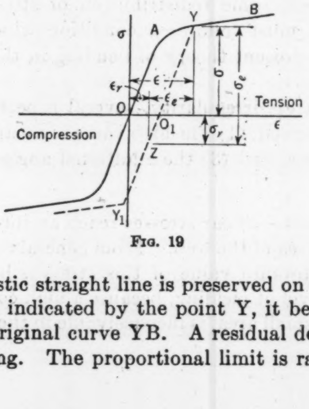


FIG. 19

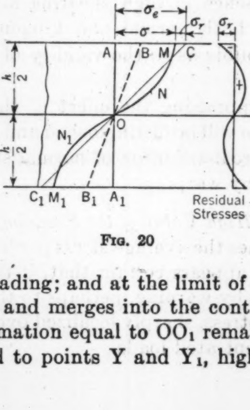


FIG. 20

new elastic straight line is preserved on reloading; and at the limit of the first loading, indicated by the point Y, it bends and merges into the continuation of the original curve YB. A residual deformation equal to $\overline{OO_1}$ remains after unloading. The proportional limit is raised to points Y and Y_1, higher than before, whereas the modulus of elasticity retains its former value. These experimental facts, together with the assumptions stated in Section 1, form the basis of the theory of bending past the elastic limit when there is a stress recession.

Imagine a beam subjected to bending, whose normal stresses in the section AA_1 Fig. 20 are represented by the curve $MNON_1M_1$. If there is a decrease in the bending moment, the assumption of plane cross sections, combined with the mechanics of stress recession, explained in Fig. 19, demands that the decrease in stresses should be linear, so that the remaining stresses correspond to the difference of ordinates between the original curve $MNON_1M_1$ and an appropriate straight line BB_1. Further reduction in moment causes the straight line BB_1 to move farther from AA_1; and, when the moment is brought to zero, the residual stresses are represented by the difference between the ordinates of the curves $MNON_1M_1$ and CC_1, whose statical effect is zero.

An important deduction is that removal of bending moment from the beam does not result in the disappearance of all stress or of all strain. If point Y in Fig. 19 represents the state of stress in the outer fibers of the beam when the loads are first applied, the removal of the moment causes a reduction in stress σ by an amount σ_e, which brings the state of stress to point Z, Fig. 19, with a residual stress σ_r and strain ϵ_r. The presence of the latter is accompanied by a residual angle change:

$$d\phi_r = 2\frac{\epsilon_r}{h}dx \qquad (34)$$

Let the bending moment in the beam created by applied loads be M, then the corresponding m-function is computed by Eq. 5. The elastic strain ϵ_e and stress σ_e superimposed on the outer fibers of the beam by a complete removal of the moment, are found by Eq. 26a, using this value of m:

$$\epsilon_e = \frac{m}{E(\kappa + \frac{1}{3})} \qquad (35a)$$

and

$$\sigma_e = E\,\epsilon_e = \frac{m}{\kappa + \frac{1}{3}} \qquad (35b)$$

The residual strain and stress in the outside fibers of the beam are:

$$\epsilon_r = \epsilon - \frac{m}{E(\kappa + \frac{1}{3})} \qquad (36a)$$

and

$$\sigma_r = \sigma - \frac{m}{\kappa + \frac{1}{3}} \qquad (36b)$$

The residual strain ϵ_r has the same sign as the original strain ϵ, whereas the residual stress σ_r proves to be negative (that is, of the opposite sign from σ). These facts follow clearly from Figs. 19 and 20.

The angle change and the deflection caused by the release of bending moment, in the case of the simple triangular moment diagram in Fig. 7, are found by substituting the expression for strain caused by the moment removal, Eq.

35*a*, in the equations for the elastic angle change and deflection, Eqs. 29*a* and 29*b*:

$$\phi_e = \frac{x\,m}{E\,h\,(\kappa + \frac{1}{3})} \qquad \text{(37a)}$$

and

$$\delta_e = \frac{2}{3}\,\frac{x^2\,m}{E\,h\,(\kappa + \frac{1}{3})} \qquad \text{(37b)}$$

The form of Eqs. 37 is not entirely convenient. Replacing x by $\frac{M}{V}$ and substituting for the moment its expression in terms of the m-function, Eq. 5, the following expressions result:

$$\phi_e = \frac{1}{2}\,\frac{A_w}{V}\,\frac{m^2}{E(\kappa + \frac{1}{3})} \qquad \text{(38a)}$$

and

$$\delta_e = \frac{h}{6}\left(\frac{A_w}{V}\right)^2 \frac{m^3}{E(\kappa + \frac{1}{3})} \qquad \text{(38b)}$$

Subtracting Eqs. 38 from the corresponding deformations brought about by creation of the moments, Eqs. 11 and 17*a*, the residual values, ϕ_r and δ_r, remaining after complete disappearance of moments, are found to be:

$$\phi_r = \frac{A_w}{V}\left[n - \frac{1}{2}\,\frac{m^2}{E(\kappa + \frac{1}{3})}\right] \qquad \text{(39a)}$$

and

$$\delta_r = \frac{h}{2}\left(\frac{A_w}{V}\right)^2 \left[u - \frac{1}{3}\,\frac{m^3}{E(\kappa + \frac{1}{3})}\right] \qquad \text{(39b)}$$

The values of the functions m, n, and u in Eqs. 39 refer to the state of stress at the end of beam B, Fig. 7, under full loading. Expressions in the square brackets do not depend on the dimensions of the beam, except for the parameter κ, and they may be considered as the new derived functions n_r and u_r, defined as follows:

$$n_r = n - \frac{1}{2}\,\frac{m^2}{E(\kappa + \frac{1}{3})} \qquad \text{(40a)}$$

and

$$u_r = u - \frac{1}{3}\,\frac{m^3}{E(\kappa + \frac{1}{3})} \qquad \text{(40b)}$$

Eqs. 40 can be used for the determination of residual deflections and angle changes in exactly the same fashion as the original functions n and u are used for computing the deformations produced by creation of the moments. For this reason, Eqs. 11, 12, 17*a*, 18, 20, and 21 hold equally true for the residual deformations in their respective conditions, if the functions n and u are replaced by the corresponding functions n_r and u_r. In the elastic range the residual functions are evidently zero. Numerical values of the functions ϵ_r, σ_r, n_r, and u_r computed for different values of the independent variable ϵ and the value of the parameter $\kappa = 1$ are presented in Table 1(*b*).

When a residual deformation occurs in a statically determinate structure, it meets with no impediment and materializes freely. The situation is different in statically indeterminate structures, in which the conditions of restraint make a free adjustment of the beam to a "no moment" state impossible. As a result, residual moments are created when the structure is unloaded, and the stresses and strains caused by the residual moments, are added to "no moment" stresses and strains. Thus, a "no loading" condition following the removal of load in a statically indeterminate beam must not be confused with the "no moment" condition. The transition from a fully loaded condition to "no loading" condition is probably always elastic, and can be analyzed by the usual formulas of the elastic theory. Since this last statement is quite important for further discussion, particularly in the analysis of moving loads, it is worthy of some elaboration. It can be expressed in the following shorthand form:

$$(\text{ALS}) - (\text{ELS}) = (\text{RS}) \quad \dots\dots\dots\dots\dots\dots (41)$$

in which (ALS) signifies the actual state of stress in the loaded structure, including moments, stresses, and strains, found by the theory presented herein; (ELS) denotes the state of stress under the same loading, found by the usual formulas of the strength of materials (although it is known that the moments so found project beyond the elastic range); and (RS) represents the residual state of stress after removal of all load. The minus sign in the left-hand side of Eq. 41 signifies the ordinary algebraic subtraction of the moments, stresses, and strains pertaining to the two states.

Eq. 41 holds true also when the beam has been pre-stressed past the elastic limit by one or several applications of loads, including movable loads, previous to the application of the load whose effect is being considered. In this case, (ALS) should signify the state created as a result of all loadings, including the last one; (RS) represents what remains after the last load has been removed; and (ELS) indicates the elastic stress condition created by the last loading alone. The details of application of this principle to movable loads will be explained in Example 3, Section 16.

13. Computation of Shape Constants of an I-Beam

The computation is illustrated by an example involving the standard I-beam (18 I 54.7) shown in Fig. 21: $h = 18.00 - \frac{1}{2}(0.46 + 0.92) = 17.31$ in.;

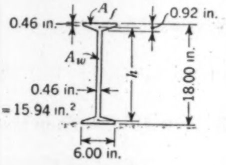


FIG. 21

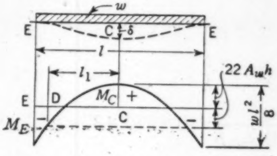


FIG. 22

$A_w = 0.46\,(17.31) = 7.96$ in.^{2}; $A_f = \frac{1}{2}(15.94 - 7.96) = 3.99$ in.^{2}; and $\kappa = \dfrac{3.99}{\frac{1}{2}(7.96)} = 1.00$.

The following examples are solved in general terms using a mild steel I-beam with $\kappa = 1$. If specific values are needed, pertaining to an 18 I 54.7, they can be obtained by substituting the foregoing values of h and A_w in the formulas derived.

14. Example 1

Given a uniformly-loaded, fixed-ended, I-beam with $\kappa = 1$ (Fig. 22), find the value of the load (referred to as the capacity load) for which the stress at the center just reaches the yield point. For this value of the load find the stress conditions at the ends and the deflection at the center. Find also the residual state (RS) brought about by removal of this load.

Because of symmetry, there is only one static unknown, the moment at the end M_E. This is larger than the moment at the center, M_C, and therefore, at some point near the end, such as point D, the moment is equal to the moment at the center.

The method of solution consists in assuming an appropriate value for M_E, or better, in assuming an appropriate value for the maximum strain at the end ϵ_E, and on this assumption determining the angle change produced on one half of the length of the beam. This angle change should be equal to zero.

From Table 1(*a*) and Eq. 5, the following conditions exist at points C and D: $1{,}000\,\epsilon = 1.1$; $\sigma = 33$ (kips/in.^{2}); $m = 44$ (kips/in.^{2}); $1{,}000\,n = 24.2$ (kips/in.^{2}); and $M = 22$ (kips/in.^{2}) $A_w\,h$. The units accompanying the expression for M actually belong to the numerical coefficient; and, in combination with the units of A_w and h, they produce the proper moment units.

Trial 1.—At point E, Fig. 22, assume $1{,}000\,\epsilon_E = 22$. Then, by Table 1(*a*), $m_E = 52.06$ (kips/in.^{2}); $1{,}000\,n_E = 87.13$ (kips/in.^{2}); and $M_E = 26.03$ (kips/in.^{2}) $A_w\,h$.

By the properties of a parabola, the length $l_1 = \overline{\mathrm{DC}}$, over which the stress is elastic, is found as follows: $l_1 = \frac{1}{2}\,l\sqrt{\dfrac{22 + 22}{22 + 26.03}} = 0.4787\,l$; then, $\overline{\mathrm{ED}} = x = 0.0213\,l$.

Assuming that the moment curve on the length ED is straight, the shearing force on this length is $V = \dfrac{(26.03 - 22)\,A_w\,h}{0.0213\,l} = 189.0$ (kips/in.^{2}) $\dfrac{A_w\,h}{l}$.

Remembering that in the elastic region $\dfrac{M}{E\,I} = \dfrac{2\,\epsilon}{h}$, the angle change occurring on the elastic part DC is $\phi_{DC} = \dfrac{2}{h}\,(10^{-3})\,[\frac{2}{3}\,(2)(1.1)(0.4787)\,l - (1.1)(0.4787)\,l] = 0.351\,(10^{-3})\,\dfrac{l}{h}$. The angle change in the length ED, by Eq. 12, is $\phi_{ED} = -\dfrac{A_w\,l}{(189.0)\,A_w\,h}\,(10^{-3})(87.13 - 24.2) = -\,0.333\,(10^{-3})\,\dfrac{l}{h}$. Then the total angle change in one half of the span length is $\phi_{EC} = 0.018\,(10^{-3})\,\dfrac{l}{h}$. Since ϕ_{EC} is positive, the values of the moment and of the strain at the end of the beam have been underestimated.

When the procedure is repeated in trial 2, assuming $\epsilon_E = 23\,(10^{-3})$, the resultant angle change is $\phi_{EC} = -\,0.058\,(10^{-3})\,\dfrac{l}{h}$. The true value of ϵ_E can be found by linear interpolation, assuming that ϵ_E varies in proportion to ϕ_{EC}. Thus, $\epsilon_E = \left(22 + \frac{18}{18 + 58}\right) 10^{-3} = 22.25\ (10^{-3})$.

The third trial made with this value of ϵ_E yields $\phi_{EC} = -\ 0.002\ (10^{-3})\ \frac{l}{h}$. Since this is negligible, the value $\epsilon_E = 22.25\ (10^{-3})$ may be considered as the true value. The corresponding value of the m-function can be found by interpolation in the design Table 1(a): $m_E = 52.06 + \frac{0.25}{2}\ (53.33 - 52.06) = 52.22$ (kips/in.²); and

$$M = 26.11\ (\text{kips/in.}^2)\ A_w\, h = C_M\, A_w\, h \ldots\ldots\ldots\ldots (42a)$$

in which $C_M = 26.11$ kips per sq in., is a coefficient. Linear interpolation has been found to work quite well, when the requisite statically unknown strain extends, as in this particular case, into the strain-hardening region. Under these circumstances, two reasonably close trials are sufficient for the solution. When the unknown strain is found in the plastic range, its variation with ϕ is not linear, and more than two trials may be needed for accurate results. In that case ϵ is very unstable.

The required intensity of the uniform loading (w) is now found by statics: $(26.11 + 22)\ A_w\, h = \frac{1}{8}\, w\, l^2$ and

$$w = 384.9\ (\text{kips/in.}^2)\ \frac{A_w\, h}{l^2} = C_w\, \frac{A_w\, h}{l^2} \ldots\ldots\ldots\ldots (42b)$$

in which $C_w = 384.9$ kips per sq in. is a coefficient.

The deflection δ at the center of the beam is found, by reference to Fig. 23, as the sum of two deflections caused by deformations of the parts ED and DC: $\overline{DC} = 0.5\, l \sqrt{\frac{22 + 22}{26.11 + 22}} = 0.4782\ l$; and $\overline{ED} = 0.0218\, l$.

The position of point F, Fig. 23, where the straight line ed cuts the axis, is found from $\bar{x} = 0.5\, l - \frac{26.11}{26.11 - 22}\ (0.0218\, l) = 0.3614\, l$.

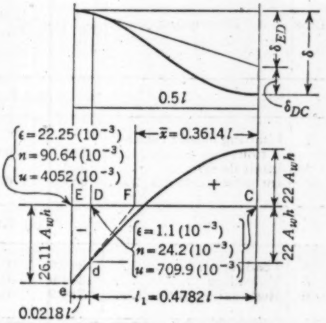


FIG. 23

The shearing force on the length ED is

$$V = \frac{4.11\ A_w\, h}{0.0218\, l} = 188.5\ (\text{kips/in.}^2)\ \frac{A_w\, h}{l} = C_V\, \frac{A_w\, h}{l} \ldots\ldots . (42c)$$

The values of n and u at point E are found from Table 1(a) by interpolation. They are stated in Fig. 23 together with the values of the same functions at point D. By Eq. 21a, $\delta_{ED} = -\left[\frac{h}{2}\left(\frac{1}{188.5}\,\frac{l}{h}\right)^2 (4{,}052 - 709.9)\ (10^{-3})\right.$

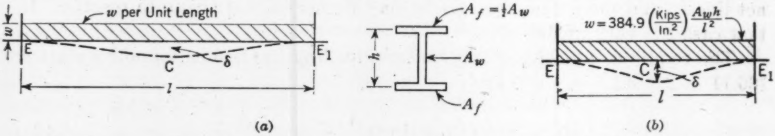
$+ \frac{1}{188.5} \frac{l}{h} (0.3614\, l)(90.64 - 24.2)\, 10^{-3} \Big] = -\, 0.1743\, (10^{-3}) \frac{l^2}{h}$. By elastic relations, $\delta_{DC} = -\, (\tfrac{2}{3}) \frac{2}{h} (1.1)\, 10^{-3} (0.4782\, l)(\tfrac{3}{8})(0.4782\, l) + \frac{1.1}{h}(10^{-3})(0.4782\, l) \times \tfrac{1}{2} \times (0.4782\, l) = 0$. Therefore,

$$\delta = -\, 0.1743\, (10^{-3}) \frac{l^2}{h} = -\, C_\delta \frac{l^2}{h} \qquad \text{(42}d\text{)}$$

The results are summarized in line 3, Table 2(*a*) under the title "Inelastic Condition." For comparison, two other sets of capacity loads and stresses are also given, one determined by the theory of limit design (line 2) and the other by the elastic theory (line 1), corresponding to the condition when the yield point has first been reached in the beam. In computing these additional data, the following values of the beam constants have been used: $E = 30{,}000$ kips per sq in. and

TABLE 2.—DEFLECTIONS, STRAINS, STRESSES, AND MOMENTS UNDER CAPACITY LOADINGS

(Units Are: σ C_W, and C_M, Kips per Square Inch; and ϵ and C_δ, Inches per Inch)


w
w per Unit Length
E
C
δ
E1
l
(a)
$A_f = \frac{1}{2} A_w$
h
A_w
A_f
$w = 384.9 \left(\frac{\text{Kips}}{\text{In.}^2}\right) \frac{A_w h}{l^2}$
E
C
δ
E1
l
(b)

Line	Description	C_w (Eq. 42*b*)	1,000 $C\delta$ (Eq. 42*d*)	(*a*) CENTER, POINT C			(*b*) END, POINT E		
				1,000 ϵ	σ	C_M (Eq. 42*a*)	1,000 ϵ	σ	C_M (Eq. 42*a*)
	(*a*) PRINCIPAL QUANTITIES								
1	Limiting elastic condition	264	0.0687	0.55	16.5	+11	1.1	33	−22
2	Limit design	352	..	..	33	+22	..	33	−22
3	Inelastic condition	384.9	0.1743	1.1	33	+22	22.25	35.49	−26.11
	(*b*) RESIDUAL QUANTITIES								
4	Loading	384.9	0.1743	+1.1	+33	+22	+22.25	+35.49	−26.11
5	Unloading	384.9	−0.1001	−0.80	−24.05	−16.04	− 1.60	−48.10	+32.07
6	Residual condition	384.9	0.0742	+0.30	+ 8.95	+ 5.96	+20.65	− 12.61	+ 5.96

$$I = \tfrac{1}{12} A_w h^2 + 2\, A_f \left(\frac{h}{2}\right)^2 = \tfrac{1}{3} A_w h^2 \qquad (43)$$

since $\frac{A_f}{\frac{1}{2} A_w} = \kappa = 1$.

The residual state of stress (RS) is determined by adding to the moments, stresses, and strains determined in Table 2(*a*) the corresponding functions found by the elastic application of the same loads in the opposite direction.

This is done in Table 2(*b*). The functions added to the original ones are stated in line 5. They are found by increasing the values pertaining to "Limiting Elastic Condition" in Table 2(*a*) in the ratio of the loads, and changing the signs. The negative sign of stresses and strains in Table 2(*b*) indicates an action opposite to the one created by the original loading.

15. Example 2

Given a fixed-ended I-beam with $\kappa = 1$, loaded with a concentrated load P at one eighth of its span length (see Fig. 24), find the value of the load for which the moment under the load just reaches the yield point. Find also the corresponding stresses and strains.

The conditions in this problem are purposely assumed so as to create a great disparity in the moments at the three critical points—that is, at the two ends and at the load. The structure is twice statically indeterminate. For this reason, the solution involves guessing at two moments or corresponding strains at once, and necessitates at least three trial assumptions instead of the two in the first problem. The moment and strain under the load are known, and the strains at the ends are assumed, using some judgment and bearing in mind the relation between the moments in the elastic range. With the conditions of strain thus known, the angle change on the entire length of the beam and the deflection of end E in relation to the tangent at end E_1 are determined and compared to zero, which value they should possess to satisfy the conditions of restraint. These deformations are found by the double application of Eqs. 12 and 21*c* in the same manner as in Example 1. The quantities entering these formulas are mostly taken from Table 1(*a*) or are computed by geometry in Fig. 24. They are all conveniently arranged in Table 3. The values of the strain functions below the proportional limit are determined by comparison with the values at the proportional limit, where $\epsilon = 1.1\ (10^{-3})$, by the following relations evident from Eqs. 26: m is proportional to ϵ, n is proportional to ϵ^2, and u is proportional to ϵ^3.

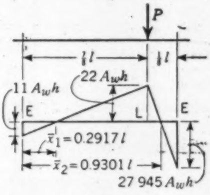


FIG. 24

TABLE 3.—TRIAL 1 (SEE FIG. 24)

Quantity	Point E		Point L		Point E_1
$1{,}000\,\epsilon$	0.55		1.1		28
m (kips/in.^{2})	22		44		55.89
$M\dfrac{1}{A_w h}\left(\dfrac{\text{kips}}{\text{in.}^2}\right)$	11		22		27.945
$1{,}000\,n$ (kips/in.^{2})	6.05		24.2		182.9
$1{,}000\,u$ (kips^2/in.^{4})	88.74		709.9		9,045
$1{,}000\,(n_0 - n_{01})$ (kips/in.^{2})		18.15		158.7	
$1{,}000\,(u_0 + u_{01})$ (kips^2/in.^{4})		798.64		9,754.9	
$V\dfrac{l}{A_w h}$ (kips/in.^{2})		37.72		399.6	
$\bar{x}(1/l)$		0.2917		0.9301	

Trial 1.—From values assembled in Table 3: $\phi = \left(\frac{18.15}{37.72} - \frac{158.7}{399.6}\right) 10^{-3}\frac{l}{h} = +\,0.0836\,(10^{-3})\,\frac{l}{h}$; and $\delta = \left\{\left[\frac{1}{2}\,\frac{798.64}{(37.72)^2} + \frac{18.15}{37.72}\,(0.2917)\right] - \left[\frac{1}{2}\,\frac{9{,}754.9}{(399.6)^2} + \frac{158.7}{399.6}\,(0.9301)\right]\right\} 10^{-3}\,\frac{l^2}{h} = +\,0.0211\,(10^{-3})\,\frac{l^2}{h}$.

TABLE 4.—RESULT OF SUCCESSIVE TRIALS, EXAMPLE 2

Trial (1)	$10^3\,\epsilon_E$ (2)	$10^3\,\epsilon_L$ (3)	$10^3\,\epsilon_{E1}$ (4)	$C\varphi = \varphi(10)^3\,h/l$ (5)	$C\delta = \delta(10)^3\,h/l^2$ (6)
One....	0.55	1.1	28	+0.0836	+0.0211
Two....	0.77	1.1	28	−0.1085	−0.0346
Three..	0.55	1.1	30	−0.0034	−0.0650

Two other similar trials are now made, changing the values of the assumed unknown strains ϵ_E and ϵ_{E1}, one at a time. The corresponding values of ϕ and δ are computed, and the results are given in Table 4.

Assuming a linear variation of ϕ and δ with ϵ_E and ϵ_{E1} in the vicinity of zero, the rates of variation are:

$$\frac{\partial\phi}{\partial\epsilon_E} = \frac{-\,0.1085 - 0.0836}{0.77 - 0.55}\,\frac{l}{h} = -\,0.873\,\frac{l}{h}\ldots\ldots\ldots\ldots(44a)$$

$$\frac{\partial\delta}{\partial\epsilon_E} = \frac{-\,0.0346 - 0.0211}{0.77 - 0.55}\,\frac{l^2}{h} = -\,0.253\,\frac{l^2}{h}\ldots\ldots\ldots\ldots(44b)$$

$$\frac{\partial\phi}{\partial\epsilon_{E1}} = \frac{-\,0.0034 - 0.0836}{30 - 28}\,\frac{l}{h} = -\,0.0435\,\frac{l}{h}\ldots\ldots\ldots\ldots(44c)$$

and

$$\frac{\partial\delta}{\partial\epsilon_{E1}} = \frac{-\,0.0650 - 0.0211}{30 - 28}\,\frac{l^2}{h} = -\,0.0430\,\frac{l^2}{h}\ldots\ldots\ldots\ldots(44d)$$

The unknown increments $d\epsilon_E$ and $d\epsilon_{E1}$, which must be added to the values of the unknowns ϵ_E and ϵ_{E1}, assumed in the first trial, to make ϕ and δ both equal to zero, are found from the two simultaneous equations:

$$-\,0.873\,d\epsilon_E - 0.0435\,d\epsilon_{E1} + 0.0836\,(10^{-3}) = 0$$
$$-\,0.253\,d\epsilon_E - 0.0430\,d\epsilon_{E1} + 0.0211\,(10^{-3}) = 0$$

From these: $d\epsilon_E = 0.101\,(10^{-3})$; and $d\epsilon_{E1} = -\,0.108\,(10^{-3})$. Then the required strains are $\epsilon_E = (0.55 + 0.101)10^{-3} = 0.65\,(10^{-3})$; and $\epsilon_{E1} = (28 - 0.108)\,10^{-3} = 27.9\,(10^{-3})$.

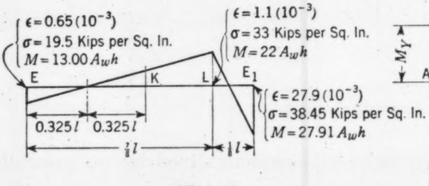

$\epsilon = 0.65(10^{-3})$
$\sigma = 19.5$ Kips per Sq. In.
$M = 13.00\,A_w h$
$\epsilon = 1.1\,(10^{-3})$
$\sigma = 33$ Kips per Sq. In.
$M = 22\,A_w h$
E
K
L
E_1
$\epsilon = 27.9\,(10^{-3})$
$\sigma = 38.45$ Kips per Sq. In.
$M = 27.91\,A_w h$
$0.325\,l$
$0.325\,l$
$\frac{7}{8}l$
$\frac{1}{8}l$

FIG. 25

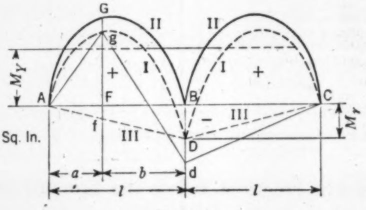


FIG. 26

The values of stresses and moments (Fig. 25) are now found by interpolation from Table 1(*a*). The load P equals the sum of the shearing forces in the two parts of the beam:

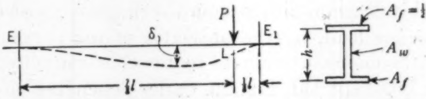
$$P = \left(\frac{13.00 + 22.00}{0.875} + \frac{22.00 + 27.91}{0.125} \right) \frac{A_w h}{l}$$

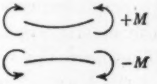
$$= 439.32 \left(\frac{\text{kips}}{\text{in.}^2} \right) \frac{A_w h}{l} = C_P \frac{A_w h}{l} \ldots\ldots\ldots\ldots\ldots (45a)$$

The maximum deflection δ occurs in the elastic region of the beam at point K, Fig. 25, where the slope is zero. It is found by the elastic formula:

TABLE 5.—DEFLECTIONS, STRAINS, STRESSES, MOMENTS UNDER CAPACITY LOADINGS

Load	Description
1	Limiting elastic condition in which P is such that the yield point is first reached in the beam.
2	Limit design.
3	Inelastic condition 1; beam loaded as in Example 2.
4	Inelastic condition 1; residual quantities after removal of load.
5	Inelastic condition 2; beam loaded, and the yield point is just reached at the far end E.
6	Inelastic condition 2; residual quantities.





			(a) End E			(b) Point L			(c) End E		
Load (description above)	$C_P = P\frac{l}{A_w h}$ (kips/in.²)	$C\delta = \delta(10)^3 \frac{h}{l^2}$	1,000 ϵ	σ (kips/in.²)	$C_M = M\frac{1}{A_w h}$ (kips/in.²)	1,000 ϵ	σ (kips/in.²)	$C_M = M\frac{1}{A_w h}$ (kips/in.²)	1,000 ϵ	σ (kips/in.²)	$C_M = M\frac{1}{A_w h}$ (kips/in.²)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
1.....	248.72	0.0230	+0.157	+4.71	−3.14	+0.275	+8.125	+5.5	+1.1	+33	−22
2.....	402.29	..	..	+33	−22	..	+33	+22	..	+33	−22
3.....	439.32	0.0915	+0.65	+19.5	−13.00	+1.1	+33	+22	+27.9	+38.45	−27.91
4.....	0	0.0515	+0.35	+10.5	−7.00	+0.575	+17.25	+11.5	+25.8	−24.55	+14.1
5.....	554.48	0.2313	+1.1	+33	−22	+23.5	+36.18	+26.51	+62	+50.06	−35.87
6.....	0	0.1905	+0.72	+21.63	−14.42	+22.84	+16.26	+13.23	+59.35	−29.54	+17.19

$$\delta = -2\,(0.65) \times (10^{-3}) \times \tfrac{1}{2}\,(0.325)\,\tfrac{2}{3}\,(0.65)\,\frac{l^2}{h} = -0.0915\,(10^{-3})\,\frac{l^2}{h} \ldots (45b)$$

These results are incorporated in Table 5, together with the conditions of loading, given for comparison.

16. EXAMPLE 3

Given a two-span continuous I-beam (Fig. 26), with $\kappa = 1$ acted on by a movable concentrated load; find the value of the load for which the stress at the support just reaches the yield point. It is assumed that the passage of the load producing this stress condition is preceded by several passages of smaller loads, whose values increase gradually to the capacity value.

A concentrated load passing over the beam creates certain maximum positive moments under the load; and in numerical value these moments exceed the negative moments produced at the middle support. Therefore, under the conditions of the problem, the inelastic deformations occur only under the loads. Curve I in Fig. 26 represents the curve of maximum positive moments. The middle sections of both spans, on which the moments exceed the yield point value $M_Y = 22\,A_w\,h$, undergo inelastic deformations, making the beam concave upward. When the load is removed, the residual deformations demand the presence of downward reactions at the outer supports to preserve the conditions of restraint. Therefore, the residual moment condition is such that it can be represented by the line III, Fig. 26. The final location of curve III has been reached gradually in the course of several previous passages of the increasing load P, and it will not be displaced farther, on a new passage, as long as the load P remains constant.

Curve II, Fig. 26, represents the curve of positive moments existing under the load, caused by the elastic application of the load P. When load P arrives at an arbitrary point F, the actual moment diagram is AgdC, Fig. 26. By the principle expressed in Eq. 41, this diagram differs from the diagram caused by the elastic application of the same load, by the ordinates of the curve III. Therefore, $\overline{\mathrm{Gg}} = \overline{\mathrm{fF}}$, and $\overline{\mathrm{Dd}}$ is the moment produced by the elastic application of the load. The value of the intercept Dd, Fig. 26, varies depending on the location of the load. Since the intercept BD is constant, the greatest negative bending moment Bd occurs at the same position of the load as in the elastic range, and by the conditions of the problem it is equal to 22 (kips/in.^{2}) $A_w\,h$.

The solution of this problem involves the use of the elastic formulas for the moments and consists of the following operations:

a. The unknown load P is assumed;
b. Using this value of P, the maximum elastic ordinate dD and the ordinates of the curves II and III are computed;
c. Curve I is found by combining the curves II and III;
d. Residual deflection of end A, Fig. 26, in relation to the tangent at the central support B, caused by the moments of curve I is computed by Table 1(*b*); and
e. The reaction necessary to bring end A back is computed, and the moments produced by it are compared with curve III, to check the assumed value of the load P.

In computing the deflection, curve I is replaced by a polygon. Although this curve does not represent a moment diagram for any single position of the load, it must be treated as if it were one. The slopes of its polygonal sides must be used as the shearing forces in the corresponding formulas for the deflection.

The elastic formulas are as follows (Fig. 26):

$$M_B\,(\text{max}) = 0.0961\,P\,l \ldots\ldots\ldots\ldots\ldots\ldots (46a)$$

and

$$M_f = \frac{P\,a\,b}{4\,l^3}\left[4\,l^2 - a(l + a)\right] \ldots\ldots\ldots\ldots\ldots\ldots (46b)$$

Trial 1.—In Fig. 26, let $P = 136\ \frac{\text{kips}}{\text{in.}^2} \times \frac{A_w\,h}{l}$. Then, by Eqs. 46, the elastic moment, $M_B\,(\text{max}) = 13.08\ (\text{kips/in.}^2)\ A_w\,h$; the residual moment at point B is $M_r = (22 - 13.08)\ A_w\,h = 8.92\ (\text{kips/in.}^2)\ A_w\,h$; and the moments of curve III, Fig. 26, are $M_{\text{III}} = -\frac{a}{l}\,(8.92)\ (\text{kips/in.}^2)\ A_w\,h$. In Table 6, the ordinates of curve I represent the difference between curves II and III. Curve I, slightly simplified, is shown in Fig. 27. The ordinates smaller than 22 $A_w\,h$ are immaterial, since they do not produce inelastic deformations. All quantities needed in computation are conveniently arranged in Table 7. The data are taken from Table 1(*b*) and Fig. 27.

TABLE 6.—Coefficients $C_M = \frac{M}{A_w\,h}$, in Kips per Square Inch, for Finding the Ordinates of Curve I

(See Fig. 26)

a/l	Curve II	Curve III	Curve I
0.266	24.37	−2.37	22.00
0.3	25.77	−2.68	23.09
0.35	27.31	−3.12	24.19
0.39	27.96	−3.48	24.48
0.40	28.09	−3.57	24.52
0.41	28.15	−3.66	24.49
0.45	28.17	−4.01	24.16
0.50	27.61	−4.46	23.15
0.537	26.79	−4.79	22.00

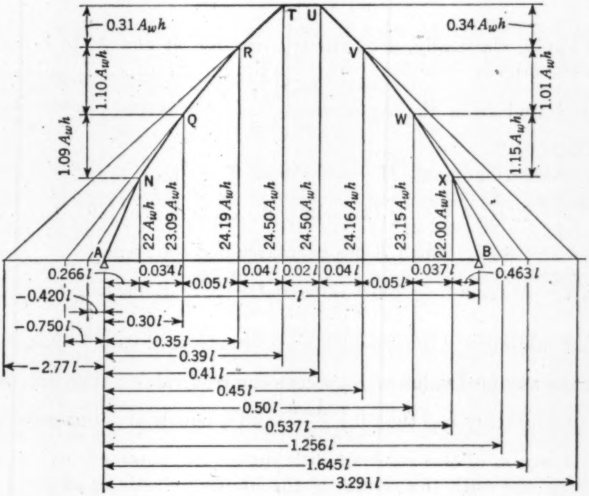

0.31 $A_w h$
T
U
0.34 $A_w h$
R
V
1.10 $A_w h$
1.01 $A_w h$
Q
W
1.09 $A_w h$
1.15 $A_w h$
N
X
22 $A_w h$
23.09 $A_w h$
24.19 $A_w h$
24.50 $A_w h$
24.50 $A_w h$
24.16 $A_w h$
23.15 $A_w h$
22.00 $A_w h$
A
B
0.266 l
0.034 l
0.05 l
0.04 l
0.02 l
0.04 l
0.05 l
0.037 l
0.463 l
−0.420 l
l
0.30 l
−0.750 l
0.35 l
0.39 l
−2.77 l
0.41 l
0.45 l
0.50 l
0.537 l
1.256 l
1.645 l
3.291 l

Fig. 27

The residual deflection of end A, Fig. 26, in relation to the tangent at middle support B is determined by applying Eqs. 21*a*, 21*b*, and 22*a* to the residual conditions. Eq. 22*a* is used for the interval TU, Fig. 27, where the strain is constant. The residual upward deflection is:

TABLE 7.—DEFORMATION DATA (SEE FIG. 27)

Point	$C_M = \frac{M}{A_w h}$ (kips/in.^{2})	m (kips/in.^{2})	1,000 ϵ	1,000 n_r (kips/in.^{2})	1,000 u_r (kips^2/in.^{4})	1,000 $(n_{rs} - n_{r1})$ (kips/in.^{2})	1,000 $(u_{rs} - u_{r1})$ (kips^2/in.^{4})	1,000 ϵ_r	$C_V = V \frac{l}{A_w h}$ (kips/in.^{2})	$\frac{x}{l}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
N	22.00	44.00	1.1	0	0					
						0.245	11.18	..	32.04	−0.420
Q	23.09	46.18	1.418	0.245	11.18					
						1.323	62.90	..	22.00	−0.750
R	24.19	48.38	2.48	1.568	74.08					
						1.006	48.93	..	7.75	−2.770
T	24.50	49.00	3.6	2.574	123.01					
						0	0	2.376	0	..
U	24.50	49.00	3.6	2.574	123.01					
						1.094	53.21	..	8.5	3.291
V	24.16	48.32	2.4	1.480	69.80					
						1.200	56.99	..	20.2	1.645
W	23.15	46.30	1.445	0.280	12.81					
						0.280	12.81	..	31.1	1.256
X	22.00	44.00	1.1	0	0					

$$\delta = 0.1994\,(10^{-3})\,\frac{l^2}{h} \quad \text{.....................} (47)$$

Now apply, elastically, a downward reaction at end A, to bring this end back to its original position. The maximum strain ϵ_e at point B, so induced, is found from the equation $2\,\frac{\epsilon_e}{h}\,\frac{1}{2}\,l\,\frac{2}{3}\,l = 0.1994\,(10^{-3})\,\frac{l^2}{h}$ as follows: $\epsilon_e = 0.2992\,(10^{-3})$.

The residual moment M_r at support B is now found by proportion: $M_r = \frac{0.2992}{1.1}\,22\,A_w\,h = 5.98$ (kips/in.^{2}) $A_w\,h$, instead of 8.92 $A_w\,h$—a discrepancy of $-2.94\,A_w\,h$. Load P has been assumed too small.

Trial 2.—Assume $P = 136.7$ (kips/in.^{2}) $\frac{A_w\,h}{l}$. Repeating the procedure of trial 1, the discrepancy in moments M_r proves to be $+0.40\,\frac{\text{kips}}{\text{in.}^2} \times A_w\,h$, showing that the assumed value of P exceeds the true value. The difference, however, is undoubtedly less than $0.1\,\frac{A_w\,h}{l}$, and a new trial is unnecessary.

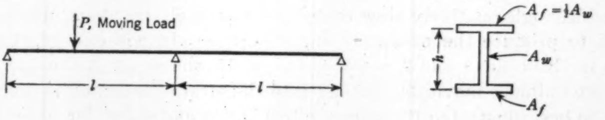
A comparison of this solution with the results obtained by the method of limit design and with the results of the limiting elastic condition is given in Table 8.

Some error is involved in this solution because an inscribed polygon has been substituted for the curved line of maximum moments. However, even if the residual deflection found on the basis of this polygonal line may be somewhat in error, the discrepancy in P should be negligible, because only a small change in P is needed to affect the deflection appreciably.

If the conditions of the problem are modified in such a way that the load P is assumed to possess its full capacity value on the first passage, rather than to be built up gradually in the course of previous passages, the problem becomes very laborious, although possible of solution. The magnitude of the capacity load found under those circumstances will be smaller.

TABLE 8.—COMPARISON OF RESULTS, EXAMPLE 3

(Units of C_P, C_M and σ Are in Kips/In.^{2})



Description	$C_P = P \frac{l}{A_w h}$ $\left(\frac{\text{kips}}{\text{in.}^2}\right)$	(a) Under Load			(b) At Center Support		
		1,000 ϵ	σ (kips/in.^{2})	$C_M = M \frac{1}{A_w h}$ $\left(\frac{\text{kips}}{\text{in.}^2}\right)$	1,000 ϵ	σ (kips/in.^{2})	$C_M = M \frac{1}{A_w h}$ $\left(\frac{\text{kips}}{\text{in.}^2}\right)$
Limiting elastic condition.	105.9	1.1	33	22	0.51	15.27	10.18
Limit design.	126.1	. .	33	22	. .	33	22
Inelastic condition.	136.7	5.9	33	24.68	1.1	33	22

17. LIMIT DESIGN

Consider a statically indeterminate structure subjected to flexure, such as one of the beams analyzed in the foregoing examples, and let the loading acting on it increase gradually in intensity. The increase in loading is accompanied by an increase in stresses, and several sections of the beam, one by one, reach the yield point. When the number of such sections at and beyond the yield point, referred to as critical sections, is one larger than the number of static unknowns, the capacity loading is reached, since a small further increase in intensity leads quickly to a large deformation.

In his theory of limit design Professor Van den Broek proposes that the capacity loading be found by assuming that the values of the bending moments at all the critical sections are equal to the value for which the yield stress is just reached. This assumption underestimates, somewhat, the moments at the critical sections reached earlier; but on the whole it is not far from truth, and (what is important) it permits finding the capacity loading by statics alone.

It is conceivable, however, that before the structure yields at the last critical section it may fail at the first critical section, and this vital point must be cleared before the theory of limit design can be accepted. This question can be answered only by an exact theory of bending, such as the one presented in this paper.

In the foregoing examples, the true capacity loadings, together with the accompanying states of stress, have been determined and compared with the limiting elastic loadings and the loadings found by the limit design. Some features of the mechanical behavior of structures, made apparent by these examples, are worthy of notice.

The moment at the first critical section does not remain constant, as the moment at the second section approaches yielding, but it increases. This increase, although relatively slow compared to the elastic stage, must still be sufficient to produce the necessary angle change. In the case of stationary loads, as in Examples 1 and 2, Sections 14 and 15, the strain at the first critical section must almost inevitably extend into the strain-hardening region. This fact can be best illustrated by reference to Fig. 22, assuming, for simplicity, an I-beam consisting mostly of flanges with practically no web. As the positive moment increases, the positive angle change increases also, demanding (by the conditions of restraint) an equal negative angle change at the support; but, as long as the flange stress at the end of the beam remains within the plastic range, the end moment M_B must remain constant, and virtually no increase in the negative angle change is possible, because all yielding is concentrated on an infinitesimal length near the end. If the stress-strain curve of the material of the beam contains no rising part corresponding to strain hardening, the beam must fail at the ends under a load no larger than the one which causes first yielding.

The presence of the web in an I-beam does not modify this condition appreciably, as may be noted from Eq. 13*b* and Table 1(*a*). Eq. 13*b* shows that the angle change occurring on a constant length x_0 of a beam subjected to a triangular moment (Fig. 7) varies as the ratio $\frac{n_0}{m_0}$ corresponding to the stress condition at the end; and this ratio, according to Table 1(*a*), remains nearly constant in the later part of the plastic range, reaching a limit which in case of $\kappa = 1$ is only some 30% higher than the value at the end of the elastic part. In the strain-hardening region, the ratio $\frac{n_0}{m_0}$ again increases quickly, so that its value, required by the conditions of deformation, is soon reached.

An important conclusion can be drawn from this discussion: It is not the plastic part of the stress-strain curve, or at least not this part alone but mostly the strain-hardening part, that makes possible a favorable distribution of moments in the region beyond the elastic stage, as visualized in the theory of limit design.

The steepness and length of the strain-hardening curve, characterized roughly by the ratio of the ultimate stress and the yield-point stress and by the percentage elongation at failure, are very significant. In mild steel these factors are quite favorable, and the strains at the points of critical moments, as a rule, do not extend beyond the very beginning of the strain-hardening range. Even in the extreme case of Example 2, characterized by a great disparity of critical moments in the elastic stage, the strain at the first critical section only reaches the value of 6.2%, which is far short of failure, when the third critical section has just entered the yielding stage. The result may be quite different, however, with some alloy steels and nonferrous metals, possessing a much smaller ratio of the ultimate stress to yield-point stress than the mild steel.

In Example 3, dealing with a movable load, the region of inelastic strain as distinguished from that in the two other examples, is spread over a large length of the span; and, as a result, an ample angle change, relieving the moments in the middle parts of the spans, is produced by only a moderate extension of strain into the plastic region.

Comparison of the capacity loadings found by different methods in the three examples shows that the limit-design values are always on the safe side, which is the direct result of underestimating the moment carried by the beam at the first critical point. This margin of difference is larger with the stationary loads than with the movable loads, and in beams with $\kappa = 1$ it seems to be usually of the order of from 7% to 10%.

The only condition that may require a special investigation in this connection is the case cited in Section 16 when the moving loads are considered to have their full values on the first passage. Limit design can offer no answer to this problem, other than the one assuming a gradual pre-stressing of the beam by smaller loads—an assumption tending to overestimate the capacity load and to make the design unsafe. This tendency is compensated for, however, by an opposite tendency, as explained; and, when appreciable stationary loads occur at the same time as moving loads, the limit design is probably always safe.

Findings of the exact theory are thus very favorable to the theory of limit design when it is applied to mild steel and when the instability failure is not impending.

It is unfortunate that continuous, or fixed-ended, steel beams are rather rare in building construction. This condition limits the practical field of applicability of the flexural theory of limit design. On the other hand, the possibility of the use of this theory in the design of beams provided with the usual web and flange-angle connections, assumed ordinarily in design as simply supported, seems to be worthy of serious study. If the connections of such beams are sufficiently strong, although quite deformable, the end moments will be brought into action eventually, after some yielding at the center, and the design on the basis of end restraints may prove economical, as well as safe. Such an investigation would have to be based on the flexural theory presented herein and on the experimental data pertaining to the strength and rigidity of the end connections.

18. CONCLUSIONS

The material presented in this paper warrants the following conclusions:

1. The proposed flexural theory provides (within the scope of its assumptions) a rigorous method of determining a complete picture of stress and deformation in statically indeterminate or determinate I-beams and channel

beams, irrespective of whether or not the material of the beam obeys Hooke's law. Both the loaded state and the residual state, remaining after the removal of the load, are susceptible to analysis.

2. Favorable redistribution of bending moments, occurring beyond the yield point and taken advantage of in the theory of limit design, is made possible partly by the yielding characteristic of steel, but mostly by its strain-hardening characteristic.

3. Flexural structures of mild steel will not fail at the point of the greatest stress before the capacity load (characterized by approach of large deflections) is reached.

4. Barring the possibility of a failure caused by instability, the value of the capacity load of a flexural structure, found by limit design is on the safe side, with perhaps some rare exceptions when moving loads are involved.

5. Limit design may prove unsafe when applied to alloy steels and nonferrous metals in which the ratio of the ultimate stress to the yield-point stress is comparatively low.

6. The question of applying limit design to steel beams provided with the usual end connections is worthy of study, with a view to placing reliance in design on the end restraints of such beams.

APPENDIX. NOTATION

The following letter symbols, adopted for use in this paper and for the guidance of discussers, conform essentially to American Standard Letter Symbols for Mechanics (ASA—Z10.3—1942), prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942. In general, a subscript C denotes "center"; E denotes "end"; e denotes "elastic"; f denotes "flange"; r denotes "residual"; u denotes "ultimate"; w denotes "web"; and Y denotes "yield."

A = area of cross section;

b = breadth, or width;

C = a coefficient with various subscripts before the expressions of different functions determined in the examples of Sections 14, 15, and 16:

$$C_M = M \times \frac{1}{A_w h} \text{ (Eq. 42}a\text{)}; \qquad C_P = P \times \frac{l}{A_w h} \text{ (Eq. 45}a\text{)};$$

$$C_V = V \times \frac{l}{A_w h} \text{ (Eq. 42}c\text{)}; \qquad C_w = w \times \frac{l^2}{A_w h} \text{ (Eq. 42}b\text{)};$$

$$C_\delta = \delta \times \frac{h}{l^2} \text{ (Eq. 42}d\text{)}; \qquad C_\phi = \phi \times \frac{h}{l};$$

d = distance or length; d in Fig. 7 is the distance from the point of zero moment, point A;

E = modulus of elasticity;
h = height, or depth;
l = span length of beam;
M = bending moment;
m = derived moment function of the material of the beam for an I-beam, Eq. 4;
m_1 = derived moment function of the material of the beam for a rectangular beam, Eq. 1;
n = derived slope function of the material of an I-beam, Eq. 10;
P = concentrated load;
q = derived shear function of the material of an I-beam, Eq. 6*b*;
u = derived deflection function of the material of an I-beam, Eq. 17*b*;
V = total shear;
w = load per unit length of span;
x = distances along the longitudinal axis of a beam; dx = an element of length x;
y = distances from the neutral axis;
δ = deflection;
ϵ = unit strain;
κ = ratio $A_f/\frac{1}{2} A_w$;
σ = unit bending and direct stress;
τ = unit shear stress; and
ϕ = angle change.

DISCUSSION

GEORGE WINTER,[6] M. ASCE.—In presenting this comprehensive and realistic solution of the problem of inelastic bending, the author has performed an unusually valuable service to the profession.

In his discussion[7] of the pioneering paper on "Theory of Limit Design" by J. A. Van den Broek, M. ASCE, the writer stated:

> "Any design theory that does not provide the means of determining the deformations corresponding to any given load up to the ultimate one is incomplete and questionable."

It is precisely this problem which is ably solved in this paper. Other attempts in this direction have been made, including the writer's sketchy treatment in the quoted discussion and some work in Great Britain. However, the paper is the first to be, at the same time, complete and practical. The extension to the strain-hardening regions and the discussion of shear stresses are particularly valuable.

From a practical point of view, the method is necessarily rather complex for use in design. The confirmatory evidence it presents for the original, simpler method of limit design is therefore doubly welcome. Although limited in design application, the author's paper should prove particularly valuable in research work in the interpretation of results of tests beyond the elastic range.

The main point of this discussion is to register a sincere request that the complete form of Table 1 be published with the closing discussion. The practical use of these methods is all but impossible without the complete tables. Indeed, as the author notes correctly, Figs. 15 and 16 are not necessarily accurate enough for numerical work. On the other hand, these tables are computed for values of κ not exceeding 1.5. Cases in which κ is larger, however, are frequent in standard wide flange sections. Thus, for example, for a 14 by 16, 426-lb wide flange section, the value of κ is of the order of 3.0. Similar values may also occur in built-up sections.

It would be of great practical value, indeed, if the author agreed to extend his tables to higher values of κ. For general use, the practical value of the paper would be greatly enhanced if it were possible to reproduce the complete and enlarged tables in the author's closing discussion.

D. C. DRUCKER,[8] JUN. ASCE.—The analysis by Professor Hrennikoff is based not only on plane sections remaining plane in the usual sense (that is, longitudinal strain proportional to distance from neutral axis), but also on the applicability of the stress-strain diagram for simple tension (Fig. 1). Unfortunately, the problem is far more complicated than any for which exact solutions of the idealized theory of plasticity have been found. It is the purpose of this discussion to indicate the necessity for a broader stress-strain relationship when vertical shear stresses are large, and to suggest that the elementary approach may seriously overestimate the load carrying capacity of short WF-beams or I-beams especially when fixed ended.

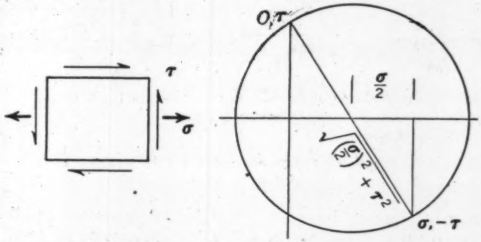
---

[6] Associate Prof., Civ. Eng., Cornell Univ., Ithaca, N. Y.

[7] *Transactions*, ASCE, Vol. 105, 1940, p. 673.

[8] Asst. Prof. of Mechanics, Illinois Inst. of Tech., Chicago, Ill.

In the case of rectangular steel beams or any beam in which the shearing stress is small compared with the bending stress, once the unstable plastic yielding of the outer fibers is complete and the strain-hardening region is reached, the author's careful work undoubtedly will give very useful and accurate information. However, WF-beams and other structural shapes are so proportioned that the vertical shearing stresses are appreciable and, if concentrated loads act near the end of a beam (Fig. 24), may control the design. The larger the ratio of depth to span, h/l, the more important the shear stress becomes.



(*a*) STATE OF STRESS (*b*) MOHR'S CIRCLE

FIG. 28.—STRESS RELATIONS AT JUNCTION OF WEB AND TOP FLANGE NEAR RIGHT END OF BEAM WITH FIXED-ENDED CONDITIONS

An element at the junction of the web and top flange near the right support of a fixed-ended beam will be subject to tensile stress σ, and shearing stress τ, as shown in Fig. 28. A ratio of 20/12 for σ/τ is somewhat low but reasonable for a fixed-ended WF-beam, although usually too low for I-beams. However, it is interesting to note that if $h/l = \frac{1}{12}$ in Example 2, the nominal shearing stress V/A_w is greater than 12 kips per sq in. when the bending stress is 20 kips per sq in. Substituting $\sigma = 20$ kips per sq in., and $\tau = 12$ kips per sq in., the maximum shear stress at the point is $\tau_{\max} = \sqrt{(\sigma/2)^2 + \tau^2} = 15.6$ kips per sq in., ignoring, as is customary, stress concentrations at the fillets and the high stresses actually present near supports or concentrated loads.

Experiments show that the plastic state in structural steel starts when the maximum shear strain energy reaches a limiting value. This criterion is more accurate but not so convenient in nominal stress analysis as the maximum shear stress basis; that is, yielding will occur when $\tau_{\max}$ equals the yield point in shear—approximately 19 kips per sq in. for a tensile yield point of 33 kips per sq in. As all the elastic stresses are proportional to the load, this means that yielding will occur when the bending stress is $20 \times \frac{19}{15.6} = 24.3$ kips per sq in., or only three fourths of the yield point in tension (33 kips per sq in.). The closer the point to the neutral axis, the smaller the ratio of the bending stress to the tension yield point.

The distribution of shear over the cross section of an I-beam or WF-beam, Fig. 29, will often greatly reduce the load required for initial yielding. Although the effect of such local yielding on the main part of the flange can only be guessed, it would be reasonable to expect a decrease in the yield strength.

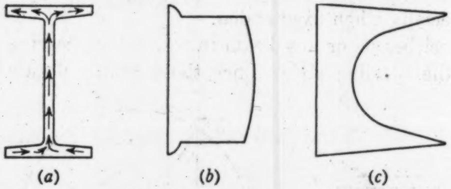


FIG. 29.—DISTRIBUTION OF SHEARING STRESS ON BEAM CROSS SECTION:

(b) NOMINAL, $-\frac{V\,Q}{I\,b}$

(c) ACTUAL, AT FIXED SUPPORTS ALONG CENTER LINE OF WEB

No matter what the ultimate effect on strength, it becomes necessary to think in more general terms than the simple tension test and so Sections 6 and 11 on shear stresses do not seem valid. The resulting shear stress of 24.5 times the average is certainly not a possibility in an I-beam, because yield must occur long before so high a stress is reached (at 19 kips per sq in. or less for the steel of the illustrative examples). As the integral of the shear stresses over the area of the cross section must give the total shearing force, the vertical shearing stresses in the zone of yielding cannot disappear; nor, correspondingly, can the bending stresses at two neighboring sections become equal in the manner shown in Fig. 18. Certainly, the distribution of the bending stresses is not given simply by the stress-strain diagrams for tension alone, but is a function of the shearing stress or strain as well as the bending strain.

BY MICHAEL R. HORNE,[9] ESQ.—

Methods of Stress Analysis Beyond the Elastic Limit.—Professor Hrennikoff's method for dealing with inelastic beams is ingenious, and has the advantage that, once the necessary tables have been compiled, it can be applied to any members of uniform section symmetrical about the plane of bending. However, it suffers from the very serious disadvantage that for complicated problems the application of Eqs. 12 and 21 becomes very involved. Many problems concerned with the elastic bending of beams may be solved quite simply by means of the three-moment theorem, and a similar method may be applied when the bending ceases to be elastic. In effect, it is possible to replace Eqs. 12 and 21 by a single relationship between the bending moments and deflections at three successive points in a beam.

Consider a beam ABC (Fig. 30) in which the bending moments at points A, B, and C are M_A, M_B, and M_C, respectively. It is assumed that between points A and B and between points B and C the bending moment diagram consists of straight lines. Let the length, depth, and web area of span AB be l_1, h_1, and A_{w1}, respectively, and let the corresponding values for span BC be l_2, h_2, and A_{w2}. If m_{AB}, and m_{BA} are the values of the function m at points

---

[9] Research Asst., Dept. of Eng., Univ. of Cambridge, Cambridge, England.

A and B for the length AB, then, from Eq. 5,

$$m_{AB} = \frac{2\,M_A}{h_1\,A_{w1}} \qquad \text{and} \qquad m_{BA} = \frac{2\,M_B}{h_1\,A_{w1}} \qquad \text{(48a)}$$

Similarly, m_{BC} and m_{CB}, the values of the function m at points B and C for the length BC, are

$$m_{BC} = \frac{2\,M_B}{h_2\,A_{w2}} \qquad \text{and} \qquad m_{CB} = \frac{2\,M_C}{h_2\,A_{w2}} \qquad \text{(48b)}$$

Let n_{AB}, u_{AB}, etc., be the values of the functions n and u corresponding to these values of m. Considering the center line of the beam, let δ_1 be the deflection at point B relative to point A, and δ_2, the deflection at point C relative to point B; δ_1 and δ_2 are both measured positive downward. Referring to Fig. 30, A′ BC′ is a straight line tangential to the center line at point B and, from the geometry of the figure,

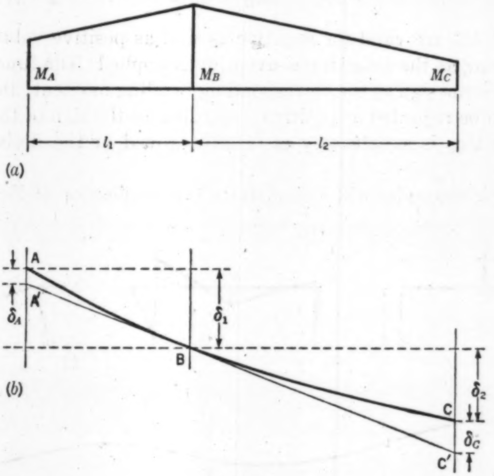


FIG. 30

$$\frac{\delta_1 - \delta_A}{l_1} = \frac{\delta_2 + \delta_C}{l_2} \qquad \text{(49)}$$

Considering span AB, Fig. 30, it is readily proved that

$$\delta_A = \frac{2\,l^2_1}{h_1}\frac{[u_{BA} - u_{AB} - m_{AB}\,(n_{BA} - n_{AB})]}{(m_{BA} - m_{AB})^2} \qquad \text{(50a)}$$

Similarly, for span BC,

$$\delta_C = \frac{2\,l^2_2}{h_2}\frac{[u_{BC} - u_{CB} - m_{CB}\,(n_{BC} - n_{CB})]}{(m_{BC} - m_{CB})^2} \qquad \text{(50b)}$$

Substituting Eqs. 50*a* and 50*b* in Eq. 49,

$$\frac{l_1}{h_1}\frac{[u_{BA} - u_{AB} - m_{AB}(n_{BA} - n_{AB})]}{(m_{BA} - m_{AB})^2} + \frac{l_2}{h_2} \times \frac{[u_{BC} - u_{CB} - m_{CB}(n_{BC} - n_{CB})]}{(m_{BC} - m_{CB})^2} = \tfrac{1}{2}\left(\frac{\delta_1}{l_1} - \frac{\delta_2}{l_2}\right) \quad ..(51a)$$

If the beam ABC is uniform throughout, $h_1 = h_2$ and $m_{BA} = m_{BC}$. Hence, there is no longer any need for the double suffixes, and Eq. 51*a* becomes

$$\frac{l_1[u_B - u_A - m_A(n_B - n_A)]}{(m_B - m_A)^2} + \frac{l_2[u_B - u_C - m_C(n_B - n_C)]}{(m_B - m_C)^2} = \frac{h}{2}\left(\frac{\delta_1}{l_1} - \frac{\delta_2}{l_2}\right) \quad ..(51b)$$

Eqs. 51*a* and 51*b* are valid for negative as well as positive values of M_A, M_B, and M_C, as long as the following convention is applied: The functions m and u are given the same sign as the corresponding bending moment, and the function n is always to be regarded as positive, regardless of the sign of the bending moment. That this is satisfactory is readily proved in the derivation of Eqs. 50*a* and 50*b*.

Two simple examples will demonstrate the application of Eq. 51.

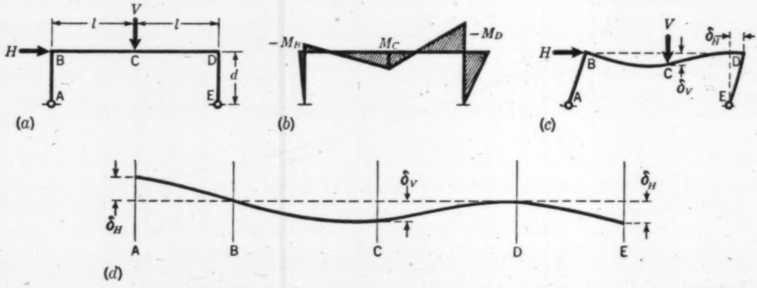


FIG. 31

Example 1.—Consider the portal frame ABDE (Fig. 31(*a*)) of uniform section throughout, supported on pins at points A and E. Let its height be d and its span $2\,l$, and let it carry a vertical load V at point C, the midpoint of span BD, and a horizontal load H applied at beam level. Let M_B, M_C, and M_D be the bending moments at points B, C, and D, respectively (Fig. 31 (*b*)), and let m_B, m_C, m_D be the corresponding values of the function m. Moments are accounted positive when they produce tension in the inner fibers of the frame. It follows from statical considerations that

$$2\,M_C - M_B - M_D = V\,l \quad \ldots\ldots\ldots\ldots\ldots\ldots (52a)$$

and

$$M_B - M_D = H\,d \qquad (52b)$$

If h is the depth of the section, and A_w the web area, then, from Eq. 50*b*,

$$2\,m_C - m_B - m_D = \frac{2\,V\,l}{h\,A_w} \qquad (53a)$$

and

$$m_B - m_D = \frac{2\,H\,d}{h\,A_w} \qquad (53b)$$

For convenience, m_B and m_C can be expressed in terms of the one unknown m_D, and, hence

$$m_B = m_D + \frac{2\,H\,d}{h\,A_w} \qquad (53c)$$

and

$$m_C = 2\,m_D + \frac{2}{h\,A_w}(H\,d + V\,l) \qquad (53d)$$

It will be assumed that all bending that results from axial loads may be neglected, so that Eq. 51*b* is applicable. The vertical deflection at point C will be denoted by δ_V and the sidesway of the frame, by δ_H. Treating the entire frame as a continuous beam (Fig. 31(*d*)), Eq. 51*b* may be applied first to the part ABC, then to the part BCD, and finally to the part CDE, thus obtaining three equations:

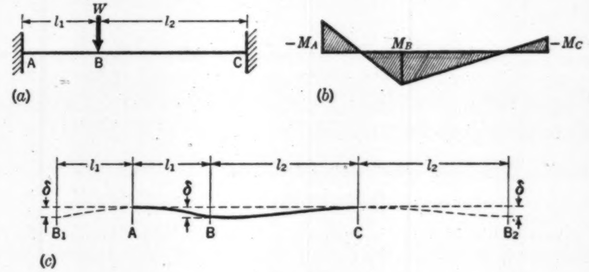


FIG. 32

$$\frac{u_B\,d}{m^2_B} + l\,\frac{[u_B - u_C - m_C\,(n_B - n_C)]}{(m_B - m_C)^2} = \frac{h}{2}\left(\frac{\delta_H}{d} - \frac{\delta_V}{l}\right) \qquad (53e)$$

$$l\,\frac{[u_C - u_B - m_B\,(n_C - n_B)]}{(m_C - m_B)^2} + l\,\frac{[u_C - u_D - m_D\,(n_C - n_D)]}{(m_C - m_D)^2} = \frac{h\,\delta_V}{l} \qquad (53f)$$

$$l\,\frac{[u_D - u_C - m_C\,(n_D - n_C)]}{(m_D - m_C)^2} + \frac{u_D\,d}{m^2_D} = -\frac{h}{2}\left(\frac{\delta_H}{d} + \frac{\delta_V}{l}\right) \qquad (53g)$$

If δ_H and δ_V are eliminated from Eqs. 53*e* to 53*g*, a further relationship is obtained between m_B, m_C, and m_D:

$$d\left(\frac{u_B}{m^2_B} + \frac{u_D}{m^2_D}\right) + l\left(\frac{n_B - n_C}{m_B - m_C} + \frac{n_C - n_D}{m_C - m_D}\right) = 0 \qquad (53h)$$

In obtaining a solution for any given values of V and H it is necessary to assume a value of m_D. Deduce the values of m_B and m_C from Eqs. 53*c* and 53*d*, and then test whether Eq. 53*h* is satisfied. When the correct values of m_B, m_C, and m_D have been obtained, the values of δ_V and δ_H may be found from Eqs. 53*e* and 53*f*.

Example 2.—In Fig. 32, consider beam AC, fixed at points A and C, and subjected to a concentrated load W at a distance of l_1 from point A and l_2 from point C. If the bending moments at points A, B, and C are M_A, M_B, and M_C, respectively, it follows from simple statics that

$$(l_1 + l_2)\, M_B - l_2\, M_A - l_1\, M_C = l_1\, l_2\, W \ldots\ldots\ldots\ldots (54a)$$

If m_A, m_B, and m_C are the corresponding values of the function m, then,

$$(l_1 + l_2)\, m_B - l_2\, m_A - l_1\, m_C = \frac{2\, l_1\, l_2}{h\, A_w}\, W \ldots\ldots\ldots\ldots (54b)$$

If δ is the deflection at point B relative to points A and C, then in applying Eq. 51*b* it is necessary that $\delta_1 = -\,\delta_2 = \delta$. Hence, two equations are available to determine the four unknowns, m_A, m_B, m_C, and δ, and it is necessary to find two more. If span AB is imagined to be reflected about end A into the position AB_1 in Fig. 32(*c*), then Eq. 51*b* may be applied to the length $B_1\,AB$, giving

$$\frac{[u_A - u_B - m_B\,(n_A - n_B)]}{(m_A - m_B)^2} = -\,\frac{h\,\delta}{2\,l^2{}_1} \ldots\ldots\ldots\ldots (55a)$$

Similarly, span BC may be reflected into the position CB_2, and the application of Eq. 51*b* to the length BCB_2 gives

$$\frac{[u_C - u_B - m_B\,(n_C - n_B)]}{(m_C - m_B)^2} = -\,\frac{h\,\delta}{2\,l^2{}_2} \ldots\ldots\ldots\ldots (55b)$$

For any given value of W it is necessary to solve Eqs. 51*b*, 54*b*, 55*a*, and 55*b* by trial and error until suitable values of m_A, m_B, and m_C are found.

The Stress-Strain Relationship of Steel in Bending.—This subject has been treated in a paper by F. G. Eric Peterson,[10] Assoc. M. ASCE, and in the subsequent discussions. As stated by J. F. Baker, Assoc. M. ASCE, and J. W. Roderick in their discussion,[11] the generally accepted theory in England is that the stress distribution is similar to that given by the tensile test when made in an autographic testing machine, care being taken to insure that the applied load is truly axial, but ignoring the effect of strain hardening, as shown in Fig. 33. Under such circumstances mild steel gives a stress-strain relationship in which the upper yield stress f_U may be between 25% and 45% in excess of the lower yield stress f_L. Professors Baker and Roderick also show that the

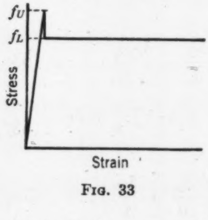


FIG. 33

---

[10] *Transactions*, ASCE, Vol. 112, 1947, p. 1201.

[11] *Ibid.*, p. 1238.

assumption of such a stress distribution accounts very well for the behavior of rectangular mild steel beams in bending. Further work by these writers, at the Engineering Laboratory of Cambridge University at Cambridge, England, in collaboration with the British Welding Research Association, has confirmed these results. The stress-strain relationship assumed by Professor Hrennikoff differs from that assumed by Professors Baker and Roderick in two respects: Professor Hrennikoff ignores the upper yield stress and takes strain hardening into account, assuming that strain hardening occurs in bending in precisely the same way as in a tension test. Most of the work done by Professors Baker and Roderick has been on beams of rectangular cross section, whereas, the paper of Professor Hrennikoff refers also to rolled beams in which local yield due to rolling stresses is certain to occur. Under these circumstances to neglect the upper yield stress is reasonable. However, the assumptions made with respect to strain hardening cannot be justified so readily. The strains at the end of the plastic range are considerable, and under such circumstances it should not be assumed too readily that the strain-hardening effect is the same in bending as in tension. It is interesting to note that the work done at Cambridge indicates that the behavior of both simply supported and continuous beams up to the loads causing large deflections may be explained satisfactorily without reference to strain hardening.

The Importance of Strain Hardening in Limit Design.—Strain hardening, obviously, has an important bearing on attempts to arrive at a design method based on the ultimate capacity of steel structures, but its importance should not be overestimated. In designing ordinary building frames by the usual elastic methods, emphasis is always placed on stresses rather than on deflections. This is because within the elastic range the strains are small. However, when an ultimate load method of design is used, the deflections become as important as the stresses, because of the greatly increased strains within the plastic range. The structure will only be useful up to that load at which the deflections become excessive. The limit method of design justifies itself because the ultimate load calculated according to the principle of equalization of moments must coincide approximately with the load at which excessive deflections first occur. This is because both the equalization of moments and the appearance of large deflections result from the same property of mild steel—its capacity to undergo plastic deformation without increase in stress. The author has now shown that the plastic yield that occurs before strain hardening begins is insufficient to cause a complete equalization of moments. Since the deflections are becoming large, at this stage, however, the ultimate capacity has been reached, so that it becomes pointless to follow the behavior of the structure further. The limit method of design achieves its purpose if it leads to a reasonable estimate of the load at which deflections begin to increase rapidly. Professor Hrennikoff's work is of particular interest since it affords a means of checking the assumption that this is actually the case. The results obtainable in this way may best be illustrated by taking a numerical example.

Example 3.—Consider a portal frame of the type shown in Fig. 31(a), in which $d = l$. Let the cross section of the member be uniform throughout, the web and flange areas being equal so that $k = 1$. Let a vertical load V first be applied so that yield stress in the extreme fibers is just reached in the center of the beam, horizontal load only being applied thereafter. With steel having the properties assumed by Professor Hrennikoff, the value of V just sufficient to produce yield stress at point C is found to be $\frac{2\,V\,l}{h\,A_w} = 140.8$ kips per sq in. The entire frame remains elastic until H has reached a value given by $\frac{2\,H\,l}{h\,A_w} = 35.2$ kips per sq in. when yield stress is just reached in the extreme fibers at point D in Fig. 31(*a*). Thereafter it is necessary to obtain solutions by the use of Eqs. 53*c* to 53*h*. The results are presented in Table 9(*a*). Cols. 2 to 7, Table 9(*a*), give the bending moments at points B, C, and D, and the corresponding extreme fiber strains, and Cols. 8 and 9 give, respectively, the vertical deflection δ_V and the horizontal deflection δ_H at the center of the beam. The bending moment at which strain hardening begins occurs first at point D at a value of $\frac{2\,H\,l}{h\,A_w} = 50.43$ kips per sq in., and subsequently at point C at a value of 58.60 kips per sq in. For the sake of comparison, results are given in Table 9(*b*), based on the assumption that the steel is capable of undergoing indefinite strain without change in stress after the attainment of yield. Where no data are given, the results are the same as in Table 9(*a*). This is equivalent to ig-

TABLE 9.—EFFECTS OF STRAIN HARDENING

Horizontal load $H\frac{2\,l}{h\,A_w}$ (kips per sq in.)	BENDING MOMENTS, $M\frac{2\,l}{h\,A_w}$ (KIPS PER SQ IN.)			EXTREME STRAINS			VERTICAL DEFLECTION, $10^3\frac{\delta\,h}{l^2}$ (IN.)	
	M_B	M_C	M_D	$1{,}000\,\epsilon_B$	$1{,}000\,\epsilon_C$	$1{,}000\,\epsilon_D$	δ_V	δ_H
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
(*a*) STRAIN HARDENING CONSIDERED								
0	−26.40	44.00	−26.40	−0.66	1.10	−0.66	0.513	0
35.20	−8.80	44.00	−44.00	−0.22	1.10	−1.10	0.513	0.586
40.00	−6.27	44.13	−46.27	−0.16	1.11	−1.44	0.515	0.677
45.00	−3.22	44.68	−48.22	−0.08	1.18	−2.41	0.531	0.796
50.00	0.56	45.96	−49.44	0.01	1.37	−11.00	0.568	0.970
50.43	0.95	46.14	−49.48	0.02	1.41	−18.00	0.574	0.990
51.03	1.51	46.40	−49.52	0.04	1.47	−18.06	0.583	1.022
55.00	5.20	48.10	−49.80	0.13	2.28	−18.49	0.660	1.236
57.20	7.20	49.00	−50.00	0.18	3.75	−18.80	0.727	1.379
58.60	8.38	49.48	−50.22	0.21	18.00	−19.13	0.816	1.517
60.00	9.38	49.78	−50.62	0.23	18.46	−19.80	1.010	1.742
65.00	12.94	50.84	−52.06	0.32	20.08	−22.00	1.597	3.824
(*b*) STRAIN HARDENING IGNORED								
0	−26.40	44.00	−26.40	−0.66	1.10	−0.66	0.513	0
50.43	0.95	46.14	−49.48	0.02	1.41	−18.00	0.574	0.990
51.03	1.53	46.42	−49.50	0.04	1.47	$-\infty$	0.584	1.022
55.00	5.50	48.40	−49.50	0.14	2.61	$-\infty$	0.679	1.267
57.20	7.70	49.50	−49.50	0.19	∞	$-\infty$	0.840	1.515

noring the effects of overstrain entirely, and complete failure occurs when equal moments sufficient to produce yield stress over the entire section have been reached at points C and D. This occurs when $\frac{2\,M_C}{h\,A_w} = -\,\frac{2\,M_D}{h\,A_w} = 49.50$ kips per sq in. and $\frac{2\,H\,l}{h\,A_w} = 57.20$ kips per sq in. Fig. 34 shows the bending moments at points C and D plotted against the horizontal load; and Fig. 35 shows the horizontal load plotted against the horizontal and vertical deflections at the center of the beam. In all cases the full lines refer to the solution taking account of the effect of strain hardening, the dotted lines indicating what happens when this is ignored. When strain hardening is taken into account the deflections begin to increase rapidly very soon after the value of $\frac{2\,H\,l}{h\,A_w}$ exceeds 57.2 kips per sq in. This represents very nearly the collapse load of the frame. Hence, the collapse load obtained by ignoring strain hardening is very nearly the same as that obtained when it is considered. The percentage differences in the various calculated quantities at $\frac{2\,H\,l}{h\,A_w} = 57.2$ kips per sq in.

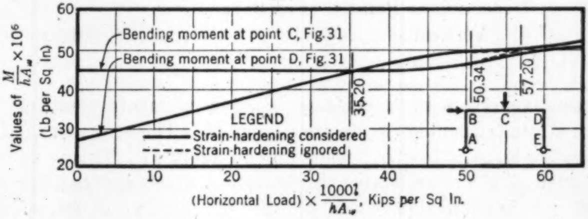


FIG. 34.—EFFECT OF STRAIN HARDENING ON BENDING MOMENTS

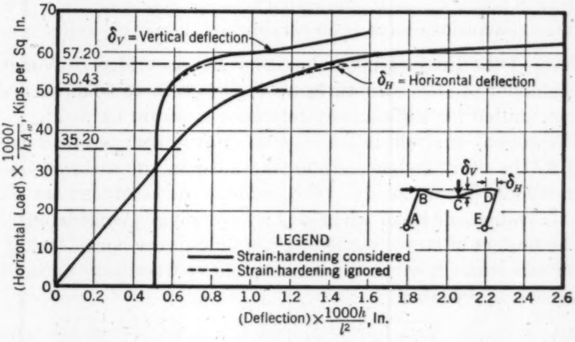


FIG. 35.—EFFECT OF STRAIN HARDENING ON DEFLECTION

are as follows:

Description	Difference (%)
Bending moment at:	
Point C, Fig. 31	1.02
Point D, Fig. 31	−1.00
Deflection at Point C (Fig. 31):	
Vertical	15.56
Horizontal	9.83

The percentage errors in calculating bending moments resulting from the neglect of strain-hardening effects are negligible. The errors in the calculated deflections are large but by no means unreasonably so. This is an interesting result in view of the considerable simplification which results when overstrain is ignored. It is particularly so in the case of a highly redundant framework, the analysis of which would involve the simultaneous solution of a number of formulas of the type of Eq. 51*b*. When overstrain is taken into account, these must all be solved by trial and error, and, obviously, with more than three such equations a solution would be virtually impossible. If overstrain is ignored, however, and if it is desired only to derive the deflections at the point of collapse, the bending moments become statically determinate everywhere, and the equations may be solved directly.

The general complexity of the behavior of rigid structures makes it difficult to draw any conclusions as to the importance of strain hardening in limit design. Three conclusions have been established, however, and are as follows:

(1) Tests at the Engineering Laboratory at Cambridge University indicate that the behavior of steel structures up to the point of collapse may satisfactorily be accounted for without any reference to strain hardening.

(2) Theoretical analysis indicates that, if the collapse load is calculated according to the limit design method, the error in the deduced values of the bending moments is very small. The deflections so deduced are somewhat in excess of the values obtained when strain hardening is considered.

(3) The neglect of strain hardening simplifies, very considerably, the stress analysis of structures beyond the elastic limit, particularly as the theoretical collapse load is approached.

It must be emphasized, however, that these conclusions are tentative. In particular, as Professor Hrennikoff indicates, the presence of semirigid joints may do much to prevent the equalization of moments, and, under such circumstances, the behavior of members in the strain-hardening range may be of considerable importance.

E. P. Popov,[12] Assoc. M. ASCE—A timely study of the theory of inelastic bending, with particular reference to mild steel, is presented in this paper by Professor Hrennikoff. Engineers engaged in the aircraft industry are re-examining the usual criteria of design based on stress and, in certain cases, are attempting to substitute in their place the criteria of excessive deflections, or the ultimate carrying capacity of the flexural members. Historically, it is of interest to note that the early elasticians were not satisfied with the direct proportionality of stress to strain dictated by the Hooke-Mariotte law. B. de Saint Venant,[13] in his notes to M. Navier's book, thus considers a generalized stress-strain relation and obtains the resulting stress distribution. In recent years most engineers are familiar with the assumption of parabolic stress distribution in concrete.[14]

---

[12] Asst. Prof., Civ. Eng., Univ. of California, Berkeley, Calif.

With the demand for rapid analytical solutions, which required simple formulas, straight-line stress distribution became universally adopted and stress became the criterion of design. This probably also followed from researches on fatigue, where stresses and stress concentrations play the major role in determining the serviceability of a member. Gradually, it became apparent that in a large number of cases reversal of loads does not occur. Hence, it is argued, stress is not the best criterion for the selection of a member. This explains the renewed activity in researches in plastic bending, and attempts are made to determine the ultimate carrying capacity of members (limit design).

All problems of plastic bending in earlier and present analyses are based essentially on the Bernoulli-Euler assumption that plane sections, before bending, remain plane after bending. Hence, it follows that strains vary linearly from the neutral axis. This assumption has a reasonable experimental support, as shown by A. Nádai,[15] based on experiments of C. V. Bach, E. Meyer, G. H. McCullough,[16] Joseph Marin, Assoc. M. ASCE, R. L. Cotterman,[17] Jun. ASCE, and others. With this basic assumption the solutions obtained and reported differ merely in form. Professor Hrennikoff chooses certain nomenclature and derives certain expressions, which, in turn, are evaluated for a particular stress-strain diagram.

In 1943 F. P. Cozzone[18] derived formulas and graphs for the plastic-range design of several materials. In his presentation the stress-strain diagram is treated as an equivalent trapezoid, having the same moment about the vertical axis as the actual stress-strain diagram. Earlier, S. Timoshenko,[19] using the conjugate beam method, and utilizing the notion of reduced modulus of elasticity, devised a semigraphical method of determining stresses and deflections. All these schemes are identical in principle and differ only in form. The solutions, obtainable by the use of these methods, should be in substantial agreement. For pure flexure, or for cases where shear is small, it appears that these solutions are in good agreement with previous tests[17,18,20] and may be used with confidence.

---

[13] "A History of the Elasticity and Strength of Materials," by I. Todhunter and K. Pearson, Cambridge Univ. Press, Cambridge, England, 1893, Vol. II, Pt. I, pp. 115–119.

[14] "Reinforced Concrete Construction," by G. A. Hool, McGraw-Hill Book Co., Inc., New York, N. Y., 1927, Vol. I, p. 60.

[15] "Plasticity," by A. Nádai, McGraw-Hill Book Co., Inc., New York, N. Y. 1931, p. 160.

[16] "An Experimental and Analytical Investigation of Creep in Bending," by G. H. McCullough, *Transactions*, A. S. M. E., 1933, p. 55.

[17] "Experiments on Plastic Bending for Aluminum Alloy 17S–T," by J. Marin and F. D. Cotterman, *Proceedings*, A. S. T. M., Vol. 43, 1943, pp. 581–599.

[18] "Bending Strength in the Plastic Range," by F. P. Cozzone, *Journal of The Aeronautical Sciences*, May, 1943, pp. 137–151.

[19] "Strength of Materials," by S. Timoshenko, D. Van Nostrand Co., New York, N. Y., Pt. II, 1941, pp. 362–373.

[20] "The Bending Modulus of Cast and Forged Sections," by N. L. Willett and L. Bonza, *Lockheed Report No. 2772*, Burbank, Calif., 1942.

To simplify his analytical treatment for I-beams and WF-beams Professor Hrennikoff also introduces a ratio κ, which is defined as a ratio of flange area to one half the area of the web. This simplification also appears reasonable, but the situation differs with shear.

The writer seriously questions whether the simple procedure developed in the paper for shearing stresses in beams is in agreement with known facts about the plastic behavior of ductile materials. Shearing stress concentrations show in Fig. 17 and, again, shearing stress distribution, shown by the dotted line in Fig. 18 of the paper, are excessively high and appear to be strangely localized. Referring to Mr. Cozzone's paper,[18] in which the same basic assumption is utilized, the shearing stress distribution is shown to be

$$\tau = \frac{V\,Q}{I\,b}\left(\frac{1 + R\left(\frac{A\,C}{Q} - 1\right)}{1 + R\,(k - 1)}\right) \ldots\ldots\ldots\ldots\ldots (56)$$

In Eq. 56 τ, V, b, and I have the same meanings as in the paper; A is the area between the fiber in question and the extreme fiber; Q is the first moment of A about the neutral axis; c is the distance to the extreme fiber; k is the section form factor ($1 \leq k \leq 2$); and

$$R = \frac{df_0}{df_m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (57)$$

which is the rate of change of intercept stress f_0 with respect to the maximum stress f_m. These latter quantities require some additional comment.

In the Cozzone analysis, in order to avoid algebraic difficulties, the actual stress-strain diagram that corresponds to any maximum stress f_m in a beam is replaced by a trapezoid. The ordinary tension test diagram is used as a basis. Let Oab in Fig. 36 represent such an actual tension test diagram. Then, for a particular moment that a beam carries, if the maximum stress is found to be f_m in the usual way, the stress distribution in that beam will be similar to area Oaf_m. The moment of the area Oaf_mA about the vertical axis through the origin is proportional to the beam's flexural capacity at stress f_m. To simplify further derivations, a trapezoid Of_0f_mA is then found such that its moment around the vertical axis through point O will be equal to the moment previously found. Thus, for a particular value of f_m there is a particular corresponding

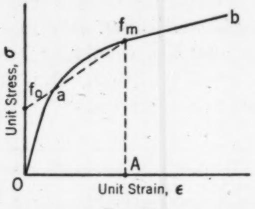


FIG. 36

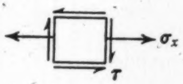


FIG. 37

value of stress intercept f_0. As the applied moment to the beam is increased, f_m increases and f_0 changes correspondingly.

In Eq. 56, R is defined as the rate of change of intercept stress f_0 with respect to the maximum stress f_m. During the plastic stage (that is in the interval between ϵ_{Y1} and ϵ_{Y2} in Fig. 13), f_m remains stationary, whereas f_0 may vary slightly. Thus, Eq. 56 reduces to

$$\tau = \frac{V\,Q}{I\,b} \qquad (58)$$

which is a well-known expression for the shearing stress distribution in the elastic range developed by D. I. Jouravsky[21] in 1855.

Thus, the analyses of Professor Hrennikoff and Mr. Cozzone in the plastic range lead to very different results. It is the writer's opinion that in the plastic range problems concerned with shear cannot be handled in such an elementary way. The flow criterion for ductile materials, such as the steel considered in Fig. 13, is expressed well by the Hencky-Mises-Huber relation.[22] In it the shear distortion energy under a compound state of stress is equated to the shear distortion energy in simple tension at the yield point of the material, σ_{yp}. In bending, with shear, the state of stress on an element in a beam is represented in Fig. 37, and the plastic flow criterion becomes:

$$\sigma_{yp} = \sqrt{\sigma^2_x + 3\,\tau^2} \qquad (59)$$

in which, as will be noted, the yield of the material is dependent on both the normal and the shearing stress.

It appears that at present no complete solution for the problem involving shear is available. However, some studies indicate[23] that, even with shear, the assumption of plane sections remains essentially valid; but shearing stresses contribute to distortion. An attempt to derive a complete solution based on elastico-plastic considerations has been made by Anton Eichinger,[23] based on some earlier work at the Swiss Federal Testing Laboratories.[24] His solution is not in an applicable form to a practical problem, yet it clearly indicates the interdependence of normal and shearing stresses in the plastic range. Thus, the analyst cannot just "rotate" the tension test diagram to obtain stress distribution in the plastic range where shear occurs. The problem involves several variables and hence it is much more complicated than the one indicated by Professor Hrennikoff.

BERNARD A. VALLERGA,[25] JUN. ASCE.—In the "Introduction" Professor Hrennikoff states that "* * * a simpler case of the same problem, involving a rectangular beam, has been treated, in a somewhat different way, by S. Timoshenko.[3]" An analysis of Professor Timoshenko's method of reduced modulus will show that the two methods are identical in so far as the determination of the resisting moment of any section is concerned. Application of the additional simplifying limitation No. 2, as listed by the author, to Professor Timoshenko's more general expression will result in identical equations for bending moment. Hereinafter is a demonstration of the preceding statement for the two cases considered.

---

[21] "Howe Bridge Trusses," by D. I. Jouravsky, Rept. of Petersburg–Moscow Railroad, Pt. II, Supplement III, St. Petersburg, Russia, 1855, p. 130.

[22] "Strength of Materials," by S. Timoshenko, D. Van Nostrand Co., New York, N. Y., Pt. II, 1941, p. 479.

[23] "Biegung mit Querkraft, ausserhalb des Gebietes der rein elastischen Formanderung," by Anton Eichinger, 2d Cong., International Assn. for Bridge and Structural Eng., *Final Report*, Berlin, 1939, English Ed., pp. 43–47.

[24] Discussion by M. Ros and A. Eichinger of "La Stabilité de L'Âme et des Ailes des Barres Comprimées," by F. Friederich Bleich, 1st Cong., International Assn. for Bridge and Structural Eng., *Final Report*, Paris, 1932, pp. 144–145.

[25] Lecturer in Civ. Eng., Univ. of California, Berkeley, Calif.

Rectangular Section.—Professor Timoshenko shows that, for a material not following Hooke's law, the resisting moment is given by the general expression,

$$M = \frac{E_R\, I}{r} \tag{60}$$

in which E_R is the reduced modulus and r is the radius of curvature of the beam due to the moment M. For a rectangular section,

$$E_R = \frac{12}{\Delta^3} \int_{\epsilon_2}^{\epsilon_1} \sigma\, \epsilon\, d\epsilon \tag{61}$$

in which Δ is the sum of the absolute values of ϵ_1 and ϵ_2. For the case of a mild steel whose compressive stress-strain diagram is assumed to be the same as the tensile stress-strain diagram (limitation No. 2),

$$\epsilon_1 = \epsilon_2 = \epsilon_0 \tag{62a}$$

and

$$\Delta = 2\, \epsilon_0 \tag{62b}$$

Therefore,

$$E_R = \frac{3}{\epsilon^3_0} \int_0^{\epsilon_0} \sigma\, \epsilon\, d\epsilon \tag{63}$$

However,

$$\frac{1}{r} = \frac{2\, \epsilon_0}{h} \tag{64}$$

and

$$I = \frac{b\, h^3}{12} \tag{65}$$

Substitution of Eqs. 63, 64, and 65 in Eq. 60 gives

$$M = \tfrac{1}{2}\, b\, h^2 \frac{1}{\epsilon^2_0} \int_0^{\epsilon_0} \sigma\, \epsilon\, d\epsilon = \tfrac{1}{2}\, m_1\, b\, h^2 \tag{66}$$

I-*Section.*—Professor Timoshenko's method can be expanded to include I-beams. Thus,

$$M = \frac{E_R\, I_w}{r} \tag{67}$$

in which, for an I-section, the general expression for reduced modulus is

$$E_R = \frac{12}{\Delta^3}\left(\int_{\epsilon_2}^{\epsilon_1} \sigma\,\epsilon\,d\epsilon + \frac{b'-b}{b}\int_{\epsilon'_1}^{\epsilon_1} \sigma\,\epsilon\,d\epsilon + \frac{b'-b}{b}\int_{\epsilon'_2}^{\epsilon_2} \sigma\,\epsilon\,d\epsilon\right) \ldots (68)$$

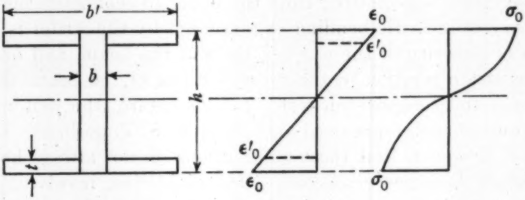


FIG. 38

If limitation No. 2 is applied to this case, together with the additional approximation that the stress does not vary across the flange (see Fig. 38), Eq. 68 becomes

$$E_R = \frac{3}{\epsilon^3_0}\left[\int_0^{\epsilon_0} \sigma\,\epsilon\,d\epsilon + \frac{b'-b}{b}\,\sigma_0\,\frac{\epsilon^2_0 - (\epsilon'_0)^2}{2}\right] \ldots\ldots\ldots (69)$$

From the geometry in Fig. 1 it is evident that

$$\epsilon_0 - \epsilon'_0 = \frac{2\,t}{h}\,\epsilon_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (70)$$

Substituting Eq. 70 in Eq. 9 after factoring the difference of the squares gives:

$$E_R = \frac{3}{\epsilon^3_0}\left[\int_0^{\epsilon_0} \sigma\,\epsilon\,d\epsilon + \frac{(b'-b)t}{\frac{1}{2}\,b\,h}\,\sigma_0\,\epsilon_0\,\frac{\epsilon_0 + \epsilon'_0}{2}\right] \ldots\ldots\ldots (71)$$

Introducing into Eq. 71 the factor κ as defined by Professor Hrennikoff, and making the further approximation,

$$\epsilon_0 + \epsilon'_0 = 2\,\epsilon_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (72)$$

then

$$E_R = \frac{3}{\epsilon^3_0}\left(\int_0^{\epsilon_0} \sigma\,\epsilon\,d\epsilon + \kappa\,\sigma_0\,\epsilon^2_0\right) \ldots\ldots\ldots\ldots\ldots (73)$$

Finally, by substituting Eqs. 64, 65, and 73 in Eq. 67, the bending moment becomes

$$M = \tfrac{1}{2}\,b\,h^2\left(\frac{1}{\epsilon^2_0}\int_0^{\epsilon_0} \sigma\,\epsilon\,d\epsilon + \kappa\,\sigma_0\right) = \tfrac{1}{2}\,m\,A_w\,h \ldots\ldots\ldots (74)$$

which is the same as Eq. 5.

Alexander Hrennikoff,[26] Assoc. M. ASCE.—Although disappointed with the extent of the response to his paper, the writer is grateful to those few who did contribute to the discussion. The writer appreciates the favorable comments of Professor Winter and agrees with him on the need for a complete set of functions of m, n, and u for the interpretation of test results. A large part of these tables was omitted from the paper to economize on space.

The theory of inelastic bending as presented by the writer is based on the assumption of linearity of the cross sections of the beam, and on the identity of the stress-strain relation in the cross sections of the beam with that in a simple tension and compression test. In this regard, the writer has followed in the footsteps of such recognized authorities as S. Timoshenko and A. Nádai. It is possible, however, that these assumptions do not always hold—in which case the theory becomes erroneous. One such case, involving a mild steel beam, with normal stresses in the plastic range, was discussed in the paper.

Professor Drucker in his thought provoking discussion questions the validity of the assumed stress-strain relation in still another case. The core of his discussion, as interpreted by the writer, is as follows: If in a mild steel beam both the normal and the shearing stresses in all parts of the cross section are within the elastic range, but the principal shearing stress reaches or even exceeds the yield value, then a very large shear deformation would tend to take place in the plane of principal shear. This condition would also result in a very large normal deformation in the cross section, for which no allowance has been made in the theory, since the normal stresses still remain within the elastic range. If, however, this large deformation does not materialize (being arrested by continuity with the adjacent material), then some readjustment in the normal stresses and strains in the cross-sectional planes is bound to occur, and the assumed stress-strain relation also becomes incorrect.

Although this reasoning is undoubtedly plausible, the extent of the resultant irregularity is uncertain and may very well be small. The writer feels that Professor Drucker's criticism, in order to be accepted as valid, should be supported by an appropriate quantitative analysis or by experimental evidence.

The objections of Professor Drucker apply only to beams made of material endowed with the property of yielding (for example, mild steel), and only in the range of yielding. For this reason, the writer disagrees with Professor Drucker concerning the validity of Section 6 of the theory presented. This section, dealing with the general theory of shearing stress for any material of the beam, is perfectly valid within the limits of the writer's assumptions.

The absence of any explanation concerning the unusual shape of the shearing stress diagram in Fig. 29(*c*) at the fixed end of an I-beam is regretted. Since the shearing stresses in the cross section of a beam are determined by statics as the difference of the normal stresses in the two adjacent cross sections, the diagram indicates that the normal stresses in flanges increase very fast, and that in the web they decrease toward the fixed support. This condition would suggest that the fixation of the end of the beam has been accomplished by holding the flanges and leaving the web free. If this explanation is correct, the diagram merely reflects the effect of some peculiar end conditions induced artificially in the beam and has no bearing on the theory presented.

---

[26] Prof., Civ. Eng., Dept. of Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

The question of possible incorrectness of the theory when applied to the mild steel beams in the plastic region of stress, as a result of disturbances produced by the shear effects, is also discussed at length by Professor Popov, who makes several references to other investigations along similar lines.

In the reference to the work of F. P. Cozzone, Professor Popov points out a striking disagreement between the expressions for the shearing stresses derived by Mr. Cozzone and the author. Fortunately, the cause of this disagreement is not difficult to find.

In the first place, the general equation (Eq. 56) for the shearing stress in the plastic region does not reduce to Eq. 58. As Professor Popov points out, in this region $df_m = 0$ and df_0 is distinct from zero, which makes R infinity, and thus reduces Eq. 56 to the form:

$$t = \frac{V\,Q}{I\,b}\,\frac{\dfrac{A\,C}{Q} - 1}{k - 1} \quad \text{.....................} (75)$$

This error does not however account for the full extent of the differences considered herein.

All the quantities entering Eq. 75 refer to the geometry of the cross section, the only exception being the term V—the shearing force. Thus, according to this equation, the only characteristic of the loading affecting the shearing stresses in the cross section of the beam is the shearing force. That this conclusion is false in the plastic region of normal stresses in the cross section is evident from Fig. 18. With the normal stresses as indicated in Fig. 18, and quite apart from any particular value of the shearing force present, the conditions of statics demand that there be no shearing stresses on either horizontal or vertical planes in the outer parts of the cross sections, where the normal stresses in the two sides of the element shown are equal, which means that all the shearing stresses must be concentrated near the center of the section. Again by statics these shearing stresses near the center must be the greater, the smaller the region over which they are distributed. Thus, it is not only the shearing force, but also the general state of strain, or, in other words, the bending moment, which determine the state of shearing stress on the vertical and horizontal planes in the plastic region of normal stresses in mild steel I-beams—which is definitely not the conclusion which may be formed from consideration of Eq. 75.

The error of Mr. Cozzone's formula is caused by the crudeness of his assumption of the trapezoidal distribution of normal stresses. The free body diagram, similar to Fig. 18, for an element of beam on the basis of this assumption is shown in Fig. 39. The incorrect manner of distribution of normal stresses on the sides of the element of the beam makes it imperative that there be shearing stresses on the vertical and horizontal planes at all points of the beam, and that these stresses must be independent of any conditions of loading other than the value of the shearing force.

Mr. Cozzone's theory is very crude also in some other respects, and it is totally unsuitable for analysis of statically indeterminate beams stressed beyond the elastic limit.

Although Mr. Cozzone's Eq. 75 is thus unquestionably erroneous, it does not follow that the writer's Eq. 25c and his other equations for the shearing stresses based on this equation are necessarily correct. They are only as correct as the underlying assumption with regard to the normal stresses, and thus the doubts expressed by Professors Drucker and Popov are not unfounded.

In spite of these objections the author still believes that, although the state of strain of a mild steel beam in the plastic region may depart appreciably from the theory presented, the resultant error in stresses is probably small, and the effect of this irregularity on the angle changes, and through them on the statically unknown quantities and the general state of stress and deformation in the statically indeterminate beams, is probably only minor. It should be possible to check this contention by tests.

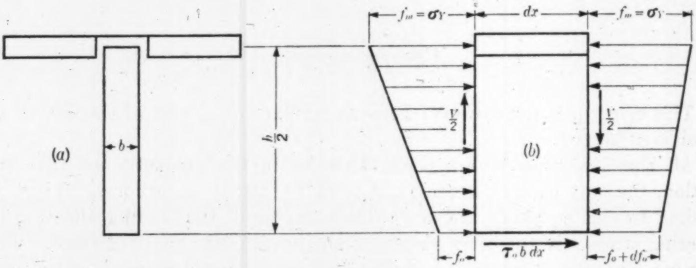


FIG. 39.—THEORY OF INELASTIC BENDING WITH REFERENCE TO LIMIT DESIGN

Mr. Vallerga in his discussion traces the similarity between the method presented by Professor Timoshenko and the writer's theory, thereby raising by implication the question of the originality of the latter. In this connection the writer freely admits that the conception expressed by the symbol m_1 was borrowed from the work of Professor Timoshenko, and the idea expressed by the symbol m represents only a simple elaboration of the former conception, as indeed is shown by Mr. Vallerga. The writer however is not aware of anything in Professor Timoshenko's work, or in any other source, suggesting the equally important ideas implied in the symbols n, u, and q; the extension of the theory into the field of stress recession; and also the general manner of treatment of several problems analyzed by these conceptions—all this comprising at least 90% of the paper.

Mr. Horne has made a thorough study of the theory; he made some further elaboration of it, and solved some additional interesting examples, throwing more light on the behavior of beams beyond the elastic limit. The author, however, disagrees with him on the usefulness of the inelastic version of the three-moment theorem (Eq. 51*b*). In the first place, Mr. Horne is not justified in deriving this formula for a straight beam and then applying it, without further elaboration, to beam ABC (Fig. 31) made of two parts at right angles to each other. Furthermore, Eq. 51*b* merely complicates the matter, instead of making it simpler. Thus, Mr. Horne applies Eq. 51*b* three times, obtaining three cumbersome equations (Eqs. 53*e*, 53*f*, and 53*g*), from which by elimination of the variables Eq. 53*h* is derived, which he uses in the analysis. Mr. Horne overlooks the fact that an equation equivalent to Eq. 53*h* can be written directly without any intermediate steps by the general formulas in Fig. 11. This equation would be different from Eq. 53*h* only in that it would be based on the basic equations (Eqs. 11 and 17*a*) for ϕ and δ. In order to get exactly the same form of the equation as Eq. 53*h* (which by the way is unnecessary), Mr. Horne would have to write first the expressions for ϕ and δ that are equivalent to those assembled in Fig. 11, but that are based on Eqs. 13*b* and 18—rather than on Eqs. 11 and 17*a*.

The writer has been much interested in Mr. Horne's discussion on the theory of limit design and on the effect of ignoring the strain hardening. He is in full agreement with Mr. Horne's general outlook on the subject, and with his conclusion that the load beyond which the deformations begin to increase very fast (which can be looked upon as the failure load) depends only very little on whether the strain hardening is operating or whether the yield stress continues indefinitely on further increase in strain. Thus, in Example 3 the value of the failure load H, allowing for strain hardening, is found to be 58.6, whereas a similar value without the strain hardening is 57.2, indicating a negligible difference.

There is, however, an important aspect of the situation which was emphasized strongly by the writer and which was completely ignored by Mr. Horne, although his data present an admirable commentary on it. Table 9 shows that, when strain hardening is ignored, the local unit strain at point D becomes infinite when the value of load H is only 51.03, which is considerably lower than the failure value, 57.2, previously mentioned. Infinite strain can mean only one thing—actual physical failure of the structure at point D. Thus, if strain hardening is absent, the structure fails long before the equilization of moments at points C and D takes place. That which saves the structure from failure is the strain hardening. Thus, Mr. Horne's conclusion (1) and his several statements—to the effect that the behavior of steel structures up to the point of collapse (he means by collapse, large deflections) may be satisfactorily accounted for without any reference to strain hardening—most certainly is not borne out by the facts. A theory disregarding strain hardening indicates an imminent physical failure at a comparatively low value of the load, and does not explain the most outstanding fact—why this early failure does not actually occur.

Thus, it is only with the knowledge that strain hardening does occur and, in the hope that its extent will be sufficient to prevent an early failure, that it is possible to compute the limiting value of the load and the corresponding deformations without any regard for strain hardening. This hope for beneficial assistance of strain hardening is probably nearly always justified when dealing with mild steel structures, but how true it would be in relation to other materials is questionable. There is strong suspicion that with some aluminium alloys, characterized by a very small and slow rise of stress beyond yielding, the theory of limit design in some cases may prove invalid. In presenting the data to serve as a warning for the too ready and unquestioned acceptance of the theory of limit design, Mr. Horne has rendered a good service to the profession.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2335

REFLECTIONS ON STANDARD SPECIFICATIONS FOR STRUCTURAL DESIGN

By Alfred M. Freudenthal,[1] Assoc. M. ASCE

With Discussion by Messrs. I. Oesterblom, John M. Hayes, and Alfred M. Freudenthal

Synopsis

Standard specifications for structural design determine the effectiveness with which an engineer utilizes his materials; they affect the safety of structures; and they exert considerable influence upon the manufacture of materials. So powerful a tool should be handled in a manner consistent with scientific and engineering progress.

This paper attempts to rationalize the formulation of such standard specifications. It elaborates on their fundamental aspects and advocates the adoption of the scientific method in their preparation. Two examples, dealing with design live loads for highway bridges and with fatigue strength of butt welds, illustrate the arguments and procedure.

Introduction

Rational and scientific methods for designing engineering structures are comparatively new. Until the last one hundred and fifty years of recorded history, engineering was practiced as an art. Up to that time great engineering works were created by artists, whose genius of intuition was guided by sound empirical knowledge acquired through individual experience. For a long time engineering and science progressed at a different level and at a different pace. Scientists of that time, such as Euler, Hooke, Poisson, and others, whose investigations, subsequently, had a most decisive influence upon structural design and analysis, worked on an entirely different level; it took many decades before their theoretical achievements affected the engineer's methods of design. The lag between the progress of science and that of engineering—between research and practice—is a typical phenomenon of technical development; the gradual reduction of this interval indicates the increasingly effective coordination of scientific and technical progress.

Note.—Published in February, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Lecturer on Bridge Eng., Hebrew Inst. of Tech., Haifa, Palestine.

Thus, engineering is being gradually transformed from a profession based on tradition and individual experience to one relying more and more on the methodically selected and classified collective experience provided by that eminently socializing agent, science. However, with this shifting of emphasis, engineering has not become an exact science—it remains a social activity. Evidently, then, it cannot dispense with subjective judgment which, while supplementing the judgment, based on objective evidence and knowledge, must not be allowed to supersede it. Only where the objective approach has not been discerned will it be necessary to rely upon subjective judgment alone.

ORGANIZATION OF ENGINEERING SCIENCE

Knowledge is obtained directly by perception or indirectly by argument. Since such knowledge as is perceived or experienced directly through sensations, by observation or measurement, furnishes the premises of that part derived by argument, experience represents the sole source of knowledge. From the premises of direct experience attempts are made to justify some degree of rational belief about all sorts of inferences and conclusions by perceiving logical relations connecting them. Terms such as "certain" and "probable" describe the different degrees of this belief according to the volume of experience supporting it. The purpose or aim of these inferences and conclusions is to organize, for subsequent application, factual knowledge—using interpretation, generalization, and abstraction—mostly in the form of data or mathematical functions describing the regularities and recurrences of the observed phenomena.

Fundamentally, it makes no difference, however, whether the formulation is mathematical or verbal. There is no particular virtue in mathematical functions or in numbers as such. If they are not really representative, they can be even more misleading than verbal statements; because, psychologically, the number or formula is bound to give the impression of accuracy.

Much engineering knowledge is still descriptive, although its presentation is mathematical. As in the case of the genuinely descriptive marine or foundation engineering, there is always an undisguised, immediate, and real experience as the basis of every generalization. The development toward an exact science by using mathematical abstractions and mathematical language, however, leads occasionally to the ascendency (or domination) of the mathematical form over the real physical content—to an overvaluation of the mathematical exactness inherent in an expression of physical reality and, consequently, to a serious distortion of perspective. To avoid the pitfalls of the mathematical approach to engineering problems, it is essential to realize and to check on its limitations.

The significance attributed to information expressed by numbers or mathematical functions is an indication of the level of scientific organization of experience. At the level of descriptive science, experience is purely qualitative and, therefore, not measurable. Most knowledge has not advanced beyond this stage. A higher level is reached when methods of measuring assumedly relevant, recurrent phenomena have been developed and when the resulting figures and relations are used to devise a quantitative, although empiric, classification of experience. Numerical data have still no absolute significance; they are useful only as far as they are suitable to delimit certain classes of phenomena, and their relative significance is determined by the rigor of the classification which they express. The number or the mathematical function reaches a significance of its own only at the highest level of organization of knowledge—at that of exact science, where the relations between sets of causes and resulting effects have been so reliably established that their expression in mathematical language is justified.

Only a small part of technical science has attained this level; even in structural engineering, the methods of which are relatively exact, data are frequently utilized that do not even belong to the empiric level—as when specifying a definite number for elongation at rupture of structural steel. Although this number may be determined by a well-defined measurement belonging to the sphere of exact physical science, it does not represent more than an arbitrary conventional classification of the material with regard to a desired structural performance, based on the assumption of a vague correlation with such performance. At best it can be used only for purposes of practical designation of the material; it therefore belongs to a stage of descriptive science where measurements of phenomena already play an important part, but where rational although empiric, correlation with essential physical properties is still nonexistent. Such data should not be invested with a meaning beyond what can be justified by the physical reality behind them.

However, even where mathematical presentation is justified, the data or the function can be expected to fit the observed reality only as closely as by the margin of uncertainty inherent in the process of sampling and observation and by the margin of error detectable with the instruments used. Hence, with both the extension of the observational range and the perfection of instruments, the accuracy and reliability of such data and mathematical formulas will undergo continual improvement. Moreover, the regularity and recurrence of physical phenomena, quantities, and constants will never be perfect in a numerical sense. This fact requires the replacement of the conventional concepts of one-valued functional correlation by concepts embodying the uncertainty inherent in any observation, abstraction, or presentation of physical phenomena—where both constants and laws are considered to be of a statistical character and where the closest approach to constancy which a physical quality may attain is understood to be such that it may be represented by a frequency distribution for which all assignable causes of variation have been eliminated.

Ultimately it is of no consequence whether the fluctuations are real, objective, and inherent in the quality considered or are apparent and caused by deficiencies in the procedure of observation or measurement. Thus, for example, the strength of a material, all things being equal, should be a fairly regular and recurrent physical quality. No two samples, however, will show exactly the same test strength, except by pure chance. Fluctuations may result from differences in the quality of the material forming the individual samples and from lack of uniformity within the samples. They may, however, result from the method by which the samples have been taken and prepared; from the procedure of testing; from inaccuracies of observation and measurement; or from a combination of these. The difficulties increase with increasing complexity of the phenomenon or the quality considered or of the sampling and testing procedure. Whereas, for instance, the results of the static tension test of steel are fairly uniform and interpretable, those of fatigue tests display irregularities and frequently become inconclusive. The question, then, is whether such lack of uniformity is an inherent characteristic of the fatigue strength of the material or is the result of the sampling and testing operation.

Generally engineering research is not so much concerned with the behavior of individual specimens of a certain group as with the behavior of the group itself. "Weight" or "strength" are abstractions not intended to represent the real measurable quality of an individual sample, but rather of the "bulk" of the material or of the statistical "universe" of all members of the group—such, for example, as the structural elements or shapes manufactured to one and the same specification in a number of plants. Similarly, "load" in a structural sense refers to the "universe" of states of loading which, for a specified group of structures, may be expected to occur. Since the behavior of a "bulk" or "universe" can be represented only by a distribution curve of statistical frequencies, any law describing the relation of real physical qualities will, to a certain extent, be statistical even if, fundamentally, its character is that of an exact law.

The Standard Specification

These basic aspects of engineering science have a particular bearing on the preparation of standard specifications for structural design. The designer relies on the specifications for all relevant information about the fundamental assumptions and premises of his design. He cannot and should not be expected to ascertain, select, and appraise, for each structure individually, the basic facts and conditions, such as load or permissible stresses, by which both the safety and the economy of the structure will be determined. This is because only in exceptional cases will he command the wide knowledge and experience required for such selection and for the appraisal of all implications. With the increasing complexity of engineering problems he will have to rely increasingly on the organized collective experience of the profession and to substitute the considered objective judgment of the group for his individual opinion.

The standard specification represents this judgment: It is the result of a collective effort of the leading groups within the profession and attempts to convey the quintessence of factual knowledge and engineering experience; it is arrived at by the application of objective methods of research and by rational interpretation of the results. Its purpose is to relieve the designer of the responsibility of selecting the premises and assumptions of the analysis of an individual structure, and to do so by establishing the fundamental relations and conditions pertaining to the design of large groups of structures having certain features in common, such as the material, the type of load, or the structural shape. This procedure has so far been successful: The number of structures designed, rather effectively, by engineers with no more than preliminary experience is considerable.

However, the rapid expansion of the engineer's knowledge and the complexity of his tasks increase the scope and difficulties of the problems to be solved and of the decisions to be reached collectively. The soundness and correctness of these decisions, embodied in and conveyed by the respective standard specifications, is therefore becoming ever more important. Because of the extensive and differentiated knowledge required, the questions arise as to whether the methods by which collective decisions have been reached are sufficiently effective and as to whether subjective judgment and belief, unsupported by adequate objective evidence, are not allowed to play too important a part.

Standard specifications determine the efficiency of the utilization of engineering materials; they decisively affect the safety of structures; and they exert considerable influence upon the manufacture of technical materials. Thus, there is every reason to require that so powerful a tool be handled not merely with judicious care and caution, but in a manner consistent with engineering progress. This is particularly important because the utmost effectiveness in the utilization of basic materials is no longer a matter of individual judgment, but one of national importance.

Among the subjects of fundamental character covered by specifications for structural design are: (1) Design loads; (2) resistance of structural members and shapes and their permissible stresses; and (3) performance, selection, and quality control of structural materials. These are interdependent and must be coordinated. Thus, a specified permissible stress has real meaning only in conjunction with the specification of the design load and the method of structural analysis.[2] The conventional procedure by which standards for design loads, for permissible stresses, and for classifying and selecting materials are devised independently and frequently by different groups—administrative bodies specifying loads; professional societies dealing with matters of structural analysis and strength; and organizations engaged in the testing of manufactured products handling the selection of materials and control requirements—should be replaced by a process coordinating all relevant data and information. Only such a process would result in balanced standard specifications.

A rational approach to any problem requires the appraisal of the meaning of the terms and concepts, since meanings often are not what they appear at first sight. A very effective tool for this purpose is P. W. Bridgman's "operational analysis"[3]—namely, making an analysis of what is actually done when using the concept leads to a real understanding of its meaning. Using this method for the analysis of the concept of "standard specification for design," it is found that the "operation" consists in applying organized fundamental knowledge, derived from past experience and presented in a form most suitable for practical use, to the design of new structures which are expected to serve a definite purpose during a period extending into a more or less distant future. Application of past experience to the design of structures for future service requires prediction of future development (mainly of load and traffic volume), of changes of resistance, and of the state of maintenance and of the deterioration of the structure. These latter are matters for conjecture and represent, therefore, essentially problems for subjective judgment.

---

[2] "The Safety of Structures," by Alfred M. Freudenthal, *Transactions*, ASCE, Vol. 112, 1947, p. 125.

[3] "The Intelligent Individual and Society," by P. W. Bridgman, The Macmillan Co., New York, N. Y. 1938, pp. 74–75.

The "operational" definition shows that the first step in preparing specifications is the collection of past experience. To apply such experience effectively, the salient and recurrent phenomena should be selected and classified to derive from them the relevant relationships either by statistical inference or by induction. Each of the two approaches leads to a different kind of knowledge: Reasoning by induction, justified only if the factual evidence is conclusive, leads to knowledge of principles and to definite functional relationships of general validity; statistical inference, dealing with arguments which are not conclusive, but about which a certain degree of belief may rationally be entertained because it is supported by an appropriate amount of direct evidence, leads to knowledge of facts only and to correlations between certain variables. The reliability of these approaches is determined by the amount of evidence available; the range of validity is restricted to the original range of observation.

It is not always easy to draw a line between the two approaches because of the rôle played by the concept of chance. "Chance" may be subjective as well as objective. Subjective chance is a measure of ignorance of principles, the knowledge of which would enable one to predict the occurrence of certain events. If, however, such events are brought about by a coincidence of causes and circumstances so numerous and so complex that knowledge of principles leading to prediction is altogether out of mental reach, this is defined as objective chance. Because it is often difficult to determine whether the chance happenings entering into experience have a subjective or an objective character, the boundary between logical induction and statistical inference remains indistinct.

"Past Experience" and Experiment

The data that represent "past experience" are obtained either (1) by statistical evaluation of observations of phenomena which bear directly on the quality considered (such as traffic records or wind velocity records for the determination of service loads); (2) by observations of the behavior of existing structures (service records); or (3) by the interpretation of the results of tests or experiments designed to supply the desired information. The testing of full-scale models of structures is mostly impracticable because of their size and the complexity of service conditions—the cost of reproducing which is prohibitive. "Past experience" will, therefore, have to be based upon tests of suitably designed and selected comparatively small specimens. Such tests will be rationally interpretable if they have structural significance—that is, if the specimens tested can be considered as models, having been designed in accordance with the relevant laws of mechanical (not merely geometrical) similitude. Tests will result in the acquisition of relevant factual evidence only if:

(*a*) They can be rationally correlated with the problem under consideration;
(*b*) The characteristics observed are real physical properties, measurable in fundamental units, so as to make the tests easily reproducible and comparable with similar experiments;

(*c*) No oversimplification has been introduced by neglecting or eliminating variables or groups of variables, the relevance of which has not been reliably ascertained; and

(*d*) They have been repeated frequently enough to control the variabilities by statistical methods in order to recognize their physical significance.

Conventional tests are mostly comparative; their structural significance is doubtful or vague, as is their correlation with actual structural performance. With regard to structural materials, modern technological skill in securing specified properties is as yet not equaled by ability to discern, to specify rationally, and to control the properties which are desirable with regard to the contemplated performance in a structure.

Since past experience is to be applied to present construction, the expected behavior of the structure must be predicted on the strength of such experience. This requires the introduction of control: Making sure that what is being done is what is intended following available experience and knowledge. Obviously, such control cannot be performed by direct observation of the relevant characteristics, such as the actual resistance of the structure, but only by observing and recording one or a number of easily measurable qualities which, singly or jointly, can be considered representative. The selection of representative qualities and of adequate testing methods is, therefore, the first step in the establishment of control. The second is the selection of representative samples and the specification of their number and of the manner in which they are to be taken, so as to enable rational conclusions about the quality of the "bulk" or the "universe."

CORRELATION

The possibility of correlating the results of tests of a comparatively small number of samples with the quality of the "bulk" or "universe" is conditional on the imposition of adequate, uniform, and permanent control of the variabilities of the relevant characteristics upon each and every stage of the manufacturing process of the structure, from the production of the raw material to the final process of assembly.

When devising representative control tests, one has to rely on: (1) A strict functional relationship between the relevant characteristic and the observed property, (2) a statistical correlation, or (3) merely an intuitive qualitative correlation. In practical application, the difference between relationships (1) and (2) is one of grade rather than one of principle—a strict functional relationship usually being considered the limit of perfection of a statistical correlation. Also, the number of standard tests based upon a functional relationship between the representative and the represented qualities is surprisingly small, and most tests are based on an intuitive correlation. The general application of statistical methods in analyzing and drawing up standard specifications is one of the conditions of the rational utilization of materials.

Statistical methods have two different functions: One descriptive, concerned with the concise presentation of the variabilities of certain characteristics of observed phenomena; and the other inductive, with the aim of extending and utilizing such description and presentation in the generalization and the prediction of the variabilities in the future. It is essential not to lose sight of the distinction since, frequently, having found a complete and satisfactory method of description, the engineer becomes less careful about the transitional arguments by which he attempts to justify the subsequent use of this description for purposes of generalization and prediction.

The ability to discriminate between the descriptive and the inductive use of statistics, as well as between functional, statistical, and intuitive correlation of phenomena, acquired by frequent and judicious application of statistical methods, is important. The engineer thus acquires not merely the habit of separating assignable causes of variation of a phenomenon from chance variabilities, but also the faculty of discriminating between subjective and objective chance causes, without losing sight of the arbitrary nature and of the expediency involved in all technical applications. He will also learn necessary caution when generalizing from a limited amount of experience. Particularly in the preparation of standard specifications will this intuitive familiarity with the possibilities as well as with the pitfalls of the statistical methods prove important.

As an illustration of the preceding arguments an attempt will be made to establish a rational approach to the drafting of two typical standard specifications for structural design. Since the data on which the examples could be based are not exhaustive, these examples are suggestive of the method applied rather than conclusive with regard to the results.

Example 1. Live Load For Medium and Long-Span Highway Bridges

The design live load for short-span highway bridges or bridge members is represented by the vehicle with the most severe concentration of weight—that is, by a fully loaded truck traveling at high speed. However, for medium and long spans the effect of the individual load concentration is small compared with the effect of the total weight of the traffic properly distributed over the respective influence line; a uniformly distributed design load will therefore represent actual loading conditions. This design load is affected by the character of the traffic; by its distribution, density, and speed; by the number of traffic lanes; by the expected period of service; and by probable changes in the traffic characteristics during the period of service. Information about the heaviest types of vehicles is insufficient, and the fact that the existing specifications convey only this information—apart from a more or less arbitrary impact factor—is a striking illustration of their inadequacy.

The types of vehicles using the modern highway are the passenger car, the passenger bus, the single truck, and the power unit or tractor with semitrailer or trailer. Denoting the loaded weights of the various groups of vehicles by $W_1,\cdots,W_n$, their empty weights by $W'_1,\cdots,W'_n$, the average frequencies of occurrence of the various groups by $F_{V1},\cdots,F_{Vn}$, the average frequencies of occurrence of the loaded vehicles of a certain group by $F_{L1},\cdots,F_{Ln}$, and the total number of vehicles which can be accommodated on all lanes of the bridge by n and their over-all length by L_n, the design live load, (w_{LO}) can be expressed by:

$$\begin{aligned} w_{LO} = \frac{1}{L_n} [F_{V1} F_{L1} W_1 + F_{V1} (1 - F_{L1}) W'_1 + \cdots \\ F_{Vn} F_{Ln} W_n + F_{Vn} (1 - F_{Ln}) W'_n] \ldots\ldots\ldots\ldots\ldots (1) \end{aligned}$$

Every term of this expression is to be considered a frequency distribution. Eq. 1 represents the static weight of the design load, the vehicles being placed along the traffic lanes close to each other. The real traffic load is, however, not static but travels over the structure at a certain speed, exerting dynamic influences in addition to its static weight. At the same time the distances between the consecutive vehicles will increase with their traveling speed, resulting in a reduction of the specific load. If both the load-increasing effects and the load-reducing effects of the traveling speed are combined into one factor, I, for which the conventional term "dynamic increment" may be used, then dynamic design load, $w_L = w_{LO}\,(1 + I)$.

An expression may now be introduced for the load-increasing effects and load-reducing effects as functions of the traveling speed, v^2. Let the impact increment and its range of fluctuation be a function of the number of vehicles n:

$$I'(v) = \bar{I}'(v) \left(1 \pm \frac{1.0}{\sqrt{n}} \right) \dots\dots (2a)$$

in which

$$\bar{I}'(v) = \frac{1.50}{(1 + V_o/3v)\,\sqrt{n}} \dots\dots (2b)$$

Also, average spacing between consecutive vehicles (center to center),

$$l(v) = l_o(v)(1 \pm 0.20) \dots\dots (3a)$$

in which $V_o = 10$ miles per hr; and $L_o = 40$ ft; and

$$l_o(v) = \frac{L_o}{1 - 0.15\,v/V_o} \dots\dots (3b)$$

The chance fluctuations of $l(v)$ about its average $\bar{l}(v)$ can be assumed to reach ±60%, so as to include zero distance between consecutive vehicles as an extreme range.

Hence,

$$(1 + I) = [1 + I'(v)]\,\frac{L_n}{n\,l(v)} \dots\dots (4)$$

In this expression $I'(v)$, L_n, and $l(v)$ are frequency distributions. The quantity $\frac{1 + I'(v)}{l(v)}$, reaches its maximum for $v = 11.5$ miles per hr. This can be considered the critical traveling speed of the design load for bridges, accommodating more than two traveling trucks per lane, corresponding to spans exceeding 75 ft.

The weight distribution diagrams of the principal type of vehicle, the single truck, can be deduced from traffic surveys[4] and studies[5] conducted by the United States Public Roads Administration. These surveys also show that the proportion of trucks of all types (including passenger buses) is normally between 10% and 25% of the total number of vehicles and that the proportion of empty trucks is between 0.35 and 0.40 of the total number of trucks. Studying the relation between the weight and length of trucks, it may be concluded that the ratios of the average weights of single trucks, tractor-semitrailers, and tractor-trailers are approximately 1:1.5:2.5 although the ratios of the respective lengths are about 1:1.5:1.9. Hence, the specific loads of single trucks very nearly equal those of tractor-semitrailer combinations; the tractor-trailer combinations are about one third heavier. The number of these combinations, however, does not exceed an average of 15% of the total number of trucks.

---

[4] *Public Roads*, Vol. 15, 1934, pp. 243–247; Vol. 16, 1935, pp. 17–31, 68–74, 225–237, and 238–239; and Vol. 17, 1936, pp. 113–127.

[5] "A Study of the Weights and Dimensions of Trucks," *ibid.*, Vol. 16, 1935, pp. 37–52.

Distribution diagrams of the gross weight of both loaded and empty single trucks and of their over-all length have been deduced from the surveys and observations of the Public Roads Administration and are presented in Fig. 1.

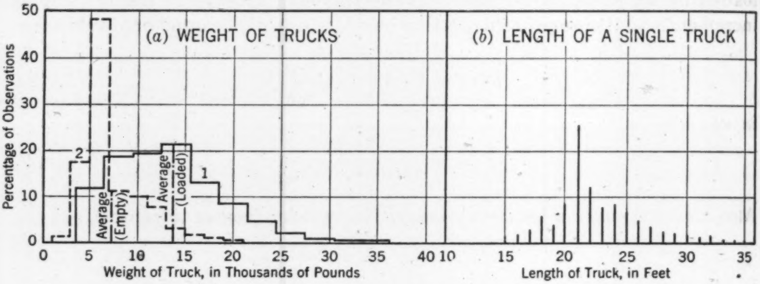


FIG. 1.—CURVES OF FREQUENCY DISTRIBUTION FOR THE PRINCIPAL TRAFFIC CHARACTERISTICS

Computations from the observations including values of σ and k give the following: From frequency distribution of weight (Fig. 1(*a*)):

Symbol	Curve 1 (loaded trucks)	Curve 2 (empty trucks)
$\bar{W}_1$	13,800 lb; $\bar{W}'_1$	7,200 lb
σ (standard deviation)	5,900 lb	3,200 lb
k (skewness)	+0.5	+1.5

$\bar{W}_1$ and W'_1 (maximum range, with a probability of 0.999)—

Computation..... $W_1(\max) = \bar{W}'_1 + 3.5\sigma$ $W'_1(\max) = W'_1 + 3.9\sigma$
Result............. 34,500 lb 19,700 lb

and, from frequency distribution of length (Fig. 1(*b*)):

$$\bar{L} = 22.0 \text{ ft}$$
$$\sigma = 3.6 \text{ ft}$$
$$k = +1.0$$

and the maximum range—

$$L\ (\max) = \bar{L} + 3.7\,\sigma = 35.5 \text{ ft (probability 0.999)}$$

No distribution diagrams are available for weights and lengths of passenger cars; their variations, however, are comparatively small and a specific load of 400 lb per ft of traffic lane with a range of fluctuation of some 25% may be considered as representing actual conditions quite fairly. The average length of passenger cars may be assumed as 15 ft; no difference need be made between loaded and empty cars.

To establish the (average) design load according to Eq. 1, the frequency values in Table 1 should be introduced. These average frequency ratios deduced from observations, have been amended to provide for future development of traffic. The loads of trucks as presented in Fig. 1 are increased by 50% to anticipate rapid development of heavy cargo hauling on highways; likewise, the specific loads of passenger cars are increased by 20%, although even this figure appears excessive in view of the probable increased use of light materials in the construction of cars.

TABLE 1.—VALUES OF FREQUENCY PERCENTAGES OF GROUPS OF VEHICLES

Line	Traffic	F_V[a]	F_L[b]
1	Single trucks, buses, and tractor-semitrailers.....	0.25	0.70
2	Tractor-trailers..........	0.05	0.80

[a] Percentage of the total number (n) of vehicles.
[b] Average percentage of loaded vehicles in a group.

Introducing these modifications and considering the observed frequency distribution of load intensities as the "a priori" probability of the occurrence of these intensities in future—a procedure dating back to Aristotle who stated that "the probable is what for the most part happens"—the static design load w_{LO} and the "dynamic" design load $w_{LO}(1 + I)$ have been evaluated from Eqs. 1 to 4 and the frequency distributions of Fig. 1. The results are presented in Fig. 2. The individual average design values and the ranges of fluctuation pertaining to a probability of 0.001 that values will fall outside this range have

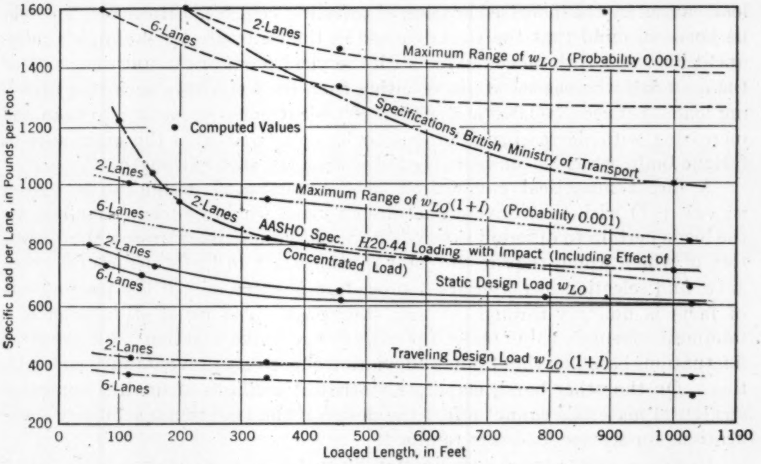


FIG. 2.—DESIGN LIVE LOADS FOR HIGHWAY BRIDGES

been computed for a two-lane structure and a six-lane structure of a length accommodating 3, 9, and 27 vehicles, respectively, per lane.

In establishing the maximum upper range of fluctuations, on which the safety factor pertaining to the design load depends, variations caused by fluctuations of the weight of vehicles according to Fig. 1(*a*), variations caused by fluctuations in the ratio of trucks, and variations caused by fluctuations in the ratio of empty vehicles have been evaluated individually and statistically superimposed. Thus, in the case of eighteen vehicles on the bridge for instance, the static design load w_{LO} comprises one loaded tractor-trailer, four loaded trucks, and one empty truck, and twelve passenger cars, all of average weight; the maximum load due to fluctuations of weight results from the same grouping of vehicles of maximum weight. The maximum variation due to fluctuations of the ratio of trucks results in a group consisting of three loaded tractor-trailers and one empty tractor-trailer, six loaded trucks and two empty trucks, and six passenger cars, all of average weight; the range of variations due to fluctuations in the ratio of empty trucks is determined by replacing the empty truck in the design-load group by a loaded one. Different groupings of vehicles result in varying lengths of the structures accommodating them.

In Fig. 2 it is shown that, contrary to conventional assumptions, the static load w_{LO} considerably exceeds the "dynamic" load $w_{LO}(1 + I)$. This is because the influence of the dynamic load increment is less than the reduction of the load caused by the increased spacing of traveling vehicles. However, it should be borne in mind that the strain caused in the structure by the highly infrequent load w_{LO} is to be compared with the yield point or the static strength of the material, whereas the strain resulting from the frequently occurring traveling load—between 500,000 and 10,000,000 occurrences according to span and increasing with decreasing spans—is to be compared with the endurance or fatigue limit, which is considerably lower than the static strength.

A direct numerical comparison of the computed design loads w_{LO} or $w_{LO}(1 + I)$ with the conventional design loads would be misleading because the loads pertain to different safety factors and permissible stresses, the evaluation of which would require the joint consideration of loads, structural resistance, and selection of material—a procedure possible only if the presentation of facts is uniform throughout and statistical. The usual stipulation of a minimum strength value instead of an average value together with ranges of fluctuations based on the standard deviation effectively prevents such coordination. On the other hand, elaborate statistical methods of quality control of structural materials cannot affect the design if the traditional arbitrary specifications for service loads are retained.

However, even perfect coordination by systematic application of statistical methods will fall short of its purpose if it is one of form only. The importance of analyzing the compatibility of the correlated qualities cannot be emphasized strongly enough. There is as little sense in correlating the strain resulting from a frequently repeated, rapidly moving load with the yield limit of a material, as there is in correlating the strain produced by a steady or very slowly moving load with the fatigue limit.

Even without numerical comparison, however, it appears from Fig. 2 that the conventional values of design loads of highway bridges are substantially higher than could be justified from the factual evidence and a rather unfavorable appraisal of future developments.

EXAMPLE 2. FATIGUE STRENGTH OF BUTT WELDS IN ORDINARY BRIDGE STEEL

To establish the "dependable" fatigue strength of butt welds in ordinary carbon bridge steel, the Committee on Fatigue Testing of the Welding Research Council of the Engineering Foundation has initiated an extensive program of tests of butt-welded specimens, prepared under fairly controlled commercial conditions.[6] From the test results, values are deduced for:

> "*** the unit fatigue strength which may be dependably expected in butt-welded joints commercially produced under standard specifications; to which dependable strength a designer will of course apply whatever factor of safety he deems appropriate for his particular condition."[7]

A study of the test results and of the conclusions shows that, because of the lack of a strictly rational, statistical interpretation of these results, their full implication has been missed, thus partly depriving an otherwise excellent and valuable test series of its merits. In the following independent analysis of the results the full information will be presented. This will lead to the establishment of a rational meaning of the term "dependable," as well as to an estimate of a more objective value of the safety factor than that which the designer "deems appropriate." Contrary to the procedure adopted in the Welding Research Council report, it has not been found necessary to introduce arbitrary criteria for eliminating certain test results only:

> "*** because they would in each case affect the average in a direction for which it is felt that they have too weak a basis in fact as compared with the other data in the same group."[8]

The results of tests on one hundred and eighty-six commercially fabricated, butt-welded specimens—5 in. by $\frac{7}{8}$ in. in critical cross section—are summarized in condensed form in Tables 2, 3 and 4. Data relative to the actual stress divided by the number of load cycles sustained (s_F/N), taken from individual tests have been converted into the approximate fatigue strength s_{FO} at N_o = 100,000 cycles and N_o = 2,000,000 cycles by using the empirical equation:

$$s_{FO} = s_F \left(\frac{N}{N_o} \right)^{\nu} \quad \text{.........................(5)}$$

in a manner similar to that adopted in the report of the Welding Research Council.[6] Exponent ν was derived from tests by establishing the frequency distribution of the number of cycles N endured to failure under various stress

---

[6] "Fatigue Strength of Butt Welds in Ordinary Bridge Steel," *Report No. 3*, Committee on Fatigue Testing, Welding Research Council, New York, N. Y., 1943.

[7] *Ibid.*, p. 2.

[8] *Ibid.*, p. 6.

TABLE 2.—OBSERVATIONS TO DETERMINE THE FATIGUE STRENGTH OF BUTT WELDS (SPECIMENS 5 IN. BY $\frac{7}{8}$ IN., ORDINARY STRUCTURAL STEEL) STRESSED FROM ZERO TO TENSION

Test No.	N	UNIT STRENGTH[a] s_{FO}	
		$N_e =$ 100,000	$N_e =$ 2,000,000
(1)	(2)	(3)	(4)
(a) APPLIED LOAD FROM 0 TO +30,000 LB PER SQ IN.[a]			
Series X:			
1	202,600	35,400	17,500
2	220,700	36,100	17,900
3	591,200	45,500	22,500
Series Y:			
4	132,500	32,100	15,900
5	147,700	32,900	16,300
6	228,000	36,400	18,100
Series Z:			
7	112,800	30,800	15,400
8	123,000	31,500	15,600
9	134,500	32,100	15,900
Series XX:			
10	226,600	36,300	18,000
11	246,700	37,200	18,300
12	285,700	38,400	19,000
Series P:			
13	209,500	36,000	17,700
14	277,800	38,500	18,900
15	324,700	40,000	19,200
Series R:			
16	181,200	34,500	17,100
17	359,400	40,500	20,100
18	485,500	43,400	21,500
Series A:			
19	254,600	37,300	18,500
20	447,800	42,800	21,100
21	521,200	44,100	21,900
Series B:			
22	275,700	38,000	18,800
23	401,100	41,600	20,600
24	438,300	42,500	21,000
Series C:			
25	266,600	37,800	18,700
26	331,000	39,700	19,700
27	468,500	43,100	21,300
Series E:			
28	83,000	28,700	14,200
29	83,400	28,800	14,200
30	136,700	32,300	16,000
(b) APPLIED LOAD FROM 0 TO +25,000 LB PER SQ IN.[a]			
Series X:			
31	300,000	32,300	16,100
32	488,300	36,200	17,900
33	548,500	37,300	18,400
34	1,060,800	43,400	21,500
Series Y:			
35	603,500	38,100	18,900
36	631,700	38,500	19,100
37	678,300	39,200	19,400
Series Z:			
38	177,800	28,600	14,100
39	179,700	28,700	14,200
40	1,052,100	43,200	21,500
Series XX:			
41	984,400	42,700	21,100
42	1,103,200	43,900	21,600
43	1,154,900	44,500	22,000
(b).—(*Continued*)			
Series P:			
44	391,800	34,400	17,100
45	507,800	36,700	18,200
46	986,100	42,700	21,100
Series R:			
47	105,700	24,700	12,500
48	1,300,800	45,700	22,600
49	1,405,100	46,400	23,000
50	1,471,000	47,000	23,000
Series A:			
51	278,400	31,800	15,700
52	399,300	34,600	17,100
53	558,400	37,400	18,500
Series F:			
54	525,600	36,900	18,200
55	550,100	37,300	18,500
56	802,200	40,700	20,200
Series B:			
57	362,300	33,800	16,800
58	372,100	34,000	16,900
59	441,000	35,400	17,500
Series C:			
60	576,600	37,800	18,700
61	781,000	40,600	20,100
62	1,051,500	43,500	21,500
Series D:			
63	164,700	28,100	13,900
64	198,100	29,400	14,500
65	353,000	33,700	16,600
Series E:			
66	334,800	33,200	16,400
67	364,000	33,900	16,800
68	475,500	36,100	17,900
Series S:			
69	212,000	29,800	14,800
70	443,100	35,400	17,500
71	1,089,200	43,900	21,700
Series T:			
72	394,800	34,500	17,100
73	680,500	39,300	19,400
74	1,055,600	43,700	21,500
Series U:			
75	161,800	28,000	13,800
76	178,200	28,600	14,100
77	244,700	30,900	15,200
Series K:			
78	246,100	31,000	15,300
79	257,200	31,300	15,500
80	425,500	35,200	17,400
Series L:			
81	563,000	37,500	18,600
82	581,400	37,900	18,700
83	1,330,800	45,800	22,800
(c) APPLIED LOAD FROM 0 TO +20,000 LB PER SQ IN.[a]			
Series A:			
84	451,300	28,500	14,100
85	499,700	29,500	14,500
86	852,000	33,100	16,400

TABLE 2.—(*Continued*)

Test No.	N	Unit Strength[a] s_{FO}	
		$N_e =$ 100,000	$N_e =$ 2,000,000
(1)	(2)	(3)	(4)
(*c*).—(*Continued*)			
Series B:			
87	1,611,200	38,500	19,000
88	2,739,100	43,500	21,000
89	4,543,600	49,100	24,300
Series C:			
90	954,800	34,000	16,800
91	1,530,700	37,900	18,800
92	1,882,400	39,700	19,700
Series D:			
93	1,365,900	37,000	18,300
94	1,669,400	38,700	19,200
95	1,691,200	38,800	19,200
Series E:			
96	724,800	31,900	15,700
97	910,600	33,600	16,600
98	1,265,700	33,300	18,000
(*c*).—(*Continued*)			
Series U:			
99	1,548,300	38,200	18,800
100	4,354,400	48,700	24,000
Series K:			
101	574,600	30,100	14,900
102	811,000	32,700	16,200
103	1,228,000	36,100	17,800
Series L:			
104	1,305,300	36,500	18,100
105	1,508,600	37,900	18,700
106	2,609,900	42,900	21,300
Totals	....	3,912,350	1,934,950

[a] s_{FO} equals the fatigue strength, in pounds per square inch, computed from the applied load and the observed number of load applications sustained prior to failure. A positive sign denotes tension.

amplitudes (see Fig. 3) and computing the value of ν from the averages. This procedure is contrary to that of the Welding Research Council report in which a value of $\nu = 0.13$ was assumed on the basis of previous, and not entirely relevant, tests.

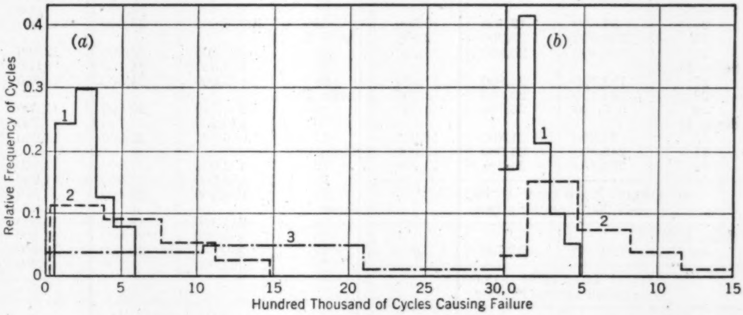


FIG. 3.—FREQUENCY DISTRIBUTION OF THE NUMBER OF CYCLES N REQUIRED TO CAUSE FAILURE

The applied procedure appears fully justified since the apparently inconsistent values of N show a unimodal distribution which, however widespread and skew, is fairly well reproducible by the second approximation of the Normal Law.[9] Thus, with the resulting values of $\nu = 0.235$ for pulsating stresses and $\nu = 0.205$ for reversing stresses, the approximate values of the fatigue strength

---

[9] *Manual on Presentation of Data*, A.S.T.M., 1937, pp. 24–27.

TABLE 3.—OBSERVATIONS TO DETERMINE THE FATIGUE STRENGTH OF BUTT WELDS (SPECIMENS 5 IN. BY ⅞ IN., ORDINARY STRUCTURAL STEEL) UNDER FULL STRESS REVERSAL

Test No.	N	UNIT STRENGTH[a] s_{FO}	
		N_e = 100,000	N_e = 2,000,000
(1)	(2)	(3)	(4)
(a) APPLIED LOAD FROM +20,000 TO −20,000 LB PER SQ IN.[a]:			
Series X:			
1	165,700	22,100	12,000
2	361,600	26,000	14,000
Series Y:			
3	84,700	19,300	10,400
4	184,000	22,600	12,300
5	329,900	25,500	13,800
Series Z:			
6	42,600	16,800	9,100
7	134,200	21,300	11,500
8	192,700	22,400	12,500
Series XX:			
9	279,500	24,700	13,400
10	391,500	26,400	14,300
11	402,700	26,600	14,400
Series P:			
12	42,300	16,800	9,000
13	67,200	18,500	8,900
14	78,200	19,000	10,200
Series R:			
15	83,900	19,300	10,400
16	143,600	21,500	11,600
17	434,700	27,000	14,700
Series F:			
18	160,100	22,000	11,900
19	161,500	22,100	11,900
20	197,300	23,000	12,500
Series G:			
21	68,000	18,500	10,000
22	119,200	20,800	11,300
23	132,200	21,200	11,500
Series S:			
24	85,300	19,400	10,500
25	225,900	23,700	12,800
26	238,400	23,900	13,000
Series T:			
27	152,700	21,800	11,800
28	188,100	22,800	12,300
29	305,100	25,100	13,600
Series L:			
30	104,200	21,800	10,900
31	245,300	24,000	13,000
32	261,300	24,300	13,200
(b) APPLIED LOAD FROM +19,000 TO −19,000 LB PER SQ IN.[a]			
Series XX:			
33	429,100	25,700	13,900
(c) APPLIED LOAD FROM +18,000 TO −18,000 LB PER SQ IN.[a]			
Series L:			
34	306,700	22,700	12,200
35	632,900	26,200	14,200
36	1,051,300	29,100	15,800
(d) APPLIED LOAD FROM +16,000 TO −16,000 LB PER SQ IN.[a]			
Series X:			
37	202,800	18,500	10,000
38	393,700	21,200	11,200
39	400,600	21,300	11,500
Series Y:			
40	737,500	24,100	13,100
(d).—(*Continued*)			
Series Y:			
41	833,500	24,700	13,400
42	1,254,200	26,800	14,600
Series Z:			
43	56,200	14,200	7,900
44	244,200	19,200	10,400
45	597,600	23,100	12,200
46	1,634,300	28,400	15,400
Series XX:			
47	263,000	19,500	10,600
48	336,800	20,500	11,100
49	545,500	22,700	12,300
Series P:			
50	118,000	16,600	9,000
51	197,000	18,400	10,000
52	305,100	20,100	10,900
Series R:			
53	83,400	15,400	8,300
54	85,600	15,500	8,400
55	148,200	17,800	9,400
56	935,900	25,300	13,700
Series D:			
57	80,700	16,700	8,200
58	155,800	17,500	9,500
59	246,000	19,400	10,400
Series F:			
60	504,400	22,300	12,100
61	603,500	23,100	12,500
62	616,300	23,200	12,600
Series G:			
63	345,300	20,600	11,200
64	727,100	24,000	13,000
65	1,006,000	25,600	13,900
Series S:			
66	613,000	23,200	12,600
67	733,900	24,100	13,000
68	805,800	24,500	13,300
Series T:			
69	300,700	20,800	11,300
70	440,200	21,700	11,700
71	1,059,400	25,900	14,000
Series U:			
72	452,000	21,800	11,800
73	470,000	22,000	11,900
74	955,200	25,400	13,800
Series L:			
75	214,100	18,700	10,100
76	359,900	20,800	11,300
77	530,000	22,500	12,200
78	663,000	25,900	12,800
(e) APPLIED LOAD FROM +15,000 TO −15,000 LB PER SQ IN.[a]			
Series L:			
79	404,400	20,000	10,800
80	718,900	22,500	12,200
Totals	....	1,758,900	949,300

[a] s_{FO} equals the fatigue strength, in pounds per square inch, computed from the applied load and the observed number of load applications sustained prior to failure. A positive sign denotes tension.

TABLE 4.—FREQUENCY DISTRIBUTIONS OF FATIGUE STRENGTH, s_{FO}, TESTS IN TABLES 2 AND 3

Normal law	(a) ZERO TO TENSION						(b) FULL REVERSAL					
	$N_o = 100{,}000$			$N_o = 2{,}000{,}000$			$N_o = 100{,}000$			$N_o = 2{,}000{,}000$		
	Loading range (kips)	No.	Frequency	Loading range (kips)	No.	Frequency	Loading range (kips)	No.	Frequency	Loading range (kips)	No.	Frequency
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
5.4	23.9–29.1	9	8.5	12.05–14.55	11	10.4	14.10–17.26	7	8.8	7.55– 9.29	7	8 8
24.2	29.1–34.3	25	23.6	14.55–17.05	23	21.7	17.26–20.42	16	20.0	9.29–11.03	17	21.3
39.9	34.3–39.5	42	39.5	17.05–19.55	43	40.5	20.42–23.58	31	38.8	11.03–12.77	30	37.5
24.2	39.5–44.9	23	21.7	19.55–22.05	22	20.7	23.58–26.74	22	27.4	12.77–14.51	22	27.4
5.4	44.9–49.9	7	6.7	22.05–24.55	7	6.7	26.74–29.90	4	5.0	14.51–16.25	4	5.0

s_{FO} have been computed and presented in Tables 2 and 3. Statistical evaluation of the results and presentation of this fatigue strength for $N_o = 100{,}000$ cycles and $N_o = 2{,}000{,}000$ cycles in the form of a frequency distribution both for zero-to-tension amplitudes and for full-stress reversals (Tables 2 and 3 and Fig. 4) give the ranges of variation about the average values $\bar{s}_{FO} \pm C\,\sigma$ where

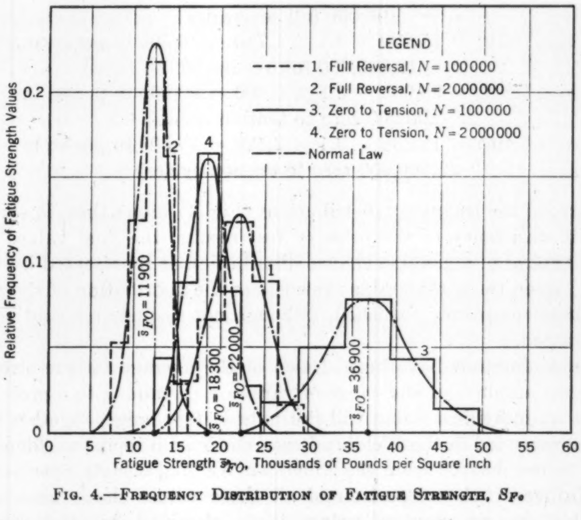


FIG. 4.—FREQUENCY DISTRIBUTION OF FATIGUE STRENGTH, S_{Fo}

C denotes the range constants pertaining to selected probabilities of occurrence of values outside these ranges.

The routine computations have been omitted from Tables 2 and 3, and the following computations will serve as examples to demonstrate the procedure:

The last item of Col. 3, Table 2, is the total of the 106 values of s_{FO} computed from individual tests. The average value then is equal to 3,912,350/106 = 36,900 lb per sq in. A column of differences from the mean is then computed for each test, and another column for squared values. The sum of this last column being $\Sigma(s_{FO} - \bar{s}_{FO})^2 = 2{,}883.8 \times 10^6$, the standard deviation is $\sigma = \sqrt{\dfrac{2{,}883.8 \times 10^6}{106}} = \pm\ 5{,}200$ lb per sq in. To derive values of the skewness of the distribution k a third column $(s_{FO} - \bar{s}_{FO})^3$ is computed for each test and the total $\Sigma(s_{FO} - \bar{s}_{FO})^3$ is found to be $1{,}896.8 \times 10^9$. Thus, $k = \dfrac{(F - \bar{F})^3}{106 \times \sigma^3}$ $= k = \dfrac{1{,}896.8}{106 \times 5.2^3} = +\ 0.13$.

Similarly, for $N_o = 2{,}000{,}000$ cycles in Col. 4, Table 2: $\bar{s}_{FO} = 18{,}300$ lb per sq in., $\sigma = 2{,}500$ lb per sq in., and $k = +\ 0.09$; for $N_o = 100{,}000$ cycles (full reversal) in Col. 3, Table 3, $\bar{s}_{FO} = 22{,}000$ lb per sq in., $\sigma = 3{,}160$ lb per sq in., and $k = -\ 0.21$; and, for $N_o = 2{,}000{,}000$ cycles (full reversal) in Col. 4, Table 3, $s_{FO} = 11{,}900$ lb per sq in., $\sigma = 1{,}740$ lb per sq in., and $k = -\ 0.24$.

Referring to Fig. 4, computations for minimum strength ($C = 3.1$ pertaining to probability 0.001) give:

$$s_{FO1}\,(\text{min}) = 22{,}000 - 3.1 \times 3{,}160 = 12{,}200 \text{ lb per sq in.}$$
$$(100{,}000 \text{ full reversals})$$
$$s_{FO2}\,(\text{min}) = 11{,}900 - 3.1 \times 1{,}740 = 6{,}500 \text{ lb per sq in.}$$
$$(2{,}000{,}000 \text{ full reversals})$$
$$s_{FO3}\,(\text{min}) = 36{,}900 - 3.1 \times 5{,}200 = 20{,}800 \text{ lb per sq in.}$$
$$(100{,}000 \text{ zero to tension cycles})$$
$$s_{FO4}\,(\text{min}) = 18{,}300 - 3.1 \times 2{,}520 = 10{,}500 \text{ lb per sq in.}$$
$$(2{,}000{,}000 \text{ zero to tension cycles})$$

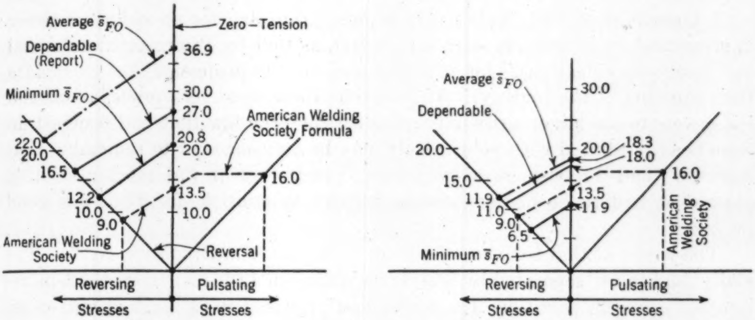
The form of the frequency distributions (Fig. 4) shows that, in spite of the apparent inconsistency of the primary test results, the final values are of a regularity typical of a genuine chance distribution as represented by the Normal Law. Even the considerable skewness of the distribution of the values of N has almost completely vanished, although not a single relevant result has been eliminated.

In Fig. 5 diagrams have been drawn of the average fatigue strength for various stress amplitudes and range of variation pertaining to a probability of 0.001 that an individual value will fall outside this range, together with the values recommended in the Welding Research Council report as "dependable" and with values derived from the American Welding Society Specification for Welded Bridges.[10] A comparison of all these values shows that the term "dependable" is meaningless unless it is classified by the appropriate probability.

To compare the ranges of variation of the fatigue strength of butt welds with those of the static strength s_T of the parent material, this strength has been computed, by statistical evaluation of the data contained in the Welding Research Council report,[6] as $s_T = 61{,}000 \pm 8{,}500 = 61{,}000\,(1 \pm 0.14)$. Expressed in terms of the average static strength the fatigue strength of butt welds has, therefore, the average values given in Table 5.

---

[10] "Specifications for Welded Highways and Railway Bridges," Am. Welding Soc., 1941, p. 18, formula 7.



(a) 100000 CYCLES (b) 2000000 CYCLES

FIG. 5.—FATIGUE STRENGTH AS A FUNCTION OF AMPLITUDE OF STRESS CYCLES, IN KIPS PER SQUARE INCH

TABLE 5.—FATIGUE STRENGTH OF STATIC TENSILE STRENGTH, s_T.

Cycles	$N_e = 100{,}000$	$N_e = 2{,}000{,}000$
Zero to tension....	$0.61\ s_T$	$0.30\ s_T$
Full reversal......	$0.36\ s_T$	$0.195\ s_T$

The statistical difference between the specific maximum ranges of variation of the fatigue strength of butt welds about the average value (which is roughly ± 45%) and of the static strength of the parent material represents the range of variation caused entirely by the effects of welding and of fatigue. This range is therefore $\sqrt{(0.45)^2 - (0.14)^2} = \pm 43\%$. According to the definition of the objective safety factor,[2] it is required that in the design of butt welds in fatigue this factor be $\dfrac{1 - 0.14}{1 - 0.43} = 1.5$ times greater than the factor pertaining to the design of plain sections in static tension—if, all other things being equal, failure is to be prevented in both cases with equal probability.

CONCLUSION

The far-reaching effects of standard specifications in the fields of structural design and of production of engineering materials make it essential that in the preparation of such specifications the barrenness of conventionality and routine be evaded by the scientific attitude to engineering problems, and a spirit of vigorous critical analysis should replace traditional complacency. Nothing must be taken for granted and every relation assumed must be carefully examined by the most appropriate method, be it logical induction or statistical inference. Only thus will the impact of modern scientific development make itself felt in the engineer's everyday work.

DISCUSSION

I. Oesterblom,[11] M. ASCE.—It is most gratifying for an old practitioner in structural engineering to see a paper such as that by Professor Freudenthal on "design specifications," for it shows how far the profession has gone since the beginning of the century. At that time there were very few controls and these were in the larger cities only. Worse yet—the few that had been set up were heavily tainted with politics. It may be amusing now to remember that concrete "had no business aboveground," and when it was most graciously allowed it had to be heavy—always heavier than brick, for that was good politics.

The writer at that time was designing a big stable in Brooklyn, N. Y. The stalls "had to be" separated by 8-in. brick walls—or 12-in. concrete; the 4-in. reinforced concrete partitions the writer had proposed were laughed out of existence. Nothing was mentioned about how the concrete was to be made, however, so that into the walls ultimately went all kinds of refuse salvaged from the lot, including some stable manure.

That time is gone now, seemingly, because the authorities are willing to listen to such excellent rationalization as is presented by the author; and this again raises the very important question: Who were the men that made the most recent codes and to what extent were they governed by the law of probabilities? This is a very pertinent question. Even today one may examine official schedules of permissible live loads, and find that there are great variations even for cities in which they should be the same. Any recent assembly of loads will show variations to 100% where there should have been no variations.

In the light of the present paper, therefore, it is proper to ask: How, and by whom, were the codes made? Were they made by politicians or by practical builders? Were professional engineers also consulted? If engineers were consulted, to what extent were their opinions respected and followed? If they were not consulted, why not? Also what was done to tell the world that engineering judgment should have been included—for the sake of equity and safety for those who "foot the bills."

Bridge builders, as a rule, have been quite free from political interference. Therefore, Example 1 is well chosen; but there should have been another similar example for the common building loads in regard to which more rationalization is sorely needed. Usually there is traffic load on all floors, with people temporarily at rest also included. This load varies for different kinds of service; but it is usually very light. It is the incidental loads of equipment that run the specified loads so high—and keep the code makers in such deep doubt that their guesses seem like a chapter from "Alice in Wonderland." Can it be stated now that the Freudenthal method has given designers the means to combine the light traffic loads with the heavy equipment loads into a rational average—one for each type of building? If this be so, it would be interesting to review the principal classes of building service and study how the rational live loads compare with their political and regional predecessors.

---

[11] Engr., Carbide and Carbon Chemicals Corp., South Charleston, W. Va.

By the way, that problem is also a challenge to the author—of both interesting and practical significance.

Example 2 is not of equal significance in that it deals with permissible stresses for which the variability largely has been eliminated by more definite and better controlled processes of manufacture. This might not yet be true for a butt weld; it certainly will be true soon. If it were not so, one may well ask if the probability method is in order. If a load were assumed too light, it would not be serious, but if a weld strength were too low at a critical point the result might be disastrous—and to include such a weak point in the setup for probability will not save a structure when it is cursed by such an element of possible breakdown.

JOHN M. HAYES,[12] ASSOC. M. ASCE.—Very pointedly, this paper emphasizes the importance of standard specifications for structural design and clearly advocates the adoption "of the scientific method in their preparation." Too often design practice, as reflected in standard specifications, has lagged behind research, theory, and the study of the action of existing structures. There seems always to be somewhat of a constant conflict between theory and practice. This is futile, since only an integral fusion of both will produce successful results. The great English statesman, Benjamin Disraeli, has stated, "A practical man is a man who practices the errors of his forefathers."

Normally, theory precedes practice. However, this is not to belittle practice. Hardy Cross, M. ASCE, has written:[13]

> "* * * a designer of engineering structures upon which the integrity of property values and the safety of human life depend has no right to accept theoretical results unless he finds for them some support in experience or in common sense."

This seems paradoxical, but it only emphasizes the complexity of the problem.

Specifications for structural design should reflect practices based on sound thinking, thoroughly blended with theory and experience. This experience should be tempered with information derived from a complete study of structures in service. Every engineer should recognize his obligation to gather data on the actions of structures in service at every opportunity and disseminate this information to the profession. In common with practitioners in every field, the structural designer must rely on authentic information from other engineers and related workers. This necessitates complete cooperation and coordination of all fields of engineering.

A structural designer must not only be able to interpret standard specifications and apply them to routine design problems, but must also have a working understanding of the theory and practice on which they are based. He must be thoroughly conversant with their limitations, and, therefore, should be able to cope with special problems that arise in design. Too often, specifications reflect personal prejudice and blind admiration for past practice.

---

[12] Dist. Bridge Engr., Public Roads Administration, Little Rock, Ark.

[13] "Limitations and Application of Structural Analysis," by Hardy Cross, *Engineering News-Record*, October 17, 1935, p. 535.

The writer can do no more than to lend emphasis to the author's closing statement:

> "Nothing must be taken for granted and every relation assumed must be carefully examined by the most appropriate method, be it logical induction or statistical inference. Only thus will the impact of modern scientific development make itself felt in the engineer's everyday work."

Alfred M. Freudenthal,[14] M. ASCE.—In discussing a previous paper by the writer,[2] to which the present paper might be considered a sequel, one of the participants pointed to the surprising lack of interest among engineers in the fundamentals of their profession. That there appears to be no reason to revise this statement is obvious by the reluctance to discuss, and thus help to solve, one of the most fundamental problems facing the structural designer—that of the design specification.

This classification of the problem is hardly an exaggeration; there is no single problem of greater importance or of greater consequence in the daily work of the individual structural designer than that of being able to work on the basis of rational and sound design specifications, which are devised to help him to use the material at his disposal with the maximum efficiency consistent with adequate safety, without, at the same time, affecting the expediency of his design procedure by the introduction of cumbersome and time consuming methods of analysis. The average designer, however, is ready to discuss in greatest detail, and apparently without ever tiring, every possible aspect of the analysis of elastic structures. He does not spare his time in attempting to eliminate whatever inaccuracy still remains in his methods of analysis by devising procedures of higher precision.

Nevertheless, he invariably turns away whenever it is pointed out to him that the stage in his analysis that could bear some investigation is that which follows moment computation. This is the stage in which, after having obtained statical characteristics such as, for instance, the bending moment for a unit load (by rigorous application of methods of analysis devised on the bases of the best scientific and engineering thought of the last two centuries), he (*a*) multiplies the moment by the specification load, a value generally arrived at by the guessing of one or a number of administrators, and then, in order to obtain the required section, he (*b*) divides the result by the permissible stress. This stress value usually represents a rather elusive compromise between the tendency to improve the utilization of the material and the conflicting tendencies to do this without affecting the existing manufacturing processes of the material—without changing established design practices and without, in general, giving the matter too much thought as a fundamental engineering problem.

The question might be asked whether such a procedure constitutes adequate design. However, it appears that this is a rather inappropriate question to ask and, apparently, not many engineers care enough to reply. Why this is so is hard to understand; but there is undoubtedly an intrinsic fascination in playing with numbers, and considerably less fascination in getting behind the numbers to find out how they were arrived at, what they mean, or if they mean anything at all.

---

[14] Visiting Prof. of Theoretical and Applied Mechanics, Graduate School, Univ. of Illinois, Urbana, Ill.

It is gratifying to find such wholehearted endorsement of a proposed line of thought as that expressed by Messrs. Oesterblom and Hayes in their comments. In connection with Mr. Oesterblom's very thoughtful discussion, one particularly interesting aspect of the problem emerges. There appears to be one point at which even the most rational engineering mind becomes reluctant to face the consequences of his own way of thinking—this is, when the mechanical strength or the permissible stress of materials is being discussed. Mr. Oesterblom considers Example 2 of the paper, dealing with strength of butt welds, to be of less significance

> "* * * in that it deals with permissible stresses for which the variability largely has been eliminated by more definite and better controlled processes of manufacture. This might not yet be true for a butt weld; it certainly will be true soon. If it were not so, one may well ask if the probability method is in order. If a load were assumed too light, it would not be serious, but if a weld strength were too low at a critical point the result might be disastrous—and to include such a weak point in the setup for probability will not save a structure when it is cursed by such an element of possible breakdown."

This is an admirably concise expression of the (apparently subconscious) apprehension when considering comparative effects of "overload" and "understrength." The variability of the load, leading to a possible overload, is accepted as an indisputable fact. However, if the variability of strength is mentioned, some cord of subconscious fear of "possible breakdown" is struck. Mr. Oesterblom's is not an isolated reaction, but one shared by almost every member of the profession with whom the writer has had the privilege to discuss his views on structural safety.

In his previous paper[2] the writer tried, apparently without great success, to argue the following points:

a. The safety factor is a factor of correlation of "load" and "resistance" (or "strength").

b. By correlation of these two design characteristics, the safety factor automatically compensates for the effects of both ignorance and uncertainty; even if, for the sake of argument, it were considered possible to eliminate "ignorance" (both subjective and objective), the objective uncertainty inherent in all—even the most carefully planned and performed—human activities and observations remains ineliminable. This applies to both load and strength.

c. It is, therefore, equally impossible to predict an absolute minimum of strength, even if the highest level of control were imposed on the manufacturing process of the material or to predict an absolute maximum load. Thus the only rational specification of either load or strength is to specify a value of relatively high frequency of occurrence, with such a range of extreme fluctuation that the probability of occurrence of values outside this range is negligibly small, considering the expected number of loads during the period of service of the structure.

d. It is the probability method that makes it at all possible to correlate an extremely improbable "overload" with an extremely improbable "understrength," by computing, instead of arbitrarily selecting, a safety factor in such a manner that a coincidence of both extreme conditions becomes practically impossible.

e. In view of the preceding arguments design loads and permissible stresses are closely interrelated; thus, in establishing design specifications they must not be considered except in conjunction, precisely because both load and strength can only be specified in the form of statistical frequency distributions. That in this respect there is no difference in principle between the strength of a butt weld and that of a steel section manufactured and rolled under the usual controlled conditions becomes evident by comparing Fig. 4 of the paper under discussion with Fig. 3 of the writer's previously published paper on safety.[15]

There is no alternative approach which can give the designer some confidence that, by rationally specifying an overload not to be exceeded more frequently than once in 10,000 loading cycles, and a minimum strength value to be attained in at least 9,999 of 10,000 tests, the probability of coincidence of extremes leading to disastrous breakdown has been reduced to less than 1 in 100,000,000. It is a chance which, considering the total number of load cycles in civil engineering structures, is beyond any practical range. The writer is particularly grateful to Mr. Oesterblom for having raised the point and thus provided an opportunity for its renewed consideration.

There is no doubt that, as Mr. Hayes points out, the problem is too complex to be solved by the exclusive application of rational methods. It is in this light that the cited remarks by Hardy Cross, Hon. M. ASCE, are very much to the point. So far, however, it is rather the reverse—that is, rational methods have only been admitted in stress analysis. In devising the procedure of how to apply the results of this analysis to the design of real structures, the reliance is mostly on subjective judgment, "experience," and "common sense." It is to plead for the admission, on at least equal terms with subjective judgment and prejudice, of rational analysis in this stage of design—the stage of drawing up of standard specifications—that the writer has presented his arguments in a manner that may appear to place too little emphasis on the existing and very real limitations of the rational approach in engineering.

---

[15] "The Safety of Structures," by Alfred M. Freudenthal, *Transactions*, ASCE, Vol. 112, 1947, p. 134, Fig. 3.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2336

THIN-SHELLED DOMES LOADED ECCENTRICALLY

BY WALTER C. VOSS,[1] ASSOC. M. ASCE, DEAN PEABODY, JR.,[2] M. ASCE, AND HOWARD R. STALEY[3] AND ALBERT G. H. DIETZ,[4] ASSOC. MEMBERS, ASCE

SYNOPSIS

A shallow thin-shelled model dome of plaster of Paris was subjected to a concentrated load successively placed between the springing and the crown. Strains were measured at near-by and distant stations on the upper surface. Compression and transverse tests of the plaster gave data concerning moduli of elasticity and Poisson's ratio whereby stresses were computed from the recorded strains. Sand loads were distributed uniformly over an eighth segment and over a quarter segment to study the transition from a concentrated load to a uniformly distributed load over the entire surface.

A mathematical analysis was made to determine the stresses caused by a concentrated load applied at the crown of a shallow thin-shelled spherical dome. Stresses of any magnitude are essentially local ones, so this analysis applies approximately to an eccentric load applied at some distance from the springing or any hole in the dome. As far as the test of one model dome can authorize, this series of tests seems to justify the use of the mathematical analysis to compute maximum stresses in a shallow dome.

INTRODUCTION

The absence of information on the effects of eccentric concentrated loads on thin-shelled domes with small rise prompted these tests. Domes of this type were already in existence and were being built without accurate data in regard to the stresses carried by load concentrations.

NOTE.—Published in October, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Head, Dept. of Bldg. Eng. and Constr., Mass. Inst. Tech., and Consultant, Cambridge, Mass.

[2] Associate Prof., Dept. of Bldg. Eng. and Constr., Mass. Inst. Tech., Cambridge, Mass.

[3] Asst. Prof., Dept. of Bldg. Eng. and Constr., Mass. Inst. Tech., Cambridge, Mass.

[4] Asst. Prof., Dept. of Bldg. Eng. and Constr., Mass. Inst. Tech., Cambridge Mass.

To check laboratory and theoretical studies against loading data from actual structures, it was decided to build a plaster model of a large dome, actually erected, to a scale of one to eight. The base of the spherical dome was 8 ft in diameter; its rise was 1 ft ($\frac{1}{8}$ d); and the thickness of the shell was 0.25 in. The radius of the sphere was 8.5 ft.

This exploratory test was made in 1941, before the mathematical analysis had been initiated; and it was not considered wise at that time to assume that an eccentric concentrated load would produce measurable strains only in the surface close to the area of load application. Consequently, sixty-four gage stations were located on the dome surface with two wires connected to each station. The assumption was made that the axes of principal strains for highly strained stations would be at, or near, the meridional and circumferential axes of the dome. A mathematical analysis by Eric Reissner[5] confirms this assumption (see Appendix I) and indicates that any divergence in test data would probably be slight. It would be necessary to add sixty-four more wired connections to obtain a third strain at each station to complete a rosette.

The inner form of the dome was not dropped far enough to permit strain readings on the under surface. An attempt was made to read deflections to 0.001 in. with a transit, but no deflections of this magnitude were recorded. If subsequent tests are made, it is desirable that the number of stations be reduced so that three strains at each station may be taken to complete a rosette, that stations be located on both the upper and lower surfaces near the area of load application, and that accurate measurement of deflections be attempted near the load application.

Strains were measured in the meridional and circumferential directions at each station. After the completion of the concentrated load tests, the distributed sand load tests were made, the sand load remaining several days on the dome while the time effect on the strains was measured. Finally, the dome was tested to failure by a concentrated load applied about three tenths of the meridian length from the springing.

Notation.—The letter symbols used in this paper are defined as they occur, and are assembled for convenience of reference in Appendix II.

PART I. EXPERIMENTAL DETERMINATION OF STRAINS

GENERAL

Objective.—The objective of these tests was to study the effect of concentrated loads on a spherical dome. Test data on models or full-sized domes are lacking for such loads, especially for loads that are not applied at the central axis. As nearly as can be determined, theoretical analyses were not in existence prior to the work done by Mr. Reissner.[5] Test data for eccentric loadings will be valuable for establishing the extent of the surface strained and the comparative magnitude of these strains, and for a comparison with the mathematical analysis.

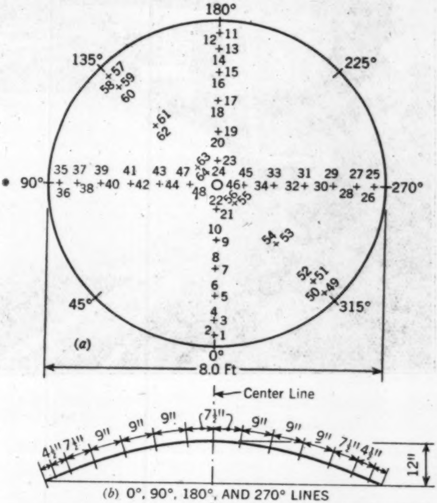
Eccentric distributed loading is desirable to supply the transition from an eccentric concentrated load to a symmetrical uniformly distributed load.

---

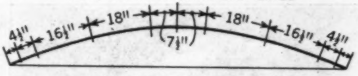
[5] "Stresses and Small Displacements of Shallow Spherical Shells," by Eric Reissner, *Journal of Mathematics and Physics*, February, 1946, p. 80.

Construction of the Model.—A rigid wooden ring was mounted in a frame in an even plane. A recess left on the top inner part of this ring acted as a support and also took the reaction of the model when loaded. The bottom of this recess acted as a guide for the templet during the operation of fabricating the base mold on which the dome shell was cast. The top of the recess was used as a guide for forming the dome shell. The base mold and dome shell were formed by rotating a metal templet about a central shaft. The shell was cast with the usual heavier ring, or haunch. Before testing, the base mold was lowered by jacks.

Strain Gages.—Electrical resistance wire gages of the type developed by Professors A. C. Ruge, Assoc. M. ASCE, and A. V. deForest of the Massachusetts Institute of Technology at Cambridge were adopted as best meeting the requirement for precision over a short gage length. Gages were placed at the sixty-four stations shown in Fig. 1. At each station, strains were measured in the circumferential and meridional directions. The 128 gages were connected to a central selector panel which allowed any one gage to be read at will. For the initial tests, the gages were straight wires, approximately 3 in. long, and were cemented to the surface, as shown in Fig. 2. Subsequently, hairpin gages were used at, or near, the loaded stations. These hairpin gages measured the strain for a 1.5-in. length. In the autumn of 1940 the shorter gages, now in use, were not on sale.



(b) 0°, 90°, 180°, AND 270° LINES



(c) 135° AND 315° LINES

FIG. 1.—LOCATION OF STRAIN GAGE STATIONS ON THE DOME

LOADING PROCEDURE

Concentrated Loads.—For most of these tests a concentrated load of 40 lb was used. Lead blocks, 4 in. square, rested on four soft rubber pads, each $\frac{1}{8}$ in. thick and $1\frac{1}{4}$ in. square. The pads were separated by $\frac{1}{2}$-in. clearance, so that it was 3 in. out-to-out for the pads, whereas the actual loaded area was 6.25 sq in. The loads were applied on the 0° meridian or at other points between the 45° and 315° meridians.

The test to failure was made by applying a concentrated load at Station 5,6 on the 0° meridian (see Fig. 1). In this case, a lever system was employed which consisted of an arm swinging about a pivot through the center of the

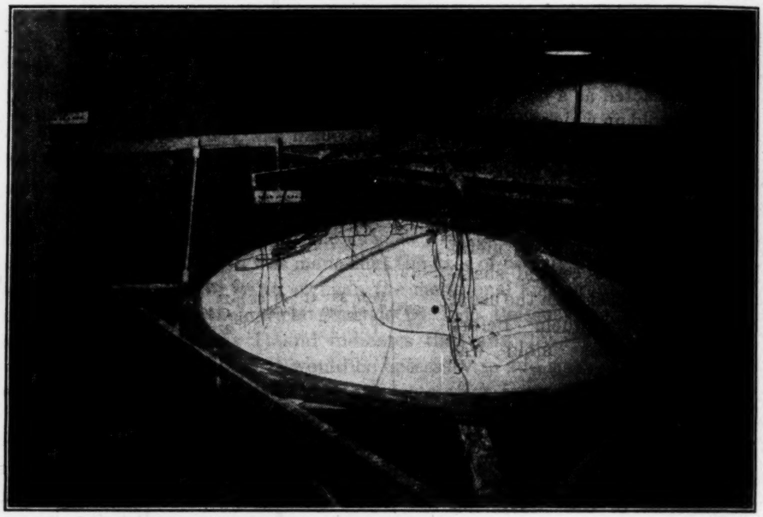


FIG. 2.—TEST TO FAILURE

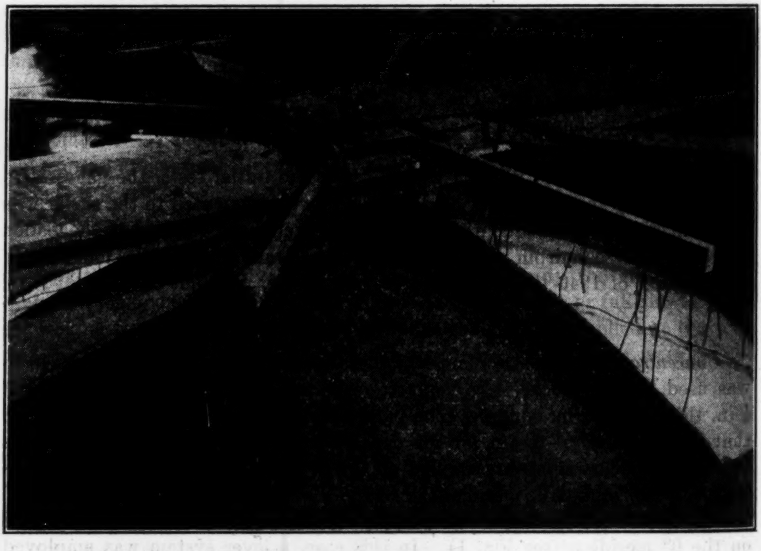


FIG. 3.—DISTRIBUTED LOAD OVER ONE QUARTER OF THE SURFACE

dome. This arm had a load platform at the far end, and rested on a leg bearing on a shoe placed on the rubber pads (Fig. 2).

Sand Loads.—A form of plywood and sheet metal rested on the dome shell and confined the sand to a definite area (Fig. 3). Dry sand was spread uniformly over the segment until the desired weight was obtained.

Poisson's Ratio and the Modulus of Elasticity

Compression tests were made on plaster cylinders, 2 in. in diameter and 4 in. high. Gages similar to those used on the dome were installed to give longitudinal and circumferential strains. From the data, the modulus of elasticity and Poisson's ratio were computed.

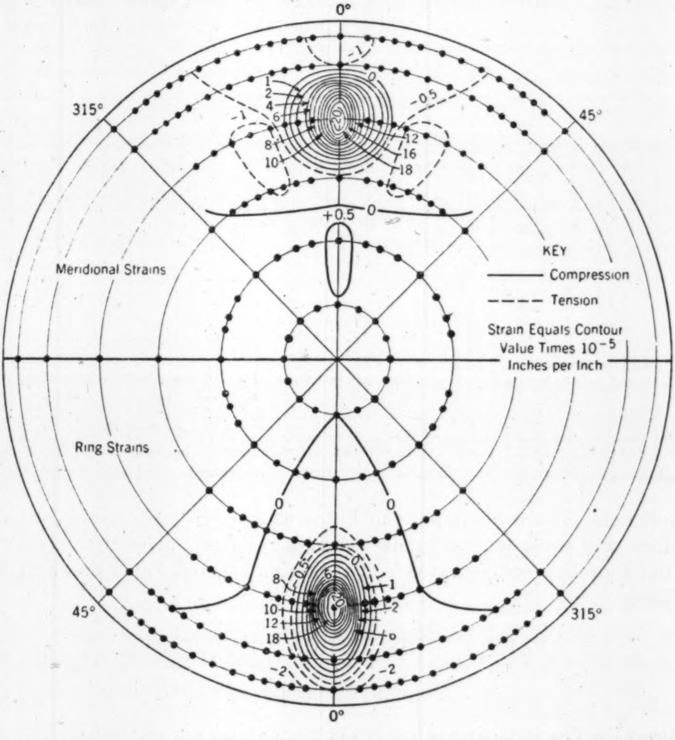


Fig. 4.—Strain Contours for a Concentrated Load of 40 Lb at Station 5,6

Transverse tests were also made on small pieces of the broken dome. Assuming uniformly varying stresses and strains, fiber stresses and the modulus of rupture were computed, as well as the modulus of elasticity. For each specimen a plot of the load-strain data can be represented by two straight lines at slightly different slopes. However, the comparison of moduli of elasticity with the compression moduli (presented subsequently in Table 6) has been made as the stress varies by tenths of the ultimate.

EXPERIMENTAL DATA AND RESULTS

The complete report on which this paper is based (including Mr. Reissner's complete mathematical treatment, which is reported briefly in Appendix I) can be consulted at Engineering Societies Library as well as at the library of the Massachusetts Institute of Technology.

Concentrated Loads.—The strains measured for a given position of the concentrated load were assembled in tables such as Tables 1(*a*) and 1(*b*). These strain data were plotted to show the variation along a meridian and along circumferences. The contour plots in Fig. 4 show equal compression strain by full lines, and equal tension by dashed lines. Table 2 gives the data for the test to failure by a concentrated load at Station 5,6, and Fig. 2 shows the type of failure.

TABLE 1.—STRAINS RESULTING FROM 40-LB LOADS APPLIED ON

(The Number Times 10^{-6} is the Strain in Inches Per Inch.

Station No.[a]	(*a*) STRAINS AT VARIOUS STATIONS														
	Distance (in Inches) of Load (in														
	0 (1)	4 (2)	4 (3)	4 (4)	4 (5)	6 (6)	6 (7)	8 (8)	8 (9)	10 (10)	10 (11)	13 (12)	13 (13)	16 (14)	16 (15)
Load[b]	40	55	40[c]	40	40	40	40	40	40	40	40	40	40	40	40
Cols. averaged	....	....	....	....	....	....	....	....	....	....	....	....	....	....	....
1......	− 1.0	− 3.35	−2.44	....	....	....	....	....	....	....	....	....	....	....	....
2......	− 1.5	− 1.4	−1.0	....	....	....	....	....	....	....	....	....	....	....	....
3......	+ 2.94	− 1.0	−0.75	−0.5	−0.5	−1.25	−1.25	−1.0	−1.0	−1.0	....	−0.5	....	−0.5	....
4......	− 1.0	− 0.75	−0.55	−0.25	−0.5	−0.75	−0.75	−0.75	−0.75	−0.75	....	−0.6	....	−0.3	....
5......	+21.65	+ 1.95	+1.42	+1.25	+1.96	−0.75	−0.6	−1.0	−0.75	−0.5	0	0	....	+0.5	....
6......	+20.69	+10.06	+7.33	+6.60	+6.79	+2.93	+2.35	+0.4	0	−1.0	−1.0	−1.1	−1.15	−0.95	−1.0
7......	− 0.5	− 1.15	−0.85	−1.0	−1.25	−1.0	−1.25	−1.0	−1.0	−0.5	....	0	....	+0.5	....
8......	− 0.75	− 1.35	−1.0	−1.0	−0.8	−0.5	−1.0	−0.5	−0.75	−1.0	....	−1.0	....	−0.75	....

[a] See also Stations Nos. 9 to 62, inclusive, in the continuation of Table 1. [b] Load, in pounds. [c] Values in the 26-in. distance, on the 315° line and on the 45° line, were identical. The average strain in Col. 30 is the aver-

PART II. DISCUSSION OF RESULTS

CONCENTRATED LOADS

Strains.—The strains were measured on the upper surface of the dome for various concentrated loads applied at a gage station, or applied at measured distances on a circumference through the gage station. Thus, it was possible to record, with a minimum of stations, the strains at definite distances from the loading point. For certain loadings several tests were made on the same circumference. In these cases the readings have been averaged. Most of the loads were constant at 40 lb, but in certain cases loads of 20 lb, 55.0 lb, 55.1 lb, and 56.1 lb were used. The corresponding strain data were revised to a common base load of 40 lb, assuming strains to vary directly with the low loads used.

It was not feasible to locate gage points on the under surface of the dome because the form was dropped only a few inches and only the gage stations near the outer haunch could have been reached.

The averaged strain data are also shown on the contour plots, using contour lines to connect points of the same strain value. These contours (see Fig. 4) were located by scaling the plotted strain values. Of necessity many of these contour lines were sketched in to connect rather widely dispersed points of equal strain. However, close to the loaded areas, they are accurate.

THE Circumference at the Level of Station 5,6 in Fig. 1
Elongation Is Negative and Shortening Is Positive)

					(b) Average Strains for Stations Loaded Similarly										Station No.
Pounds) from Station 5,6:															
19 (16)	19 (17)	22 (18)	22 (19)	26[d] (20)	0 (21)	4 (22)	6 (23)	8 (24)	10 (25)	13 (26)	16 (27)	19 (28)	22 (29)	26[d] (30)	
40	40	40	40	40	40 Col. 1	40 Cols. 3, 4, and 5	40 Cols. 6 and 7	40 Cols. 8 and 9	40 Cols. 10 and 11	40 Cols. 12 and 13	40 Cols. 14 and 15	40 Cols. 16 and 17	40 Cols. 18 and 19	40 Col. 20	Load
....	....	....	....	....	− 0.1	−2.44	....	....	....	....	....	....	....	....	1
....	....	....	....	....	− 1.5	−1.0	....	....	....	....	....	....	....	....	2
....	....	....	....	....	+ 2.94	−0.58	−1.25	−1.00	−1.00	−0.5	−0.5	....	....	....	3
....	....	....	....	....	− 1.0	−0.43	−0.75	−0.75	−0.75	−0.6	−0.3	....	....	....	4
+0.5	....	+0.5	....	....	+21.65	+1.54	−0.68	−0.88	−0.25	0	+0.5	+0.5	+0.5	....	5
−0.75	−0.75	−0.45	−0.4	−0.3	+20.69	+6.91	+2.64	+0.2	−1.0	−1.13	−0.98	−0.75	−0.43	−0.3	6
....	....	....	....	....	− 0.5	−1.03	−1.13	−1.0	−0.5	0	+0.5	....	....	....	7
....	....	....	....	....	− 0.75	−0.93	−0.75	−0.63	−1.0	−1.0	−0.75	....	....	....	8

Col. 3 are the strain readings in Col. 2 revised to a common base corresponding to a 40-lb load. [d] The strains at age of the two.

The common characteristic of these plots is the large compressive strain immediately under the loaded area in both meridional and circumferential directions. The strains decrease rapidly outside the loaded area; farther away small tensile strains usually occur, often followed by equally small compressive strains. The deformations suggest those observed if one presses a finger against the surface of an inflated toy balloon. Except for the area close to the loaded point, the magnitude of the strains is very small. Near the loaded area, the meridional contours are approximately circular whereas the circumferential contours are elliptical. The reader should note that "meridional" and "circumferential" refer to the central dome axis and not to an axis through the center of the load.

The strains are comparatively small for a load applied at Station 1,2 near the outer ring, or haunch (see Fig. 5). The stiffer haunch influences the strain distribution. As the position of application of the 40-lb load is moved up the meridian toward the crown of the dome, the strains increase to a maximum at

Station 21,22, which was the station nearest to the hole at the crown, the increase being about 40% more than the maxima at Station 3,4. The numerical values of the maximum strains in the meridional and circumferential directions at the same station are approximately equal.

Stresses.—The stress-strain plots of the compression and transverse tests indicate that the plaster behaves as an elastic material for stresses of less than 700 lb per sq in. To convert strain data to stresses, average values of the modulus of elasticity E (1,770,000 lb per sq in.) and of Poisson's ratio μ (0.20) are used. Knowing the meridional strain ϵ_m and the circumferential strain ϵ_c, under the load, the corresponding stresses σ_m and σ_c can be computed by the elastic relations:

$$\sigma_m = \frac{E}{1 - \mu^2} (\epsilon_m + \mu \, \epsilon_c) = 1{,}844{,}000 \, (\epsilon_m + 0.2 \, \epsilon_c) \ldots\ldots\ldots (1a)$$

and

$$\sigma_c = \frac{E}{1 - \mu^2} (\epsilon_c + \mu \, \epsilon_m) = 1{,}844{,}000 \, (\epsilon_c + 0.2 \, \epsilon_m) \ldots\ldots\ldots\ldots (1b)$$

TABLE 1.—(*Continued*)

Station No.	(*a*).—(*Continued*)			(*b*).—(*Continued*)	
	Distances.—(*Continued*):				
	0	4	4	0	4
	(1)	(2)	(3)	(21)	(22)
Load[b]...	40	55	40[c]	40	40
Cols. averaged...	....	....	....	Col. 1	Cols. 3, 4, and 5
9.......	−1.75	−1.75	−1.27	−1.75	−1.27
10.......	+0.75	+0.6	+0.45	+0.75	+0.45
11.......	....	0	0	....	0
12.......	....	+0.25	+0.2	....	+0.3
13.......	....	0	0	....	0
14.......	....	+0.4	+0.3	....	+0.3
15.......	....	−0.25	−0.2	....	−0.2
16.......	....	0	0	....	0
17.......	....	−0.5	−0.35	....	−0.35
18.......	....	0	0	....	0
19.......	....	−0.75	−0.55	....	−0.55
20.......	....	−0.15	−0.1	....	−0.1
21.......	0	−0.5	−0.35	0	−0.35
22.......	+0.25	0	0	+0.25	0
23.......	....	−0.4	−0.3	....	−0.3
24.......	....	−0.25	−0.2	....	−0.2
41.......	....	−0.75	−0.55	....	−0.55
42.......	....	−0.75	−0.55	....	−0.55
49.......	....	−0.75	−0.55	....	−0.55
50.......	....	−1.25	−0.9	....	−0.9
51.......	....	−1.0	−0.75	....	−0.75
52.......	....	+0.5	+0.35	....	+0.35
53.......	....	+0.5	+0.35	....	+0.35
54.......	....	−0.5	−0.35	....	−0.35
55.......	....	−0.5	−0.35	....	−0.35
56.......	....	−0.75	−0.55	....	−0.55
61.......	....	−0.75	−0.55	....	−0.55
62.......	....	−0.25	−0.2	....	−0.2

[b] Load, in pounds. [c] Values in Col. 3 are the strain readings in Col. 2 revised to a common base corresponding to a 40-lb load.

Using the averaged maximum strains under the load, the maximum stresses for a 40-lb load applied at the successive gage stations are given in Table 3. These maximum stresses increase as the load is applied nearer the crown of the dome. At any given station, the maximum meridional and circumferential stresses may be regarded as equal. The computed stresses at Station 21,22 are about 16% of the compressive strength of the plaster cylinders tested in compression (3,850 lb per sq in.).

Stress Comparison with the Mathematical Analysis.—Equations for the normal stress in a thin shell of a shallow dome, due to a concentrated load applied over a small circular area at the crown, are presented in Appendix I. The assumption is made that the load is applied relatively distant from any boundary or hole in the shell. Eqs. 6 in Appendix I give the values of the

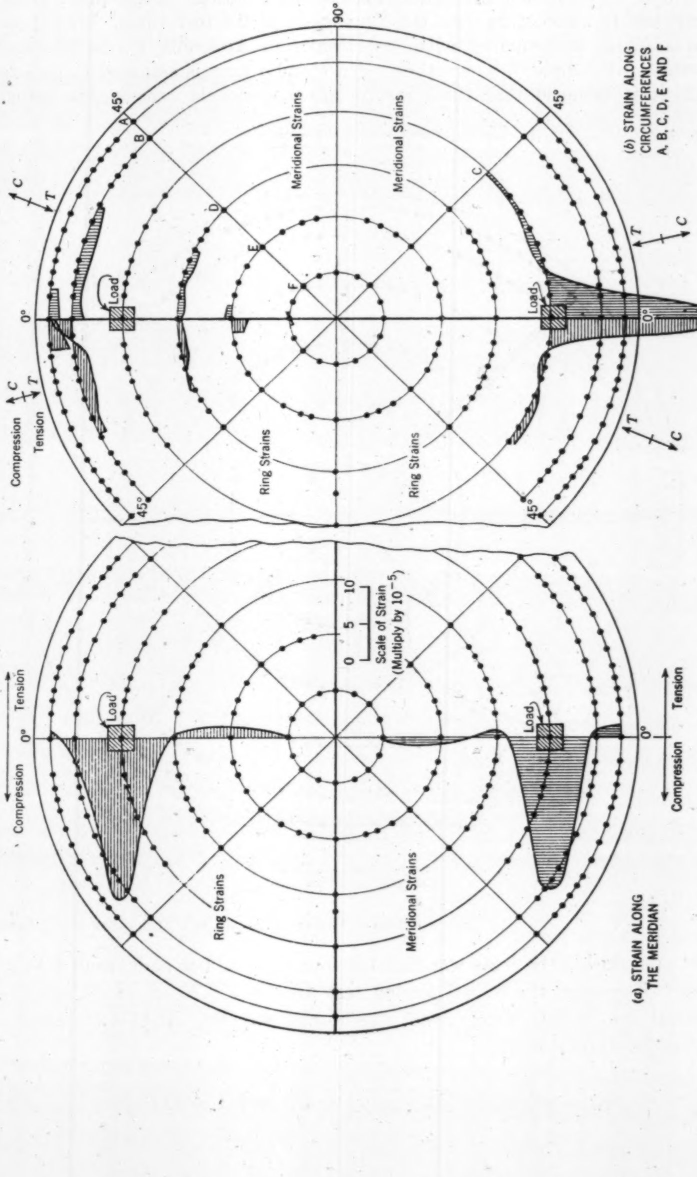


FIG. 5.—ANALYSIS OF STRAIN VARIATION FOR A CONCENTRATED LOAD OF 40 LB AT STATION 5,6

normal stresses in the meridional and circumferential directions at the upper and lower surfaces of the shell due to direct stress and bending. The equations are then applied to a structure with the dimensions of the test dome. The theoretical analysis is compared with test results in Appendix I (see heading, "Illustrative Example"), where the loaded circular area is assumed to be 6.25 sq in. It is believed that these results are comparable because the actual loaded area during the tests was through four thin rubber pads, each 1.25 in. square. Therefore, the actual loaded test area was 6.25 sq in. Using the numerical results thus determined, the normal stresses, direct and bending, at the upper surface equal:

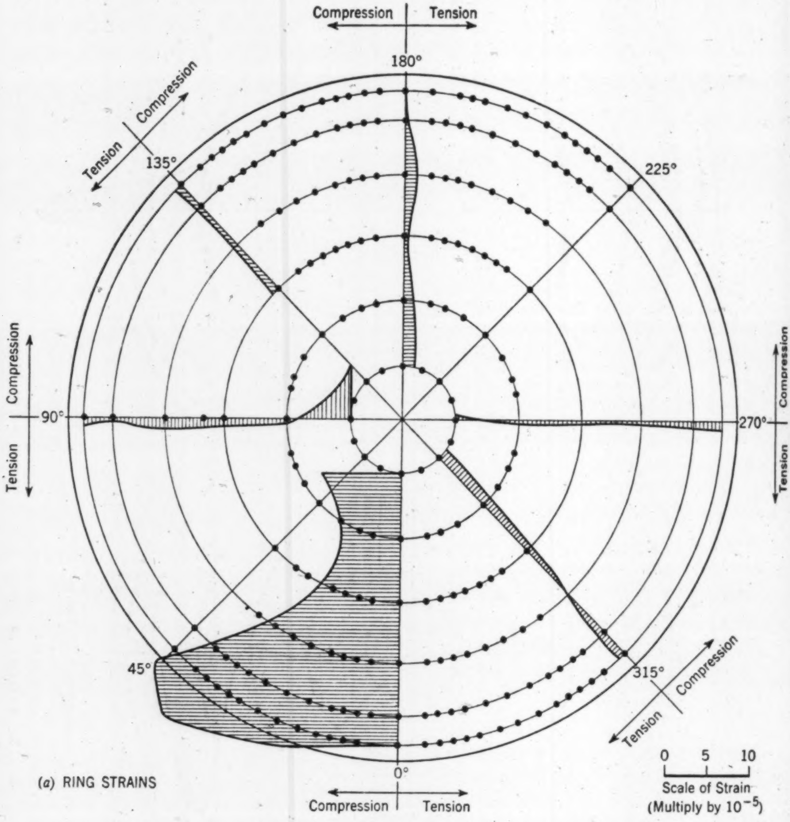


FIG. 6.—STRAIN VARIATION ALONG THE MERIDIANS FOR A SAND LOAD

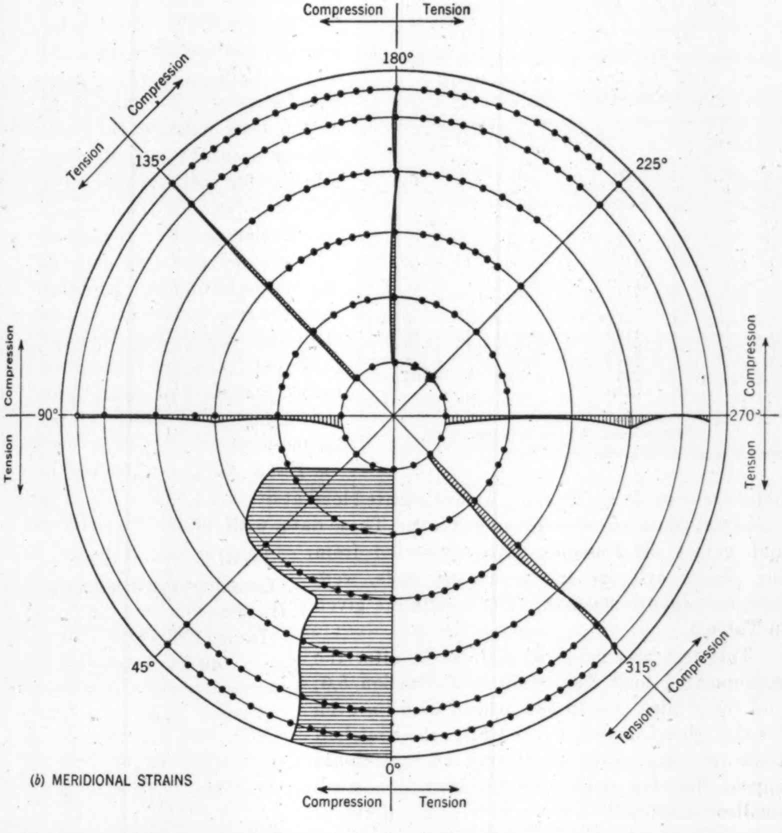
$$\sigma_m = \frac{P}{\pi \, \xi^2_c \, h^2} [\sqrt{3(1 - \mu^2)}(1 - C_1) + 3(1 + \mu) \, C_2] \dots\dots\dots (2a)$$

and

$$\sigma_c = \frac{P}{\pi\, \xi^2_c\, h^2}\left[\sqrt{3(1-\mu^2)}(1-C_1) + 3(1+\mu)\, C_2\right] \ldots\ldots\ldots (2b)$$

in which $\xi_c = 0.5146$, $C_1 = 0.9050$, and $C_2 = 0.1728$. By computation:

$$\sigma_m = \sigma_c = (0.1938 + 0.7477)\,\frac{P}{h^2} = \frac{(0.9415)\,P}{(0.25)^2} \ldots\ldots\ldots\ldots (3)$$



(*b*) MERIDIONAL STRAINS

of 267 Lb on an Eighth Segment Centered about the 0° Meridian

If $P = 40$ lb, $\sigma_m = \sigma_c = 124 + 479 = 603$ lb per sq in. (compression). This value is close to the average of the test stresses at Station 21,22. The stresses on the upper surface are larger than the resultant stresses on the under surface. It appears that the theoretical analysis approximates closely the maximum stresses given by the test data, even though Station 21,22 is about 6.5 in. from the edge of a hole, approximately 2 in. in diameter, at the crown of the test dome. The results from only one test dome are not conclusive, but they establish the possibility that the theoretical analysis may be used with safety to design shallow domes subjected to a concentrated load.

If Fig. 7(*a*), presented subsequently, is examined, it will be noted that the curves of stress variation are similar to the plots of strain variation from the test data. Measuring along the meridian the theoretical stresses should equal zero about 4.95 in. out from the center of the loaded area, for meridional stresses, and about 7.15 in. out, for circumferential stresses. The strain data of the tests indicate that zero stresses occur, in general, about 8 in. out for both stresses. The difference between the theoretical circular area and the actual four squares probably explains the nonconformance. In either case, the compressive stresses on the upper surface have reduced to zero at a distance beyond the loaded boundary of about twice the diameter, or width, of the loaded area.

TABLE 2.—TEST TO FAILURE, WITH A CONCENTRATED LOAD ON A 3-IN. SQUARE, APPLIED BY LEVER SYSTEM AT STATION 5,6

(The Number Times 10^{-5} Is the Strain in Inches per Inch. Elongation Is Negative and Shortening Is Positive)

Line No. (1)	Load (lb) (2)	Station 5 (3)	Station 6 (4)
1	37.6	+ 3.9	+ 4.4
2	66.6	+ 15.6	+ 17.5
3	96.5	+ 28.35	+ 31.0
4	124.7	+ 41.0	+ 45.8
5	153.0	+ 56.5	+ 59.2
6	181.7	+ 73.9	+ 79.4
7	211	+ 90.8	+ 95.8
8	280	+110.0	+118.6
9	37.6[a]	+ 11.2	+ 12.6
10	178.3	+ 76.1	+ 84.2
11	236	+108.9	+117.0
12	264.5	+127.1	+137.9
13	293	+160.3	+165.5
14	322[b]	....	....

[a] Load release. [b] Load at failure.

Test to Failure.—The test to failure was made by applying a concentrated load at Station 5,6 (Fig. 1). The load-strain data are recorded in Table 2. These data, with moduli of elasticity and values of Poisson's ratio estimated from the plaster cylinder and transverse tests, were used to compute stresses. The results are given in Table 4.

TABLE 3.—MAXIMUM COMPRESSIVE STRESSES (IN POUNDS PER SQUARE INCH) CREATED BY A 40-LB LOAD

Station (Fig. 1)	σ_m (1)	σ_e (2)
1,2	164	167
3,4	437	436
5,6	462	476
7,8	471	453
9,10	545[a]	545[a]
21,22	596	631

[a] Strains assumed to be equal in meridional and circumferential directions.

This test was made several weeks after the concentrated load test of 40 lb at Station 5,6; and the strains due to the load of 37.6 lb, even for the reloading (see Table 4(*b*)), are about half those recorded earlier at Station 5,6. It would appear that the strains for the lower loads are smaller than would be expected from the previous tests. During the period of testing, the plaster had dried thoroughly. However, with the greatest load applied (293 lb), the computed stresses approach the compressive strength of the moist cylinders, giving about 94% of the cylinder strength for a load that was 91% of the load at failure.

The stresses computed by the mathematical analysis in Appendix I approximated the stresses obtained from the measured strains only for the load applied at Station 21,22. For the 40-lb load applied at Station 5,6, the average stress from the strain data was 469 lb per sq in., or 77.8% of the theoretical stress of 603 lb per sq in. Using this percentage reduction of the theoretical stress applied to the second loading, a comparison is afforded by Table 4(*b*) of the stresses from strain data (Col. 6) and the corrected theoretical stresses.

TABLE 4.—MAXIMUM COMPRESSIVE STRESSES AT THE UPPER SURFACE

Load (lb)	E (number times 10^6)[a]	Poisson's ratio μ	MAXIMUM COMPRESSIVE STRESS (POUNDS PER SQUARE INCH)				Percentage of cylinder strength[c]	Load (percentage of ultimate)[d]
			Meridional, σ_m	Circumferential, σ_c	Average			
					Strain data	Formula[b]		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
			(*a*) INITIAL LOADING					
37.6......	1.77	0.20	96	88	92	....	2.4	11.7
66.6......	1.77	0.20	380	352	366	....	9.5	20.7
96.5......	1.77	0.20	676	636	656	....	17.1	30.0
124.7......	1.77	0.20	995	925	960	....	25.0	38.7
153.0......	1.77	0.20	1,300	1,260	1,280	....	33.3	47.5
181.7......	1.76	0.21	1,746	1,670	1,708	....	44.3	56.4
211	1.76	0.23	2,164	2,096	2,130	....	55.3	65.5
280	1.69	0.24	2,600	2,484	2,542	....	66.1	87.0
			(*b*) LOAD RELEASED AND RELOADED					
37.6......	1.77	0.20	274	254	264	441	6.9	11.7
178.3......	1.76	0.22	1,858	1,740	1,799	2,090	46.8	55.5
236	1.71	0.23	2,564	2,450	2,507	2,770	65.2	73.3
264.5......	1.67	0.25	3,020	2,880	2,950	3,100	76.6	82.3
293	1.65	0.26	3,662	3,598	3,630	3,440	94.3	91.0

[a] Pounds per square inch. [b] By Eq. 4*a* (see text). [c] Col. 6 $\times \frac{1}{3,850}$. [d] Col. 1 $\times \frac{1}{322}$.

The values in Col. 7 were computed by the formula:

$$\sigma = 0.778 \frac{0.9415}{0.25^2} P \qquad (4a)$$

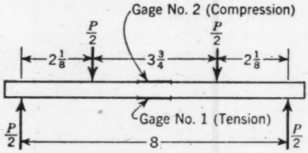
The theoretical stress of 441 lb per sq in. for a load of 37.6 lb is the value that would be expected from the strain data of the initial tests at Station 5,6. As the dome approached failure this same ratio (0.778) of test data to theoretical analysis was approached for the strain data of Table 2. The dome failed close to the load predicted by stress computations for the upper surface.

However, the computed stresses for the lower surface are tensile and

$$\sigma_m = \sigma_c = (0.7477 - 0.1938) \frac{P}{h^2} \qquad (4b)$$

For a load of $P = 293$ lb, this tensile stress equals $\sigma_m = \sigma_c = 0.778 \times 0.5539 \times 16 \times 293 = 2,020$ lb per sq in., as compared with the compressive value of 3,440 lb per sq in. on the upper surface (Col. 7, Table 4(*b*)). The transverse tests of Table 5 indicate a modulus of rupture for the plaster of about 750 lb per sq in., the failure being tensile. It would seem that the effect of shrinkage and plastic flow is such that the computed tensile stresses on the under surface are not realized. Until greater experience is gained, the high computed tensile stresses are an added incentive to use conservative allowable stresses and to disregard such stress reduction factors for position of the load as the value 0.778 used in Eq. 4*a*. Tests to observe variations in modulus of elasticity are presented in Table 6.

TABLE 5.—TRANSVERSE TESTS OF BEAMS CUT FROM DOME FRAGMENTS



Specimen number	Widths	Depths
1	1.98	0.306
2	2.00	0.295
3	2.02	0.256

(*a*) PHYSICAL DATA, IN INCHES (AGE OF SPECIMENS, SIXTY DAYS)

Specimen No.	Gage No.	Load P, in Pounds								
		0	2.205	4.410	6.615	8.820	11.025	15.435	19.845	22.050
(*b*) STRAIN, IN INCHES PER INCH, IS THE NUMBER MULTIPLIED BY 10^{-5}										
1	1	0	−4.78	−9.88	−14.74	−19.90	−25.08	−35.80	−46.5	−53.3
1	2	0	+4.74	+9.61	+14.29	+18.88	+33.88	+33.40	+43.5	+48.6
2	1	0	−5.07	−10.27	−15.51	−20.75	−26.46	−37.20	−49.5	....[a]
2	2	0	+5.03	+10.32	+15.35	+20.60	+25.77	+36.25	+45.3	....[a]
3	1	0	−6.15	−12.42	−18.70	−24.94	−30.94	−45.20	....[b]	....
3	2	0	+6.06	+12.25	+18.16	+24.07	+30.40	+42.85	....[b]	....
(*c*) FIBER STRESS, IN POUNDS PER SQUARE INCH, ASSUMING THAT STRESS AND STRAIN AT A SECTION VARY UNIFORMLY										
1	....	0	75.8	151.5	227.5	300	379.5	530.5	682	758
2	....	0	80.8	161.9	242.9	320	404.0	566.0	728	....
3	....	0	106.2	212.5	318.8	425	531.0	744.0	....	....
(*d*) MODULUS OF ELASTICITY, IN POUNDS PER SQUARE INCH (USING AVERAGE STRAINS), IS THE NUMBER MULTIPLIED BY 10^{6}										
1	....	....	1.59	1.56	1.56	1.55	1.55	1.535	1.515	1.485
2	....	....	1.60	1.57	1.57	1.55	1.55	1.54	1.535	....
3	....	....	1.74	1.72	1.73	1.735	1.73	1.69	....	....

[a] Specimen 2 broke at a load of 22.05 lb. [b] Specimen 3 broke at a load of 17.64 lb.

Continued Loading.—A time-load test was made with 40 lb applied at Station 7,8 for 5 days. During this interval the compressive strains directly under the loaded area gave increased values. The strains at the other stations tended to become tensile, this effect being most marked from 1 day to 5 days. Between the initial and the 24-hour readings, the maximum increase in compressive strain occurred at Station 7,8, followed by a slight steady decrease for the next 4 days. From 0 days to 1 day a few of the other stations also showed an increase of compressive strain which was overcome by the decrease for the next 4 days.

Summary.—Concentrated loads about one eighth of the ultimate load were applied at stations on a meridian varying in position from close to the outer haunch to close to the central hole. In every test the greatest stresses occur at the upper surface of the shell directly under the load, and they increase as the load is moved toward the crown. The stresses for a load close to the crown check closely those obtained by mathematical analysis (see Appendix I). The compressive stresses on particles in the upper surface decrease rapidly as the distance increases away from the loaded area. The stresses located twice the width of the loaded area from the loaded boundary have ceased to be compressive. Beyond this perimeter all stresses are comparatively small in magnitude. A concentrated load has only a local effect as far as stress intensities are concerned.

If a concentrated load remains in place for several days, the strains increase about 15% under the load during the first day. Thereafter, there is a slow decrease; at 5 days the strains are about 5% greater than the initial readings. Elsewhere in the dome the strains showed small elongations due to plastic flow.

TABLE 6.—SUMMARY OF AVERAGE TESTS FOR MODULUS OF ELASTICITY IN POUNDS PER SQUARE INCH (NUMBER TIMES 10^6)

Stress[a] (%)	Transverse tests	COMPRESSION TESTS	
		Secant	Tangent
10	1.64	1.77	1.77
20	1.62	1.78	1.79
30	1.62	1.78	1.72
40	1.61	1.76	1.65
50	1.61	1.76	1.59
60	1.60	1.71	1.59
70	1.60	1.69	1.59
80	1.59	....	....
90	1.58	....	....

[a] Percentage of the ultimate strength.

DISTRIBUTED LOADS

Eighth Segment Loaded.—The sand load of 267 lb on an eighth segment corresponds to an intensity of 40 lb per sq ft, or 0.278 lb per sq in., as compared with the concentrated load of 40 lb on 6.25 sq in. or 6.4 lb per sq in. Table 7 gives strain data at the time of load application for continued loading to 1.7 days. Fig. 6 shows the plotted strain data at the time of load application. The sand load was centered on the 0° meridian, and all other recorded stations were outside the loaded area. The initial circumferential strains (Fig. 6(*a*)) were relatively large along the 0° meridian with maximum values at Stations 3 and 5 of about the magnitude of those recorded for the concentrated load of 40 lb at Station 21,22. Elsewhere in the dome the circumferential strains were of small tensile magnitudes, except for the innermost Station 47 on the 90° meridian and three stations on the 315° meridian. The meridional strains (Fig. 6(*b*)) were also compressive on the 0° meridian, but the maximum values occurred nearer the crown of the dome at Stations 10 and 22. Elsewhere the strains were very low tensile values. Using these strain data, with a modulus of elasticity of 1.77×10^6 lb per sq in., and a Poisson ratio $\mu = 0.20$, the computed stresses along the 0° meridian are given in Table 8(*a*).

None of these stresses, caused by a load of 267 lb, exceed those obtained

TABLE 7.—STRAINS RESULTING FROM SAND LOADS CENTERED ON THE 0° RADIUS, INCLUDING THE EFFECT OF CONTINUOUS LOADING

(The Number Times 10^{-6} Is the Strain in Inches Per Inch. Elongation Is Negative and Shortening Is Positive)

Station No. (Fig. 1)	(a) 267 LB ON AN EIGHTH SEGMENT			(b) 534 LB ON A QUARTER SEGMENT				
	Duration of Loading, in Days:							
	0 (1)	0.85 (2)	1.7 (3)	0 (4)	0.8 (5)	1.8 (6)	2.8 (7)	4.8 (8)
1	+13.2	+ 4.75	+ 0.8	+ 5.4	−1.0	+0.05	+1.2	−0.55
2	+11.8	+ 7.3	+ 2.65	− 0.1	−295	−1.05	+0.2	−0.95
3	+28.2	+14.4	+ 6.75	+ 0.95	+0.7	+2.3	+4.0	+3.1
4	+10.9	+11.75	+ 7.35	+ 1.9	+2.1	+3.0	+4.7	+4.7
5	+29.15	+ 7.6	+ 4.4	+ 0.4	+1.4	+2.8	+3.9	+3.1
6	+11.1	+ 8.0	+ 5.4	− 2.75	−0.1	+1.3	+2.2	+1.45
7	+11.85	+ 6.6	+ 4.85	+ 2.85	+1.25	+2.6	+4.1	+2.8
8	+ 8.8	+ 7.5	+ 7.0	+ 2.9	−1.3	+0.75	+1.6	+2.1
9	+ 6.9	+ 5.35	+ 4.25	+ 0.8	−2.1	+0.15	+2.3	−1.9
10	+17.1	+ 9.1	+10.6	+ 4.9	+1.05	+3.1	+4.45	+4.5
11	− 0.1	− 0.1	+ 0.95	+ 0.25	−2.4	−1.45	−1.1	−0.9
12	− 0.25	− 1.2	− 0.8	+ 0.75	−2.35	−3.15	−3.2	−0.55
13	− 0.4	− 1.0	− 0.5	+ 0.2	−2.25	−2.05	−1.5	−0.45
14	+ 0.2	− 0.55	− 0.85	+ 1.05	−1.0	−1.8	−1.35	+0.6
15	− 1.55	− 2.45	+ 0.4	+ 1.55	−1.15	−0.45	−2.95	−0.2
16	− 0.05	− 0.8	+ 0.15	+ 1.05	−0.85	−1.15	−0.9	+0.5
17	− 0.3	− 0.9	+ 0.4	− 0.75	−2.4	−2.0	−1.4	−0.25
18	+ 0.15	− 0.25	+ 1.0	+ 1.0	−0.6	−0.45	+0.05	+3.0
19	− 1.25	− 1.7	0	− 1.0	−2.45	−1.95	−2.05	−1.15
20	+ 0.3	0	+ 0.95	+ 1.6	+0.05	−0.15	+0.05	+1.25
21	+ 9.25	+ 6.4	+ 4.8	− 0.7	+0.2	+2.6	+4.8	+3.8
22	+14.3	+ 5.5	+ 6.0	+ 0.65	+0.3	+2.8	+4.35	+3.35
23	− 1.5	− 1.7	− 0.35	− 1.5	−3.25	−3.3	−3.0	−2.0
24	+ 0.2	+ 0.25	+ 1.2	+ 0.5	−0.75	−1.25	−1.1	+0.55
25	− 0.9	− 1.65	− 1.1	− 1.4	−1.7	−1.1	−1.9	−0.9
26	− 0.5	− 1.25	+ 1.5	− 1.35	−3.1	−4.2	−4.45	−0.15
27	− 0.65	− 0.8	− 0.8	− 1.3	−3.0	−4.0	−3.2	−0.7
28	+ 0.2	− 0.25	+ 1.15	− 0.35	−1.0	−2.05	−1.0	+1.85
29	− 0.25	− 1.05	+ 0.15	− 0.4	−2.35	−2.85	−2.3	−0.25
30	− 1.2	− 1.05	− 0.3	+ 0.85	−1.0	−1.9	−1.5	+1.05
31	− 0.1	− 1.8	+ 1.0	0	−2.1	−2.7	−2.65	−0.4
32	− 0.15	− 0.6	− 0.15	0	−1.6	−2.0	−1.65	+0.5
33	− 0.1	− 0.8	− 0.3	0	−1.8	−2.2	−1.85	−0.25
34	− 0.1	− 0.45	+ 0.05	− 0.4	−1.75	−2.2	−1.9	−0.35
35	− 0.9	− 1.1	− 1.0	+ 0.15	−2.0	−1.5	−1.2	−0.75
36	− 0.2	− 1.25	− 1.4	+ 0.6	−2.15	−3.65	−3.75	−0.1
37	− 0.25	− 0.8	− 2.15	+ 0.65	−2.15	−2.6	−3.1	−0.3
38	− 0.25	− 0.9	− 2.2	+ 1.5	−1.05	−2.2	−2.55	+0.7
39	− 0.9	− 0.85	0	+ 3.4	−2.25	−1.9	−1.25	−0.1
40	− 0.4	− 0.1	+ 0.45	0	−0.85	−0.95	−0.25	+1.05
41	− 0.7	− 0.85	− 0.3	− 0.2	−2.3	−2.45	−2.35	−0.1
42	− 0.7	− 0.95	+ 0.85	0	−1.5	−1.5	−1.6	+0.35
43	− 0.3	− 0.7	+ 0.55	0	−1.65	−1.5	−1.35	+0.2
44	− 0.35	− 0.5	+ 0.65	− 1.0	−1.75	−1.75	−1.4	0
45	+ 0.4	0	+ 0.3	+ 1.75	−0.65	−0.65	+0.1	+2.0
46	− 0.65	− 1.75	− 1.5	− 1.0	−3.5	−3.6	−3.75	−1.45
47	+ 5.8	+ 0.5	+ 1.35	+ 1.45	0	−0.95	−0.75	+0.2
48	− 1.05	− 1.45	− 0.6	− 0.7	−2.8	−3.9	−4.15	−2.95
49	− 0.85	− 1.25	− 0.6	− 5.75	−4.5	−1.6	0	−1.1
50	− 0.1	− 1.45	− 1.0	− 1.5	−3.6	−2.55	−0.85	−0.45
51	− 0.6	− 1.35	+ 0.2	− 2.95	−1.6	+0.7	+1.15	+0.5
52	+ 0.55	− 0.25	0	+ 3.8	+1.1	+1.35	+2.9	+4.4
53	+ 0.95	+ 0.15	+ 0.6	− 2.0	−1.1	+0.65	+1.9	+1.55
54	− 1.0	− 2.3	− 1.4	+ 2.94	−1.6	−0.9	+0.15	+0.9
55	+ 1.5	+ 0.75	+ 1.32	0	+0.6	+2.85	+4.8	+5.4
56	− 0.15	− 0.85	− 1.25	+11.95	−2.95	−1.2	−0.4	+0.8
57	− 0.7	− 0.9	− 0.75	− 0.65	−2.05	−2.1	−1.75	−0.3
58	0	− 1.2	− 0.45	0	−2.6	−4.8	−3.7	0
59	− 0.5	− 1.3	− 0.4	− 0.65	−2.45	−2.95	−2.5	−0.4
60	− 0.1	− 0.55	+ 0.35	0	−1.45	−2.55	−1.75	+1.05
61	− 0.65	− 0.85	+ 0.15	− 1.0	−1.9	−2.05	−1.95	−0.25
62	− 0.3	− 0.45	+ 0.65	+ 0.4	−0.9	−1.45	−1.1	+1.25
63	....	....	....	....	....	....	....	....
64	− 0.15	+ 0.2	+ 1.0	− 0.4	−1.05	−1.6	−1.45	+0.1

by the concentrated load of 40 lb at Station 21,22. If the meridional stress, σ_m, at each station is plotted according to its position on the meridian, the individual values do not deviate greatly from their average. This is not true for the circumferential stresses, σ_c, which greatly exceed their average from a location close to Station 1,2 (and the haunch) to a position about halfway up the meridian; beyond this point the circumferential stresses are much lower than their average.

TABLE 8.—COMPARISON OF STRESSES ALONG THE 0° MERIDIAN (POUNDS PER SQUARE INCH) FOR A DISTRIBUTED SAND LOAD OF 267 LB ON A GIVEN SEGMENT
(Compression Is Positive)

Station (see Fig. 1)	(a) EIGHTH SEGMENT		(b) QUARTER SEGMENT	
	σ_m	σ_c	σ_m	σ_c
(1)	(2)	(3)	(4)	(5)
1,2	267	287	18	99
3,4	305	560	39	25
5,6	312	579	−3	−49
7,8	206	252	64	63
9,10	341	191	93	33
21,22	298	224	9	−11
Average....	(288)	(349)	....	....

Continued loading for 0.85 day gives a marked decrease in compressive strains along the central 0° meridian. The amount of depression had decreased but the general shape was unchanged, since the stations of greater strain had proportionately greater reductions. Elsewhere in the dome there was an increase in tensile strains, with the changes again proportional to the magnitude of the initial strain. It appears that the depression under the load was reduced, whereas elsewhere the dome shell bulged outward.

In the interval from 0.85 to 1.7 days there was further readjustment. The circumferential compressive strains on the 0° meridian again decreased roughly in proportion to the strain value, whereas the meridional strains increased at Stations 10 and 22 with a considerable decrease at Stations 2 and 4 near the haunch. Elsewhere in the dome, outside the loaded area, the change in strain was largely a reduction of tensile strain, so that about half of the final strains were compressive. It appears that the depression under the load was again reduced; elsewhere the previous bulge was also decreased.

The total effect of continued loading from 0 to 1.7 days was to reduce the compressive strains under the load on the 0° meridian. The circumferential strains were reduced from 38% to 94%; and the meridional, from 20% to 68%. The greater strains had the greater percentage reduction so that the strains at 1.7 days tended to approach a common value more closely. The stations outside the loaded area, for the most part, had net shortenings so that compressive strains were increased and tensile strains were reduced. Of the twenty-five circumferential stations outside the loaded area, twenty one recorded small initial tensile strains of less than 1.55×10^{-5} upon application of the load. The other four were close to the hole at the crown of the dome. At the end of 1.7 days eleven had tensile, twelve compressive, and two zero strains, compared with the unloaded dome. Similarly, for the twenty-six meridional stations outside the loaded area, the initial strains were: Nineteen tensile, six compressive, and one zero. Four of the compressive strains were on the 180° meridian, the other two were near the haunch. At the end of 1.7 days these twenty-six stations divided into twelve with tensile, thirteen with compressive, and one with zero, strains.

The effect of plastic flow on this plaster dome appears to be a readjustment by which the strains on the upper surface under the loaded area are markedly reduced, with a tendency to approach more uniform values. Stations outside the loaded area show an increase in tension from 0 day to 0.85 day and then a greater decrease, so that the general net result is a decrease in the original tensile values. However, stations on the circumference including Station 3,4 have final stresses indicating that the restraint of the near-by haunch causes bending.

The foregoing statement covers the observed readings on the upper surface. A complete knowledge of the effect of plastic flow cannot be obtained unless data in regard to the strains on the under surface and the change of deflection are also known. The area under the load is in bending as well as direct stress and the plastic readjustment is complex. It is possible that the dome is no longer a perfect spheroid elsewhere, and has been displaced horizontally as well as vertically.

Quarter Segment Loaded.—The load of 534 lb also corresponded to an intensity of 40 lb per sq ft, or 0.278 lb per sq in. The sand load centered on the 0° meridian, and its boundaries were the 45° and 315° meridians. Compared with eighth-segment loading, the initial strains were much reduced. Using a modulus of elasticity of 1.77×10^6 lb per sq in. and a Poisson ratio of 0.2, the computed stresses along the 0° meridian are given in Table 8(*b*).

At the boundary on the 315° meridian the circumferential strains on the top surface were tensile and the meridional strains were compressive. Outside the loaded area the strains were all of small magnitude, whereas their reversals suggested wave action in the shell.

This sand load remained in place for 5 days. Strain readings to measure plastic flow were taken at approximately 1, 2, 3, and 5 days. The readings varied considerably from day to day with many reversals of strain from compression to tension and vice versa. Many readings were as small as the probable error of reading. On the center line of the loaded area (0° meridian), the circumferential strains indicated that proximity to the haunch affected Station 1 and the hole at the crown may have affected Stations 9 and 21. The haunch apparently affected the meridional strains as far out as Station 6. Neglecting the readings at Station 1,2, at the end of 2.8 days the average circumferential strain was $+ 3.8 \times 10^{-5}$ and the individual values were fairly uniform; at 4.8 days the average strain was $+ 2.2 \times 10^{-5}$, but the values at Stations 9 and 21 departed considerably from this average. Similarly, the meridional strains averaged $+ 3.5 \times 10^{-5}$ at 2.8 days and $+ 3.2 \times 10^{-5}$ at 4.8 days.

At the boundary of the sand load on the 315° meridian, the circumferential strain curve preserved its general shape as time passed but steadily shifted to the compression side, so that initial strains that were wholly tensile by 2.8 days became wholly compressive. The 4.8-day readings were slightly lower. The meridional strains were wholly compressive for the initial readings (except for those at Station 50) with the greatest strain recorded for all stations at Station 56 near the crown (11.95×10^{-5}). There was a marked shift at 0.8 day so that all strains were tensile, except those at Station 52 near the haunch.

Thereafter the general pattern was preserved but the shift was toward small compressive strains.

The general effect of plastic flow on the upper surface of the shell along the center line meridian of the loaded area was to produce fairly uniform compressive strains which oscillate above and below $+ 3 \times 10^{-5}$ between 1.8 days and 4.8 days, except for the station close to the haunch. These strains were usually greater than the initial strains. At the boundary of the sand load (315° meridian) the strains reversed, the circumferential strain becoming compressive, whereas the meridional strain became tensile at 0.8 day, and then the strains slowly returned to low compressive values.

Outside the loaded area the initial strains had low values, some compressive, some tensile. From 0.8 day to 2.8 days they all shifted toward greater tensile values, remaining fairly constant during this time interval. At 4.8 days the strains all tended to approach zero strain: Seventeen with final compressive values (maximum $= + 3.0 \times 10^{-5}$), twenty four with final tensile values (maximum $= - 2.95 \times 10^{-5}$), and two with zero strain.

Uniform Load on Entire Dome Surface.—The tendency for the stresses to decrease as a larger area is loaded is reasonable. If the same sand load of 40 lb per sq ft is assumed to be spread over the entire surface, the circumferential and meridional stresses can be computed, assuming the shell to be a membrane.[6] In the part of the shell close to the haunch, the circumferential stress equals 40 lb per sq in.; and the meridional stress, 60 lb per sq in. No allowance was made for bending stresses caused by the proximity to the larger and more rigid haunch; but a load four times that on the quarter segment produced direct stresses of the general magnitude of those computed in Table 8(*b*) for the quarter segment.

GENERAL SUMMARY

The distributed load tests were made to clarify the transition from an eccentric concentrated load to a uniformly distributed load. They emphasize the fact that progressive concentration causes increasingly larger strains under the loaded area.

The discussion of concentrated loads is based on a load of 40 lb bearing on an area of 6.25 sq in. The distributed loads were 6⅔ × 40 on an eighth segment and 13⅓ × 40 on a quarter segment. The maximum stresses for the eighth-segment loading were comparable to those obtained for the concentrated load in the position producing the greatest recorded strains. The maximum stresses for a quarter-segment loading were about a sixth as much. The part of the dome well outside the loaded area had negligible stresses.

The readjustments due to plastic flow tended to reduce the strains in the upper surface of the loaded area. Elsewhere, the strains tended to become tensile in the first day of quiescent loading. Thereafter the strains outside the loaded area tended to approach zero. During these adjustments the restraining effect of the haunch was noticed at near-by stations.

---

[6] "Analytic Solution of Masonry Domes," by David C. Coyle, *Transactions*, ASCE, Vol. 88, 1925, p. 102.

ACKNOWLEDGMENT

This investigation was planned and executed for the Preload Company. The test was made on a plaster model to a reduced scale of a large dome designed and erected by this company.

APPENDIX I. MATHEMATICAL ANALYSIS

BY ERIC REISSNER

To facilitate the analysis of the test data of the model dome, theoretical equations were derived[5] for the direct stresses and bending stresses as components of the total normal stress in a shallow spherical dome treated as a thin shell. The equations apply to a total load P distributed uniformly over a circular area with its center at the apex of the dome. Let "ber," "bei," "ker," and "kei" be used to designate tabular functions. Also let $\xi_v = x/l$ and $\xi_c = r_o/l$ denote a dimensionless variable and a dimensionless constant, respectively (see Fig. 7). For small values of ξ_v:

$$\operatorname{ber} \xi_v \approx 1 - \frac{\xi^4_v}{64} \qquad \text{(5a)}$$

$$\operatorname{bei} \xi_v \approx \frac{\xi^2_v}{4} - \frac{\xi^6_v}{64 \times 36} \qquad \text{(5b)}$$

$$\operatorname{ker} \xi_v \approx -\log_e \xi_v + 0.1159 + \frac{\pi}{16} \xi^2_v \qquad \text{(5c)}$$

and

$$\operatorname{kei} \xi_v \approx -\frac{\xi^2_v}{4} \log_e \xi_v - \frac{\pi}{4} + (0.1159 + 1) \frac{\xi^2_v}{4} \qquad \text{(5d)}$$

Equations for maximum stresses, σ, are: Direct—

$$\sigma_m (0) = \sigma_c (0) = \frac{P}{h^2} \frac{\sqrt{3(1 - \mu^2)}}{\pi \xi^2_c} (1 - C_1) \qquad \text{(6a)}$$

and bending—

$$\sigma_m (0) = \sigma_c (0) = \pm \frac{P}{h^2} \frac{3(1 + \mu)}{\pi \xi^2_c} C_2 \qquad \text{(6b)}$$

TABLE 9.—FORMULAS FOR THE DETERMINATION OF ARBITRARY DESIGN CONSTANTS

C_1	C_2	D_3	D_4	Value
ber ξ_c	bei ξ_c	−ker ξ_c	−kei ξ_c	1
−bei ξ_c	ber ξ_c	kei ξ_c	−ker ξ_c	0
ber′ ξ_c	bei′ ξ_c	−ker′ ξ_c	−kei′ ξ_c	0
−bei′ ξ_c	ber′ ξ_c	kei′ ξ_c	−ker′ ξ_c	0

in which subscript m denotes "meridional"; and subscript c denotes "circumferential." To use Eqs. 5 and 6, the following data must be known: The concentrated load P; the radius r_s of the spherical dome shell; the thickness h of the shell; the radius r_c of the circular loaded area, equal to the actual loaded area; and Poisson's ratio μ. For concrete $\mu = 0.20$ is a reasonable value. The arbitrary constants C_1 and C_2 in Eqs. 6 are determined by solving the equations in Table 9.

The procedure consists in first determining the values of the arbitrary constant $\xi_e = r_e/l$, the length parameter l being defined as:

$$l = \frac{\sqrt{h\,r_s}}{\sqrt[4]{12(1-\mu^2)}} \qquad \text{(7)}$$

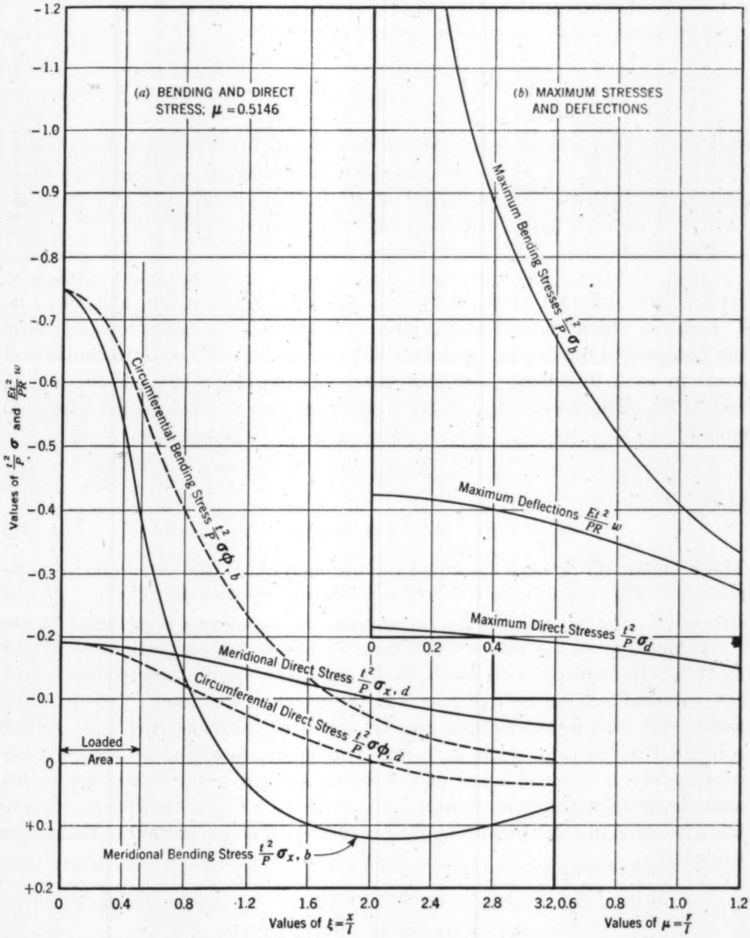


FIG. 7.—DESIGN VALUES OF DIMENSIONLESS FACTORS ξ_v AND ξ_e

With this value of ξ_e enter Fig. 7(a) and read the maximum bending stress σ_b and the maximum direct stress σ_d. Fig. 7(a) is not applicable for $\xi_e < 0.2$ or $\xi_e > 1.3$; but Eqs. 5 can be solved directly or by interpolation in Table 10.

Illustrative Example.—A numerical example was solved and discussed in detail in the original analytical report.[5] For this study the dimensions of the model were used—namely: $r_s = 102$ in.; $h = 0.25$ in.; and $\mu = 0.20$. By Eq. 7, then, $l = 2.7410$; and $\xi_v = x/l = 0.3648\,x$. The load was assumed to act on three different areas—$\pi\, r^2_c = 1.5625$ sq in., 6.25 sq in., and 16 sq in. The case in which the loaded circular area equals 6.25 sq in. approaches closely the test case of 6.25 sq in. composed of four squares with outer boundaries of 3 in. in each direction. For this case: $\xi_c = 0.5146$; $C_1 = 0.9050$; and $C_2 = 0.1728$. The maximum combined stress at the upper surface of the shell equals:

TABLE 10.—VALUES OF FUNCTIONS OCCURRING IN THE SOLUTION OF THE EQUATIONS OF THE FLAT SPHERICAL DOME PROBLEM

x	ber x	bei x	ker x	kei x	$\frac{\text{ber}' x}{x}$	$\frac{\text{bei}' x}{x}$	$\frac{\text{ker}' x}{x}$	$\frac{\text{kei}' x}{x}$
0	1.0000	0	$+\infty$	−0.7854	0	0.5000	$-\infty$	$+\infty$
0.2	1.0000	0.01000	1.7331	−0.7581	−0.00250	0.5000	−24.615	1.1146
0.4	0.9996	0.04000	1.0626	−0.7038	−0.01000	0.4999	−5.8802	0.7738
0.6	0.9980	0.8998	0.6931	−0.6374	−0.02250	0.4997	−2.4276	0.5803
0.8	0.9936	0.1599	0.4529	−0.5664	−0.3999	0.4989	−1.2342	0.4488
1.0	0.9844	0.2496	0.2867	−0.4950	−0.06245	0.4974	−0.6946	0.3524
1.5	0.9211	0.5576	0.05294	−0.3314	−0.1400	0.4868	−0.1961	0.1971
2.0	0.7517	0.9723	−0.04164	−0.2024	−0.2465	0.4585	−0.05330	0.1099
3.0	−0.2214	1.9376	−0.06703	−0.05115	−0.5233	0.2935	+0.00716	0.03068
5.0	−6.2301	0.1160	−0.01148	+0.01118	−0.7691	−0.8708	0.00344	−0.00016

$$\sigma_m = \sigma_{m,d} + \sigma_{m,b} \quad \text{(8a)}$$

and

$$\sigma_c = \sigma_{c,d} + \sigma_{c,b} \quad \text{(8b)}$$

These maximum stresses are equal for the meridional and circumferential directions. Fig. 7 shows, not only the maximum stresses, but the variation of stress over, and beyond, the loaded area as the loaded area is increased in size. Table 10 gives the values of the functions used in the equations of Table 9 to obtain the constants C_1 and C_2. These stress equations are derived for a concentrated load on a circular area at the crown of the dome. The fact that the loaded area is not circular can be neglected, provided the area is small compared to the total dome surface. The equations can also be applied approximately, if the load is not at the crown, as long as the loaded area is not near the springing, a hole, or other discontinuity. These results are approximate because an eccentrically applied gravity load is not normal to the surface loaded.

APPENDIX II. NOTATION

The following letter symbols, adopted for use in the paper, and in its discussion, conform essentially to American Standard Letter Symbols for Mechanics (ASA—Z10.3—1942), prepared by a Committee of the American

Standards Association, with Society participation, and approved by the Association in 1942:

C = an arbitrary constant; formulas for the determination of C_1, C_2, D_2, and D_4 are listed in Table 9;
c = subscript denoting "circumferential";
D = an arbitrary constant, such as D_2 and D_4 in Table 9;
d = diameter;
E = modulus of elasticity;
h = thickness of shell;
l = a length parameter defined by Eq. 7;
m = subscript denoting "meridional";
P = a concentrated point load, or the resultant of a distributed load assumed to be acting at the centroid of pressure;
r = radius, with subscript c to denote radius of area of concentrated loading, with subscript s to denote radius of spherical shell;
x = perpendicular distance from the axis of symmetry of a spherical shell;
ϵ = strain, with subscripts m and c to denote "meridional" and "circumferential," respectively;
μ = Poisson's ratio;
ξ = dimensionless quantity, with subscripts c and v to denote a "dimensionless constant" and "dimensionless variable," respectively; and
σ = unit stress, with subscripts b and d to denote "bending" and "direct" stress, respectively, and subscripts m and c to denote "meridional" and "circumferential," respectively.

The various tabular functions "ber," "bei," "ker," and "kei" are defined by Eqs. 5. These operators primed denote first differentials.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2337

UNDERGROUND CONDUITS—AN APPRAISAL OF MODERN RESEARCH

By M. G. Spangler,[1] M. ASCE

With Discussion by Messrs. Wilson V. Binger, Jacob Feld, Zdeněk Bažant, Jr., Anson Marston, George E. Shafer, E. F. Kelley, Bailey Tremper, and M. G. Spangler.

Synopsis

The increasingly successful application of scientific principles of mechanics to the analysis of soil has characterized engineering development during the first twoscore years of the twentieth century. A number of problems that involve the soil as an integral part of the finished structure and which a generation or two ago were considered by many to be too complex to be subject to orderly classification and solution are yielding to the onslaught of organized research.

One notable example of problems in this category concerns the design of underground structures such as conduits, sewers, drains, culverts, water mains, and gas lines. Conduits of this general type have been used by mankind for at least 3,000 years, being found among the earliest examples of the practice of the engineering arts; but not until about 1910 was serious effort directed toward the development of a rational method for determining the magnitude and character of the loads to which underground conduits were subjected in service due to the earth overburden and other load sources. In those early days, Anson Marston, Past-President and Hon. M. ASCE, then director of the Engineering Experiment Station at Iowa State College in Ames, inaugurated a research program which has been continuously active under his direction. The purpose of this paper is to review Dean Marston's theory of loads on, and supporting strengths of, underground conduits and to record the experimental background, together with working values of many of the constants required for the practical application of the theory.

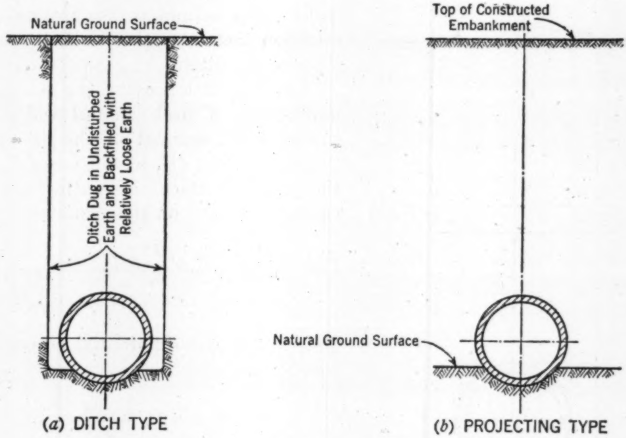
Note.—Published in June, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Research Associate Prof., Civ. Eng. Dept., Iowa Eng. Experiment Station, Iowa State College, Ames, Iowa.

Load Classification of Underground Conduits

Underground conduits, in general, may be divided into two main classes on the basis of construction conditions under which they are installed—that is (1) ditch conduits and (2) projecting conduits.

(1) "Ditch conduits" are structures installed and completely buried in narrow ditches in relatively passive or undisturbed soil. Examples of this class of conduits are sewers, drains, and water mains (see Fig. 1(*a*)).


Natural Ground Surface
Ditch Dug in Undisturbed Earth and Backfilled with Relatively Loose Earth
Top of Constructed Embankment
Natural Ground Surface

(*a*) DITCH TYPE (*b*) PROJECTING TYPE

Fig. 1.—Essential Elements of Typical Conduits

(2) "Projecting Conduits" are structures installed in shallow bedding with the top of the conduit projecting above the surface of the natural ground, and then covered with an embankment, as shown in Fig. 1(*b*). Railway and highway culverts are good illustrations of this class of conduits. Conduits installed in ditches wider than about two or three times their maximum horizontal breadth may also be treated as projecting conduits.

Notation

The letter symbols in this paper are defined where they first appear, in text or by illustration, and are assembled for reference in Appendix I. Discussers are requested to conform to this usage.

Loads on Ditch Conduits

When a conduit is placed in a ditch not wider than about two or three times its outside breadth, and covered with earth, the backfill will tend to settle downward. This downward movement or tendency for movement of the soil in the ditch above the pipe produces vertical frictional forces or shearing stresses along the sides of the ditch which act upward on the prism of soil within the ditch and help to support the backfill material. Assuming the cohesion between the backfill material and the sides of the ditch to be negligible, the magnitude of these vertical shearing stresses is equal to the active lateral pressure exerted by the earth backfill against the sides of the ditch, multiplied by the tangent of the angle of friction between the two materials. This assumption of negligible cohesion is justified because: (*a*) Even when the ditch is dug in, and backfilled with, cohesive material, considerable time must elapse before effective cohesion between the backfill material and the sides of the ditch can develop after backfilling; and (*b*) the assumption of no cohesion yields the maximum probable load on the conduit. The maximum load may develop at any time during the life of the conduit due to heavy rainfall or other causes which may eliminate or greatly reduce cohesion between the backfill and the sides of the ditch.

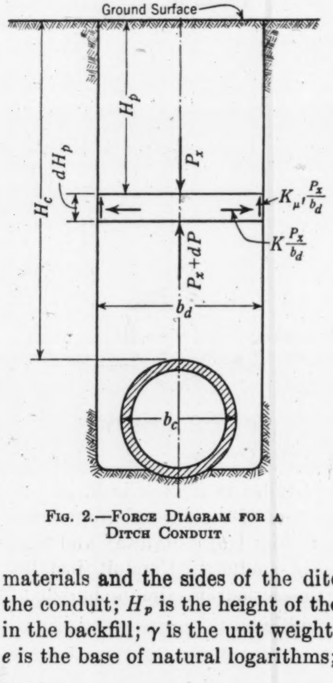


FIG. 2.—FORCE DIAGRAM FOR A DITCH CONDUIT

Let Fig. 2 represent the section of a unit length of ditch conduit and consider a horizontal element of the fill material having a height dH_p at a distance H_p below the ground surface. Equating[2] the vertical forces acting on this element:

$$P + dP + 2\,K\,\mu' \frac{P}{b_d}\,dH_p = P + \gamma\, b_d\, dH_p \,..\,(1a)$$

This is a linear differential equation, the solution for which is

$$P = \gamma\, b^2_d \frac{1 - e^{-\alpha H_p}}{2\,K\,\mu'}\,\ldots\ldots(1b)$$

in which P equals the total vertical pressure on a horizontal plane in the backfill, in pounds per linear foot; μ' is the coefficient of sliding friction between the fill materials and the sides of the ditch; b_d is the width of ditch at the top of the conduit; H_p is the height of the ground surface above any horizontal plane in the backfill; γ is the unit weight of filling material, in pounds per cubic foot; e is the base of natural logarithms; and, to simplify typography:

$$\alpha = \frac{2\,K\,\mu'}{b_d}\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2a)$$

The ratio K of active lateral pressure to vertical pressure, is defined as

$$K = \frac{\sqrt{\mu^2 + 1} - \mu}{\sqrt{\mu^2 + 1} + \mu}\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2b)$$

in which μ is the coefficient of internal friction of the filling material. The coefficient μ' may be equal to or less than μ, but it cannot be greater.

When $H_p = H_c$, Eq. 1*b* gives the total vertical pressure within the ditch at the level of the top of the pipe. The proportion of this total pressure that will be carried by the conduit will depend upon the relative rigidity of the conduit and of the fill material between the sides of the conduit and the sides of the ditch. In the case of very rigid pipes, such as burned clay, concrete, or heavy cast-iron pipe, the side fills may be relatively compressible and the pipe itself will carry practically all the load, P. On the other hand, if the pipe is a relatively flexible, thin-walled pipe and the side fills are thoroughly tamped in at the sides of the pipe, the stiffness of the side fills may approach that of the conduit and the load on the structure will be reduced by the amount of load the side fills are capable of carrying.

---

[2] "The Theory of Loads on Pipes in Ditches and Tests of Cement and Clay Drain Tile and Sewer Pipe," by A. Marston and A. O. Anderson, *Bulletin No. 31*, Iowa Eng. Experiment Station, Ames, Iowa, 1913, p. 32.

For the case of rigid ditch conduits with relatively compressible side fills, the load will be:

$$W_c = C_d\,\gamma\,b^2_d \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3a)$$

For the case of flexible pipes and thoroughly compacted side fills having the same degree of stiffness as the pipes, the load will be:

$$W_c = C_d\,\gamma\,b_c\,b_d \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3b)$$

in which W_c is the total load on the conduit, and:

$$C_d = \frac{1 - e^{-\alpha H_c}}{2\,K\,\mu'} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3c)$$

Eq. 3*a* has been verified many times by experiment and by observation of the performance of actual sewers and drains. The results of two of Dean Marston's early load experiments on rigid pipes are shown in Fig. 3, together with loads calculated by Eq. 3*a*.

Evaluation of Eqs. 3*a* and 3*b* may be simplified by the use of Fig. 4 in which values of C_d for various values of H_c/b_d have been plotted for several kinds of filling materials having different coefficients of internal friction.

The width of ditch, b_d, is the actual width of a normal, parallel-sided ditch. In case the ditch is constructed with sloping sides, or the conduit is placed in a subditch at the bottom of a wider trench, experiments[3] have shown that the width of ditch at or slightly below the top of the pipe is the proper width to use when determining the load.

These ditch conduit formulas (Eqs. 3*a* and 3*b*), with proper selection of the physical factors involved, give the maximum loads to which any particular conduit may be subjected in service. On the other hand, because of the development of cohesion, any particular conduit may escape the maximum load for a long time, sometimes until its removal for other causes than load failure. Experiments and field observations show that the load on a conduit at the time the fill is completed is usually less than it will be at some later time. This condition accounts for the fact that sewers and other conduits which have been observed to be structurally sound immediately upon completion are sometimes found to be cracked some months or years later.

---

[3] "Loads on Pipe in Wide Ditches," by W. J. Schlick, *Bulletin No. 108*, Iowa Eng. Experiment Station, Ames, Iowa, 1932.

LOADS ON PROJECTING CONDUITS

Projecting conduits, as defined and as the name implies, are placed with their tops projecting some distance above the natural ground surface. They may be of any shape, such as circular, rectangular, or elliptical; may be made of any material, such as concrete, burned clay, cast iron, corrugated metal, or wood; and may possess any degree of rigidity from the very rigid concrete pipes and monolithic box culverts to the very flexible, lightweight corrugated metal pipes.

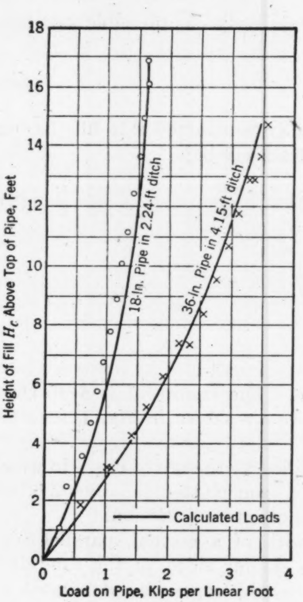


FIG. 3.—MEASURED LOADS ON DITCH CONDUITS

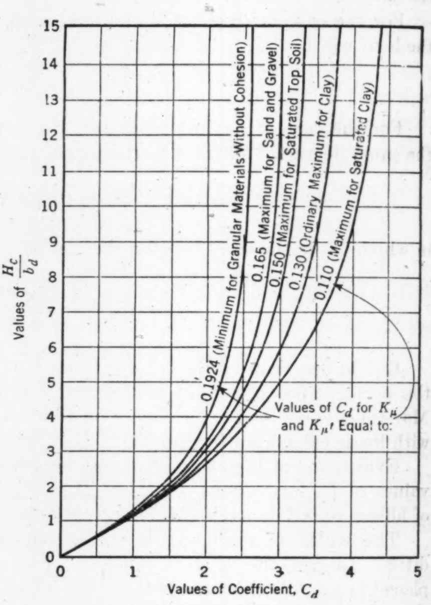


FIG. 4.—COMPUTATION DIAGRAM FOR DITCH CONDUITS

As a basic case for studying the action of an embankment over this type of conduit, consider first a rigid structure resting on an earth foundation which settles under the load the same amount as the natural ground adjacent to the structure (see Fig. 5). Then in the embankment built over the conduit, there are three masses or prisms of soil, one of which, called the "interior prism," is directly over the conduit between the vertical planes tangent to the sides. The other two masses of soil, called the "exterior prisms," are those on each side of the structure adjacent to the tangent vertical planes.

It is evident that the height of the interior prism will be less than that of the exterior prisms by the amount which the conduit projects above the natural ground. In accordance with the well-known phenomenon that a high prism of soil will settle more than a lower prism, there is a tendency for the exterior prism to settle more than the interior prism and for friction forces or shearing stresses to be exerted along the tangent vertical planes bounding the interior prism. These shearing stresses are transmitted to the conduit, making the load on the structure greater than the weight of the interior prism of soil. The magnitude of the shearing stresses (again neglecting cohesion) is assumed to be equal to the active lateral pressure at these planes, multiplied by the coefficient of internal friction of the fill material. It is recognized that definite shearing planes between the interior and exterior prisms of soil do not actually exist in an earth embankment and that, in all probability, the shearing stresses are transferred from one prism to another through more or less narrow zones of the filling material. Nevertheless, actual vertical shearing planes are assumed for convenience in developing the theory, and load measuring experiments indicate that the assumption is valid.

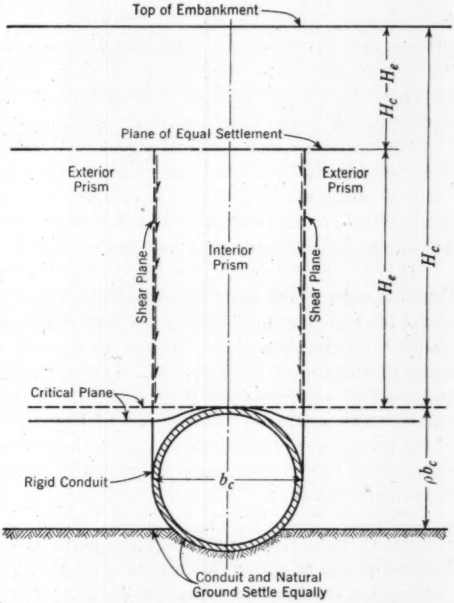


FIG. 5.—BASIC CASE FOR CONSIDERING THE ACTION OF AN EMBANKMENT OVER A PROJECTING CONDUIT

If the embankment is not very high, these shearing stresses may extend upward from the conduit completely to the top of the embankment. In the case of higher fills, the shearing stresses will not extend to the surface, but will terminate at some horizontal plane between the top of the conduit and the top of the embankment, known as the "plane of equal settlement." The distance between the top of the conduit and this plane is called the "height of equal settlement, H_e."

The plane of equal settlement is defined as the horizontal plane in the embankment at and above which the settlements of the interior and exterior prisms of soil are equal. Above the plane of equal settlement there is no tendency for relative movement between the three adjacent prisms and no shearing stresses are generated along the boundaries of the interior prism above this plane, whereas, below it, relative movements do have a tendency to occur and shearing stresses are developed.

A plane of equal settlement in this basic case results from the transfer of pressure, by shear, from the exterior prisms to the interior prism. Since the vertical deformation of a prism of material due to its own weight is a function of the height as well as characteristic of the material, normally the summation of deformations from the bottom of a prism upward will be at a greater rate in a high prism than in a lower one if the prisms act independently of one another. In the case of projecting conduits, however, the exterior prisms of soil transfer a part of their vertical pressure to the interior prism and the rate of summation of vertical deformations will be reduced in the exterior prisms and increased in the interior prism because of this stress transfer. Therefore, the total summation of deformations in the interior prism will approach that in the exterior prisms, and the height at which the deformations become equal is the height of equal settlement.

The existence of a plane of equal settlement was first announced by Dean Marston in 1922[4] on the basis of pure mathematical reasoning and a formula for evaluating its height was developed at that time. Since then the actual existence of such a plane has been demonstrated by measurements of the settlements and of the loads on a number of experimental conduits.

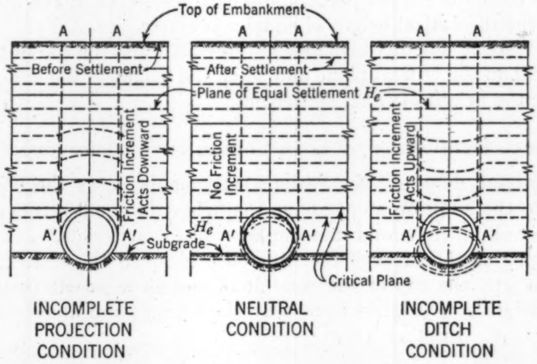
This basic settlement situation is practically always modified by two additional factors which must be considered in the development of a general theory of fill loads. The first of these is the settlement of the undisturbed subgrade under the exterior prisms adjacent to the conduit; and the second, the settlement of the top of the conduit. The settlement of the subgrade adjacent to the conduit augments the downward movement of the two exterior prisms of earth. The settlement of the top of the conduit (which is equal to the sum of the settlement of its foundation and the distortion or shortening of its vertical dimension) has the tendency to neutralize this action by reducing the relative movement between the interior and exterior prisms. Indeed, if the conduit is sufficiently flexible, if it is placed on a very yielding foundation, or if both conditions hold true, the top of the conduit may settle enough to permit the interior prism to move downward a greater amount than the exterior prisms. When this occurs, the direction of the induced shearing stresses is reversed and the stresses are subtractive from the weight of the prism of earth over the conduit. The rate of summation of settlements in relation to height above the conduit for the interior prism is less than that in the exterior prisms due to the reduced pressure. Nevertheless, there still may be a plane of equal settlement, even if the direction of relative movements is reversed, because the total settlement is augmented by the settlement of the top of the conduit, which causes the entire interior prism to move downward.

In this connection it is convenient to define a "critical plane" which is the horizontal plane in the fill material at the level of the top of the conduit at the beginning of construction of the embankment and before settlements have begun to develop. With this definition in mind, the aforementioned facts concerning direction of the induced shearing stresses may be simplified by stating that, when the critical plane settles more than the top of the conduit, the shearing stresses act downward on the interior prism and that, when it settles less, they act upward.

---

[4] "Second Progress Report to the Joint Concrete Culvert Pipe Committee," by Anson Marston, April 7, 1922 (mimeographed).

A neutral or transition case occurs when the top of the conduit settles downward an amount just equal to the settlement of the critical plane. When this case occurs, the plane of equal settlement is right at the top of the conduit and coincident with the critical plane; the interior and exterior prisms of soil move downward equally throughout their full height and no shearing stresses are induced, with the result that the load on the structure is equal to the weight of the prism of soil directly over it. Some typical settlement situations affecting loads on projecting conduits are illustrated in Fig. 6. (The solid horizontal lines represent imaginary planes of embankment before settlement. The broken lines represent the same planes after settlement.[5])



INCOMPLETE PROJECTION CONDITION

NEUTRAL CONDITION

INCOMPLETE DITCH CONDITION

FIG. 6.—TYPICAL SETTLEMENT CONDITIONS AFFECTING LOADS ON PROJECTING CONDUITS

In the mathematical analysis of loads on projecting conduits, the net effect of all these settlement factors, both as to magnitude and direction of the relative movements of the three prisms of soil, is combined into an abstract ratio known as the "settlement ratio," which is defined as the ratio of the difference between the settlement of the critical plane and the top of the conduit, to the deformation of the fill material adjacent to the conduit within the vertical distance between the top of the conduit and the natural ground line. Dean Marston defined this ratio on the basis of the settlements of the various elements which are produced by that section of the fill above the plane of equal settlement.

For purposes of calculating the external vertical loads on projecting conduits, the field conditions affecting the loads are conveniently grouped into four subclassifications based on the magnitude of the settlement of the interior prism relative to that of the exterior prisms and, therefore, the direction of the shearing stresses, and the height of the embankment in relation to the height of equal settlement. These four conditions are: (1) The complete projection condition, (2) the incomplete projection condition, (3) the complete ditch condition, and (4) the incomplete ditch condition.

---

[5] "Investigation of Loads on Three Cast Iron Pipe Culverts Under Rock Fills," by M. G. Spangler, *Bulletin No. 104*, Iowa Eng. Experiment Station, Ames, Iowa, 1931, p. 26, Fig. 11.

They are defined as follows:

(1) The "complete projection condition" prevails when the top of the conduit settles less than the critical plane and when the height of the embankment is less than the theoretical height of equal settlement.

(2) The "incomplete projection condition" prevails when the top of the conduit settles less than the critical plane and when the height of the embankment is greater than the height of equal settlement.

(3) The "complete ditch condition" prevails when the top of the conduit settles more than the critical plane and when the height of the embankment is less than the theoretical height of equal settlement.

(4) The "incomplete ditch condition" prevails when the top of the conduit settles more than the critical plane and when the height of the embankment is greater than the height of equal settlement.

MATHEMATICAL DERIVATION OF LOAD FORMULAS FOR PROJECTING CONDUITS

Dean Marston defines[4,6] the various settlements, y, as the "* * * increments due to the addition at or above the height of equal settlement of any [the same for all] incremental uniform layer of fill materials." A settlement ratio is defined as the ratio of the difference between the settlement of the critical plane and the top of the conduit of the fill adjacent to the conduit within the height $\rho\, b_c$ (Fig. 5) caused by the fill material above the plane of equal settlement. In other words:

$$\delta = \frac{(y_m + y_g) - (y_f + y_c)}{y_m} \qquad \text{(4a)}$$

in which y_g is the settlement of the embankment subgrade adjacent to the conduit, in feet; y_f is the settlement of the flow line of the conduit; y_c is the deflection of the conduit—that is, the shortening of its vertical dimension; and y_m is the deformation of the filling material adjacent to the conduit within the height $\rho\, b_c$ in feet—that is:

$$y_m = \frac{(H_c - H_e)\,\gamma}{E_f}\,\rho\, b_c \qquad \text{(4b)}$$

in which E_f is the modulus of deformation of the filling material. The settlement of the "critical plane," in this notation is $y_m + y_g$ and the settlement of the top of the conduit is $y_f + y_c$. In Eq. 4*a*, δ is negative for the "incomplete" ditch condition and positive for the "incomplete" projection condition.

Considering first the complete ditch and projection conditions (Fig. 7(*a*)), in which $H_c < H_e$, the vertical forces on any thin horizontal element of the interior prism may be equated as follows:

$$P + dP = P + \gamma\, b_c\, dH_p \pm 2\, K\, \mu\, \frac{H_p}{b_c}\, dH_p \qquad \text{(5a)}$$

---

[6] "The Theory of External Loads on Closed Conduits in the Light of the Latest Experiments," by Anson Marston, *Bulletin No. 96*, Iowa Eng. Experiment Station, Ames, Iowa, 1930.

The solution of this equation is

$$P = \gamma\, b_c^2 \frac{e^{\pm \beta H_p} - 1}{\pm\, 2\, K\, \mu} \ldots\ldots\ldots\ldots\ldots\ldots (5b)$$

in which μ is the coefficient of internal friction of the filling material, and to simplify typography (compare Eq. 2a):

$$\beta = \frac{2\, K\, \mu}{b_c} \ldots\ldots\ldots\ldots\ldots\ldots (5c)$$

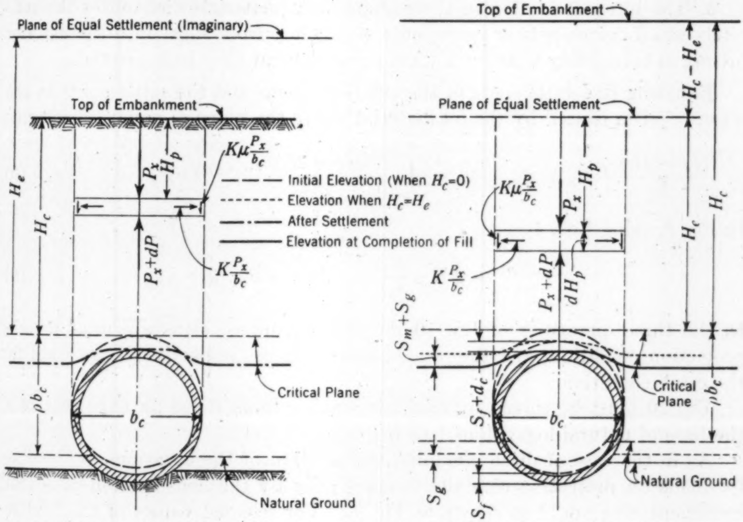
At the top of the conduit, $P = W_c$ and $H_p = H_c$; therefore:

$$W_c = C_c\, \gamma\, b_c^2 \ldots\ldots\ldots\ldots\ldots\ldots (6a)$$

in which the load coefficient for projecting conduits is represented by:

$$C_c = \frac{e^{\pm \beta H_c} - 1}{\pm\, 2\, K\, \mu} \ldots\ldots\ldots\ldots\ldots\ldots (6b)$$

in which H_c is the height above the top of a conduit. In Eq. 6b the plus signs are used for the complete projection condition and the minus signs for the complete ditch condition.


Plane of Equal Settlement (Imaginary)
Top of Embankment
H_e
H_c
P_x
H_p
$K\mu\frac{P_x}{b_c}$
P_x+dP
$K\frac{P_x}{b_c}$
Initial Elevation (When H_c=0)
Elevation When H_c=H_e
After Settlement
Elevation at Completion of Fill
Critical Plane
ρb_c
b_c
Natural Ground
Top of Embankment
H_c-H_e
Plane of Equal Settlement
$K\mu\frac{P_x}{b_c}$
P_x
H_p
H_e
H_c
$K\frac{P_x}{b_c}$
P_x+dP
dH_p
S_m+S_g
S_f+d_c
b_c
Critical Plane
ρb_c
Natural Ground
S_g
S_f

(a) COMPLETE PROJECTION CONDITION (b) INCOMPLETE PROJECTION CONDITION

FIG. 7.—FORCES ACTING ON A PROJECTING CONDUIT

Considering the incomplete ditch and the incomplete projection conditions (Fig. 7(b)), in which $H_c > H_e$, and again equating the vertical forces on a thin horizontal element:

$$P + dP = P + \gamma\, b_c\, dH_p \pm 2\, K\, \mu \frac{P}{b_c} dH_p \ldots\ldots\ldots\ldots (7a)$$

When $H_p = 0$, $P = (H_c - H_e)\,\gamma\,b_c$ and the solution of this differential equation (Eq. 7*a*) is:

$$P = \frac{\gamma\,b^2_c}{2\,K\,\mu} \pm \frac{\gamma\,b^2_c}{2\,K\,\mu}\,e^{\pm\beta H_p} + (H_c - H_e)\,\gamma\,b_c\,e^{\pm\beta H_p} \quad \text{(7b)}$$

At the top of the conduit, $P = W_c$ and $H_p = H_e$; therefore, in Eq. 6*a*, the load coefficient C_c becomes:

$$C_c = \frac{e^{\pm\beta H_e} - 1}{\pm\,2\,K\,\mu} + \left(\frac{H_c}{b_c} - \frac{H_e}{b_c}\right) e^{\pm\beta H_e} \quad \text{(8)}$$

As before, the plus signs are applicable to the incomplete projection condition and the minus signs apply to the incomplete ditch condition.

To derive an expression for evaluating H_e, the following two conditions are assumed:

a. The internal friction in the embankment materials distributes the infinitely small increments or decrements of pressure from shear into the interior prism below the plane of equal settlement in such manner that their effect on settlement may be assumed to be substantially the same as for uniform vertical pressure; and

b. The internal friction in the embankment materials distributes the infinitely small decrements or increments of pressure from shear into the exterior prisms so completely that their effect on settlement may be neglected.

Equating the settlements in the exterior prisms and the settlements in the interior prism caused by the fill material above the plane of equal settlement:

$$\frac{(H_c - H_e)\,\gamma}{E_f}\,H_e + y_m + y_g = \int_0^{H_e} \frac{(H_c - H_e)\,\gamma}{E_f}\,e^{\pm\beta H_p}\,dH_p + y_f + y_c \quad \text{(9)}$$

In Eq. 9, using Eqs. 4:

$$e^{\pm\beta H_e} \mp 2\,K\,\mu\,\frac{H_e}{b_c} = \pm\,2\,K\,\mu\,\delta\,\rho + 1 \quad \text{(10)}$$

In the three quantities of Eq. 10 the +, −, and + signs are used for the projection condition; and the −, +, and − signs, respectively, are used for the ditch condition.

Eq. 10 must be solved by trial, since it contains H_e as an exponent of *e*, the base of natural logarithms (see Eq. 5*c*).

As in the case of ditch conduits, the solution of these various expressions for loads on projecting conduits is made easy by the construction of a load coefficient diagram,[7,8] as shown in Fig. 8. For selected values of the "settlement" ratio and the "projection" ratio ($\delta\,\rho$) and given values of H_c/b_c the proper value of the coefficient C_c is read from the curves and substituted in Eq. 6*a*. Comparisons between actual weighed loads in carefully conducted experiments and loads computed by Dean Marston's load theory are shown in Fig. 9.

---

[7] "The Theory of External Loads on Closed Conduits in the Light of the Latest Experiments," by Anson Marston, *Bulletin No. 96*, Iowa Eng. Experiment Station, Ames, Iowa, 1930, p. 14, Fig. 3.

[8] "The Supporting Strength of Rigid Pipe Culverts," by M. G. Spangler, *Bulletin No. 112*, Iowa Eng. Experiment Station, Ames, Iowa, 1933, p. 46, Fig. 18.

Several alternate methods of determining the value of H_e have been investigated, including an analysis in which the settlement ratio was based on the total settlements of the various elements of the fill and of the conduit caused by the total height of embankment. The loads obtained by this alternate analysis were nearly the same as those obtained by Dean Marston's original analysis, being somewhat less for high values of $\frac{H_e}{b_e}$ and $\delta\,\rho$.

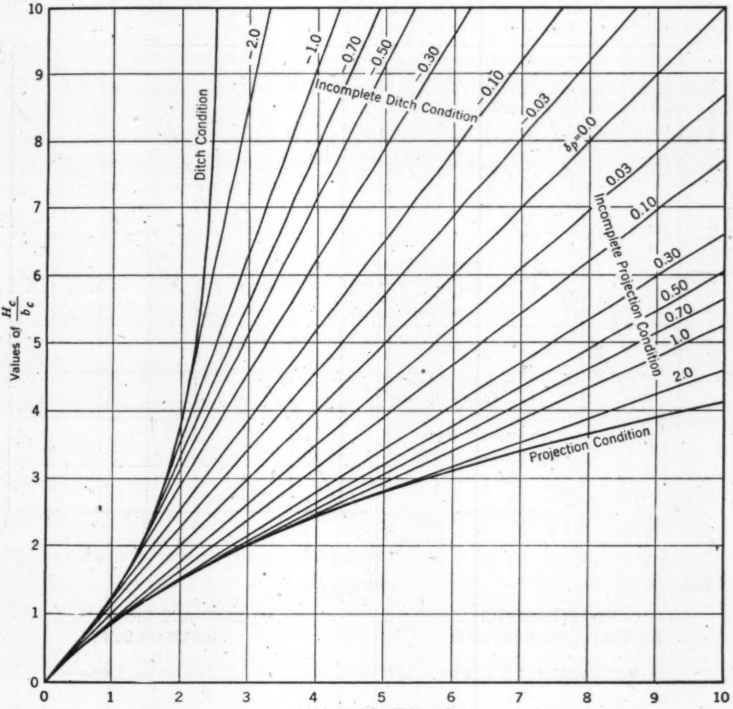
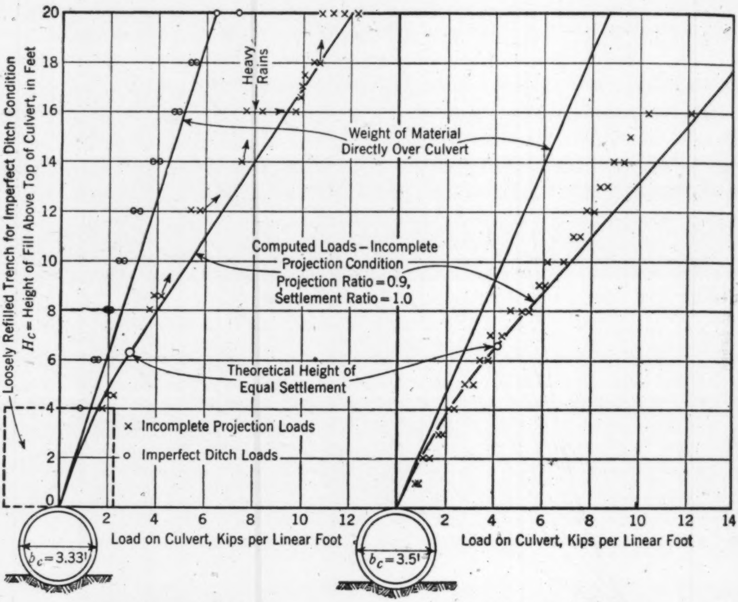


FIG. 8.—COMPUTATION DIAGRAM FOR EXTERNAL LOADS ON CONDUITS ($K\,\mu$ = 0.1924)

Working values of μ, the coefficient of internal friction of the embankment material, may be determined by soil tests in a particular case. However, Dean Marston has shown that the variation in load on a projecting conduit is relatively small for wide variations of the coefficient of internal friction. For example, when this coefficient varies from about 0.3 to 1.0, corresponding to angles of friction from about 17° to 45°, the product $K\,\mu$ varies from about 0.17 to 0.1924 and back to 0.17. Since the load is dependent upon the product $K\mu$ and not on μ alone, a value of $K\mu = 0.19$ may reasonably be chosen as a safe working value for most cases of projecting conduit loads.

TIME EFFECT ON LOADS

To study the effect of time on the loads produced on culverts by earth embankments, three test culverts supported on weighing devices were built and loaded with a 15-ft embankment late in the fall of 1927, and observations of the loads on these structures have been made periodically since that date. This load history shows that the maximum load was reached in April of the following spring after the fill was completed, after which the load fell off somewhat, reaching a low point about 10% below the maximum in the following October and November. Since that date, the load has fluctuated between these approximate limits, reaching the maximum for each year early in the spring and the minimum late in the fall.



(a) IMPERFECT DITCH
Top Soil Fill, 96 Lb per Cu Ft

(b) SAND AND GRAVEL FILL
126.5 Lb per Cu Ft

FIG. 9.—COMPARISON BETWEEN ACTUAL WEIGHED LOADS ON PROJECTING CONDUITS

Eq. 10 for H_e has been derived for the assumed case in which the fill material has a uniform modulus of deformation throughout; and it is emphasized that the value of H_e is independent of the value of this modulus. In other words, even if the fill material is thoroughly compacted during placement and a high value of the compression modulus is attained, the direction of the relative settlements of the interior and exterior prisms and the height of equal settlement will not be affected thereby.

The settlement ratio is wholly rational in the theoretical analysis of loads on projecting conduits. However, except in unusual cases it will not be practicable to predict the value to use in the design of a given structure on the basis of individual study and analysis of the soil at a proposed site. It has been deemed advisable and appropriate, therefore, to consider the settlement ratio as a semi-empirical factor and to determine proper design values on the basis of the observed performance of projecting conduits in service.

Rather extensive field measurements of the settlement characteristics of some actual culverts under the highways of Iowa and southern Minnesota, which have been conducted by the Iowa Engineering Experiment Station in cooperation with the Public Roads Administration, Federal Works Agency, tentatively indicate the following working values for the settlement ratio—for rigid culverts on:

Foundation material	δ
Rock or other unyielding material............	+ 1.0
Ordinary earth.................................	+ 0.5 to + 0.8

Observations of the settlement ratio have been made on a few flexible type culverts in actual service, but the number of specimens is limited. These measured values ranged from 0 to + 0.7 and they varied directly with the values of the modulus of passive pressure of the side-fill materials. It appears, therefore, that high quality densification of the side fills around a flexible pipe, for the purpose of increasing the modulus of passive pressure, tends to increase the load on the pipe. However, the ultimate deflection of the pipe is decreased by this practice, since the increase in supporting strength of the pipe more than offsets the effect of the increased load.

Unusual installation conditions may cause exceptional values of the settlement ratio and of the load to develop. For example, a case has come to the writer's attention where a highway fill was built across a salt marsh underlain with stiff clay at about 10 ft or 12 ft below the surface. A concrete pipe culvert was built prior to construction of the fill. To reduce the expected settlement of the culvert and to increase its supporting strength, the pipes were laid in a well-designed concrete cradle, which in turn was supported on piles driven into the clay stratum. Shortly after the fill was completed this culvert collapsed, even though it was constructed of excellent reinforced concrete pipe. Although no settlement data are available, it seems probable that the fill material at the sides (the "exterior prisms") settled a very large amount due to the soft spongy character of the marsh bed, whereas the culvert resting on piles driven to the stiff clay could not settle. Under these conditions the critical plane probably settled much more than the top of the culvert, causing a very high positive value of the settlement ratio and a very high load to develop.

IMPERFECT DITCH CONDUITS

In the early days of Dean Marston's researches on conduit loads, he was impressed by the very high loads that may develop on projecting conduits when the conditions are such that large shearing forces add to the weight of the prism of soil directly over the conduit, and he strove to devise a method of construction that would reduce or eliminate these shearing forces, or would possibly reverse their direction so that they would act benevolently as in the case of ditch conduits. With this objective in mind, he developed the imperfect ditch method of construction in which the soil on both sides and above the conduit for some distance above its top is thoroughly compacted by rolling, tamping, or any other suitable method. Then a ditch is constructed in this compacted fill by removing the prism of material directly over the conduit. The ditch is refilled with very loose compressible material, after which the embankment is completed in a normal manner.

The purpose of this method of construction is accomplished by creating a condition wherein it is certain that the prism of material directly over the conduit will settle more than the adjacent prism; the ditch in the artificially compacted material must be deep enough and the refilling material must be loose enough to insure this action. Straw or other highly compressible material may be used as part of the ditch backfill to augment the settlement of the interior prism. The efficacy of this method of construction was demonstrated by a series of experiments conducted by Dean Marston from 1919 to 1921, in which the load on a conduit was measured, first, when the fill was constructed in the ordinary manner for a projecting conduit, and, second, when the fill was constructed by the imperfect ditch method. The loads measured in these experiments are shown in Fig. 9(*a*).

SURFACE LOADS

In addition to external loads imposed by the filling material around and above underground conduits, these structures are also subjected to loads resulting from highway, railway, or airplane traffic or from other types of loads applied at the surface and transmitted through the soil to the underground structure. Such loads are of major importance when a conduit is placed under a trafficway with a relatively shallow covering of earth.

John H. Griffith, M. ASCE, was among the first to suggest the applicability of the Boussinesq solution[9] for the distribution of stress in a semi-infinite elastic solid to various problems of stress distribution in soils in a report prepared for a subcommittee on soils of the United States Bureau of Standards of which he was chairman. Extensive experiments[10,11] on both ditch and projecting conduits subsequent to this suggestion have shown that a concentrated surface load, such as a truck wheel, is transmitted through the soil covering to the underground structure substantially in accord with the Boussinesq solution.

---

[9] "Revised Report of Subcommittee on Soils," U. S. Bureau of Standards, *Proceedings*, ASCE, Vol. XLVI, 1920, pp. 916–941.

[10] "The Causes of Structural Failures of Sewers," by Anson Marston and M. G. Spangler, *Proceedings*, Am. Soc. of Municipal Engrs., Vol. 38, 1933, p. 236.

[11] "Experimental Determinations of Static and Impact Loads Transmitted to Culverts," by M. G. Spangler, Clyde Mason, and Robley Winfrey, *Bulletin No. 79*, Iowa Eng. Experiment Station, Ames, Iowa, 1926.

From these facts, the load on an underground conduit due to a concentrated surface load may be expressed as:

$$W_P = \frac{1}{L} F_i \, C_t \, P_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

in which W_P is the average load on an underground conduit due to a concentrated surface load, in pounds per linear foot; P_0 is a concentrated surface load, in pounds; F_i is an impact factor; L is the length of the conduit, in feet; and C_t is a coefficient for load on an underground conduit due to a concentrated surface load.

The coefficient C_t may be calculated by dividing the area of the top of the conduit or its horizontal projection, $L\,b_c$, into a number of small areas as illustrated in Fig. 10, and summing up the total pressure on all these small areas, according to the formula:

$$C_t = a \sum \frac{3}{2\,\pi} \frac{H^3_c}{H^5_s} \ldots (12a)$$

in which a is the area of an element, in square feet; H_c is the vertical height from the top of the conduit to the embankment surface, in feet; and H_s is the slant height, in feet, from the center of each element to the point of application of the load; that is: $H_s = \sqrt{x^2 + y^2 + z^2}$, in feet.

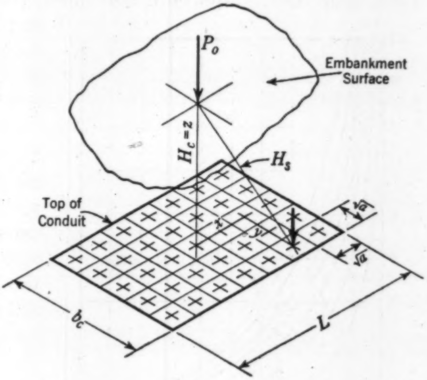


FIG. 10.—COMPUTATION OF THE CONCENTRATED SURFACE LOAD TRANSMITTED TO AN UNDERGROUND CONDUIT

Early in 1929 D. L. Holl integrated this expression for C_t and obtained:

$$C_t = 1 - \frac{2}{\pi} \left\{ \left[\sin^{-1} H_c \sqrt{\frac{\dfrac{L^2}{4} + \dfrac{b^2_c}{4} + H^2_c}{\left(\dfrac{L^2}{4} + H^2_c\right)\left(\dfrac{b^2_c}{4} + H^2_c\right)}} \right] - \frac{\dfrac{L\,b_c\,H_c}{4}}{\sqrt{\dfrac{L^2}{4} + \dfrac{b^2_c}{4} + H^2_c}} \left(\frac{1}{\dfrac{L^2}{4} + H^2_c} + \frac{1}{\dfrac{b^2_c}{4} + H^2_c} \right) \right\} \ldots\ldots\ldots (12b)$$

Eq. 12*b* can be evaluated[12] readily by the table prepared by N. M. Newmark,[13] M.ASCE, for determining stresses in the soil beneath uniformly loaded rectangular foundations.

---

[12] "A Method of Computing Live Loads Transmitted to Underground Conduits," by M. G. Spangler and Richard L. Hennessy, *Proceedings*, Highway Research Board, National Research Council, Vol. 26, 1946, p. 179.

[13] "Simplified Computation of Vertical Pressures in Elastic Foundations," by Nathan M. Newmark, *Circular No. 24*, Eng. Experiment Station, Univ. of Illinois, Urbana, Ill., 1935.

Values of C_t for conduits 3 ft long of various widths, b_c, are shown[14] in Fig. 11 for various heights of fill up to 10 ft. When fill loads and surface loads are combined to obtain the design load on an underground conduit, the minimum load will be found to occur when the height of fill is relatively low over the top of the structure.

The impact factor, F_i, is equal to unity when the surface load is static. When it is moving, as in the case of truck or airplane wheels, the value of F_i may vary widely depending on speed of the vehicle, vibratory action, wing uplift, and most important, on roughness characteristics of the roadway surface. Experiments[9] have indicated design values of F_i ranging from 1.5 to 2.0 for highway traffic on an unsurfaced roadway.

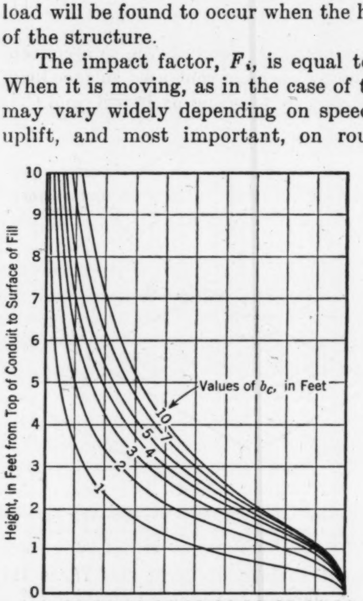


FIG. 11.—CONCENTRATED SURFACE LOAD COEFFICIENTS FOR UNDERGROUND (L = 3 Ft)

Professor Griffith gave extensive consideration to problems of the transmission of stress through soils and the applicability of the Boussinesq classical solution to them. In 1929 he developed a generalized expression[15] for the intensity of vertical pressure in a soil mass due to a concentrated surface load in the form:

$$P_z = \frac{\nu}{2\pi} P_0 \frac{\cos^{\nu+2}\theta}{z^2} \quad \ldots . (13)$$

in which P_z is the intensity of pressure at a point in the soil mass; P_0 is the concentrated load applied at a point on the surface; θ is the angle formed with the vertical by the radius vector from the point of application of the surface load to the point considered; z is the vertical distance from the surface to the point; and ν is a disposable parameter, often referred to as the "concentration" factor or "dispersion" factor.

Professor Holl[16] has utilized Eq. 13 in the solution of a number of problems dealing with the transmission of surface loads through soil masses.

THE SUPPORTING STRENGTH OF UNDERGROUND CONDUITS

Underground conduits are constructed in a wide variety of shapes and of many different structural materials. In general, the load on a conduit is independent of its shape and the material of which it is made. In contrast, the supporting strength of a conduit is intimately dependent upon its shape and the kind and quality of material of which it is made. The remainder of this paper will be devoted to a discussion of the supporting strength of a number of commonly used types of conduits.

---

[14] "The Theory of External Loads on Closed Conduits in the Light of the Latest Experiments," by Anson Marston, *Bulletin No. 96*, Iowa Eng. Experiment Station, Ames, Iowa, 1931, p. 19, Fig. 5.

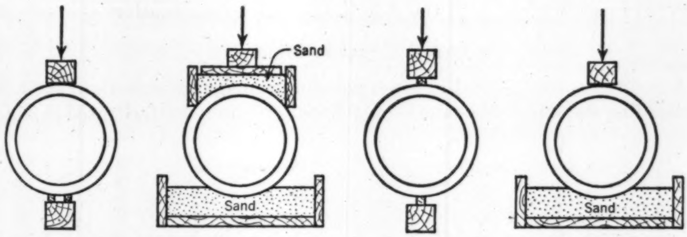
[15] "Dynamics of Earth and Other Macroscopic Matter," by John H. Griffith, *Bulletin No. 117*, Iowa Eng. Experiment Station, Ames, Iowa, 1934.

[16] "Stress Transmission in Earths," by D. L. Holl, *Proceedings*, Highway Research Board, National Research Council, Vol. 20, 1940, p. 709.

Monolithic Arch and Box Culverts or Sewers.—These monolithic types of reinforced concrete structures may be designed satisfactorily by any current procedures for analysis of rigid frame structures. Numerous measurements of the distribution of the vertical load on both rectangular and circular shaped culverts due to the earth overburden have indicated that it is essentially uniformly distributed over the width of the conduit and may be so considered for design purposes. Culverts may or may not be subjected to active lateral pressure on all or part of their side-wall areas, depending on the local situation.

Some laboratory experiments[17] on the supporting strength of monolithic box culverts indicate that the actual supporting strength exceeds the calculated strength until the concrete of the top slab cracks, after which the actual strength very closely approximates the calculated strength.

Rigid Circular Pipes.—Rigid circular pipes, which are usually pre-cast of such materials as burned clay, plain or reinforced concrete, or cast iron, are difficult to analyze by principles of mechanics; and, since they are usually relatively small structures, their supporting strength can be most easily determined by testing a representative group of specimens in the laboratory. Several methods of testing short sections of circular pipes have been devised[18] —the two-edge bearing, the three-edge bearing, the sand bearing, and the Minnesota bearing tests—the details of which are shown in Fig. 12. Of these, the three-edge bearing test (Fig. 12(*a*)) is the simplest and most easily performed. At the same time it gives accurate and uniform results, and for these reasons it is widely used in tests to determine the strength of pipes, although some engineers prefer the sand bearing test (Fig. 12(*b*)) because of the wider distribution of both the applied load and reaction.



(*a*) THREE-EDGE BEARING (*b*) SAND BEARING (*c*) TWO-EDGE BEARING (*d*) MINNESOTA BEARING

Fig. 12.—Typical Bearings for Laboratory Tests of Pipes

The test load and reaction on the pipe are distributed differently in each of these types of tests and the ultimate load or supporting strength will likewise be different. It is convenient to express the supporting strength of a pipe in terms of its strength when tested by the three-edged bearing method, taken as unity, by a load factor or strength ratio which is defined as the ratio of the strength of a pipe under any stated condition of loading to its three-edge bearing test strength. The following values of load factor or strength ratio for the other three test methods shown in Fig. 12 are applicable:

---

[17] "The Supporting Strength of Reinforced Concrete Box Culverts," by George C. Pagels, thesis presented to Iowa State College, Ames, Iowa, in 1935 in partial fulfilment of the requirements for the degree of Master of Science.

[18] "The Supporting Strength of Rigid Pipe Culverts," by M. G. Spangler, *Bulletin No. 112*, Iowa Eng. Experiment Station, Ames, Iowa, 1933, p. 13, Fig. 5.

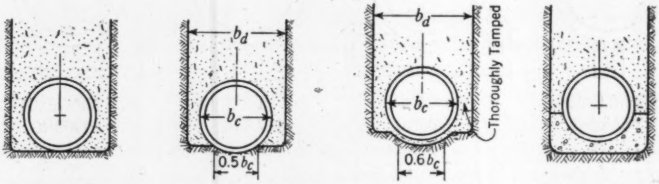
Test	Load factor
Sand bearing	1.5
Two-edge bearing	1.0
Minnesota bearing	1.1

Likewise when a pipe is placed and loaded in an actual field installation, the loads and reactions will be distributed much differently than in the test loading; and, in certain cases, lateral pressures will be exerted against the sides of the pipe thereby increasing its ability to support vertical loads.

A wide variety of bedding conditions affecting the load and reaction distribution and the lateral pressure situation may be encountered in sewerage, water works, and culvert construction practice; and a wide variety of supporting strengths of a given conduit may be obtained simply by varying the installation conditions. It is feasible to establish and define classifications of bedding conditions covering a range of practical attainments and determine a load factor for each classification which, when multiplied by the three-edge bearing laboratory strength of the pipe, will give the safe supporting strength for pipes installed in accordance with the definition of that classification.

DITCH CONDUIT BEDDINGS

For ditch conduits used in sewerage and drainage construction, the following bedding classifications have been defined[19,20,21] and are illustrated in Fig. 13.



(*a*) NOT PERMISSIBLE (*b*) ORDINARY (*c*) FIRST CLASS (*d*) CONCRETE CRADLE

FIG. 13.—TYPICAL BEDDING SPECIFICATIONS FOR SEWERS AND OTHER DITCH CONDUITS

Bedding conditions for cast-iron pipe in water works construction will be discussed subsequently.

---

[19] "The Supporting Strength of Sewer Pipe in Ditches and Methods of Testing Sewer Pipe in Laboratories to Determine Their Ordinary Supporting Strength," by Anson Marston, W. J. Schlick, and H. F. Clemmer, *Bulletin No. 47*, Iowa Eng. Experiment Station, Ames, Iowa, 1917.

[20] "Supporting Strength of Drain Tile and Sewer Pipe Under Different Pipe-Laying Conditions," by W. J. Schlick, *Bulletin No. 57*, Iowa Eng. Experiment Station, Ames, Iowa, 1920.

[21] "Concrete Cradles for Large Pipe Conduits," by W. J. Schlick and James W. Johnson, *Bulletin No. 80*, Iowa Eng. Experiment Station, Ames, Iowa, 1926.

1. "Impermissible" bedding (Fig. 13(*a*)) is that method of bedding ditch conduits in which little or no care is exercised to shape the foundation to fit the lower part of the conduit exterior and to refill all spaces under and around the conduit.

2. "Ordinary" bedding (Fig. 13(*b*)) is that method of bedding ditch conduits in which the conduit is placed with "ordinary" care in an earth foundation shaped to fit the lower part of the conduit exterior with reasonable closeness for a width of at least 50% of the conduit breadth. The remainder of the conduit is surrounded to a height of at least 0.5 ft above its top by granular materials, shovel placed and shovel tamped to fill completely all spaces under and adjacent to the conduit—all under the general direction of a competent engineer.

3. "First-class" bedding (Fig. 13(*c*)) is that method of bedding ditch conduits in which the conduit is set on fine granular materials in an earth foundation carefully shaped to fit the lower part of the conduit exterior for a width of at least 60% of the conduit breadth. The remainder of the conduit is entirely surrounded to a height of at least 1.0 ft above its top by granular materials carefully placed by hand to fill, completely, all spaces under and adjacent to the conduit. The fill is tamped thoroughly on each side and under the conduit as far as practicable in layers not exceeding 0.5 ft thick—all under the direction of a competent engineer represented by a competent inspector constantly present during the operation.

4. "Concrete-cradle" bedding is that method of bedding conduits in which the lower part of the conduit exterior is set in plain or reinforced concrete, of suitable thickness, under the lowest part of the conduit exterior and extending upward on each side of the conduit for a greater or less proportion of its height.

The load factor for each of these bedding classifications has been determined experimentally to be:

Figure No.	Bedding	Load factor
13(*a*)	Impermissible	1.1
13(*b*)	Ordinary	1.5
13(*c*)	First class	1.9
13(*d*)	Concrete cradle	2.2 to 3.4

PROJECTING CONDUIT BEDDINGS

In culvert construction practice, rigid pipes are often installed as projecting conduits and the fill material may exert an active lateral pressure against the sides of the pipes, which tends to increase the supporting strength of the structure. The supporting strength of a projecting conduit, therefore, is a function of both the distribution of the vertical load and the reaction on the pipe, and the magnitude and distribution of the active lateral pressure on sides which are exposed to the embankment filling material.

Because of the large number of possible combinations of reaction distributions and effective lateral pressure distributions, the experimental determination of load factors for projecting conduits has been supplemented by analytical studies of the stress situation in pipe rings under various combinations of loads and lateral pressures.[22] By definition, the load factor is

$$F_p = \frac{R_c}{R_{eb}} \qquad (14a)$$

in which F_p is the load factor; R_c is the supporting strength of the pipe in any stated loading condition; and R_{eb} is the three-edge bearing supporting strength of the pipe.

Except for the normal variations in physical properties of the material of which culvert pipes are made, it is sufficiently accurate to assume that the outer fiber stress at the point and at the time of rupture will be the same whether the pipe is loaded in a testing machine or in an actual embankment. Therefore, the thin ring elastic theory and the flexure formula may be used to express the outer fiber stress in terms of the loads on the pipe in the field and laboratory loadings; and, by equating these expressions, the load factor is readily determined. When the field loading situation is such that the pipe cracks first at the bottom, this process leads to the formula:

$$F_p = \frac{1.431}{X_P - X_A \kappa_t} \qquad (14b)$$

in which X_P is a parameter which is a function of the distribution of the vertical load and vertical reaction; X_A is a parameter which is a function of the area of the vertical projection of the pipe on which the active lateral pressure of the fill material acts; and κ_t is the ratio of the total lateral pressure to the total vertical load.

When the load and reaction situation causes the pipe to crack first at the top (which is usually the case when pipes are bedded in concrete cradle), X'_P and X'_A should be substituted for X_P and X_A in Eq. 14*b*. Values of the parameters X_A and X'_A for various projection ratios are as follows:

Projection ratio, p	Values of X_A	Values of X'_A
0	0	0.150
0.3	0.217	0.743
0.5	0.423	0.856
0.7	0.549	0.811
0.9	0.655	0.678
1.0	0.638	0.638

As in the case of ditch conduits, it is convenient to name and define several classes of bedding conditions for projecting conduits and to determine a value of X_P or X'_P for each class. These classes of bedding are defined as follows:[23]

1. "Impermissible projection bedding" is that method of bedding projecting conduits in which little or no care is exercised either to shape the foundation surface to fit the lower part of the conduit exterior or to fill all spaces under and around the conduit with granular materials. This type of bedding also includes the case of conduits on rock foundations in which an earth cushion is

---

[22] "The Supporting Strength of Rigid Pipe Culverts," by M. G. Spangler, *Bulletin No. 112*, Iowa Eng. Experiment Station, Ames, Iowa, 1933.

[23] *Ibid.*, p. 15, Fig. 6.

provided under the conduit, but is so shallow that the conduit, as it settles under the influence of vertical load, approaches contact with the rock.

2. "Ordinary projection bedding" is that method of bedding projecting conduits under embankments in which the conduit is bedded with "ordinary" care in an earth foundation shaped to fit the lower part of the conduit exterior with reasonable closeness for at least 10% of its over-all height. The remainder of the conduit is surrounded by granular materials, placed by shovel to fill all spaces completely under and adjacent to the conduit—all under the general direction of a competent engineer. In the case of rock foundations, the pipes are bedded on an earth cushion, having a thickness under the pipe of not less than 0.5 in. per foot of height of fill, with a minimum allowable thickness of 8 in.

3. "First-class projection bedding" is that method of bedding projecting conduits, having a projection ratio not greater than 0.70, in which the conduit is carefully bedded on fine granular materials in an earth foundation carefully shaped to fit the lower part of the conduit exterior for at least 10% of its over-all height. The earth filling material is thoroughly rammed and tamped, in layers not more than 6 in. deep, around the conduit for the remainder of the lower 30% of its height—all under the direction of a competent engineer, represented by a competent inspector constantly present during the operation. In the case of rock foundations, the pipes are bedded in an earth cushion having a depth as provided under ordinary projection bedding.

4. "Concrete-cradle projection bedding" is that method of bedding conduits in which the lower part of the conduit exterior is bedded in a cradle, constructed of 2,000-lb concrete, or better, having a minimum thickness under the pipe of one fourth of the nominal internal diameter and extending up the sides of the pipe for a height equal to one fourth of the outside diameter. The values of X_P for these bedding classes are as follows:

Type of projection bedding	Value of X_P
Impermissible	1.310
Ordinary	0.840
First class	0.707
Concrete cradle (X'_P)	0.505

Working values of the load factor F_P, computed for rigid culvert pipes installed as projecting conduits on ordinary, first-class, and concrete-cradle beddings, with various projection ratios and a value of the settlement ratio of 0.7 are shown in Fig. 14. The general correctness of the foregoing analytical method of determining load factors has been established by a series of experiments in which ten pipe culverts of four kinds of material—plain concrete, reinforced concrete, burned clay, and cast iron—were loaded by earth embankments and their supporting strengths determined. Then samples of pipes representative of each of the ten culverts were tested in the laboratory by the three-edge bearing test and the ratios of the field and laboratory strengths were obtained.[24]

---

[24] "The Supporting Strength of Rigid Pipe Culverts," by M. G. Spangler, *Bulletin No. 112*, Iowa Eng. Experiment Station, Ames, Iowa, 1933, p. 53.

CONCRETE-INCASED CLAY PIPE

Circumstances may arise under which an engineer may wish to increase the strength of clay pipes by incasing them in concrete. Some extensive experiments[25] in which commercial vitrified salt-glazed clay pipe was incased in plain concrete and the combined structure tested in both sand and three-edge bearings have indicated that the test strength is only slightly greater than the sum of the individual supporting strengths of the clay pipe and of the incasement. The action of the incased pipe was very nearly that of two independent but concentric rings—the supporting strength of the combined structure being very much less than if the two materials had acted as a unit in resisting the stresses due to the test load. As a result of these experiments, computations have been made of the three-edge bearing supporting strength of A.S.T.M. (American Society for Testing Materials) standard sewer pipe when incased in concrete of various thicknesses. The results of these computations are shown in Fig. 15.

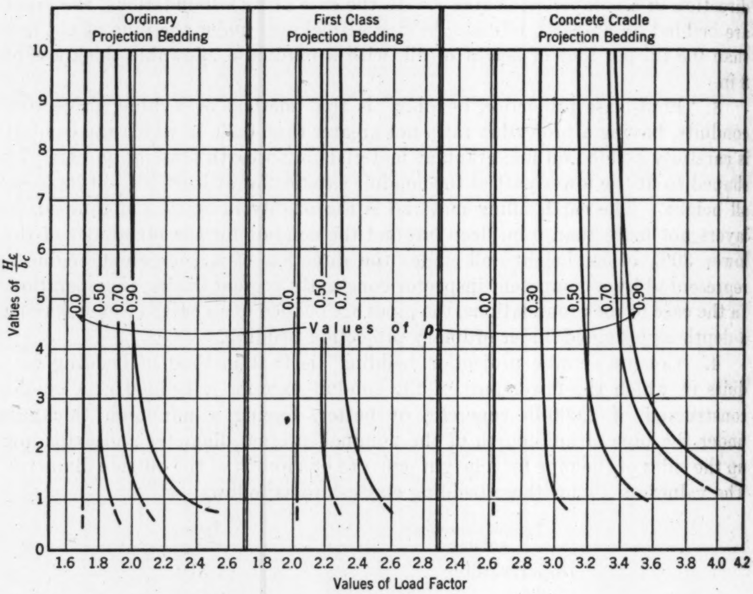


FIG. 14.—WORKING VALUES OF THE LOAD FACTOR FOR PROJECTING CONDUITS ($\delta = 0.70$)

THE SUPPORTING STRENGTH OF CAST-IRON PIPE WHEN SUBJECTED TO INTERNAL PRESSURE

When used for the transportation of water, gas, or other fluids, cast-iron pipe conduits not only are subjected to the ring stresses produced by the external loads due to backfilling and surface loads, but in addition must resist the stresses produced by the internal fluid pressure as well. Since 1939 completed studies[26, 27, 28] on the supporting strength of this kind of pipe under combined loads and pressures indicated an approximately parabolic relationship between the magnitude of the external load and the internal pressure which is required to cause failure of the pipe. The external loads may be those for any loading condition, although usually it will be most convenient to test the pipe in three-edge bearings and then apply the proper strength ratio or load factor to calculate the equivalent supporting strength under any particular field condition of installation.

---

[25] "Supporting Strength of Concrete-Incased Clay Pipe," by W. J. Schlick, *Bulletin No. 93*, Iowa Eng. Experiment Station, Ames, Iowa, 1929.

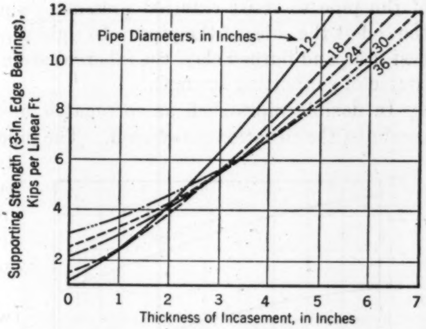


FIG. 15.—SUPPORTING STRENGTHS (AT 70° F) OF STANDARD A.S.T.M. SEWER PIPE, INCASED IN 3,000-LB CONCERTE

Safe design values of the strength ratio have been determined for cast-iron pit-cast pipes installed under six different field conditions, as follows:

Condition	Trench	Backfill
A	Flat bottom	Untamped
B	Flat bottom	Firmly tamped under and around pipe
C	Pipe laid on blocks	Untamped
D	Pipe laid on blocks	Firmly tamped under and around pipe
E	Shaped bottom	Untamped
F	Shaped bottom	Firmly tamped under and around pipe

Quoting W. J. Schlick,[29] M. ASCE:

> "The safe design values of the strength ratio for these field conditions are shown for nominal pipe diameters from 4″ to 60″ [in Fig. 16]. The values shown in this figure have been adopted by Sectional Committee A21 of the American Standards Association, with whom this research work was conducted on a cooperative basis. They are based directly upon the experimental results although those adopted for conditions B and F have been reduced from the experimental values in the belief that the tamped earth beddings in the laboratory tests were more carefully made and therefore more effective than is safe to assume for field conditions."

---

[26] "A Proposed New Method for Determining Barrel Thicknesses of Cast Iron Pipe," by T. H. Wiggin, M. L. Enger, and W. J. Schlick, *Journal*, A.W.W.A., May, 1939, p. 841.

[27] "American Recommended Practice Manual for Computation of Strength and Thickness of Cast Iron Pipe" (A21.1), A.S.A., 1939.

[28] "Supporting Strength of Cast-Iron Pipe for Water and Gas Service," by W. J. Schlick, *Bulletin. No. 146*, Iowa Eng. Experiment Station, Ames, Iowa, 1940.

[29] *Ibid.*, p. 8, Fig. 2.

Flexible Pipes

In general, it may be stated that underground conduits derive their capacity to support the earth above them from two sources: First, the inherent strength of the pipe to resist external pressures; and, second, the lateral pressure of the soil at the sides of the pipe. In rigid pipes such as those made of concrete, cast iron, and burned clay, the inherent strength of the pipe is the predominant source of supporting strength.

In flexible pipes such as corrugated metal culverts and thin-walled steel conduits, the situation is reversed. The pipe itself has relatively little inherent strength, and a large part of its capability of supporting vertical load must be derived from the passive pressures induced as the sides move outward against the soil.

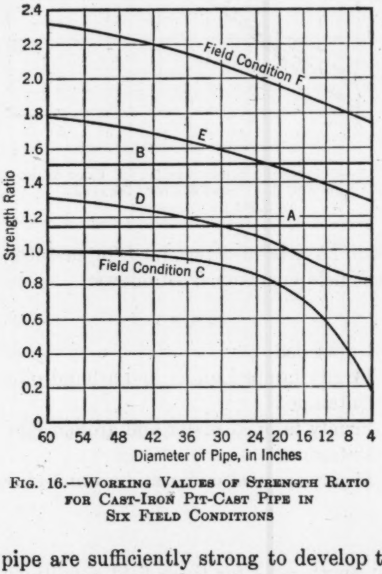


Fig. 16.—Working Values of Strength Ratio for Cast-Iron Pit-Cast Pipe in Six Field Conditions

Another major difference between the rigid types of conduits and the flexible types is that the latter fail by deflection rather than by rupture of the pipe walls, as do the former. The action of a flexible pipe under embankment loading is one of large deflection change unaccompanied by rupture of the pipe wall; and, for this reason, the design of the flexible types of conduits should be based upon the deflection of the ring rather than upon stress in the side walls as in the case of the rigid types. This statement is predicated on the assumption that the longitudinal joints in the flexible pipe are sufficiently strong to develop the pipe wall as a continuous ring.

Laboratory tests[30] of short sections of corrugated metal culvert pipes of various diameters and gages of metal have indicated that the elastic theory of flexure as applied to thin rings by Claude W. L. Filkins and Edwin J. Fort,[31] M. ASCE, is applicable to these flexible conduits, even if the deflections and changes in radius of the elastic ring are relatively very large. Therefore, if the loads and pressures acting on a flexible pipe are known or can be assumed confidently, the deflection of the pipe can be determined by this theory within the elastic limit of the pipe material. Typical results of laboratory measurements of diameter changes under three-edge loading conditions are shown in Fig. 17.

---

[30] "The Structural Design of Flexible Pipe Culverts," by M. G. Spangler, *Bulletin No. 153*, Iowa Eng. Experiment Station, Ames, Iowa, 1941.

[31] "Stresses and Deflections in Circular Rings Under Various Conditions of Loading," by Claude W. L. Filkins and Edwin J. Fort, *Transactions*, Assn. of Civ. Engrs. of Cornell Univ., Ithaca, N. Y., Vol. IV, 1896, pp. 99–112.

A number of field loading experiments on corrugated metal pipe culverts, in which the vertical and lateral pressures and the deflections of the pipes were measured have led to the following conclusions regarding soil pressures and deflections of flexible conduits:

1. The vertical load may be determined by Dean Marston's theory of loads on conduits; it is distributed approximately uniformly over the breadth of the pipe.

2. The vertical reaction is equal to the vertical load and is distributed approximately uniformly over the width of bedding of the pipe.

3. The horizontal pressure on each side of the pipe is distributed parabolically over the middle of 100° of the pipe, and the maximum unit pressure (which occurs at the ends of the horizontal diameter of the pipe) is equal to the modulus of passive resistance of the fill material multiplied by one half of the horizontal deflection of the pipe.

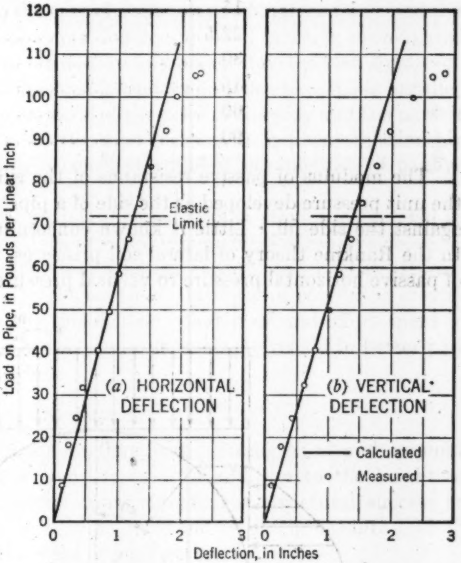


FIG. 17.—LOAD-DEFLECTION DIAGRAMS FOR A 42-IN. CORRUGATED CULVERT PIPE ON THREE-EDGE BEARINGS

4. The pipe may continue to deform for some period of time after the maximum fill load has developed, due to continuing deformation of the side fills under lateral pressure. The assumed loading is shown graphically[32] in Fig. 18. The expression for horizontal deflection resulting from these hypotheses is:

$$\Delta = F_d \frac{F_k W_c r^3}{E I + 0.061 \epsilon r^4} \quad \dots\dots\dots\dots\dots\dots (15)$$

in which Δ is the horizontal deflection of the pipe, in inches; F_d is the deflection lag factor; F_k is a bedding constant, depending upon the bedding angle; W_c is the vertical load per unit length of the pipe, in pounds per inch; r is the mean radius of the pipe, in inches; E is the modulus of elasticity of the pipe material, in inches; I is the moment of inertia per unit length of cross section of pipe wall, in inches^4 per inch; and ϵ is the modulus of passive resistance of the enveloping soil, in pounds per square inch per inch.

---

[32] "The Structural Design of Flexible Pipe Culverts," by M. G. Spangler, *Bulletin No. 153*, Iowa Eng. Experiment Station, Ames, Iowa, 1941, p. 26, Fig. 17.

Values of the bedding constant, F_k, for various values of the bedding angle are:

Degrees	Bedding constant, F_k
0	0.110
15	0.108
22.5	0.105
30	0.102
45	0.096
60	0.090
90	0.083

The modulus of passive resistance of the side filling material is defined as the unit pressure developed as the side of a pipe moves outward a unit distance against the side fill. Little is known concerning the nature of this modulus. In the Rankine theory of lateral soil pressures, the limiting value of the ratio of passive horizontal pressure to vertical pressure which a granular soil without cohesion can develop is shown to be the reciprocal of the active pressure ratio. However, this theory does not give a clue as to the amount of movement required to develop the limiting value of passive pressure, and it would seem that the actual passive pressure may be any value less than the maximum, depending on soil characteristics and amount of movement of the sides of the pipe.

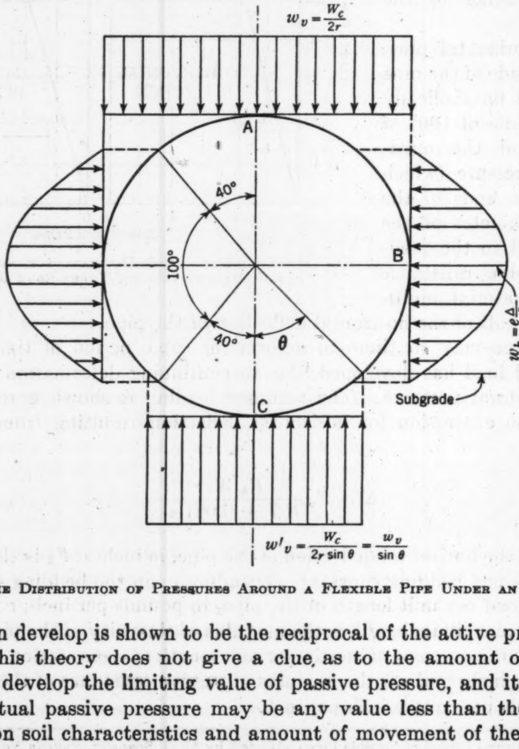

$w_v = \frac{W_c}{2r}$
A
40°
100°
B
40°
θ
$w_h = e\frac{\Delta}{2}$
Subgrade
C
$w'_v = \frac{W_c}{2r \sin \theta} = \frac{w_v}{\sin \theta}$

FIG. 18.—THE DISTRIBUTION OF PRESSURES AROUND A FLEXIBLE PIPE UNDER AN EARTH FILL

Several facts concerning the value of the modulus of passive resistance of side fills adjacent to experimental flexible culvert pipes have been observed. First, this modulus varies widely with soil characteristics. Measurements have indicated a value for a gravel material approximately three times as great as for a loam material. Also in the case of a silty clay soil material, the modulus of passive resistance was approximately doubled by hand tamping as compared to the value for the same material in a relatively loose, shovel-placed condition. The second fact that has been observed is that when a flexible pipe is installed without struts the horizontal movement of the sides of the pipe and the passive resistance pressure at the sides of the pipe increase in a linear relationship with the height of fill, indicating a constant value of the modulus of passive resistance pressure as the height of fill is built up.

Field observations of several flexible pipe culverts in service, completed in 1946, have indicated that the settlement ratio, the modulus of passive pressure, and the deflection lag factor are all intimately interrelated. These studies also indicate that the structural performance of this type of conduit is greatly influenced by the character of the side fills and that time and effort spent in compacting the soil at the sides of the pipe will be amply repaid in better performance of the structure as a whole.

SUMMARY

Dean Marston's theory of loads and supporting strengths of underground conduits serves to highlight the many factors, in addition to the height of fill and the conduit strength, which contribute to the structural success or failure of such structures. It is a guide to sound design procedure and to good construction practice in this field of engineering.

For example, the theoretical analysis of ditch conduits indicates that the width of ditch has a marked effect upon the load which the conduit must carry and field observations tend to verify this indication. Good practice, therefore, requires that the design width of ditch be held to a practical minimum value and that this design width be adhered to in construction.

In the case of projecting conduits, such factors as the projection ratio and the settlement ratio have an important bearing on the load which the soil overburden produces on the structure, and knowledge of the characteristics of the site which influence these ratios is necessary before an accurate load analysis can be made. These characteristics are concealed to a very great extent when an embankment is built so that it is usually not possible to determine the load on a projecting conduit after it is completed merely by observing the height of fill and the character of the soil covering. Likewise, the bedding characteristics of a conduit, and the conditions governing the development of lateral earth pressures on the structure, both of which influence its field supporting strength, cannot be determined readily after construction is completed. It is not feasible, therefore, to base one's judgment of the safe height of fill which can be constructed over a given type of conduit in one location, on the height of fill which the same type of conduit may be successfully supporting in another location, unless it is known that all factors which influence load and supporting strength are the same in both locations.

APPENDIX. NOTATION

The following letter symbols, adopted for use in this paper and in its discussion, conform essentially with ASCE *Manual of Engineering Practice No. 22*, "Soil Mechanics Nomenclature":

a = an area element (see Fig. 10);
b = horizontal breadth or width, with subscripts denoting:
    c = conduit;
    d = width of ditch at top of conduit;
C = load coefficient, with subscripts denoting:
    c = projecting conduits;
    d = ditch conduits;
    t = load on the underground conduit transmitted by concentrated load on the surface;
E = modulus of elasticity of the pipe material; Young's modulus;
E_f = modulus of deformation of the filling material;
e = base of natural logarithms;
F = factor, with subscripts denoting:
    d = deflection lag;
    i = impact;
    k = bedding constant;
    p = load;
H = height of fill, with subscripts denoting:
    c = height to ground surface above top of conduit;
    e = height to the plane of equal settlement from the top of a pipe; this is the "height of equal settlement";
    p = height to top of backfill from any horizontal plane;
    s = slant height from the center of an elementary area in Fig. 10 to the point of application of a load;
I = moment of inertia per unit length of cross section of pipe wall;
K = ratio of lateral pressure at a point to vertical pressure; hydrostatic pressure ratio;
L = longitudinal length of the conduit;
P = total vertical pressure on a horizontal plane in the backfill, or within the interior prism:
    P_0 = the pressure of a concentrated load at the surface;
    P_z = total at any point distant z from the surface;
R = supporting strength or resistance of a pipe, with subscripts denoting:
    c = strength in any stated loading condition;
    eb = strength for the case of three-edge bearing;
r = mean radius of the pipe;
W = weight or total load, with subscripts denoting:
    c = load on an underground conduit;
    P = load transmitted by a concentrated surface force;
w = load per unit area, with subscripts denoting:

h = horizontal unit load;

v = vertical unit load;

X = a parameter symbol, denoting by subscripts a function of:

A = the area of the vertical projection of the pipe on which the active lateral pressure of the fill material acts;

P = the distribution of the vertical load and vertical reaction; When the load and reaction situation (Eq. 14*b*) causes the pipe to crack first at the top the parameter symbol is primed, thus: X';

y = vertical deflection or vertical settlement, corresponding to Δ = horizontal deflection, the subscripts denoting:

c = deflection of the conduit or the shortening of its vertical dimension;

f = settlement of the flow line of the conduit;

g = settlement of embankment subgrade adjacent to the conduit;

m = deformation of the filling material adjacent to the conduit within the height $\rho\, b_e$;

z = distances along the z-axis, also vertical distance from the surface to a given point;

α = a substitution symbol involving b_d introduced to simplify typography (see Eq. 2*a*);

β = a substitution symbol involving b_c introduced to simplify typography (see Eq. 5*c*);

γ = unit weight of filling material;

Δ = horizontal deflection;

δ = a settlement ratio defined by Eq. 4*a*;

ϵ = modulus of passive resistance of the enveloping soil;

θ = angle formed with the vertical by the radius vector from the point of application of the surface load to the point considered (Eq. 13), also the angle to the intersection of a pipe trench with the ground surface (Fig. 18);

κ_t = ratio of total lateral pressure to total vertical load;

μ = coefficient of internal friction between fill materials; μ' = coefficient of sliding friction between the fill materials and the sides of the ditch;

ν = concentration factor or dispersion factor; a disposable parameter; and

ρ = projection ratio; ratio of the vertical distance between the top of the conduit and the natural ground surface adjacent to the conduit, to b_c; the conduit projection is $\rho\, b_c$.

DISCUSSION

Wilson V. Binger,[33] **Assoc. M. ASCE.**—Verification of the Marston theory of loads on underground conduits is given by Professor Spangler for heights of fill up to 20 ft (see Fig. 9). The writer believes it will be of value to present similar data, with which he is familiar, for heights up to 50 ft.

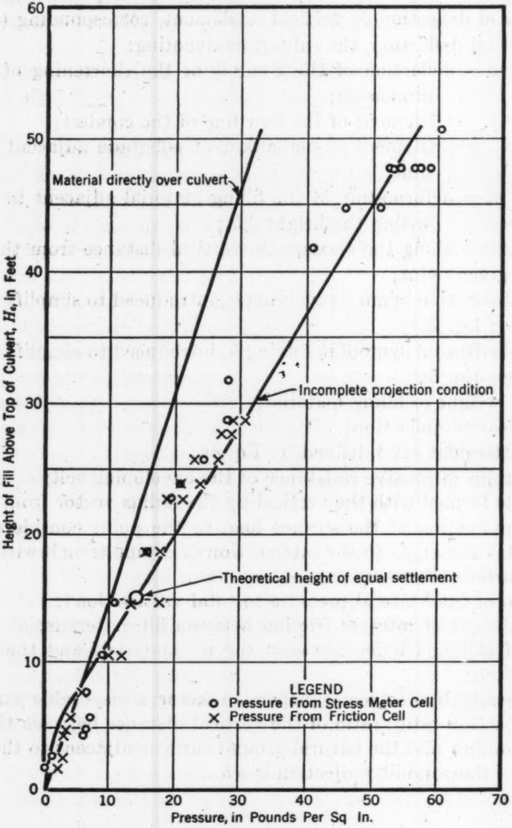


Fig. 19.—Pressures Versus Height of Fill Over Culvert for Full Load and for Incomplete Projection Condition

Relocation of a section of the Panama Railroad during the Panama Canal Third Locks Project involved construction of a reinforced concrete box culvert under a fill approximately 50 ft high. This culvert, built in early 1942, was 9 ft wide, 10 ft 10 in. high, and approximately 240 ft long. It was founded partly on weathered rock; and, where soft organic clay unsuitable as a foundation had been removed to depths of from 7 ft to 10 ft, on a traffic-compacted backfill of residual clay. Pressure cells of two different types were installed on this culvert during its construction in order to determine the relative efficiency of each. This work was carried out as a first step in determining the proper type of device for use in measuring the earth pressures against the structures of the Third Locks Project.

---

[33] Chf., Soils and Geology Branch, Missouri River Div., Corps of Engrs., Omaha, Nebr.

Two types of pressure-measuring devices were used. The first, three of which were installed, was the friction type developed by Professor W. C. Huntington, M. ASCE, whereas the second, two of which were installed, was the stress-meter type developed by Professor Roy Carlson, Assoc. M. ASCE. All cells were calibrated in the laboratory before installation in the culvert. One friction cell and one stress-meter cell were mounted in the wall of the culvert in order to measure side pressures; the results of these measurements are not within the scope of this discussion. The other two friction cells and one stress-meter cell were installed in the roof of the culvert under the center of the fill. After completion of the culvert, a 2-ft layer of red-brown clay common to the region was hand placed over the cells and compacted with wooden hand tampers. The rest of the fill, principally a fine-grained sandstone, was then placed and compacted in 2-ft layers by the normal traffic of three 12-cu yd scrapers and a 12-ton bulldozer.

Readings were taken on all cells once a day until the fill had reached its maximum height and the readings had become fairly constant. Above a fill height of 28.6 ft, the load on the friction cells exceeded the capacity of the spring balance used to determine the pressure. With the aid of a larger pulley to increase the torque applied to the cell, readings were continued but did not prove consistent. To determine the weight of the fill which was placed over the culvert, density samples were taken for each 2-ft layer as construction would permit. From these tests the average density of each layer was obtained and the pressure of the fill directly over the culvert determined.

Fig. 19 shows the pressure on the culvert plotted against the height of the fill, in a graph similar to Fig. 9. Computed pressures were determined for the "incomplete projection condition" by Eqs. 6*a*, 8, and 10, with the following constants and assumptions: $K \mu = 0.19$; $\delta = 0.5$; $\rho = 1.2$; $\gamma = 93$ lb per cu ft; and $b_c = 9$ ft.

Jacob Feld,[34] **M. ASCE.**—The casual reader of this paper might reach the conclusion that the problem of underground conduits was studied solely at the Iowa State College Experimental Station in Ames. The writer does not desire, in any way, to reduce the credit for the continuous attack on this problem, which was initiated in 1910 by Anson Marston, Hon. M. ASCE, which was aided from the early 1920's by W. J. Schlick, M. ASCE, and which has been continued since 1926 by the author. This paper is an authoritative summary of consistent practical results obtained from a program of research on one problem under a continuity of direction.

---

[34] Cons. Engr., New York, N. Y.

However, reference should be made to similar work performed during the same period. From 1923 to 1928, for instance, the experimental work conducted at the University of North Carolina in Chapel Hill consisted of full-size load measurements, together with the Goldbeck cell readings on culvert pipe of various materials (including pipe filled with concrete), which were under 50% projection and full projection conditions.[35] In 1923, William Cain sent copies of notes on the theoretical determination of vertical pressures (and, later, notes on stresses and deflections of pipe culverts) to several fellow workers in the field of soil mechanics. In a modified form, these notes were attached to the experimental report as Appendix I and Appendix II.[36] Mention should also be made of George L. Fowler's test on the load-carrying capacity of, and expected loads on, corrugated iron culvert pipe.

From a practical standpoint, construction methods have changed so radically from 1910 (or even 1923) to 1947 that many of the problems have become purely academic. To cite a few examples: Pipe trenches are seldom dug by hand, and back-hoe trenchers are used for depths as great as 25 ft in all types of soil. Trenches can be provided with substantially vertical sides and with fairly uniform widths, thus eliminating the benched hand-dug deep trench of greater widths at the surface, or the ragged clamshell excavation of indefinite width and shape. The ditch type, shown in Fig. 1(*a*), can be provided economically in widths from 20 in. to 54 in., with the added correction that the ditch can be backfilled better than "with relatively loose earth." In one airport project, in the clay fill area, the backfilling of the drainage trenches was so well done that they became relatively hard lines in the subgrade and showed up prominently during the final subbase rolling.

In regard to the projecting type of conduit (Fig. 1(*b*)), the use of heavy equipment to place and compact fills with the probability of damage and displacement of pipe laid on the natural ground makes it much more economical to place the fills first, and to trench through them for the drainage lines afterward. This type of installation is rapidly disappearing from the scene.

A modification of "first class" bedding shown in Fig. 13(*c*) is becoming a common practice because of the reduction of scarce hand labor. Excavation is made to the depths required for a concrete cradle, allowing some 4 in. below the pipe. The trench bottom is then lined with a layer of sand or other granular material, into which the pipe is bedded. The sand and the backfill required to cover the pipe by at least 1 ft (or by a depth equal to the pipe diameter) are tamped with special care. The load factor of such bedding should be close to 2.0, which strength value can be obtained without the experienced labor necessary to shape the ditch bottom, and without bringing in different labor trades to form and concrete a cradle base.

For combining the authoritative data covering all cases of design for underground conduits, the author deserves the thanks of the profession, which will look to this paper as the standard reference on drainage and piping installed below ground levels by open excavation methods.

---

[35] "Earth Pressure Experiments on Culvert Pipe," by G. M. Braune, William Cain, and H. F. Janda, *Public Roads*, November, 1929, pp. 153–165.

[36] *Ibid.*, pp. 165–175.

ZDENĚK BAŽANT, JR.,[37] **ESQ.**—In this interesting review of the test results obtained at the Iowa Engineering Experiment Station at Ames, the author states that rigid circular pipes of reinforced concrete are difficult to analyze by the principles of mechanics. The writer's opinion differs on this point, because he has used this type of analysis for the design of reinforced concrete pipes since 1944 with good results.

For ditch conduits the author uses a design load factor (under the heading, "The Supporting Strength of Underground Conduits") "* * * which is defined as the ratio of the strength of a pipe under any stated condition of loading to its three-edge bearing test strength." For concrete cradle (under the heading, "Ditch Conduit Beddings") he gives the load factor values varying from "2.2 to 3.4." The designer has a choice within these broad limits, but elsewhere he can find more precise data and can determine different load factors for various dimensions of cradles.[21]

If a form of cradle is designed for which values have not been published, the empirical load factor is not known; however, it can be calculated theoretically. The pipe is considered a circular arch fixed into the cradle. In the writer's opinion it is possible to use this method of calculation for all forms of cradles and to eliminate the empirical load factor completely. A comparison can be made of the accuracy of this assumption for the dimensions of the cradles tested.[21] As an example for comparison, pipes 24 in. in diameter have been chosen. In the Iowa tests the pipes were tested with a 120° upper sand bearing, which causes uniform load. Theoretical representation of this load on a circular arch is shown in Fig. 20.

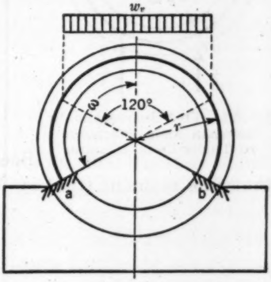


FIG. 20.—PIPE CONSIDERED AS CIRCULAR ARCH FIXED INTO CONCRETE CRADLE

The maximum moment occurs at points a and b, where the first cracks occur and

$$M_a = M_b = M_e - H\,(r\cos\omega - t_o) - C = n_1\,w_v\,r^2 \ldots\ldots\ldots (16)$$

in which

$$M_e = -\,w_v\,r^2\,\frac{\sqrt{3}}{2}\left(\sin\omega - \frac{\sqrt{3}}{4}\right) \ldots\ldots\ldots\ldots\ldots (17)$$

$$H = -\,w_v\,r^2 \; \frac{r\,\frac{\sqrt{3}}{2}\left(\sin^2\omega - \frac{\sqrt{3}}{2}\sin\omega + \frac{1}{4}\right) + t_o\left(\sqrt{3}\cos\omega + \frac{3}{4}\,\omega - \frac{5\,\pi}{12} - \frac{3\,\sqrt{3}}{8}\right)}{r^2\,\omega + r^2\sin\omega\cos\omega - 4\,t_o\,r\sin\omega + 2\,t_o^2\,\omega} \ldots (18)$$

$$t_o = \frac{r\sin\omega}{\omega} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

---

[37] Associate Prof. of Foundations, Univ. of Technical Sciences, Prague, Czechoslovakia.

and

$$C = -\frac{w_v r^2}{2\omega}\left[\sqrt{3}\cos\omega - \frac{\sqrt{3}}{2}\left(\omega - \frac{\pi}{3}\right) + \frac{\pi}{6} - \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{2}\right] \ldots (20)$$

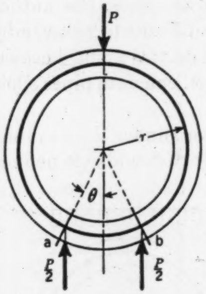


FIG. 21.—PIPE CONSIDERED AS CIRCULAR ARCH SUBJECTED TO THREE-EDGE BEARING

For three-edge bearing (Fig. 21),

$$M'_a = M'_b = \frac{P\,r}{2\,\pi}\left[\cos\theta - (\pi - \theta)\sin\theta + 1\right] = n_2\,P\,r \ldots\ldots (21)$$

From Eqs. 16 and 21, in which n_1 and n_2 are absolute numbers for definite ω and θ,

$$F_p = \frac{w_v\,r\,\sqrt{3}}{P} = \frac{\dfrac{M_a\,\sqrt{3}}{n_1\,r}}{\dfrac{M'_a}{n_2\,r}} \ldots\ldots\ldots (22a)$$

Because the pipe cracks occur in both cases under the same moment (that is, $M_a = M'_a$),

$$F_p = \frac{n_2\,\sqrt{3}}{n_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22b)$$

Numerical results are given in Table 1. The distance ab for three-edge bearing is assumed to be 3 in., thus making $\theta = 5° 44'$. The load factor is expressed for ultimate load, from which the allowable load is obtained by applying a safety factor. The load factor computed for the first crack cannot be used for the comparison, because it is not proportional to the moments obtained by analysis. However, if Table 1 proves the usefulness of the writer's methods, simplifications are made by neglecting the normal thrust, the effect

TABLE 1.—COMPARISON OF TEST RESULTS WITH THEORETICAL CALCULATIONS FOR 24-IN. DIAMETER PIPES

Type of cradle[a]	Thickness of upper part of cradle	ω	n_1[b]	n_2[b]	LOAD FACTOR, F_p	
					Observed for ultimate load	Calculated by Eq. 22b
(1)	(2)	(3)	(4)	(5)	(6)	(7)
4....	$\frac{b_s}{8}$	138° 35'	0.159	0.269	2.74	2.93
5....	$\frac{b_s}{4}$	120° 0'	0.131	0.269	3.25	3.56
8....	$\frac{b_s}{2}$	90° 0'	0.101	0.269	4.20	4.62

[a] "Concrete Cradles for Large Pipe Conduits," by W. J. Schlick and James W. Johnson, *Bulletin No. 80*, Iowa Eng. Experiment Station, Ames, Iowa, 1926, p. 39. [b] Absolute numbers, neglecting sign.

of thickness of pipe, the influence of dead load of pipe, and the elastic fixing of pipe into the cradle.

Thus, the circular arch concept can be used for designing pipes on concrete cradles. Despite the elaborate appearance of the computation, the design is actually simple, because the n-coefficients can be calculated for each angle ω and for each type of load.

Security against the first crack is obtained with the writer's method by using a safety factor fixed experimentally for each kind of pipe so that tension stresses do not exceed the selected allowable value. Further investigation is needed to ascertain the exact manner in which the rigid cradle causes concentration of loads on the pipe. It is quite possible that it increases the load coefficient C. However, despite the present lack of data the writer considers this effect negligible.

ANSON MARSTON,[38] PAST-PRESIDENT AND HON. M. ASCE.—In this paper Professor Spangler has made a good, concise presentation of the writer's theory of external loads on conduits under fill materials. In the paragraph containing Eqs. 4, he quotes the writer as defining[4,6] "* * * the various settlements, y, as the * * * increments due to the addition at or above the height of equal settlement of any [the same for all] incremental uniform layer of fill materials." This definition is so worded because the calculation coefficient C_e in the load equation (Eq. 8) for projection conduits contains the value at height H_e of the coefficient C_e in Eq. 6b—thus providing for all fill loads (including their time lags) due to settlements (also including time lags) below H_e. To clarify its character further the plane at H_e in the writer's theory should be termed the "plane of equal additional settlements." It should be noted that all inequalities in the settlements of fill materials in the vertical prism over the culvert and those in the masses of "alongside" embankment materials, which develop during the construction work up to the height H_e, are leveled up in grading.

Following Eq. 8 in the paper, the author states two conditions that are used in deriving the writer's equation for evaluating H_e:

> "*a*. The internal friction in the embankment materials distributes the infinitely small increments or decrements of pressure from shear into the interior prism [that over the culvert] below the plane of equal [additional] settlement in such manner that their effect on settlement may be assumed to be substantially the same as for uniform vertical pressure; and
>
> "*b*. The internal friction in the embankment materials distributes the infinitely small decrements or increments of pressure from shear into the exterior prism so completely that their effect on settlement may be neglected" (but the "alongside" embankment fill materials are just masses of indefinite horizontal extent, not distinct vertical prisms of determinable—or assumable—width).

In the foregoing quotations the interpolations in brackets and parentheses are the writer's. Assumption *a* is the same as that previously found to be true (for ditch conduits) for all values of b_d investigated.[2,3] The late Milo S. Ketchum,[39] Hon. M. ASCE, verified it for distances of at least 40 ft between the walls of bins filled with granular materials. In each case, the horizontal distance over which uniform pressure due to shear is transmitted is that between two vertical planes, one on each side, along which slippage occurs because of settlement. In the cases of ditch conduits and culverts, the planes are those of the two sides of the vertical prism of fill materials overhead. For bins filled with granular materials the two planes of slippage are the side walls of the bins.

---

[38] Dean Emeritus of Eng., Iowa State College, Ames, Iowa.

[39] "Design of Walls, Bins and Grain Elevators," by Milo S. Ketchum, McGraw-Hill Book Co., Inc., New York, N. Y., 3d Ed., 1919, pp. 310–311.

In reality, assumption *b* is the same as assumption *a*, but on each "alongside" there is no definite, opposite, vertical plane of settlement slippage in a typical embankment to stop the uniform distribution of pressure caused by the shear, due to the addition of the embankment materials above H_e before it has spread so widely as to have only negligible effect on settlement.

Concerning assumption *b*, the writer stated, in 1922,[4] in announcing the discovery of the "plane of equal additional settlement":

> "The only possible real test of the reliability of the new theory of loads on culverts from embankment materials (as proposed above) is comparison of the theoretical loads with the loads actually weighed in carefully conducted experiments with culverts"

—and he proceeded to show extremely close agreements of the final weighed and theoretical loads on the culverts under the 20-ft top soil (1919–1920) embankment and the 16-ft sand and gravel (1922) embankment. After 1922 similar close agreement between weighed and theoretical loads has been found in nine more experiments at the Iowa Engineering Experiment Station (IEES) with different culverts under different embankments; and the correctness of the writer's theory has been so completely verified in addition by detailed field studies of more than five hundred culverts under highway and railway embankments, that it is entitled to be admitted unreservedly.

However, as stated by Professor Spangler in the text following Eq. 10, at least two possible modifications of the theory have been suggested. The most appealing of these is the one based on the assumption of a "plane of equal total settlements." This, as investigated by the author for the case of the IEES "permanent" concrete pipe culvert (constructed in 1927 and still under observation), gives different values for H_e for different values of H (beginning 5 ft too high as proved by the actual settlement measurements), and gives values for the culvert load 17% higher than that weighed at $H = 15$ ft and 11% higher than that computed for the writer's theory for $H = 73$ ft. This is not the method that Professor Spangler found (as stated, following Eq. 10) to give computed loads "* * * nearly the same as those obtained by Dean Marston's original analysis, being somewhat less for high values of $\frac{H_c}{b_c}$ and $\delta\,\rho$."

Any paper on the theory of culvert loads should include mention and discussion of the twelve culvert pipe load experiments made at the University of North Carolina at Chapel Hill, from 1924 to 1927, in two series.

First Series, 1924 to 1926.—Three tests were made with 30-in. cast-iron pipe sections (32 in. in outside diameter) projecting 50% above subgrade. In the first test, the embankment was of sand, 20 ft high; in the second test, the embankment was of clay, 14 ft high; and in the third it was of clay, 11 ft high, in which a ditch, 43 in. wide and 48 in. deep, was dug during construction directly over the pipe and then loosely refilled (copying a similar IEES experiment in 1920–1921 with a sand embankment). All embankments were kept covered by tarpaulins during construction "except during actual work on the fill or during pressure observations," and the clay embankments were rolled at each 1-ft layer above 3 ft, using a 4-ton roller. The first and second embankments were removed promptly on completion. The third was continued in place from February, 1925, to April, 1926. From August, 1925, the load decreased 33%, doubtless because of "* * * the drying and shrinking of the initially saturated clay, resulting from the summer's unusually low rainfall."[40] It then began a slow increase, which was still in progress when it was removed.

Second Series, 1926–1927.—In the second series of tests six 29-in. to 32-in. (inside diameter) pipe culverts (from 30.2 in. to 34 in. in outside diameters), and three 20-in. to 21-in. culverts (from 20 in. to 21.15 in. in outside diameters) were tested, successively (with 100% projection above subgrade), on the same load-weighing apparatus, under sand embankments 12 ft high above the culvert tops, which were kept covered by tarpaulins as for the first series. Each embankment was removed promptly on completion. The culvert pipes were, respectively, of thin, smooth iron, corrugated metal, steel tubes, cast-iron pipe, reinforced concrete pipe, and solid concrete plugs.

The final detailed report[35] on these tests was made by the late G. M. Braune and H. F. Janda, Members, ASCE, with appendices by the late William Cain,[36] M. ASCE, long eminent for retaining wall pressure and column-formula mathematics. The support and weighing arrangements for the North Carolina tests were quite different from those adopted in the IEES experiments cited in Professor Spangler's paper.

The pipes were all mounted direct, without any bedding beneath, on what was practically like a solid rock formation. During the second series bank construction, some fill materials worked under; but a central strip from 60° to 80° wide was left bare.[36] The weighing apparatus was housed in a reinforced concrete structure on solid rock, covered by a reinforced concrete slab extending 10 ft, 7 in. on each side of the vertical prism over the pipe. This made the values of y_g and y_f each equal to 0 in the settlement ratio formula—

$$\delta = \frac{(y_m + y_g) - (y_c + y_f)}{y_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

so that

$$\delta = 1 - \frac{y_c}{y_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Because they did not recognize that the curved, lower parts of their plotted load curves proved the existence of planes of equal additional settlements at some heights H_e above the tops of the pipes, or that there were any settlement ratios δ, the North Carolina investigations made no measurements or discussions of y_m, or any other embankment settlements except y_c.

---

[40] "Earth Pressure Experiments on Culvert Pipe," by G. M. Braune, William Cain, and H. F. Janda, *Public Roads*, November, 1929, p. 157.

The use of tarpaulins and the prompt removal of embankments eliminated the time load lags, which other experiments and field studies prove would have added from 15% to 30% to the recorded culvert loads in the North Carolina tests.

The writer has made a thorough restudy of the detailed results of the North Carolina culvert tests, and finds that, adding proper percentages for time load lags, using the known value 1.00 for δ in the cases of the solid concrete plugs, and semi-empiric values of δ for the other culverts, the North Carolina culvert test results conform closely to results calculated by the writer's theory.

Neither Dean Braune[35,41] nor Professor Cain finally derived a definite comprehensive culvert load theory from the North Carolina results, although both toyed considerably with Dean Braune's idea[42] that they confirmed

> "* * * the theory on culvert pipes, namely, that the pressures vary inversely as some power of the deflection of the pipe, and values of the ratio, *K*, have been obtained ranging between limits above and below unity."

A similar idea was stated in his final report. The "*K*" referred to was the ratio of the culvert load to the weight of the fill materials directly overhead. The idea that the culvert load might equal $K\ w\ b\ H$ was suggested by the writer in 1921 in his "First Progress Report," but was promptly abandoned by him in 1922, upon his discovery of the "plane of equal additional settlements."[4]

Professor Cain has stated[43]

> "All of the preceding results show that the deflection of the pipe is a leading factor in the theory of the vertical pressure actually exerted on the pipe"

and again[44]—"The value of *K* can only be obtained by experiment for a particular earth * * *."

GEORGE E. SHAFER,[45] ASSOC. M. ASCE.—This summary of valuable research on underground conduits should stimulate more practical use of the proved but too little accepted methods of determining the loads that drainage structures must withstand. For more than twenty years the writer has followed and helped disseminate the results of various research on culvert loads and design. Field experience during this period justified the feeling that much money has been wasted because engineers have not made use of the data available.

The writer had the pleasure of cooperating with Anson Marston, Past-President and Hon. M. ASCE, in the derivation and verification of the equations for "complete" and "incomplete" ditch condition discussed in the forepart of

---

[41] "Earth Pressures on Culvert Pipes," by G. M. Braune, *Public Roads*, January, 1927, p. 222.

[42] *Ibid.*, p. 226.

[43] "Earth Pressure Experiments on Culvert Pipe," by G. M. Braune, William Cain, and H. F. Janda, *Public Roads*, November, 1929, p. 169.

[44] *Ibid.*, p. 175.

[45] Chf. Engr., Armco Drainage and Metal Products, Inc., Middletown, Ohio.

Professor Spangler's paper. Referring to the section, "Flexible Pipes," the writer and his associates were perhaps the first to recognize that flexible pipes do not fail as soon as the metal is stressed beyond the elastic limit, but continue to function structurally until the deflection resulted in a change in shape to the extent of reversing the curvature in the top or bottom of the structure. If one agrees with the author that structural failure of corrugated metal structures is due to excessive deflection, the problem then is to determine a method of predicting deflection.

Referring to Fig. 22, a flexible pipe is in equilibrium when the inherent strength of the pipe, R_I, plus the side support, W_H (developed in the earth as passive pressure due to outward movement of the sides of the pipe), is equal to the vertical load, W_c, on the pipe and its accompanying reaction.

How much a pipe deflects depends on all three factors—the larger the values of R_I and W_H the smaller will be the vertical deflection, y_c, and the larger the value of W_c, the greater that deflection. Therefore,

$$y_c = f\left(\frac{W_c}{R_I + W_H}\right) \quad \text{(25)}$$

Eq. 15 is in this form and includes those three factors.

Prior research on the design of flexible structures resulted in an empirical equation for deflection. This research should be considered, so that the results of the two equations can be compared. In 1926, after it was fully recognized that flexible structures fail by change in shape (excessive deflection), the writer and his associates set out to determine: (1) The law of deflection, so that structural performance could be predicted, and (2) the maximum permissible deflection, so that some part of the maximum could be used as a safe design deflection.

As to the law of deflections, it was known that deflection would vary directly as some power of the fill height, H, and the diameter, D, and inversely as the thickness of metal, t. Thus,

$$y_c = k\,\frac{H^m D^n}{t^s} \quad \text{(26)}$$

Using the deflection data from the American Railway Engineering Association (A.R.E.A.) investigation,[46] wherein the deflections were accurately measured as the fill was slowly made (permitting partial settlement to take place), along with measurements taken on numerous large diameter, high fill, unstrutted structures under both highway and railroad service, the values of k, m, n, and s were determined.

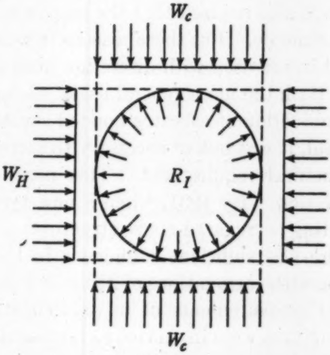


FIG. 22.—CONDUIT LOADING DIAGRAM

---

[46] "Culvert Load Determination," *Bulletin No. 284*, A.R.E.A., 1926.

The maximum deflection before failure occurred was readily determined by inspecting numerous, large diameter pipe installations with shallow cover of the "horse and buggy" days in California. With better roads and flatter grades, many of these structures, because of their good material and structural performance, were left in place and extended to permit the fill height to be increased. In some cases, the additional cover was sufficient to cause considerable additional deflection. The maximum recorded was 22% at a localized point. The average safe maximum deflection was determined as 20% of the vertical diameter. Using the conservative factor of safety of 4, the design deflection was established at 5%.

During the thirty years prior to 1926, when the empirical equation for riveted corrugated pipe was derived, the use of corrugated pipe was steadily increasing. By 1926 there were in use approximately 100,000,000 ft of average size corrugated pipe. With acceptance and use, gage tables based on experience had been compiled. Each large producer had his own gage table; however, all were practically the same. A new gage table based on the empirical equation using 5% deflection checked very closely with the old gage tables then in use, which were based on experience. Both were based on average installation conditions.

At that time, especially in certain parts of the United States, there were some questions regarding the durability of light gage, metal drainage structures. Gages that satisfied structural requirements often had to be increased because of durability requirements. For this reason, a gage table based solely on the equation was impractical. One of the objectives of the A.R.E.A. investigation[46] was a rational design for flexible pipe. The objective was never obtained; the empirical equation likewise fell short of the A.R.E.A. objective. It was also realized that the empirical equation was new and somewhat radical in principle. For these reasons it was not published. Like a new and untried tool it was used with discretion until its merit was established.

With the more general use of asphalt coated and paved invert pipe, plus the introduction of asbestos bonded sheets to insure adhesion of the asphalt to the metal, it was not so necessary to increase the gage of metal over and above the structural requirement. This permitted a more rigid use of the empirical equation. By 1931, when a new type of large diameter metal pipe made by bolting corrugated curve structural plates together in the field was introduced, there was sufficient confidence in the original empirical equation for riveted pipe with corrugations of $2\frac{2}{3}$ in. by $\frac{1}{2}$ in. so that a new equation was derived from field measurements of unstrutted structural plate pipe as soon as sufficient structures were in service to represent a good average. The equation was later revised as more installations were made and a new equation was derived for field strutted structures. These equations served to predict deflection. The bolted longitudinal seam was designed separately. Thousands of riveted and bolted structures have been designed on the basis of the empirical equations with almost faultless structural performance.

None of the equations have been published—not because they were considered trade secrets, but because the men who derived them knew their limitations.

The equations have just one application—to determine the required thickness of metal for a given diameter of pipe and height of cover or dead load. A tabular form giving the required gage for all structurally safe combinations of diameter and fill height is a gage table. Such tables, based on all three equations, have been available since the equations were derived. In some cases, light gages have been modified because of durability requirements; and, in others, to take care of various live load conditions. Since a gage table is far more convenient to use than an equation, the public or profession has not suffered because the exact equations were not known. The form of the equation and the fact that the gage tables were based on an empirical equation were published in 1937.[47]

When Professor Spangler proposed his investigation on the structural design of corrugated pipe various members of the industry took interest and cooperated. Those familiar with the empirical design hoped a rational design that would take into account installation conditions, type of soil, and other factors would result, so that, when possible, a more economical design could be used. When the investigation was completed and Eq. 15 was published, its use was investigated. The apparent weakness of the equation at that time necessitated the selection of the proper value of ϵ (modulus of passive resistance of the soil) to use for various soils. Later, the writer tried to use the equation to verify some live load tests and extend them into a gage table for airport loading. Again it was difficult to find the proper value of ϵ.

Perhaps the best way to discuss the application of Eq. 15 is to compare the permissible heights of cover for a range of diameters and gages with those obtained by the empirical equation. In Table 2, the heights of cover for Eq. 15 are based on $\Delta = 0.1\ r$, or 5% deflection, with $F_d = 1.5$, $F_k = 0.1$, $E = 30{,}000{,}000$, and $\epsilon = 20$. In general, the two equations agree fairly well for certain combinations of diameter and gage, but vary widely for other combinations. Eq. 15 does not give as much value to difference in gage as does the empirical equation and as is warranted by experience. It also permits higher fills for large diameters and light gages.

TABLE 2.—COMPARISON OF FILL HEIGHTS, IN FEET, OBTAINED BY EMPIRICAL EQUATION AND BY EQ. 15

Gage	24 IN. IN DIAMETER		48 IN. IN DIAMETER		72 IN. IN DIAMETER	
	Empirical	Eq. 15	Empirical	Eq. 15	Empirical	Eq. 15
16	26	24.2	7.2	15.9	3.5	21.7
12	62	36.6	18	17.4	8.8	22.2
8	140	53.7	39	19.6	18.5	22.2

Those engaged in the manufacture and distribution of flexible drainage structures see a definite need for a rational method of design. Not because design based on experience, such as the empirical equation, is wrong, but because it is possible that the full economy of flexible construction is not utilized in all cases. Furthermore, engineers prefer a rational approach to any problem, even though they use empirical methods in the solution of many problems. The rational approach also leads to a clearer understanding of the basic principles involved.

---

[47] "Handbook of Culvert and Drainage Practice," Armco Culvert Mfrs. Assn., Middletown, Ohio, 1937.

With this desire for a rational design in mind, the writer makes the following comments on each factor in Eq. 15 in the hope that by constructive criticism a solution will be evolved that will form the basis of all flexible conduit design.

Referring back to Eq. 25, the simplified form of Eq. 15, the writer has already implied agreement with the Marston theory for determining the vertical load on flexible structures. The part of Eq. 15 involving inherent strength R_I is based on the elastic theory of a closed ring. Referring to Fig. 23, this means that the load-deflection relation is a straight line and may coincide with the actual deflection curve from two-edge bearing tests for small deflections, or may deviate appreciably when the ring is stressed beyond the elastic limit.

Regardless of whether the load for two-edge bearing is determined from the equation,[30]

$$y_c = 0.149 \frac{P r^3}{E I} \quad \text{(27)}$$

or for the factor of Eq. 15 corresponding to R_I in Eq. 25, the result is the same, when F_d and F_k are assumed equal to 1.5 and 0.1, respectively. Therefore, the strength that is built into the pipe can be determined by Eq. 27. It is realized that Eq. 27 does not apply beyond the elastic limit of the metal, yet the comparison is made on the basis of 5% deflection, slightly beyond the limit, because that deflection has become standard practice in the design of corrugated metal pipe. In Table 3, values of P or W are given for 5% deflection and for a representative range of diameters and gages. The data on two-edge bearing loads does not include the Iowa tests, but does include unpublished data from several other investigations.

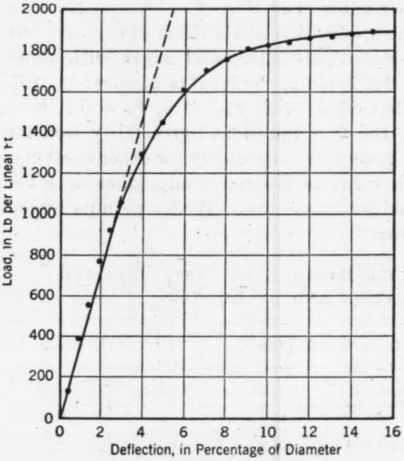


FIG. 23.—LOAD-DEFLECTION CURVE FOR 36-IN. DIAMETER CORRUGATED METAL PIPE (WALL THICKNESS, 0.112 IN.; ½-IN. CORRUGATIONS)

The loads determined by Eq. 27 for pipe 24 in. in diameter are approximately 2¼ times those from tests, and for pipe 72 in. in diameter they are approximately equal. If the effect of gage is considered regardless of diameter, Eq. 27 shows 8-gage pipe to be approximately 2¾ times stronger than 16-gage pipe, whereas the test data show this ratio to be approximately 4. These differences tend to affect the final design resulting from Eq. 15 by failing to show the full effect due to gage, regardless of diameter. This is a general comment regarding the use of Eq. 15 in actual practice. There are two other factors that should be considered with the foregoing data: (1) How much stronger is a flexible pipe in uniform vertical loading, as compared with concentrated loading, and (2) how much does uniform loading raise the elastic limit over that for concentrated loading?

TABLE 3.—COMPARISON OF INHERENT PIPE STRENGTHS[a] AS DETERMINED BY EQ. 27 AND BY ACTUAL TESTS

Gage	24 IN. IN DIAMETER		48 IN. IN DIAMETER		72 IN. IN DIAMETER	
	Iowa[b]	Test[c]	Iowa	Test	Iowa	Test
16	3,530	1,300	880	500	375	295
12	6,080	2,826	1,510	1,110	675	665
8	9,630	5,220	2,370	2,100	1,060	1,190

[a] Load in pounds per foot at 5% deflection. [b] Calculated by Eq. 27. [c] Determined by actual tests.

The usually accepted conversion factor for uniform loading from two-edge or three-edge bearing for rigid pipe is 1.5. According to the elastic theory, either by deflection of the ring or for moment in the ring, the factor should be approximately 2.0. It is the writer's opinion that a ratio of 2.0 is conservative.

Undoubtedly the elastic limit is higher (Fig. 23) for uniform loading as compared to concentrated loading, because of the more gradual and uniform change in shape or curvature under uniform load. Just how much the limit is raised is difficult to determine. If the change is small, then factors in Eq. 15 entering into inherent strength should be changed so that instead of being based on the elastic theory they would be based on actual concentrated load test data corrected for uniform vertical load. The average concentrated loads per foot of pipe to produce 5% deflection can be determined from the empirical equation,

$$P = \frac{4.825 \times 10^6 \, t^{1.40}}{D^{1.35}} \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

in which t is the thickness of the metal in inches (corrugations $2\frac{2}{3}$ in. by $\frac{1}{2}$ in.) and D is the pipe diameter in inches.

The side support part of Eq. 15 is based on certain assumptions, two of which will be discussed:

1. " * * * the horizontal pressure on the (side of a) pipe at any point bears a nearly constant relationship to the horizontal movement of the point."[30]

2. The distribution of passive pressure on the sides of the pipe is assumed symmetrical about the horizontal axis.

Considering first the assumption that horizontal pressure has a constant relationship to movement of the pipe sides, it is true that the Iowa data support this contention. Earlier research work[35] at the University of North Carolina

(Chapel Hill) seems to offer verification, except for very low fills. Referring to Fig. 24, the assumption is correct when the pressure and movement curves are straight lines. The pressure curves are straight lines, but the deflection is not a straight line for low fill heights. The deflection curve for Professor Spangler's first experiment has a similar characteristic.

If the pressure movement ratio is constant, the value of ϵ should be a constant for any soil characteristic and should not vary with fill height, or with the degree of rigidity of the pipe. There seems to be evidence that ϵ varies with fill height, or pipe rigidity—or both. Considering first the data supporting the claim that ϵ varies with fill height, in Fig. 24 the value of ϵ calculated using Eq. 15 varies from less than 30 for a 1-ft fill to more than 70 for a 12-ft fill. Similar values of ϵ for the 20-in. pipe in the same investigation are not as consistent as those plotted in Fig. 24 for the 30-in. pipe, but do vary from 28 for a 2-ft cover to 56 for an 8-ft cover.

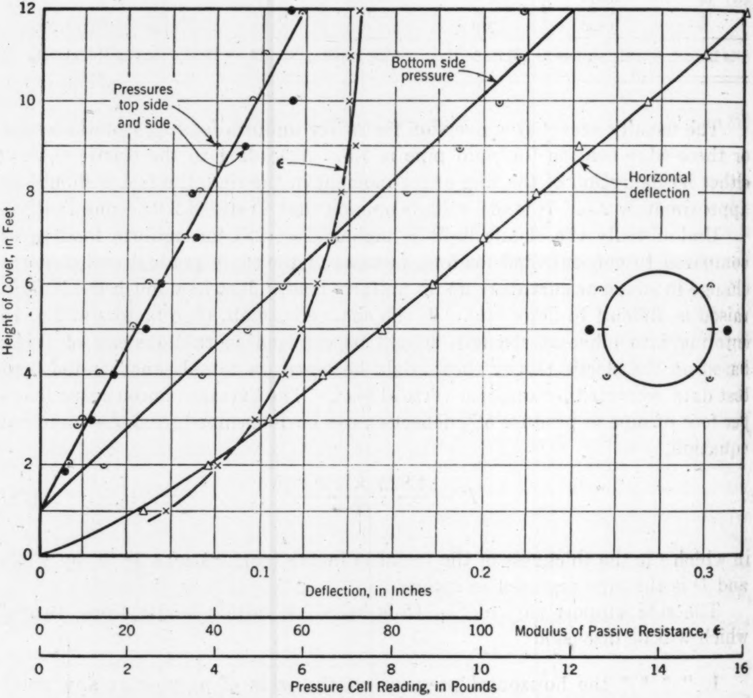


FIG. 24.—PRESSURE DEFLECTION DATA FROM UNIVERSITY OF NORTH CAROLINA STUDIES ON 30-IN. DIAMETER CORRUGATED METAL PIPE (WALL THICKNESS 0.108 IN.)

In 1928 accurate deflection measurements were taken on a rather large number of metal pipes installed in a new line of the Union Pacific Railroad Company. Original diameters were measured before backfilling, and final readings were taken nine years later. The number of unstrutted structures for which there was complete data (diameter, gage, average load, two-edge bearing strength at 5% deflection, and calculated ϵ according to Eq. 15) are shown in Table 4. Also included in Table 4 are similar data on two other investigations—those at the University of North Carolina[30] and the unpublished data on a live load test made by the writer. As further proof that ϵ may vary with the fill height, note how the average ϵ decreased with the average fill. The thirty-eight structures included in this investigation were installed under various conditions. For that reason, a comparison between individual structures is unwarranted, although a comparison between the averages for any pipe size seems permissible.

Referring to Table 4(*b*), the North Carolina tests were carefully made so that all pipes were installed and backfilled as nearly alike as possible. The three pipes listed are those that can be considered as flexible, the most flexible being a smooth pipe that had to be braced to hold the circular shape until the pipe was covered with sharp bank sand. The second and third pipes were progressively more rigid, as evidenced by the fact that the load factors (the percentage of load directly above the pipe actually carried by the pipe under 12-ft fill) were 61, 80, and 87, respectively. The values of ϵ, as calculated from Eq. 15 are 29, 45, and 85, respectively. These data tend to show that ϵ is a function of pipe rigidity. In Table 4(*c*) live load is the predominant load. All structures were carefully installed under the same installation conditions and loaded in the same manner. In this case ϵ varies with the pipe rigidity.

TABLE 4.—COMPUTED VALUES OF ϵ FROM VARIOUS TEST PROGRAMS

No. of structures (1)	Diameter (in.) (2)	Gage (3)	AVERAGE LOAD: Dead (ft of fill) (4)	AVERAGE LOAD: Live (5)	Bearing strength of pipe[a] (lb) (6)	Computed ϵ (7)	$\frac{\text{Col.6}}{\text{Col.7}}$ (8)
			(*a*) UNION PACIFIC RAILROAD COMPANY				
3	24	10	34	E60	3,950	62	64
16	36	10	17	E60	2,300	38	61
19	48	8	5	E60	2,100	18	117
			(*b*) UNIVERSITY OF NORTH CAROLINA				
1	30[b]	12	12	0	....[c]	29	....
1	30	12	12	0	1,620	47	36.3
1	30[d]	$\frac{1}{16}$ in.	12	0	....[c]	89	....
			(*c*) LIVE LOAD				
1	24	14	1	Truck with dual tires	1,780	262[e]	6.8
1	24	12	1		2,820	339[e]	8.3
1	48	12	1		1,110	149[e]	7.5
1	48	10	1		1,580	188[e]	8.4

[a] Two-edge bearing strength at 5% deflection. [b] Smooth pipe, very flexible. [c] Unknown. [d] Steel tube.
[e] Average for three different live loads (4,635 lb, 8,700 lb, and 13,695 lb) per wheel.

It will be noted that the ratio of two-edge bearing strength at 5% deflection to ϵ is much more constant than the values of ϵ.

To summarize the data in Table 4, there is evidence to show that ϵ varies with pipe rigidity more than with fill height. Nevertheless, in the Union Pacific data (Table 4(a)) for 36-in., 10-gage pipe and the 48-in., 8-gage pipe, the rigidity is approximately the same, where average ϵ varies from 38 for an average fill of 17 ft to 18 for a fill of 5 ft. This, added to the variation in ϵ shown in Fig. 24 for the same pipe under varying fill heights, leads the writer to believe that ϵ may vary both with fill height and pipe rigidity.

It is unfortunate that all the pipes in the author's third experiment, the major part of his tests wherein conditions were identical, were of approximately the same rigidity. In selecting the size and gage of pipe to use, the aim was to keep them as flexible as possible within the limits of practicability. The inherent or two-edge bearing strengths of the four pipes in the third experiment, according to the writer's data, are shown in Table 5. Had the tested pipe been of widely different degrees of rigidity, the resulting data might have indicated that ϵ is a function of pipe rigidity.

TABLE 5.—TWO-EDGE BEARING LOAD FOR PIPES OF THE AUTHOR'S THIRD EXPERIMENT

Diameter (in.) (1)	Gage (2)	Two-edge bearing load[a] (lb) (3)
36	16	750
42	14	840
48	14	685
60	1	815

[a] At 5% deflection.

Considering the second assumption, that the passive pressure on the sides of the pipe is symmetrical about the horizontal axis, the right half of Fig. 25 shows the development by Professor Spangler of a "witch-shaped" (witch of Agnesi) curve representing the locus of the horizontal movement of all points on the half circle, as the circle changes to an ellipse, their horizontal axes being coincident. In actual installations the inverts are coincident if there is no settlement. The left side of Fig. 25 shows the approximate shape of the locus based on coincident inverts and the same parabolic curve Professor Spangler used, which closely simulates the witch curve. The new center of the parabola is below the horizontal axis of the pipe by an amount equal to $y/2$. Actually, the difference in shape of the sides is not great, but the difference in shape combined with increased pressure below the horizontal could appreciably change the pressure diagram or composite loading.

Referring to the North Carolina data[35] for the 30-in. pipe, the pressure curves in Fig. 24, which were drawn to show the increase in pressure starting at a 1-ft fill, show the average pressure 45° above the horizontal "top side" to be approximately equal to the average side pressure. The average pressure 45° below the horizontal "bottom side" is more than twice the side pressure. This does not mean that the horizontal component of the measured pressure on the bottom side is greater than the horizontal component on the top side. It does, however, indicate that there can be a high pressure under the haunches of the pipe. Professor Spangler's composite loading, on which Eq. 15 is based, is shown on the right side of Fig. 26. The pressure at a point 45° below the horizontal, according to this loading, is very small. The data justified that assumption, but the writer believes that, had the pipe in the tests deflected more, or had the pipe been installed under 100% projection, the load diagram would be similar to that shown on the left side.

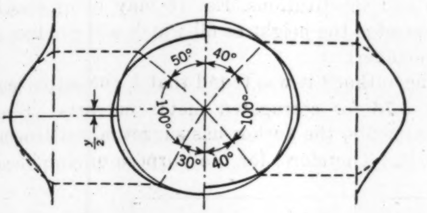


FIG. 25

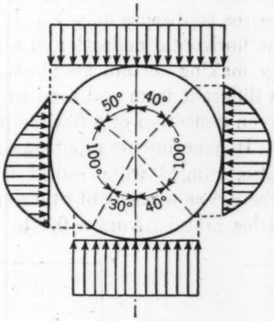


FIG. 26

As further proof of greater pressure under the haunches than that assumed by the author, the averages of all the pressure cell readings for the three flexible pipes in the 30-in. size tested at North Carolina[35] are shown in Fig. 27. These pressures are in pounds per square inch, and are the average increase in pressure due to any additional foot of fill from 0 ft to 12 ft. These data show that, with the exception of the top point, the greatest pressure is at the haunches, or 45° below the sides. Previously published data[48] show the haunches to be points of low pressure for low fills, with a trend toward more uniform pressures for higher fills. Such a revised pressure diagram would complicate the mathematical analysis but would, in the writer's opinion, more nearly fit field conditions for design deflections and the usual installation conditions, wherein the foundation is often quite yielding.

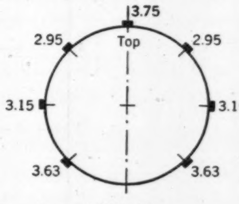


FIG. 27

To summarize, Professor Spangler's investigation and interpretation are steps in the right direction. It is unfortunate that the test pipes in the third experiment were all of virtually the same inherent strength. This, plus the fact that the range of diameters was small, apparently led to a solution that gives undue value to the side support factor for large diameters. Additional data are needed on working values of ϵ, especially under live load conditions. Whether there are sufficient test data available from all sources to revise Eq. 15, so that the results will be more consistent with actual practice, is unknown to the writer. It is emphasized again that the comments in this discussion are intended to be constructive and to encourage ideas and data that will eventually lead to a solution of a rather complex problem.

---

[48] "The Structural Design of Flexible Pipe Culverts," by M. G. Spangler, *Bulletin No. 153*, Iowa Eng. Experiment Station, Ames, Iowa, 1941, Figs. 24 and 25.

E. F. Kelley,[49] **Esq.**—This is an excellent summary of the results of the research work on underground conduits that has been carried on over a period of years by the Iowa Engineering Experiment Station at Ames, under the direction of Anson Marston, Past-President and Hon. M. ASCE. The writer desires to discuss only Eq. 15, which is presented as a method for computing the horizontal deflection of a flexible pipe, such as a corrugated metal culvert. By making certain assumptions and substitutions, Eq. 15 may be expressed in different form and used to determine the height of fill which will produce a given deflection of a flexible pipe culvert.

In experiments reported by the author[50] it was found that δ, the settlement ratio, ranged from -0.14 to -0.76 for corrugated metal culverts. Elsewhere,[51] as a result of other experiments, the author has suggested settlement ratios ranging from -0.4 to $+0.8$. Therefore, for the purpose of simplification and illustration it is considered sufficiently accurate to assume the settlement ratio equal to zero.

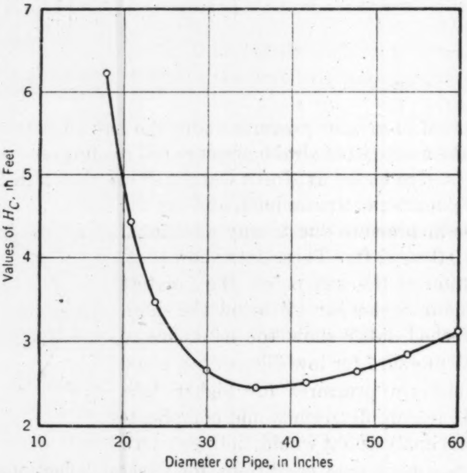


Fig. 28.—Relation Between Height of Fill, H_c, and Diameter of 16-Gage Corrugated Metal Pipe, Computed by Eq. 15 for a Deflection of 1% of the Pipe Diameter

It is apparent from Fig. 8 that when $\delta = 0$, $\delta\,\rho = 0$ and

$$C_c = \frac{H_c}{b_c} \qquad (29)$$

Also, with b_c expressed in feet and r in inches,

$$b_c = \frac{2\,r}{12}, \qquad (30)$$

---

[49] Chf., Div. of Physical Research, Public Roads Administration, Washington, D. C.

[50] "The Structural Design of Flexible Pipe Culverts," by M. G. Spangler, *Bulletin No. 153*, Iowa Eng. Experiment Station, Ames, Iowa, 1941, p. 56.

[51] "Analysis of Loads and Supporting Strengths, and Principles of Design for Highway Culverts," by M. G. Spangler, *Proceedings*, Highway Research Board, National Research Council, Vol. 26, 1946, p. 198.

Substituting Eqs. 29 and 30 in Eq. 6*a*,

$$W_e = \frac{H_e \gamma r}{6} \qquad \text{(31)}$$

Thus, if W_e is expressed in pounds per linear foot of pipe, Eq. 15 may be reduced to the form,

$$H_e = \frac{72\,\Delta\,(E\,I + 0.061\,\epsilon\,r^4)}{F_d\,F_k\,\gamma\,r^4} \qquad \text{(32)}$$

Computed values of H_e for corrugated metal pipe having diameters from 18 in. to 60 in., inclusive, are shown in Fig. 28. For the purpose of the computation it was assumed that: $\Delta = 1\%$ of the pipe diameter $(= 0.02\,r)$; $E = 30{,}000{,}000$ lb per sq in.; $I = 0.001766$ in.^{4} per in., a value computed for 16-gage corrugated metal; $\epsilon = 20$ lb per sq in. per in.; $F_d = 1.5$; $F_k = 0.10$, corresponding to a bedding angle of about 35°; and $\gamma = 120$ lb per cu ft.

It is quite generally agreed that the deflection of corrugated metal pipes in service should be limited to a maximum of 5% of the pipe diameter. However, as the author has pointed out, Eq. 15 is applicable only within the elastic range of the pipe metal and it is not known that a 5% deflection of a pipe in service is within this range. It is known that, when corrugated metal pipe is subjected to the three-edge bearing test, a deflection of 5% is generally outside the elastic range, whereas a deflection of 1% is well within it. It is for this reason that, for purposes of illustration, the relatively small deflection of 1% of the pipe diameter has been selected.

For the smaller sizes of pipe Fig. 28 shows, as would be expected, that H_e decreases as the diameter of pipe increases. However, the minimum value of H_e is obtained for a diameter of 36 in.; thereafter the value of H_e increases with increases in the pipe diameter. Such results are unreasonable and cannot be accepted as correct.

There may be one or more reasons for the questionable results shown in Fig. 28. It may be that empirical Eq. 15, although applicable within the range of the author's experiments, may not have a general application. It may be also that the writer's assumption of a constant settlement ratio or a constant value of the modulus of passive resistance, or both, may be incorrect.

These questions, and the further question as to whether a deflection of a pipe in service of the order of 5% is within the elastic range of the pipe metal, lead to the conclusion that, on the basis of present knowledge, Eq. 15 cannot be accepted for use in the design of flexible pipe culverts.

BAILEY TREMPER,[52] ESQ.—In the State of Washington there are many pipe culverts installed under embankments of from 50 ft to 75 ft in height and in at least one case of 130 ft. The record of performance of these installations ostensibly should afford an excellent opportunity to check the theories of load and supporting strength as developed by Anson Marston, Past-President and Hon. M. ASCE, and reviewed by the author.

---

[52] Materials and Research Engr., Dept. of Highways, State of Washington, Olympia, Wash.

The difficulties in evaluating the theories in the light of such experience become apparent from a study of Fig. 8. The very wide range in loads shown for varying values of projection and settlement ratios requires that these conditions be known with considerable accuracy before proper evaluation can be made.

This discussion will be confined to observations of reinforced concrete pipe culverts in service with no intention of conveying the idea that other types have not been used satisfactorily. In many cases good records are available of the depth and width of trenches in which the culverts were installed. Such trenches, however, usually were dug in steeply sloping ground to one side of the watercourse. The depth varied greatly within short distances and, characteristically, was far from uniform on the two sides of the culvert. A good estimate of projection ratio can be assumed only within wide limits. No information is available on the settlement ratio at any site. Although the three-edge bearing strength of the pipe is known within reasonably close limits, the actual care used in bedding and backfilling, and hence the supporting strength of the pipe in place, can only be assumed.

With no good basis for estimating either the loads on the culverts or their supporting strength in place, definite conclusions as to the validity of the theory are not warranted. Based on careful study of the available data of construction and inspection of the condition of the culverts, it appears, however, that satisfactory service is being obtained under heights of fill greater than that indicated by the theory, and certainly greater than conservative design, based on the theory, would indicate.

The writer, therefore, believes that certain modifications of the theory with respect to loads under high fills and the performance of reinforced concrete pipe culverts are warranted. In so far as is known, the experimental work on which the theory of load is based did not involve fills greater than 20 ft in height. Therefore, data for the curves shown in Fig. 8 for higher fills seem to have been obtained by extrapolation. When the load on the culvert resulting from uniform increments of earth to the partly completed embankment is calculated by the Boussinesq equation, the result for high fills is less than is indicated by the curves in Fig. 8. For example, for embankments having a width at the top of 40 ft and side slopes of 1 on $1\frac{1}{2}$, the loads on the culvert, calculated by the Boussinesq method, are the following percentages of those derived from the curve $\delta\, p = 0$ in Fig. 8:

Fill height, in feet	%
30	95
50	90
70	87

The author has found that the Boussinesq solution is in good agreement with measured forces transmitted by concentrated live loads. Therefore, it should be valid for distributed dead loads.

In estimating the supporting strength of reinforced concrete pipe, the author treats it as a rigid structure, making no allowance for flexibility. Actually after the first crack is formed, the pipe begins to exhibit measurable flexibility. At some point, approximately that of a 0.01-in. crack, vertical and horizontal deformations increase rapidly with increasing load during the test. When installed as culverts, the passive pressure of the adjacent soil must afford considerable resistance to deformation and must therefore contribute appreciably to the supporting strength of the pipe. Flexibility in the pipe tends to reduce the settlement ratio as well, and thus reduces the load reaching the culvert from the superimposed embankment.

Some of the pipe in the culverts, to which reference has been made herein, was found to be cracked to widths of 0.01 in., or somewhat more, when examined a few years after construction. Subsequent inspections have revealed frequent cases of autogenous healing or apparent closing of the cracks. Rarely have cracks, when measured on a second inspection, been found to be of greater width than on the first. These findings lead to a belief that cracks in well-made reinforced concrete pipe culverts, when of the order of 0.01 in., are not harmful. A report published in 1947[53] indicates that serious corrosion of the reinforcement is extremely improbable.

It is suggested, therefore, that reinforced concrete pipe culverts be designed to the full estimated supporting strength of the pipe at a 0.01-in. crack, and that its flexibility beyond this point be considered as an adequate factor of safety. It is not intended to imply that ultimate strength requirements should be eliminated from specifications, as tests to ultimate strength are necessary to establish proper functioning of the reinforcement and capacity to withstand moderate deformation without undue structural damage.

Modern methods of compacting embankments appear to make it unnecessary to install culverts in trenches made for the sole purpose of reducing the projection ratio. Usually the density and strength of the embankment can be made greater than those of the soil in its original position. With proper attention to moisture control and compaction in the areas adjoining the culvert, more favorable conditions can be produced than in a dug trench. This is so because of less width, less settlement at the sides of the culvert, and better opportunity for good bedding. A condition approaching an imperfect ditch results from the usual restriction against heavy equipment passing over the culvert until the fill is a few feet above its top. The zone immediately above the culvert is therefore likely to be in relatively loose condition. The depth of the imperfect ditch, of course, can be increased when found desirable at little additional cost.

It is believed that designs of pipe culverts under assumed incomplete projection ratios are neither economical nor necessary. With reasonable requirements for compaction of embankments, and their adequate enforcement in the vicinity of the culvert, a condition should be obtainable such that the load on the pipe is less than, or does not exceed, that of the weight of the earth directly above it.

---

[53] "The Corrosion of Reinforcing Steel in Cracked Concrete," by Bailey Tremper, *Journal*, A.C.I., June, 1947, p. 1137.

M. G. Spangler,[54] M. ASCE.—The time and effort put forth in preparing this paper have been more than compensated for by the stimulating ideas which have been expressed by the various discussers. Of particular value are the data presented by Mr. Binger relative to the pressures measured on a concrete box culvert under a 50-ft fill on the Panama Railroad. The close correlation between the measured pressures and those indicated by the Marston theory when applied to the conditions of the installation serve, powerfully, to reinforce the confidence which engineers may place in the principles involved in the theoretical development.

Mr. Binger's data are significant for two reasons: First, they show the load produced by a fill two and one-half times higher than any previous fill for which loads have been measured; and, second, the fill was compacted by modern heavy equipment during construction.

Mr. Feld is entirely justified in referring to the researches on underground conduit loads which were conducted at the University of North Carolina in Chapel Hill. In preparing any paper the matter of scope becomes highly important and in this case it was decided to limit the paper to the work which had been done at Iowa State College in Ames. Dean Marston has made an exhaustive study of the reports covering the North Carolina experiments and has concluded that the measured loads in those tests conform closely to the results calculated by the Marston theory, as he has stated in his discussion.

It is difficult to subscribe to Mr. Feld's characterization of the problems discussed in the paper as purely academic. Although it is true, as everyone knows, that methods of earth handling and construction have undergone radical changes during the past quarter of a century, the principles of stress transfer embodied in the Marston theory are ageless and will be applicable many years hence when present-day construction methods become outmoded. The ditch conduit analysis may be applied with equal validity to a hand-dug or a machine-dug trench without reference to the slope or irregularity of the sides, provided the correct ditch-width factor is used in the analysis. If the backfill material is placed in the condition which Mr. Feld describes as "better" than relatively loose earth, considerable harm may result in that the loads on the pipe may be very greatly increased as compared to the loads under the more usual manner of backfilling. In the example cited where the backfills of drainage trenches became relatively hard lines in the subgrade, there is little doubt in the writer's mind but that the load at the base of the trench was relatively high, because the transferred shearing stresses would be additive to the weight of the backfill instead of subtractive as in the more usual case.

The method of constructing projecting conduits in trenches dug through compacted embankments has distinct advantages from the standpoint of loads produced on the structure, in addition to the advantages mentioned by Mr. Feld, since placing the pipes in ditches tends to cause the shear stress increments to act upward on the prism of material in the ditch, provided the backfill is loosely placed. The California Highway Department has developed methods of construction based on this principle which appear to be efficient in reducing loads on the pipes. These procedures, together with excellent bedding and backfilling practices, have permitted a substantial increase in safe heights of fill over drainage conduits under highways in that state. However, it seems doubtful whether the practice of building an embankment first and then trenching through it to construct a drainage line can ever be widely applicable, inasmuch as climatic conditions in many localities are such that a heavy rainstorm between the time of completion of the embankment and the construction of the trench through it might be very detrimental or even disastrous. The favorable load condition produced by this method of construction is recognized, however, and it is clearly within the scope of the Marston load theory. The deeper the trench through the embankment in relation to the trench width, and the more loosely the backfill is placed in the region immediately above the conduit, the farther toward the left side of Fig. 8 the situation falls and the less will be the load on the structure.

---

[54] Research Prof., Civ. Eng. Dept., Iowa Eng. Experiment Station, Iowa State College, Ames, Iowa.

Mr. Tremper's remarks indicate that similarly good results are being obtained in the State of Washington by installing pipe culverts in sidehill ditches to one side of the watercourse. Where practicable, this procedure would appear to be better than that suggested by Mr. Feld, because drainage water could be cared for in the natural watercourse during construction of the culvert. The discussions by both Messrs. Feld and Tremper lend emphasis to the more favorable load situation that can be achieved in culvert construction when specific attention is given to creating conditions in which pressures are transferred by shear from the interior prism of soil to the exterior masses.

From Prague, Czechoslovakia, comes a very interesting analysis of concrete pipe rings from which values of the load factor for cradled pipe have been determined. There is no difference of opinion between Professor Bažant and the writer relative to the analytical determination of bending moments in a reinforced concrete pipe. This procedure is precisely that employed by the writer prior to 1933 which led to the development of Eq. 14*b*. The difficulties in design arise when one attempts to convert the calculated moments into stresses in the steel and concrete of which the pipes are made.

Professor Bažant's calculated load factors shown in Table 1 are approximately 10% greater than the average experimental values obtained by W. J. Schlick, M. ASCE, and James W. Johnson.[55] These differences are not significant and readily may be accounted for by the fact that the sand-bearing load was assumed to be uniformly distributed in the analysis, whereas it is known that the pressure in the test is greater at the center of the bearing than at the sides and that it approaches a parabolic distribution.

However, several circumstances and assumptions connected with the analysis contained in the discussion lead to the conclusion that the correlation shown in Table 1 may be more coincidental than basic. For example, it is assumed that: "The maximum moment occurs at points a and b, where the first cracks occur * * *." In the Schlick and Johnson experiments, 94% of the cradled specimens cracked first at the top of the pipe. The first cracks in the remaining 6% occurred simultaneously either at the top and bottom or at the top and sides of the pipe. Also, in every case the cradle cracked in the region below the bottom crack in the pipe and at the same time that the bottom crack occurred. In no case did the first cracks occur at the junction of the pipe wall and the cradle as assumed by Professor Bažant.

---

[55] "Concrete Cradles for Large Pipe Conduits," by W. J. Schlick and James W. Johnson, *Bulletin No. 80*, Iowa Eng. Experiment Station, Ames, Iowa, 1926, p. 39.

The comparison between computed and experimental values of the load factor for ultimate loads is probably not valid, even though the values appear to be reasonably close numerically, because at ultimate load the pipe and the cradle have long since ceased to be a continuous arch body. At ultimate loadings, the experimental pipes were cracked in many places, the cradles were broken into two parts, and the bond between the cradles and the pipes was destroyed, so that the structures were considerably different after they were cracked from the structures which Professor Bažant analyzed. For these reasons it would seem that a comparison between calculated and observed load factors at "first crack" load would be much more appropriate. Such comparisons for the three cradle types included in Table 1 show that the calculated load factors exceed the observed values by 18%, 37%, and 35%, respectively.

Mr. Shafer's discussion is a most welcome addition to the literature on the supporting strength of flexible pipe conduits. He has had wide experience in this field, has made extensive observations of the performance of this type of structure, and has brought rare good judgment to the interpretation of his observations. The writer has a very high regard for the utility of Mr. Shafer's empirical Eq. 26. It has served both the industry and the users of metal culvert pipes very well since its inception in 1926.

In the writer's opinion, however, the flexible pipe problem is far too complex to rely solely on experience for a solution. Theory is needed to supplement judgment and to guide the interpretation of the shotgun pattern of the facts gleaned in the field. The empirical formula falls short in the achievement of scientific design of flexible pipe culverts and tends to stifle progress in that direction, since it does not take into account several important variables that influence the problem. One of these is the passive resistance of the soil side fills, which is widely recognized as a highly important factor (probably the most important single factor) on which pipe deflection depends. The empirical formula throws all the effect of side pressure into the factor k, and the best that can be expected under these circumstances is the determination of "average" deflection, which may be greatly exceeded in individual cases. Very few field data are available to the profession from which to judge the correlation between actual deflections developed in the field and deflections predicted by the empirical formula. Such data as the writer has seen show a very wide range of deflections, many individual specimens being in excess of 100% above the "average," and extreme cases being from 400% to 500% above.

Another variable factor which the empirical formula submerges in the "average" is the relationship between the height of fill and the load on a culvert. It is unfortunate, but nonetheless true, that the load and the height of fill are not related by a constant factor for all culverts. This fact is clearly shown by Fig. 8. In the present state of knowledge it is difficult to predict values of the settlement ratio, which has such an important effect on the relationship between load and height of fill. Nevertheless, it is present and effective in every culvert installation and scientific progress demands that it be recognized and studied. This same comment applies to the projection ratio, except that the latter factor is more easily determined for a specific pipe installation.

For these reasons and others, the writer does not consider that the fill height data given in Table 2 are validly comparable. It would be the purest coincidence if the fill heights shown in the parallel columns were in agreement. The only valid check on Eq. 15 is a comparison of the deflections computed by the formula with actual measured values when all the factors relative to a culvert installation which influence the deflections are known or can be estimated with a reasonable degree of precision.

When Eq. 15 was developed and published, it was intended to be used to compute the deflection of a flexible pipe culvert when the diameter of the pipe and its physical properties (the height of fill, the settlement ratio, the projection ratio, the width of bedding, the unit weight of the fill, and the character of the soil as it influences passive resistance pressures) are known. Immediately after publication, many engineers turned the formula around and attempted to use it to calculate a fill height that would produce some predetermined deflection of the pipe. This practice has led to some difficulties and erroneous concepts concerning the formula. Mr. Kelley rejects the formula as giving unreasonable results, because the fill height to produce a given deflection, expressed as a constant percentage of pipe diameter, is larger for large pipe than it is for some smaller pipe sizes. His point would be well taken if it could be definitely stated that, when two pipes of different diameters have deflected under a fill an amount equal to 5% of their diameters, they have both progressed the same distance toward failure and are, therefore, on a comparable basis as far as strength is concerned. The writer does not believe this is true.

This premise opens the whole question as to the theory of failure of flexible pipes and a suitable criterion for an allowable design limit. The metal pipe industry has for many years suggested a deflection limit of 5% of the diameter as being a suitable criterion and many engineers have accepted this value. The writer has looked on the 5% limit as a good value to "shoot at," but he has come to realize that neither it nor any other constant percentage-of-diameter deflection represents the same strength situation for all diameters of pipes. As evidence on this point, the writer has investigated the pipes for which Fig. 28 was drawn by computing the fill heights to produce a constant value of bending moment at the bottom of the pipes. The equation for the moment at the bottom of a flexible pipe under field loading conditions (for a bedding angle of 35°) is[56]

$$M = 0.176\, W_c\, r - 0.166\, h\, r^2 \quad \ldots\ldots\ldots\ldots\ldots (33a)$$

---

[56] "The Structural Design of Flexible Pipe Culverts," by M. G. Spangler, *Bulletin No. 153*, Iowa Eng. Experiment Station, Ames, Iowa, 1941, p. 58.

in which

$$h = \epsilon \frac{\Delta}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33b)$$

Substituting Eq. 15 in Eq. 33*a* and using the same values of various factors that were used by Mr. Kelley, it is possible to compute heights of fill over the various diameter pipes to produce the same bending moment in each of the pipes. The bending moment in the 36-in. pipe of Fig. 28 was arbitrarily chosen as the common value for all the pipes. The results are shown in Fig. 29. On the basis of this criterion the height of fill decreases with increasing pipe diameter, which seems to be more reasonable. Since both Eq. 15 and Eq. 33 are based on the elastic analysis, the author believes that Fig. 28 throws a cloud of uncertainty on the validity of applying a constant percentage-of-diameter deflection limit to all sizes of pipe, rather than on the validity of Eq. 15.

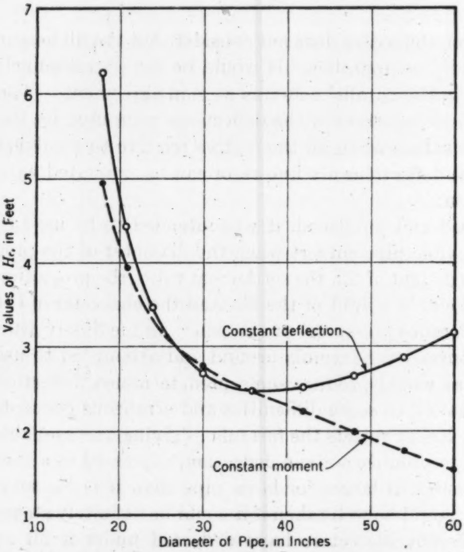


Fig. 29.—Height of Fill Versus Pipe Diameter

Mr. Shafer goes further and applies the 5% limit to pipes loaded in two-edge bearings and compares the loads to produce this deflection by elastic analysis and by actual test. Since the smaller sized pipes tested were obviously loaded beyond the elastic limit of the metal, the calculated loads to produce 5% deflection were considerably greater than the test loads. This discrepancy between calculated loads and test loads is cited by Mr. Shafer as an indication of a weakness in Eq. 15. The writer believes that the comparisons in Table 3 are wholly fortuitous and irrelevant and that Mr. Shafer's conclusions based on them are not valid.

In support of this contention, it is pointed out that, if the test specimens had been made of metal having a higher elastic limit, the difference between the loads calculated by Eq. 27 and the test loads would have been materially less. In Fig. 23, the metal in the 36-in. pipe for which load-deflection data are plotted apparently reached its elastic limit at a load of 1,056 lb per lin ft, or 88 lb per lin in. The bending moment at the top and bottom of a circular ring loaded in two-edge bearing is[57]

$$M = 0.318\, P\, r \ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

in which P is the load per unit length of pipe; and r is the radius of the pipe. By the flexure formula,

$$s = \frac{M\,c}{I} \ldots\ldots\ldots\ldots (35)$$

The elastic limit of the metal in this pipe is approximately 46,000 lb per sq in. by Eqs. 34 and 35.

Corrugated metal pipe culverts are made of many different kinds of iron and steel, ranging from commercially pure iron having a relatively low elastic limit to copper and copper-molybdenum iron or steel alloys having high elastic limits. In the research work on which Eq. 15 was based, the metal pipes furnished for the tests by four different manufacturers varied in elastic limits from 47,800 lb per sq in. to 65,500 lb per sq in. If the 36-in. pipe of Fig. 23 had been made of metal having an elastic limit of 66,000 lb per sq in. instead of 46,000 lb per sq in., the load-deflection diagram would have been approximately as shown by the dotted line in Fig. 30 and the difference between actual loads and those computed by Eq. 27 would have been much less.

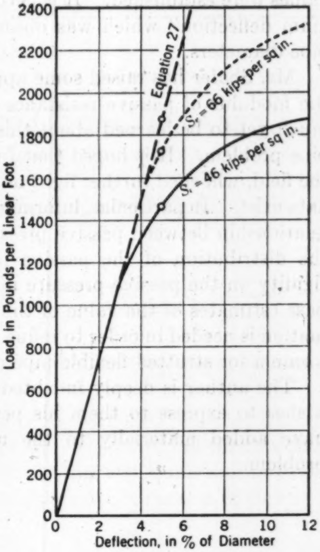


FIG. 30.—LOAD-DEFLECTION DIAGRAM FOR CORRUGATED METAL PIPE CULVERTS

Likewise, the data shown in Table 3 are intimately dependent on the elastic limits of the particular pipes tested. It is the writer's view that these data do not provide a proper basis for generalization without some detailed knowledge of the elastic limits of the metals from which the test specimens were made.

The data in Table 3 are irrelevant because the load and reaction in a two-edge bearing test are much more concentrated than in the field loading for which Eq. 15 was developed and it seems probable that the difference between computed loads and actual loads would be much less in the latter case, even when the pipe metal is stressed beyond its elastic limit at localized points—that is, the load-deflection curve under field loading would probably deviate from a straight-line elastic relationship much less rapidly than under concentrated loading. Mr. Shafer appears to agree with this view. Obviously, also, pipes made of metal having a high elastic limit would deviate less from the elastic line and would have less deflection for a given load than would pipes made from low elastic limit metal.

---

[57] "The Structural Design of Flexible Pipe Culverts," by M. G. Spangler, *Bulletin No. 153*, Iowa Eng. Experiment Station, Ames, Iowa, 1941, p. 14.

The data shown in Table 3 again raise the question of whether it is valid to apply a constant limiting deflection to all sizes of pipe. Mr. Shafer has given an interesting account of the origin of the commonly accepted 5% limit. He would do the profession a service if he would publish the data on which the values were established. It is pertinent to ask whether the "average safe maximum deflection" which was observed was the same percentage value for all pipe diameters.

Mr. Shafer has raised some appropriate questions concerning the nature of the modulus of passive resistance of soils, and the writer agrees that there is much yet to be learned about this property and its application to the flexible pipe problem. It is hoped that future research, both in the laboratory and in the field, will shed further light on this factor concerning which so little factual data exist. In particular, information is needed to determine more definitely the relationship between passive pressures and movement of the sides of the pipe, the distribution of the passive pressures, the effect of fill height and pipe rigidity on the passive pressure modulus, and soil test procedures on which to base estimates of the value of the modulus for design use. Such added information is needed in order to refine and improve Eq. 15 and to develop a parallel formula for strutted flexible pipe culverts.

The author is deeply indebted to those who have discussed the paper and wishes to express to them his personal thanks. He believes the discussions have added materially to the understanding of the underground conduit problem.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2338

SUBSIDENCE OF THE TERMINAL ISLAND-LONG BEACH AREA, CALIFORNIA

BY FREDERIC R. HARRIS,[1] M. ASCE, AND EUGENE H. HARLOW,[2] ASSOC. M. ASCE

WITH DISCUSSION BY MESSRS. JAMES GILLULY AND U. S. GRANT, JACOB FELD, AND HYDE FORBES.

SYNOPSIS

By early 1945 the subsidence taking place in and around Terminal Island, Long Beach, Calif., had reached a maximum of more than 4 ft and the rate was approaching 1 ft per yr. Therefore, to protect existing structures and to evaluate the risks in future engineering enterprises in the area, it became imperative to determine the answers to three questions:

(1) What is the cause of the subsidence?
(2) Can its future progress be predicted?
(3) What effect will it have on civil engineering works?

This paper presents answers to these questions. It now appears certain that by far the predominant influence is the effect of pumping oil from the Wilmington oil field, which coincides in location and extent with the subsiding area. Extensive data were gathered to determine the factors producing oil field settlements, and laboratory tests were performed on samples of shales from different depths to determine the relative compressibilities. Calculated theoretical subsidence was compared with observed subsidence, a correction factor was computed, and theory was then extended into a prediction as to the future course of the settlement. From these studies it is concluded that the maximum rate of settlement has probably been reached and that henceforth the rate will gradually diminish until eventually, in perhaps ten or fifteen years, the maximum subsidence will reach about 9 ft.

Elevations of waterfront structures, gravity flow pipe lines, etc., may have to be adjusted accordingly. Since differentials will be gradual, only very long or rigid structures will be subjected to warping or cracking.

NOTE.—Published in October, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Rear Admiral, CEC, U.S.N. (*Retired*); Cons. Engr., New York, N. Y.

[2] Foundation Engr., Frederic R. Harris, Inc., New York, N. Y.

NATURE OF OBSERVED SUBSIDENCE

Occurrences of differential movements in the Los Angeles-Long Beach harbor areas of California have long been recognized (1)(2)(3)(4).[3] However, comparison of records of tide gages maintained at several points in the Los Angeles and Long Beach Harbor areas for many years with observations at La Jolla and San Francisco show that the gage at berth 60, municipal pier No. 1, Los Angeles harbor, referred to as "tidal 8," has not changed in elevation since 1931 by more than about 0.1 ft.

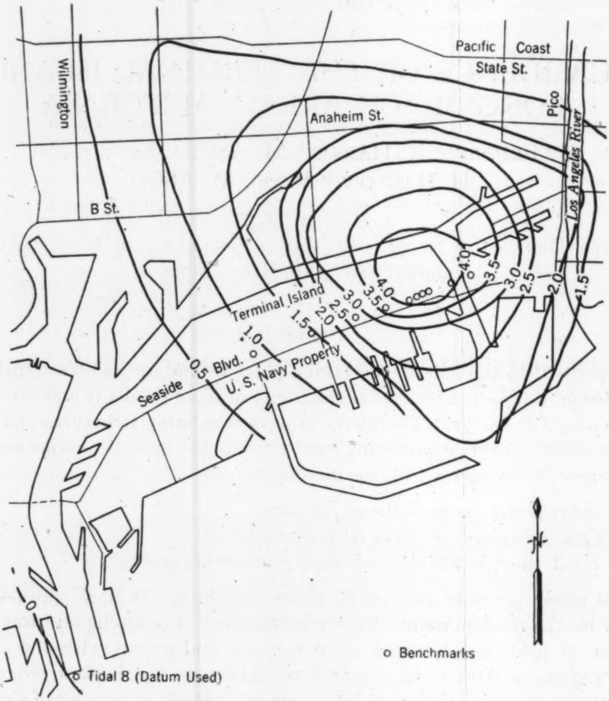

Pacific Coast
State St.
Wilmington
Anaheim St.
Pico
Los Angeles River
B St.
Terminal Island
Seaside Blvd.
U. S. Navy Property
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
o Benchmarks
o Tidal 8 (Datum Used)

FIG. 1.—CONTOURS OF SUBSIDENCE—1945

Subsidence at various bench marks in the affected area during different periods was determined, in general, by subtraction of corresponding elevations. Those for 1933 were taken as representing zero subsidence, because study of the Long Beach outer harbor tide gage record, as well as of frequent levelings by the Southern California Edison Company, reveals the 1933 leveling to be the most recent extensive survey prior to the beginning of the rate of subsidence prevailing in 1945 and 1946.

---

[3] Numerals in parentheses, thus: (1), refer to corresponding items in the Bibliography (see Appendix).

Fig. 1 shows contours of the subsidence that occurred from 1933 to 1945. Profiles taken on several dates along a representative line through the area are given in Fig. 2. Fig. 3 shows the increase of subsidence at certain bench marks plotted against time.

GEOLOGY OF THE REGION

The geologic history of the region has been amply described in previous reports and papers (5)(6). Generally, all the sediments in the upper 7,000 ft are poorly consolidated "shales" and sands of Miocene, Pliocene, Pleistocene, and recent origin. The Miocene and lower Pliocene strata are warped into a gentle anticline with its axis in a northwest-southeast direction, forming an elongated dome whose highest parts are close to the east end of Terminal Island. These strata are all highly faulted, and form the structure of the Wilmington oil field, at present one of the largest producing fields in California. The faults have been traced upward to an unconformity at the base of the Upper Pliocene, or Pico, formation, about 2,000 ft below the surface, but not above this point Varying thicknesses of beds in the underlying Repetto formation indicate that the motion along these faults was still occurring until late Pliocene times, but did not continue into later periods. The upper 1,800 ft of sediments generally contain more or less fresh water in the voids, and have been the source of fresh water supply to residents of the basin for many years.

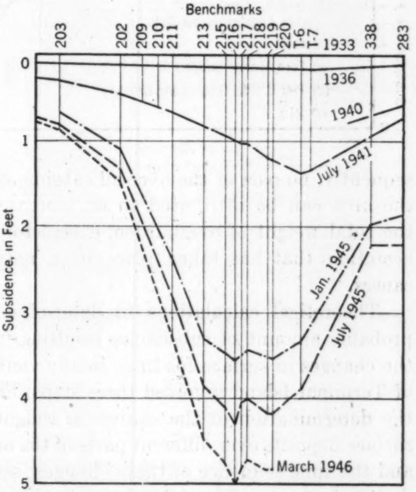


FIG. 2.—SUBSIDENCE ALONG SEASIDE BOULEVARD

FACTORS CONTRIBUTING TO THE SUBSIDENCE

At least five major sources of settlement operating in this area have contributed to the observed subsidence in varying degree. They are: (*a*) Changes in the land surface from filling and dredging; (*b*) pumping of water for industrial and domestic uses; (*c*) pumping of water during dewatering of the site of dry dock No. 1, United States naval dry docks; (*d*) pumping of oil from five major oil zones in the Wilmington oil field; and (*e*) tectonic forces. Any study of this problem resolves itself finally into an opinion based on factual observation and accepted principles of statics and mechanics, concerning the relative importance of these possible sources. Therefore, each of them is discussed.

Filling and Dredging Operations.—Development of the Los Angeles and Long Beach harbor areas has been in progress since about 1900. At that time, Terminal Island, then called Rattlesnake Island, was a low sand spit. It has been gradually enlarged by dredged fill to several times its initial size, and was eventually built up to about El. +13. However, as far as is known, no fill materials have been brought by man into this area from any distance. Consequently, no part of the over-all subsidence of the area can be attributed to an increase in the total weight of overburden, except for the accretion that has taken place from natural causes.

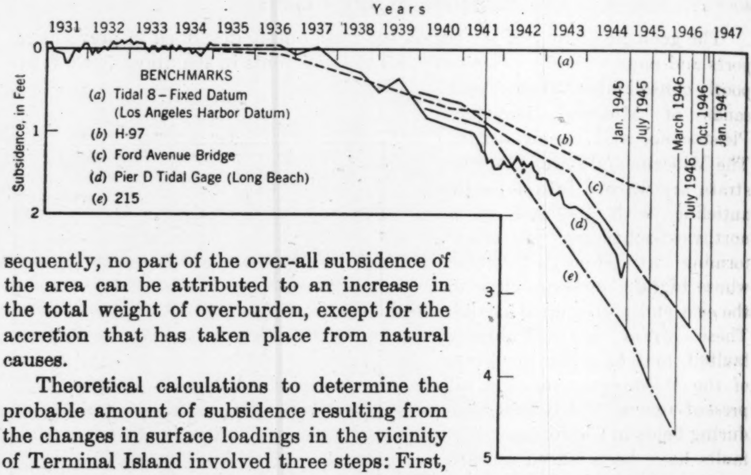


FIG. 3.—SUBSIDENCE OF TYPICAL BENCH MARKS IN PERIOD 1931–1947

Theoretical calculations to determine the probable amount of subsidence resulting from the changes in surface loadings in the vicinity of Terminal Island involved three steps: First, the determination of the change in weight of surface deposits over different parts of the area, and the time sequence of these changes; second, the total consolidation of underlying compressible strata that should ultimately occur; and, third, the influence of time effects during the consolidation, which reflect the proportion of the settlement that would cause observed movement of surface bench marks. From a study of many consolidation tests on clays and loams by several investigators (7)(8), it was concluded that, of the total subsidence measured, perhaps in the strip along the seaward side of Terminal Island, about 0.2 ft could be attributed to surface loading.

General Effect of Pumping Liquids.—If the ground-water table is lowered, all the sediments between the initial and final ground-water surfaces will experience a loss of buoyancy, and will rest more heavily on the underlying soil skeleton, which will eventually compress. The compression may not occur suddenly, since reduction in volume requires the escape of excess water in the voids. According to the theory of consolidation advanced by Karl Terzaghi, M. ASCE, the time required varies as the square of the thickness of the stratum undergoing compression. Since the more compressible strata are generally the fine-grained silts and clays, which may exist in layers between more pervious sand beds, time is needed for such a volume change before a static equilibrium is reestablished.

If two or more pervious beds are separated by continuous impervious clay strata, the water pressures in the voids of the lower bed may not conform to the ground-water table in the upper bed (9). Also, if the water pressure in the lower or confined aquifer is reduced, the pressure and level of the water in the surface aquifer will not be affected (see Fig. 4). However, the lower aquifer will undergo compression throughout its thickness.

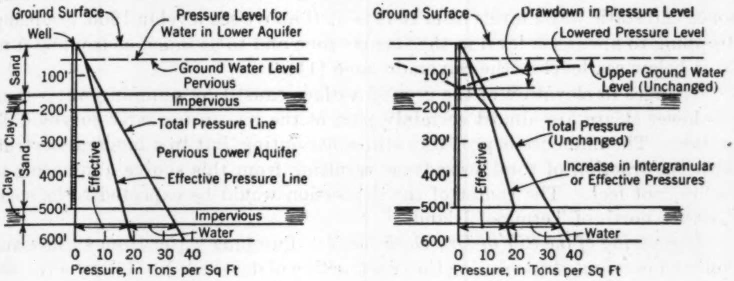

Ground Surface
Well
Pressure Level for Water in Lower Aquifer
Sand
100'
Ground Water Level
Pervious
Impervious
Clay
200'
Total Pressure Line
300'
Sand
Pervious Lower Aquifer
400'
Effective
Effective Pressure Line
Clay
500'
Impervious
Water
600'
0 10 20 30 40
Pressure, in Tons per Sq Ft
Ground Surface
Drawdown in Pressure Level
Lowered Pressure Level
Upper Ground Water Level (Unchanged)
100'
200'
Total Pressure (Unchanged)
300'
Increase in Intergranular or Effective Pressures
400'
Effective
500'
Water
600'
0 10 20 30 40
Pressure, in Tons per Sq Ft

FIG. 4.—PRESSURES IN SOILS
(*a*) Before Pumping
(*b*) During Pumping from Lower Aquifer

From the foregoing, it is seen that there are two distinct effects that may result from the pumping of a liquid from the ground:

1. Loss of buoyancy of the soil particles. This occurs if the ground-water surface is lowered, causing compression of all underlying strata.
2. Increased intergranular pressures in a pervious zone confined between impervious layers. This occurs if the liquid pressure in the confined zone is reduced, and results in compression of the confined zone and of adjacent parts of the "aquicludes" (impervious layers) at top and bottom.

The same general principles apply to the effect of pumping any liquid—whether water or oil.

Water Pumping for Industrial and Domestic Uses.—Fresh water in the Long Beach-Wilmington coastal plain is obtained from two water-bearing zones. The upper or Gaspar zone occurs in the surface deposits of recent age and consists of from 100 ft to 200 ft of alluvial deposits of sand and gravel interbedded with silt and clay. It includes both surface waters and confined waters. The lower or Silverado zone is a confined aquifer of Pleistocene deposits of sand and gravel with intervening layers of silt and clay. The zone varies from 150 ft to 500 ft in thickness. It varies in depth, being from 200 ft to 900 ft below the land surface in the Wilmington section.

The Newport-Inglewood fault forms a partial barrier to the flow of underground water. The area to the northeast of the fault was originally highly artesian, deriving pressure from water entering the pervious strata in the foothills of the mountains which surround the basin. The area to the southwest of the fault was probably only slightly artesian originally. Water-bearing zones in this area are replenished mainly by underflow across the Newport-Inglewood fault, through imperfections in the fault barrier. As a result, the effect on lowering the ground-water levels by pumping is much more noticeable in the coastal area south of the fault than in the areas to the north.

Development of ground water in the Los Angeles coastal plain began about 1870 (10). Records show that wells drawing from the Gaspar and Silverado zones had static water levels from 10 ft to 30 ft above sea level in 1903, gradually dropping to about sea level in the Gaspar zone and to as much as from 60 ft to 70 ft below sea level in the Silverado zone (11).

Changes in elevation of the ground surface caused by pumping water from the lower strata are almost certainly part of the reason for early survey difficulties. The influence is probably still contributing, but to a moderate extent, with the fraction of total subsidence resulting from this source amounting to inches, not feet. The center of the depression would be expected to lie some distance north of Terminal Island.

Dewatering of the Site of Dry Dock No. 1.—Pumping water from surface and confined pervious strata during the construction of dry dock No. 1 drew down the pressure levels in these beds from sea level to about El. −75, based on mean lower low water datum. The pumping covered a period from June, 1941, to April, 1942. Maximum drawdown in water levels did not occur until some weeks after pumping was begun. After pumping was stopped, readings on near-by wells indicated that the pressure levels rapidly rose to their initial elevations, close to mean sea level.

Water levels in wells in the upper aquifers were affected varying amounts as far north as 2 miles. Local subsidence was observed in the affected area, varying with the drawdown in pressure level, roughly in inverse proportion to the distance from the dock site. The Southern California Edison Company, whose Long Beach steam station is about 2,000 ft northeast of the dock, installed fifteen observation wells, some 35 ft deep and some 150 ft deep, to measure the reduction in water pressure level in the two principal pervious strata of the local part of the aquifer (7). Observations were also made of the motion of bench marks around the plant.

Within a radius of 2,000 ft of the dock site the ground settled as much as 0.45 ft around the casings of many oil wells. In most instances, the construction of the derrick foundations permitted the ground and concrete base slabs to settle around the casings without appreciable damage to either casings or concrete. There were a few occurrences of cracked base slabs, and damage was reported to at least one well casing. Differential movement between the ground surface and the casings was observed only during the period of pumping at the dock site. Prior to and since that time, the oil well casings have apparently behaved as if they were part of the ground surface.

To estimate how much of the total subsidence was caused by dewatering the dock site, the theoretical consolidation resulting from the effect of the observed drawdown of liquid pressure levels was computed.

From a study of the stratification and drawdown records, it appears that the substrata affected by the pumping consist of sands and gravels of the Gaspar aquifer, extending in depth from about El. −70 to about El. −270. The silty clay materials bounding, and within, the aquifer were assumed to have a coefficient of compressibility (C_c) of 0.144. Calculated contours of equal subsidence resulting from this pumping appear consistent with the measurements of motion around well casings. When pumping was stopped, in the spring of 1942, water levels rose to their previous values, and the consolidation ceased. In fact, a slight rebound, or increase in volume of the consolidated strata, occurred. Therefore, this settlement is considered to represent merely an incident in the general subsidence phenomenon in the area, and its effect was felt solely in the period from June, 1941, to April, 1942.

Pumping from the Wilmington Oil Field.—The mechanics of liquid flow during production from an oil well are generally the same as those for pumping water from a confined aquifer. The oil sands forming the equivalent of the aquifer are bounded top and bottom by impervious layers of "shale," or silt and clay compacted by the weight of the overburden. The oil in the sand voids is initially subjected to pressures which depend mainly on the depth of the strata below sea level. As oil is withdrawn from the well, there is a pressure reduction, which produces a gradient toward the well from all sides. This gradient depends on the rate of pumping, the permeability of the sand, and the extent and uniformity of the stratum as it affects the resistance to liquid replacement of the oil withdrawn. In oil wells the flow is complicated somewhat by the presence of large amounts of gas, either free or dissolved in the oil, and by the presence of at least two liquids, oil and water, with different physical properties (12).

As with pumping water from a confined aquifer, reduction in the liquid pressure in the oil sands produces a compensating increase in the intergranular or effective pressures in the soil structure within the stratum affected. Compression of the oil zones results. The amount of compression is influenced by the amount of decrease in liquid pressure and by compressibility of the materials within and bordering the affected zone, at top and bottom.

Compression of oil zones has frequently resulted in subsidence of the ground surface above ((5)(13)(14)(15) and unpublished reports). It is significant that, in the Los Angeles basin, noticeable subsidences have been recorded by first order leveling in the areas of a number of oil fields. Among these are the Wilmington, Signal Hill, Santa Fe Springs, Playa del Rey, Inglewood, and Huntington Beach fields. However, first order levels have not been run across many fields sufficiently often to detect possible subsidence. The relation between the amount of the unchecked compression of the oil zones and the corresponding surface subsidence depends on the structural strength of the sediments in between. It is quite common for extensive oil fields to experience no noticeable surface subsidence. On the other hand, subsidence of large areas from any cause is seldom noticed in inland areas. Along the seashore, the relatively constant sea level provides an automatic reminder whenever subsidence of appreciable amount occurs.

The Wilmington oil field was discovered in 1932, but little attention was paid to it until the last half of 1936, when a carefully cored and tested well revealed the presence of four major oil zones. The drilling boom started in the Los Angeles harbor area thereafter, spreading to the Long Beach harbor area in 1937 (6)(16)(17). Maximum production of 100,000 bbl of oil per day was reached in February, 1938, dropping gradually to about 82,000 bbl per day in 1940, and then rising again to around 100,000 bbl per day by 1944. The field's oil production rate in 1945–1946 is the highest in California by a wide margin. This time sequence agrees generally with the occurrence of most of the subsidence, as is apparent from a comparison between Fig. 3 and Fig. 6. The anticlinal structure of the oil field shows a remarkable correlation with the contours of equal settlement.

To ascertain the importance of oil pumpage from the Wilmington oil field in producing settlement, quantitative studies were made in 1945 based on actual measurements of the several factors governing consolidation of sediments in the oil zones. These factors consist primarily of the stratification of the sands and shales, the compressibilities of the sands and shales, and the reduction in liquid pressure of the oil and water in the voids after different periods of time.

Stratification was determined from oil well electric logs and coring records. Fairly detailed information is available and has been plotted on contour maps and profiles by the oil companies and by various other agencies interested in the field. The general structural characteristics are shown in Fig. 5. There are five major zones: Tar, Ranger, Upper Terminal, Lower Terminal, and Ford, in that order, varying in depth from about 2,200 ft at the top of the Tar zone to about 4,700 ft at the top of the Ford zone. The basement complex of igneous rock, a schist, is encountered at about 6,500 ft below mean sea level. Each zone varies in thickness to some extent; but except for faulting the differences in stratification between one point and another are remarkably small. The faulting, however, represents the most serious complication. A series of faults extending in a roughly north-south direction divide (see Fig. 5) the field into five major fault blocks, which are themselves broken by innumerable smaller faults,

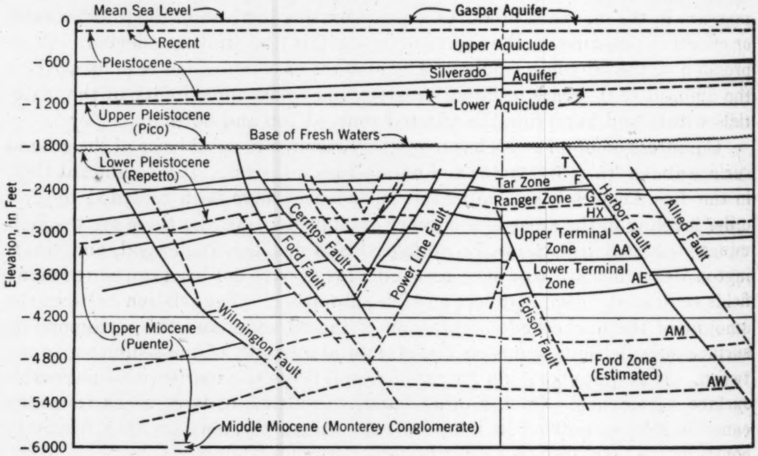

Mean Sea Level
Gaspar Aquifer
Recent
Upper Aquiclude
Pleistocene
Silverado
Aquifer
Lower Aquiclude
Upper Pleistocene
(Pico)
Base of Fresh Waters
Lower Pleistocene
(Repetto)
T
Tar Zone
F
Ranger Zone
G
HX
Harbor Fault
Allied Fault
Cerritos Fault
Ford Fault
Power Line Fault
Upper Terminal
Zone
AA
Lower Terminal
Zone
AE
Edison Fault
Upper Miocene
(Puente)
Wilmington Fault
AM
Ford Zone
(Estimated)
AW
Middle Miocene (Monterey Conglomerate)
Elevation, in Feet
0
-600
-1200
-1800
-2400
-3000
-3600
-4200
-4800
-5400
-6000

Fig. 5.—Section through Wilmington Oil Field

so that most cross sections of the field do not attempt to trace all details of the complex pattern. The locations of faults are determined mostly by inductive reasoning, through study of the oil well electric logs (18).

Fluid pressures in oil wells constitute one of the most important subjects for study by oil producing companies. Measurements of flowing and shutoff pressures are made to determine the productivity of the well, which enables estimates to be made of future production, and also data to be obtained for use in operation to insure maximum total production from the field. The usual custom is to shut off the well for 24 hours or longer before taking the measurements, to allow time for the pressure to build up close to the maximum. A pressure recording instrument is lowered into the well, and readings are taken at intervals so that a pressure-depth curve can be plotted.

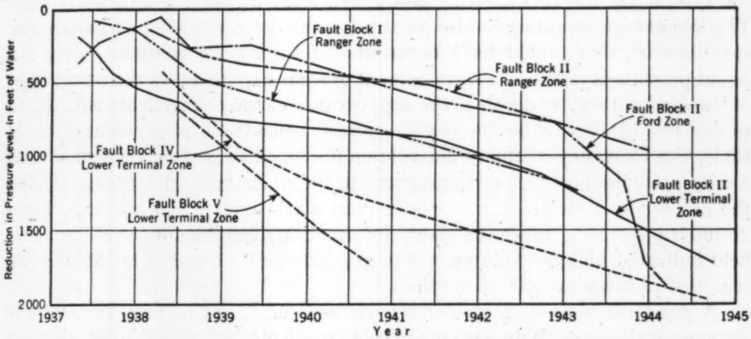


FIG. 6.—REDUCTION IN OIL ZONE PRESSURE LEVELS FOR TYPICAL WELLS

Initial pressures in the field, as determined from a large number of initial measurements on early wells, were approximately the same as if the underground liquids were composed of salt water with the surface at mean sea level. Continued pumping has generally caused a steady reduction in the pressures. If pressure at any depth is reduced to terms of pressure level, or the level at which a column of salt water would stand in static equilibrium in a tube whose lower open end is inserted in one of the confined sands of an oil zone, then it may be stated that the pumping has caused a gradual reduction in the pressure level below the initial mean sea level elevation. Typical curves showing pressure levels at various times for different wells are shown in Fig. 6. Pressure levels in the various zones are not the same, because of the impervious shales separating them. It has also been observed that the pressure levels on opposite sides of a fault are likely to differ, the displacement along the fault plane apparently having caused a breakdown of the normal sand structure and the formation of relatively impervious materials, which act as a barrier to the flow of liquids.

Pressure relations in the oil zones are complicated to some extent by the presence of gas. Initially, gas in the Wilmington field is believed to have existed in a near saturated solution, without any appreciable free gas cap. However, as pumping continued, pressure reduction has caused the formation of some free gas, which may have accumulated in gas caps at the crest of the structure and along faults in some zones. The near saturated condition of the original oil is indicated by the fact that wells originally "came in" as gushers, the gases rapidly expanding out of the solution in the casings, carrying the oil upward in the typical gas lift action. Gas production from the field is still fairly high. The free gas affects intergranular pressures generally by causing a smaller pressure increase than would have occurred had the gas been absent, since the liquid pressure is reduced. Development of a gas cap increases the unit weight of the sand within the depth of the cap, because of the change from submerged to dry weight; but, since the gas is confined beneath an impervious dome of shale, the buoyancy of the liquid volume displaced by gas of very small unit weight produces lower effective pressures in the strata below the shale. Unless the gas cap is quite thick, its presence has a relatively small effect on intergranular pressures. Since the scanty information available for the Wilmington field indicated no large volume of free gas, its effect on consolidation was disregarded in these subsidence studies.

A great many pressure measurements were made in the first five years of development of the Wilmington oil field, particularly by the large property owners. There are many smaller operators, particularly in the northwest part of the field, known as the "Townlot Area," who have made practically no pressure measurements in that section, although there has been considerable pumping since early in the field's history. Also, pressure measurements by the large operators were less frequent after the beginning of World War II. Since insufficient measurements were available for 1945, data were also included for measurements made in 1943 and 1944, which were given the same weight as the 1945 measurements, in preparing pressure contour maps for areas and zones where the later data were not available. Thus, discrepancies were introduced between calculated and observed subsidence for 1945, as will be noted hereafter.

Compressibility of sedimentary materials at the pressures existing in oil zones several thousand feet below the surface is a matter about which little information is available (19)(20)(21). To obtain data more applicable to this problem, it was decided to perform tests on actual samples of materials taken from oil wells in the Wilmington field. For several reasons, shale cores only were selected for testing. In the first place, shales, the product of silts and clays, composed of fine-grained particles, were expected to be much more compressible than sands. Previous tests had demonstrated fairly conclusively that sands, and particularly sandstones, show extremely small volume change when subjected to very high pressures. Secondly, relatively undisturbed shale cores, suitable for testing, could be obtained with much less difficulty than sands, because of their cohesive quality. Most of the sands from the Wilmington field are quite friable, and break up easily.

It was also felt that, rather than adopt triaxial tests, the procedure would more nearly duplicate the consolidation process in nature if the applied load were confined to the vertical direction, at the same time preventing lateral volume change, as is essentially the usual practice for standard consolidation tests on silts and clays. In most consolidation phenomena, where the load is applied to a relatively wide area in proportion to the thickness of the compressing stratum, the volume change is predominantly vertical. This condition can be approximated in the laboratory by subjecting a sample to increased vertical pressure, at the same time restraining it against lateral expansion.

Available samples consisted of cores drilled from various oil wells. Most of these cores were several years old, having been obtained during the period when the oil companies were exploring the field. In a few instances, fresh cores were obtained from wells where drilling was currently in progress. The cores varied in size from 1½ in. to 5 in. in diameter.

Equipment used in performing the tests was designed to accommodate two sets of samples simultaneously. The apparatus, which is pictured both in assembly and in detail in Fig. 7, employs a manually-operated screw-driven jack for applying the load, and a beam gage for measuring the load. Porous stones, placed on both ends of each 1-in.-thick sample, enabled application to, and drainage of water from, the samples. Two sets of three equally spaced 1/10,000-in. micrometer dials were used to measure the compression of the samples. In determining the reduction in thickness of each sample from the

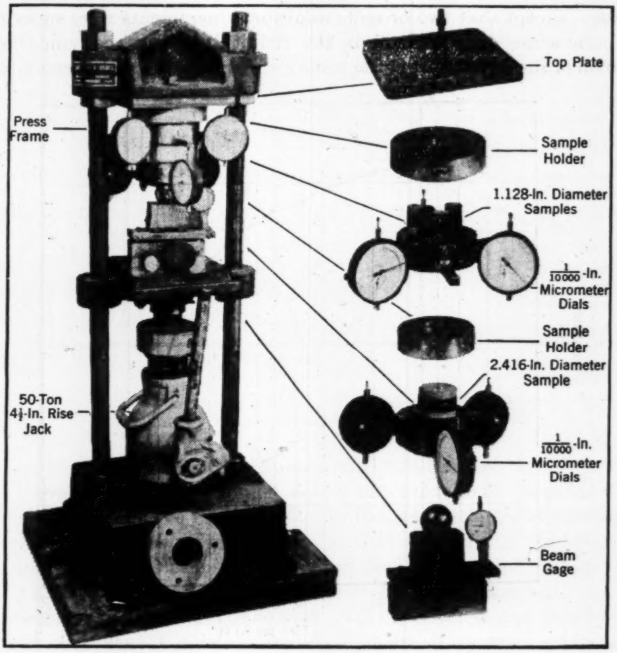

Top Plate
Press Frame
Sample Holder
1.128-In. Diameter Samples
1/10000-In. Micrometer Dials
Sample Holder
2.416-In. Diameter Sample
50-Ton 4½-In. Rise Jack
1/10000-In. Micrometer Dials
Beam Gage

FIG. 7.—ASSEMBLY AND DETAIL OF CORE TESTING APPARATUS

average of the dial readings, it was necessary to measure the deflections occurring in the porous stones and to make appropriate adjustments.

Samples to be tested were first placed in a saturated atmosphere to reestablish in them, as nearly as practicable, a saturated condition. They were then accurately trimmed in a lathe—all weathered or damaged parts being removed in the process—to fit the internal diameters of the sample holders, and the ends were faced to provide samples of the desired thickness. After machining, the samples were kept in a saturated atmosphere until ready to be placed in the testing apparatus.

Immediately after application of a small initial load, screws holding the sample holders in position during the setting-up process were released, and water was applied to the top and bottom of the samples through the porous stones, which were kept constantly wet throughout the remainder of the test. Load was applied in a series of increments, with each increment being twice its predecessor, except that two or more additional increments were superimposed on the basic series of increments in the vicinity of the preconsolidation load. On removal of the samples from the testing apparatus, the moisture content, dry density, and the specific gravity of the soil grains were determined for each sample. A typical curve of pressure versus voids ratio data obtained from the shale consolidation tests is shown in Fig. 8, for a 2.416-in. diameter sample. Typical compression-time curves are shown in Fig. 9. Each point in Fig. 9 represents the average of a set of three dial readings, expressed in terms of the

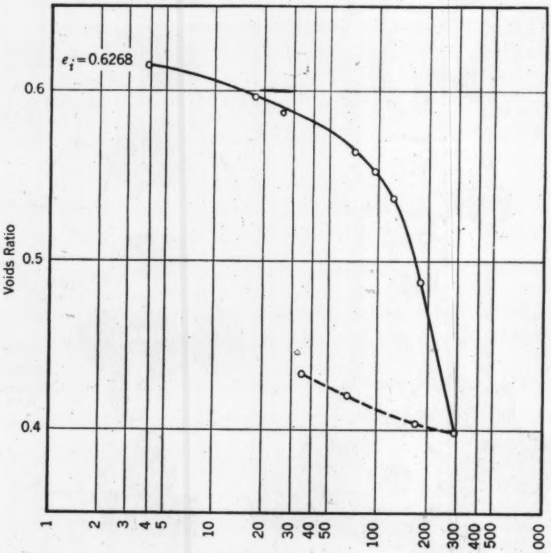


FIG. 8.—TYPICAL CONSOLIDATION TEST OF SHALE

percentage of the total deflection observed at the end of the first 1,440 min during which a particular load was applied.

For the highest pressure to which two of the samples were subjected, it was observed that some of the material was squeezed by plastic flow into the space between the porous stones and the sample ring. In these instances, in arriving at the straight-line approximation, a correction was made to the voids ratio for the last point on the curve in proportion to the amount of material displaced. This plastic flow probably would have been avoided if the diameter of the porous stones could have been made closer to that of the cylindrical openings in the testing rings. No other evidences of plastic flow were observed.

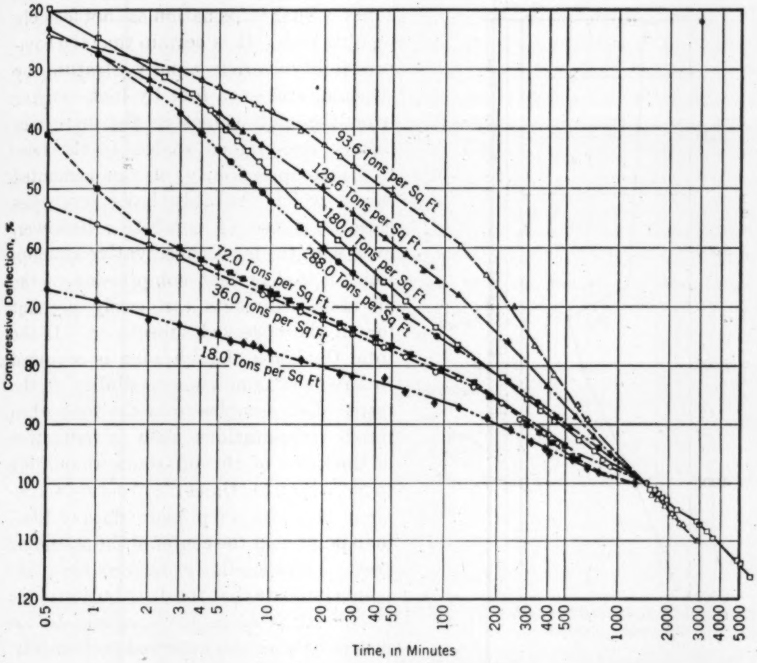


FIG. 9.—CONSOLIDATION TESTS OF SHALES

For comparative purposes, the compressibility of each sample can be defined in terms of a coefficient (C_c) determined by the slope of the assumed initial compression curve, or the change in voids ratio for a tenfold increase in pressure. The initial compression curves were assumed to be tangent to the steepest sections of the test curves, and to represent an approximation to the probable compression curves of the shales in their original condition in the ground. If these compression coefficients are plotted against depth, a definite variation is seen to exist (Fig. 10). The inference is that the shales encountered near the contact between the Lower Pliocene and the Upper Miocene deposits, represented by the

Tar, Ranger, and Upper Terminal zones, are more compressible than are materials at greater or lesser depths, under the same conditions of loading.

To compute theoretical consolidation accurately, it would be necessary to know in minute detail the variation in compressibility of all strata encountered by a vertical section through the oil zones. Such information cannot feasibly be obtained. It is certain that the compressibility varies rapidly stratum by stratum, and even foot by foot, ranging from the high values in the more uniformly fine-grained shales, to the relative incompressibility of the cemented sands, which probably have properties close to those of concrete. However, even with the most conservative assumptions, it is clear that compression of the oil zone sediments can easily account for all the observed subsidence. If the total thickness of each zone is assumed to have the same compressibility as the tested shale samples from the zone, then rough computations show a reduction in thickness of the oil zones amounting to more than ten times the observed settlement, based on the pressure data of 1945, for a point near the center of the subsiding area. Data similarly derived for other points indicate that the distribution of the consolidation corresponds closely to the distribution of the observed settlement.

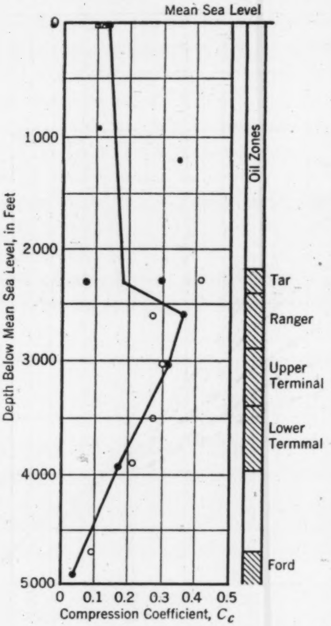


FIG. 10.—COMPRESSIBILITY OF WILMINGTON OIL SHALES IN RELATION TO DEPTH

An estimate of the future course of the subsidence was made, based on the following procedure: Ultimate drawdown of pressure levels in each oil zone was estimated, and a correction factor to be applied to the total thickness of the oil zones was computed. This factor was determined by deducting from the observed subsidence of a point near the center of the depression the computed settlement from water and dock site pumping, as of 1945. The difference was assumed to represent the effect of oil pumping to that time. A constant percentage of the thickness of each zone was chosen so as to make the calculated consolidation at that particular point equal to this difference. Using this factor, the same for all zones over the entire area of the field, together with the average compressibility coefficients, different for each zone, and the total thickness of the zones, which varied over the area of the field, consolidation was computed at some two hundred points. These computations were made using the pressure data of 1945 and also using estimated pressures that may occur in the future. Theoretical subsidence from oil pumping, in 1945, is shown by the contours of Fig. 11.

The many factors that influence the theoretical subsidence computed in this manner perhaps warrant some discussion. In the first place, the consolidation computed for the oil zones was assumed to produce an equal surface subsidence, except for smoothing out local irregularities. This assumption neglects the effect of possible lateral stress changes, which probably may result from the structural strength of the deep beds. The domed structure of these beds itself may result in a certain amount of arching action, which tends to transfer laterally the vertical loads applied near the center of the field, relieving the centrally located shales of some of the increased pressure they would otherwise experience. This action may reduce not only the local irregularities, caused by local variation in drawdown of the liquid level, but also the total surface subsidence at the center of the depression, spreading it over a wider area. However, the same errors from this source exist in the calculated 1945 consolidation as in the calculated ultimate consolidation. Since the predictions for the future are derived from the difference, the net error should be relatively small. This factor may partly explain the lack of agreement in detail between the contours for calculated oil zone consolidation (Fig. 11) and the contours of observed subsidence (Fig. 1), neglecting the relatively small and uniform consolidation from water pumping.

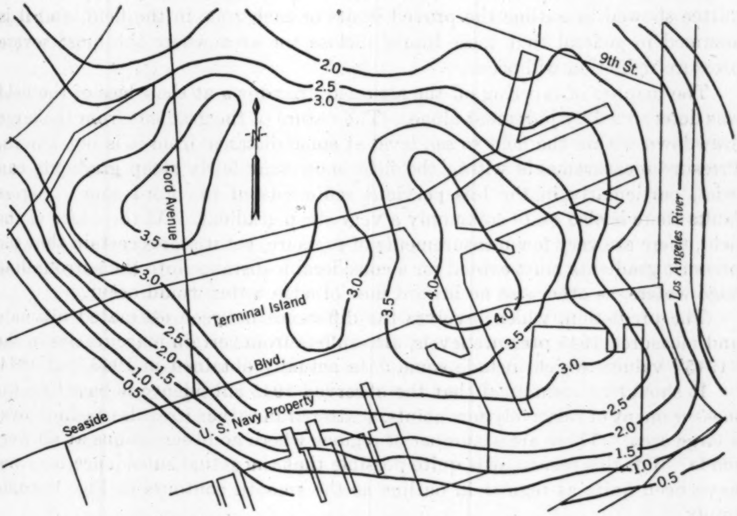


FIG. 11.—CALCULATED CONSOLIDATION OF OIL ZONES—1945

The estimates of future drawdown in each of the oil zones probably constitute the greatest single source of error in the estimate of future subsidence. They were obtained in a purely arbitrary manner, but were based to some extent on past experience and on certain limiting factors.

Leading petroleum engineers and geologists familiar with the Wilmington field agree that eventually, as a result of pumping, oil zone pressures will be greatly reduced. In some oil fields, as oil is withdrawn, edge waters encroaching from the sides of the structure replace it freely enough to prevent extreme pressure reductions in the oil. In such cases, the oil reservoirs may be completely depleted with only moderate reduction of the oil pressure level—the rate of oil removal governing to a large degree the amount of pressure reduction. In the Wilmington field, however, it appears that reasonable pumping rates cannot be maintained without gradual reduction in the liquid pressures, since experience shows that edge-water replacement is not sufficiently rapid. This may be caused by low permeabilities in the oil bearing strata, by fault barriers, or by confinement of these strata within relatively impervious layers.

With these factors in mind, it was concluded that ultimate liquid pressures may very likely be reduced to about 200 lb per sq in.—equivalent to about 400 ft of water head at the top of each zone. The distribution of this drawdown over the area of the field is an uncertain matter, but it appeared that the report prepared in 1941 by the Wilmington Field Engineering Committee contained sufficient data for reasonable approximations. Maps prepared by this committee showed in outline the proved limits of each zone in the field, and it is assumed in general that these limits enclose the area where the most severe pressure reduction will occur.

The manner of tapering off the assumed drawdown at the edges of the field was determined by judgment alone. The nature of the gradient from the large drawdown within the field to sea level at some distance from it is not known. Pressure measurements within the field show that fairly steep gradients can exist, particularly in the less pervious sediments of the Ford zone. Across faults there is also quite commonly a very steep gradient. At the edges of the field, there are very few measurements of pressure, yet it seems certain that the pressure gradients must extend for a considerable distance into the surrounding edge waters, as otherwise no inward flow of edge water would occur.

The prediction, which involves the difference between estimated ultimate and measured 1945 pressure levels, also suffers from certain inaccuracies in the "1945" values, which include some data actually obtained in 1943 and 1944.

It should be mentioned that the observed 1945 subsidence is based on the measurement of relatively few points, scattered in rather irregular fashion over a large area. There are a number of regions where no observations at all were made. For this reason, it is quite possible that the actual subsidence may not have been quite as regular in outline as the smooth contours of Fig. 1 would imply.

Compressibility of the oil zone sediments should not change appreciably with time, in the relatively brief geologic interval of this subsidence. Therefore, if the true average compressibility were known for all zones at a given point, and the increase in drawdown after 1945 were the same for all zones, future consolidation could be predicted quite accurately. However, pressure reductions to be expected in the future vary in the different zones. For example, in fault block III, near the center of the depression, a much larger proportion of the ultimate pressure reduction has already taken place in the Upper Terminal and Lower Terminal zones than has occurred in the Tar, Ranger, and Ford zones. Thus, the proportion of the total consolidation caused in each of the zones, for a given increment of pressure reduction, becomes important. This proportion depends on the total thickness of compressing materials in each zone and on the average compressibilities of those materials.

Thicknesses of each of the zones were determined principally from the contour maps prepared by the Wilmington Field Engineering Committee. These maps showed the variation in elevation of the top of each zone. The thicknesses of the Tar, Ranger, and Upper Terminal zones were taken as the differences between the elevation of the top of the zone and the elevation of the top of the next zone below. The Lower Terminal zone has not generally been productive as deep as the top of the Ford zone, and has not been pumped for this full thickness. The Lower Terminal zone was therefore taken to have an average thickness of 500 ft, based on cross sections prepared by the committee and others and on various electric logs. Similarly, the thickness of the Ford zone was taken as 750 ft. The proportion of each zone composed of shale layers varies over a considerable range, but averages about 50%.

Calculated consolidation, both for 1945 and for the future, was obtained for each zone by multiplying the full thickness of the zone by a correction factor of 8% to adjust the calculated change in thickness, for a point at the center of the depression, with the observed subsidence as of 1945. The surprisingly low ratio between observed and calculated unadjusted settlement may be explained by the facts that: (*a*) Total thickness of oil zones was used in the calculation; and (*b*) each zone was assumed to have the compressibility of the average of the tested shale samples. Thus, the presence of sands of low compressibility is disregarded as are the relatively incompressible sediments composed of shales mixed with sands, which affect the average greatly. The presence of many water-bearing strata that may not be greatly affected by pumping is also neglected, since well casings are not perforated in water zones. Also, as previously mentioned, the tested samples were chosen for their suitability for machining, were very fine-grained, and therefore were more compressible than the majority of shales examined at the oil companies' storage racks.

This method leaves much to be desired, when compared with the degree of accuracy usually sought in consolidation studies. However, it is believed to be sufficiently accurate for the purpose required, because most errors inherent in the theoretical approach are canceled by the adjustment to an already sizable consolidation, representing a major part of the possible total, which has been actually measured.

In all the calculations of oil zone consolidation, it was assumed that the time lag between the imposition of load, through reduction of the pressure level, and the corresponding decrease in voids ratio of the compressible shales has been relatively small. Inspection of the time-compression curves obtained in the laboratory tests (Fig. 9) shows that a majority of the curves do not have the same shape as typical time curves explained by the Terzaghi theory of consolidation (22).

In general, the shape of the time curves resembles the results of certain soil tests in which, because of the character and thickness of the sample, the so-called "secondary" time effects have practically concealed the "primary" effects, or those resulting from squeezing out water. In such cases, instead of the typical reversed S-shape on the semilogarithmic plot, the time curve may be straight, or even slightly concave upward.

The time required for primary consolidation is assumed to vary as the square of the thickness of the compressing layer. The time required for secondary consolidation is more or less an unknown factor, but it is generally held to be independent of the thickness of the layer. In other words, secondary consolidation may occur in nature, in a layer many feet thick, at the same rate as it occurs in the thin laboratory sample.

Those test curves that do display some similarity to the theoretical time curve indicate that for pertinent load increments 90% consolidation takes place in less than 400 min. On this basis, if the average thickness of individual shale laminations, between sand streaks, is assumed to be 2 ft, then 90% consolidation in the oil field would require about 290 days.

For the compression of oil shales considered in this study, many unknown factors influence the time rate. Even the average thickness of the compressible layers is complicated to evaluate, since there are many layers of varying thickness in each of the oil zones. Consequently, no attempt was made to calculate how many of the time effects result from primary compression, inasmuch as the laboratory curves show this influence to be minor compared to that of the secondary or plastic characteristics. The time required for these latter effects is a matter of days, instead of months or years, and it was neglected in computing theoretical consolidation at any particular time. Since the loads themselves are applied gradually, over a period of years, the error introduced by this procedure is probably not very large.

Influence of Tectonic Forces.—It is well known that, in the coastal area of California, motions of the earth's crust are constantly in progress. No attempt will be made to describe the many faults in the subsiding area and around it, or to describe the seismic history of the region (5)(13). It is obvious that in an area interlaced with active faults, where minor quakes are registered almost continuously at the near-by Pasadena Seismological Laboratory, movements along fault zones are more or less likely to occur as the result of slow relative motion of adjacent land masses.

As far as is known, there is no reliable method of estimating or timing the probable occurrence of movements resulting from tectonic forces. However, as far as evidence derived from observations at oil wells is concerned, it appears likely that movement along the many faults in the oil field area ceased in Pliocene times. During these movements, the oil field was generally lifted upward, with downward "cave-ins" of some of the central blocks in the field. A general upward movement of this section of the coast is evident also from the terraced contours of the San Pedro hills around Point Fermin. Geologically, this gradual uplift may still be in progress.

The area that might be affected by tectonic forces is likewise indeterminate. The Newport-Inglewood fault, which is the major known active fault zone passing through the Los Angeles coastal basin, lies some 4 or 5 miles northeast of the center of the subsiding area. Although it might appear that the subsiding area, which coincides with the Wilmington oil field, is a very highly faulted one, compared with other areas, it should be remembered that knowledge of the subsurface details in this area is highly developed from observations made at many hundreds of oil wells. There is no assurance that many other similarly faulted areas of the Pliocene and Miocene sediments do not lie undetected merely for lack of exploration.

In weighing the probability of tectonic forces causing a major part of the subsidence, several factors must be considered along with the known possibility of such movements in this region. One such factor is the lack of any marked increase in seismic activity in recent years in the Long Beach area. Since the rate of subsidence there has increased to a degree unprecedented in California history, if tectonic forces and movements of underlying rock masses were largely responsible, the area might properly be expected to develop a nest of shocks, as the earth stresses produced during such severe strains readjusted themselves. However, although innumerable shocks have occurred in Southern California, no such nest of shocks has been observed at Long Beach.

Another such factor is the absence of damage to oil well casings in the Wilmington field. If there had been any appreciable motion along the faults traversed by the wells, pinching and shearing off of the casings or puncturing of cement plugs should have taken place; and it seems highly improbable that all the relative motion along these inclined faults could have occurred at depths below the oil wells, without producing considerable displacements in the upper sediments. In this connection, experiences at Dominguez Hills in 1941 indicate that motions along faults of as little as a fraction of an inch were sufficient to collapse oil well casings passing through the faults (23). In many cases, motions of more than a foot were observed, shearing off casings completely—yet this condition occurred at an oil field where surface subsidence, if it took place at all, was so small as to go unnoticed. It is considered, therefore, that, although slow tectonic movements have occurred in the area in question, and probably are occurring at present, these movements are very gradual, possibly on the order of 0.04 ft per yr or less, vertically. They may be either upward or downward; and available facts are insufficient to determine which, or to establish the rate, concealed as it is beneath the very much more rapid subsidence phenomenon centered over the oil field.

CONCLUSION

Estimated ultimate subsidence, after 1933, from oil and water pumping is shown by Fig. 12, in which some of the irregularities in contours derived on a mathematical basis have been smoothed out, on the assumption that arching and other effects will in nature reduce abrupt changes and transitions in the over-all pattern. This contour map, by no means considered infallible, can serve as a guide for future construction in this area. The most vital conclusion brought out by the contours is that in all probability there will be no large-scale inundation of the easterly end of Terminal Island. Wherever present freeboards above extreme high tide level exceed 5 ft, the chances are that there will be no overtopping of the ground surface.

Nevertheless the possibility of differential movement should be considered in the design of any large structures to be built in the subsiding area. In general, since the depression in the ground surface covers a very wide area, with flat side slopes, the amount of differential motion to be anticipated in the length of most buildings will be very small. If the buildings are sufficiently flexible to withstand such motion without excessive stresses, no particular damage will result.

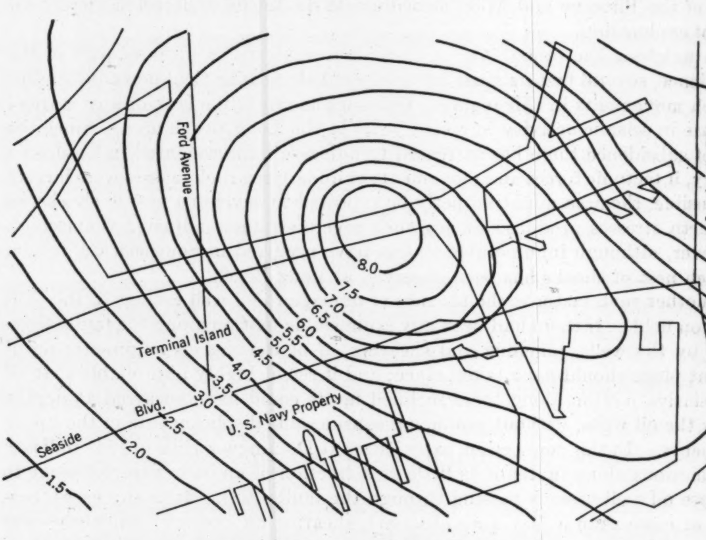


Fig. 12.—Probable Ultimate Subsidence

Continued subsidence will necessitate a restudy of the elevations and grades of certain facilities. The design of gravity sewers should be re-examined in the light of probable future differential changes in grade. Lack of proper drainage may result in some places as the subsidence continues. The same problem applies to surface grades. Such sources of future difficulties can thus be anticipated to some extent, and provisions can be made to insure continued satisfactory operation.

In conclusion, this study indicates that this vicinity should continue to serve as the site for industrial development. Present structures will continue to function satisfactorily if suitable preventive measures are adopted to avoid known possibilities of trouble; future structures can be designed to allow for probable future subsidence, which will continue for many years at a slowly reducing rate.

Acknowledgment

This paper is based on a report prepared for the Bureau of Yards and Docks, United States Navy Department, by Frederic R. Harris, Incorporated, Consulting Engineers of New York, N. Y., dated December 18, 1945, with a supple-

ment dated February 25, 1946. Invaluable assistance was rendered by R. R. Desmond, and C. M. Duke, Juniors, ASCE, who, together with the junior author, assembled the basic data examined during the investigation. Gratitude is due many individuals and agencies for their assistance in making available the necessary data—Capt. H. E. Wilson, Public Works Officer at the Terminal Island Naval Base, and members of his staff; Lt.-Comdr. A. S. Porter and Frank F. Mead and Trent R. Dames, Members, ASCE; Messrs. R. Winterburn and Butterfield of the Union Pacific Railroad Company; and many others. To all these sincere appreciation is offered.

APPENDIX. BIBLIOGRAPHY ON LAND SUBSIDENCE

(1) **"Earth Movements in California Determined from Apparent Variations in Tidal Datum Planes,"** by H. Leypoldt, *Bulletin*, Seismological Soc. of America, January, 1934, p. 63.

(2) **"Periodicity of Earth Movements in Los Angeles Harbor,"** by H. Leypoldt, *ibid.*, January, 1938.

(3) **"Surveys Show Long Beach, California, Is Moving Up, Down and Sideways,"** by J. R. McHenry, *Engineering News-Record*, March 26, 1942, p. 477.

(4) **"Variations in Levels 1919 to 1927, in Los Angeles Harbor,"** by G. F. Nicholson, *Bulletin*, Seismological Soc. of America, December, 1929, p. 200.

(5) **"Subsidence and Elevation in the Los Angeles Region,"** by U. S. Grant, *Science in the University*, Univ. of California, Berkeley, Calif., 1944, p. 129.

(6) **"Wilmington Oil Field,"** by R. Winterburn, in **"Geological Formations and the Economic Development of the Oil and Gas Fields of California,"** *Bulletin No. 18*, State Div. of Mines, Pt. 3, San Francisco, Calif., March, 1943.

(7) **"Report to Southern California Edison Company on Soil Explorations and Studies as to Effect on Edison Plants from Navy Construction Operations,"** by R. Labarre, Los Angeles, Calif., 1941 (unpublished).

(8) **"Report to Navy on Soil Explorations and Foundation Studies for Graving Dock at Fleet Operating Base,"** by D. R. Warren and F. J. Converse, Los Angeles, Calif., 1941 (unpublished).

(9) **"Compressibility of Artesian Aquifers,"** by O. E. Meinzer, *Economic Geologist*, Vol. 23, 1928, pp. 263–291.

(10) **"Development of Underground Waters in the Central Coastal Plain Region of Southern California,"** by Walter C. Mendenhall, *Water-Supply Paper No. 138*, U.S.G.S., U. S. Govt. Printing Office, Washington, D. C., 1905.

(11) **"Description of Water Wells in the Coastal Zone of the Long Beach-Santa Ana Area, Calif.,"** by J. F. Poland and others, U.S.G.S. Rept., Los Angeles, Calif., December, 1942 (unpublished).

(12) **"Ground Water,"** by C. F. Tolman, McGraw-Hill Book Co., Inc., New York, N. Y., 1937.

(13) **"Some Recent Changes of Elevation in the Los Angeles Basin of California and Their Possible Significance,"** by U. S. Grant and W. E. Sheppard, *Bulletin*, Seismological Soc. of America, April, 1939, p. 299.

(14) *Engineering News-Record*, June 10, 1937, p. 858.

(15) **"Local Subsidence of the Goose Creek Oil Field,"** by W. E. Pratt and D. W. Johnson, *Journal of Geology*, Vol. 34, 1926, pp. 577–590.

(16) **"Wilmington Oil Field,"** by W. J. Crown, in **"Summary of Operations, California Oil Fields,"** 26th Annual Rept. of State Oil and Gas Supervisor, Dept. of Natural Resources, Div. of Oil and Gas, San Francisco, Calif., 1940–1941.

(17) **"Summary of Operations—California Oil Fields,"** Annual Repts., Dept. of Natural Resources, Div. of Oil and Gas, 1932–1945, inclusive.

(18) **"Electrical Coring,"** by C. Schlumberger, in **"The Science of Petroleum,"** Oxford Univ. Press, Oxford, England.

(19) **"The Compressibility of Minerals and Rocks at High Pressures,"** by L. H. Adams and E. D. Williamson, *Paper No. 484*, Carnegie Inst. Geophysics Laboratory, Pittsburgh, Pa. (reprinted from *Journal of the Franklin Institute*, April, 1923, pp. 475–529).

(20) **"Measurements of Compressibility of Consolidated Oil-Bearing Sandstones,"** by C. B. Carpenter and G. B. Spencer, *Report No. 3540*, U. S. Bureau of Mines, U. S. Govt. Printing Office, Washington, D. C., October, 1940.

(21) **"The Effect of Pressure on the Migration and Accumulation of Petroleum,"** by F. M. Van Tuyl and A. C. Beckstrom, *Bulletin*, Am. Assn. of Petroleum Geologists, October, 1926, pp. 917–930.

(22) **"Theoretical Soil Mechanics,"** by K. Terzaghi, John Wiley & Sons, Inc., New York, N. Y., 1943.

(23) **"Los Angeles Basin Earthquake of October 1941 and Its Effect on Certain Producing Wells in Dominguez Field,"** by K. Bravinder, *Bulletin*, Am. Assn. Petroleum Geologists, March, 1942, p. 387.

(24) *Transactions*, ASCE, Vol. 111, pp. 346–349.

(25) **"Development of Underground Waters in the Eastern Coastal Plain Region of Southern California,"** by Walter C. Mendenhall, *Water-Supply Paper No. 137*, U.S.G.S., U. S. Govt. Printing Office, Washington, D. C., 1905.

(26) **"Development of Underground Waters in the Western Coastal Plain Region of Southern California,"** by Walter C. Mendenhall, *Water-Supply Paper No. 139*, U.S.G.S., U. S. Govt. Printing Office, Washington, D. C., 1905.

(27) **"Vertical Movement in the Los Angeles Region, 1906–1946,"** by Ernest J. Parkin, *Transactions*, Am. Geophysical Union, February, 1948, p. 17.

(28) *Transactions*, ASCE, Vol. 110, pp. 340–341.

(29) *Ibid.*, Vol. 112, pp. 437–438.

DISCUSSION

James Gilluly,[4] Esq., and U. S. Grant,[5] Esq.—It appears that this paper contains some fundamental assumptions that are incorrect and that are believed to have led to an unacceptable conclusion. The authors (under the heading, "Factors Contributing to the Subsidence: Pumping from the Wilmington Oil Field") state:

> "As with pumping water from a confined aquifer, reduction in the liquid pressure in the oil sands produces a compensating increase in the intergranular or effective pressures in the soil structure within the stratum affected. Compression of the oil zones results. The amount of compression is influenced by the amount of decrease in liquid pressure and by compressibility of the materials within and bordering the affected zone, at top and bottom."

The writers agree with this statement. However, they wish to point out that it is entirely unproved and highly improbable that the decrease in liquid pressure has been significant except within the oil sands. It is, therefore, gratuitous to assume that it has affected appreciably the interbedded shales.

The authors essentially attribute all the subsidence to compaction of the shales interbedded in the oil sands, and bounding them above and below. They were apparently influenced by an assumption that the sands were relatively incompressible and "probably have properties close to those of concrete." Good concrete, with a low voids ratio and permeability, can hardly be similar to porous, more or less incoherent, highly permeable oil sand, such as occurs in the Wilmington oil field, Long Beach, Calif. If these two materials were similar, there would be little oil and no appreciable flow of oil in the highly productive oil wells. Inasmuch as the authors state that the amount of compression is influenced by the "compressibility of the materials within and bordering the affected zones," it is surprising that they failed to test the compaction of the sandstones, because it is only within the sands that certainty is assured of the decline in fluid pressure on which the mechanism appealed to by them depends. They omitted any tests of the sands because they believed the shales to be much more compressible, which, of course, they are, and secondly, because it was easier to obtain undisturbed shale cores than sand cores. Both these reasons are believed irrelevant.

Tests of the sandstone cores by the writers showed them to be significantly compressible, both elastically and by plastic flow—the plastic flow accomplished presumably by movement of cement from points of high pressure to stress shadows between the grains. At any rate, whatever the mechanism of the flow, these tests indicate that the sandstones are sufficiently compressible to account for the entire subsidence without appealing to the very dubious addition of shale compression. Computations for the compression of the sandstones alone yielded a theoretical subsidence about twice the actual—a discrepancy readily explicable, as the pressure drop was necessarily measured at the well points and therefore must be far greater than the average decline through the inter-well areas of the oil sand. The authors' conclusion that the subsidence was due entirely to shale compaction is somewhat surprising in view of the fact that the writers' data and conclusions were available to them in an unpublished report to the Long Beach Harbor Department in 1945.

---

[4] Prof. of Geology, Univ. of California at Los Angeles, Los Angeles, Calif.

[5] Prof. of Geology, Univ. of California at Los Angeles, Los Angeles, Calif.

The correction factor of 50%, derived from the use of the writers' sandstone compaction data, is to be compared with the correction factor of 8% that the authors must use in order to reconcile the theoretical with the actual subsidence in accordance with their theory. In other words, the theoretical subsidence by the authors' theory is about twelve times the actual. They apply a pressure decline measured in the relatively permeable oil sands to a great thickness of shales that are known from geological evidence to have a very low permeability, and within which it is very doubtful that any appreciable reduction in fluid pressure has occurred.

It is obvious that the fluid pressure within the shales can decline only by loss of fluid from them. Drilling and electric logs of the wells show that, at Wilmington, as in most other sandstone reservoirs, the oil is confined to the sandstone beds and that the shale beds associated with them are saturated with water. That the shale beds have very low permeability, at least in the direction transverse to the bedding planes, is proved by the fact that this impermeability is responsible for the trapping of the oil through geological time. Furthermore, several water sands have been sealed off from the oil sands by these same, nearly impermeable, "aquicludes." With trivial exceptions, the casings in the Wilmingtion oil field have been perforated only in the oil sands. Accordingly, the pressure decline owing to oil production has primarily occurred within the sand beds, not within the shales. Certainly loss of pressure within the sands has set up a pressure gradient within the shales which must be losing water (and hence compacting by the mechanism advocated by Messrs. Harris and Harlow). It is highly probable, however, that this is a very slow process and that it has not contributed significantly to the subsidence.

The requirement of fluid continuity demands that only as the shales lose water may the pressure within them decline. Their extremely low permeability, transverse to stratification, indicated by the geological evidence, makes it seem improbable that the water loss could have been significant within the brief period of oil extraction—ten years. The argument need not rest, however, on these grounds; yet, this assumption would be much more likely than that made by Messrs. Harris and Harlow.

The water loss from the shales, upon which their compaction must depend, can be estimated in a way most favorable to the authors' hypothesis, by assuming that all the water produced with the oil came from the shales associated with the oil sands. This assumption is known to be incorrect, because much of the small water cut at Wilmington is produced from sand zones. If it originated in the shales, and not in the edge water or interbedded water sands, such as are revealed by electric logs, it constitutes the most favorable assumptions for their hypothesis. The data on water production are highly significant, for they show that there is no relation whatever between the amount of water produced from a given block of the oil field and the subsidence of the surface over that block.

Water production from blocks III, IV, and V of the Wilmington oil field, from its discovery to January 1, 1945, is shown in Table 1, in which the data in Col. 4 were supplied by C. L. Vickers, engineer, Long Beach Harbor Department. Data in Cols. 2 and 3 are from unpublished records in the office of the Long Beach Oil Development Company. It seems significant that the subsidence shows no relation to water production, for shale compaction is absolutely dependent on loss of fluid from the shale and this fluid must be water, not oil. These data, together with the geological evidence of very low permeability of the shales, seem to compel the conclusion that the contribution of shale compaction to the subsidence has been insignificant. This is, of course, directly opposed to the authors' hypothesis.

TABLE 1.—WATER PRODUCTION AND SUBSIDENCE IN THE WILMINGTON OIL FIELD, CALIFORNIA

Block No. (1)	WATER PRODUCTION		Volume of subsidence (cu ft) (4)	Ratio of water volume to subsidence volume (5)
	Barrels (2)	Cubic feet (3)		
III	202,642	1,134,800	114,670,000	0.0098
IV	6,305,140	35,308,000	101,923,000	0.346
V	2,675,445	14,982,500	103,322,000	0.145

It is obvious that with the decline in fluid pressure and the thicknesses of oil sands, both of which are maximal in the central parts of the oil field, their product, multiplied by any arbitrary constant called "compressibility," is necessarily a maximum in the center of the oil field. Thus, the fact that the authors found a theoretical subsidence roughly proportional to the actual is not surprising, even though they applied the mechanical parameters of a substance that probably did not undergo the postulated decline in fluid pressure, and failed to consider the mechanical properties of the sands within which the fluid pressure is known to have declined.

JACOB FELD,[6] M. ASCE.—There are so few authentic data concerning the extent and cause of surface subsidence resulting from liquid removal from the subsoil that this paper should be read with interest and studied with care. From a qualitative point of view, the authors undoubtedly show the cause of the major portion of the recorded subsidence; from a quantitative approach, the methods used do leave "much to be desired," and cannot become a model for the solution of similar problems.

A comparison of the contours when drawn to the same scale shows: In Fig. 1, the actual subsidence measured from 1933 to 1945; in Fig. 11, the calculated subsidence as of 1945; and, in Fig. 12, the probable ultimate subsidence.

This comparison indicates the following general observations:

---

[6] Cons. Engr., New York, N. Y.

1. The ultimate subsidence will be about 175% of the amounts measured in 1945; and
2. There is no apparent correlation between the data in Figs. 1 and 11, and the pattern of the predicated ultimate subsidence bears no relation to that of the computed values.

These conclusions are not consistent with the information given in other parts of the paper. The shape of the settlement time curves of the four bench marks shown in Fig. 3 does not indicate a decreasing rate of settlement during recent years. Quite the contrary conclusion can be reached from the graphs in Fig. 2, as well as Fig. 3, since the first foot of settlement required about 5 years, the second foot and third foot each occurred in $1\frac{3}{4}$ years (average), and both the fourth foot and fifth foot occurred in about 1 year—all referred to 1933 as datum.

The chief cause of such a lack of coordination seems to be the implicit reliance on the method used for computing expected subsidence. Drilling cores, several years old, when tested in the laboratory were assumed to be typical of geological zones, and were taken to have average thicknesses of 500 ft and 750 ft, respectively. The computations were corrected by multiplying the shale, sandstone, and other expected compressible layers by 8% (that is, a reduction in computed values by 92%) to make one spot agree with the measured settlement at that point. It might be more logical to assume that only one layer—the weakest—caused all the subsidence.

The discussion of actual and expected consolidation settlements from soil compression and from unwatering operations indicates values quite consistent with those measured in similar soil areas along the eastern seaboard. Filled areas covering salt marshes and silt where the soil water is affected by tides, although protected by tight bulkheads, showed settlements of about an inch for each 5 years. The area along the Harlem River in New York, N. Y., has been under observation for about 75 years, and the total subsidence in that period is more than 15 in. For nearly a year dewatering in this area to a depth of 35 ft (45 ft below ground level) caused a maximum, almost immediate, additional settlement of about 5 in. nearest the excavation, with no measurable settlement some 150 ft away. In another area of similar soil conditions about 1,000 ft from tidewater, a ground-water lowering of 45 ft (55 ft below street surface) settled the surface about 3 in. In both cases very little rebound of the surface was noted when the water table was permitted to rise to normal level.

The paper, as printed, is a summary of a much longer report which may contain more convincing proof that the quantative method used is reliable, and, if so, it is hoped that the authors will publish such proof in the closing discussion.

Hyde Forbes,[7] M. ASCE.—Subsidence is a phenomenon common to many localities in California, caused by the increase in ground-water pumping, since about 1918. In this presentation of the measurement of the phenomenon, the authors advance the conclusion that it is caused principally by the consolidation of shale members of the productive oil formations underlying the area. This latter view is open to question. The total subsidence shown in Fig. 12 is less than that known to have occurred during the same period at the southern end of San Francisco Bay (California), where there was no oil production.

---

[7] Cons. Engr. and Geologist, Palo Alto, Calif.

The paper enumerates the factors contributing to the subsidence of the area in question (with which there is no disagreement) and then places a measure on each in turn:

(*a*) About 0.2 ft is attributable to surface loading;

(*b*) The effect of pumping water from the coastal plain southwest of the Newport-Inglewood fault is summarized (under the heading, "Factors Contributing to the Subsidence: Water Pumping for Industrial and Domestic Uses") as "the fraction of the total subsidence resulting from this source amounting to inches, not feet";

(*c*) The subsidence caused by the dewatering of the dock site to El. −75, based on the theoretical consolidation resulting, was assumed to be limited to the time of such dewatering and to cease with the measurement in the spring of 1942 (using curve (*e*) in Fig. 3, this would be less than 1 ft); and

(*d*) The subsidence resulting from the consolidation of the oil measures to considerable depths because of the extractions of gases and fluid. Item (*d*) is cited as the major factor.

The writer will discuss these factors from the point of view of his knowledge of conditions, believing that the subsidence in the Terminal Island area, in particular, is due to three causes which, although incapable of exact measurement (but in order of magnitude), are: (1) The dewatering and the compacting of the sand making up the island and its continuation toward the shore below tide level; (2) the dewatering and the compacting of the alluvium of the Southern California coastal plain with the increased draft upon ground water for water supply purposes; and (3) possibly some compaction of the sands of the oil measures with the withdrawal of their fluid content under the weight of the overburden. It is doubtful that the shale members of the bedded sediments have suffered any further consolidation since 1937. It is true that the interbedded sands would provide drainage of facilities equivalent to the porous stones (of the tests performed). However, the increased loading required for consolidation of the shale does not exist, because the additional surface loading to be transmitted to depth is much less than the fluid load that has been removed from the overlying beds and sediments.

The writer's first acquaintance with Terminal Island was in the school year, 1907–1908, when the Saturday field trips of the Los Angeles (Calif.) High School science class took him to the island and to the adjacent Long Beach-San Pedro coast line. The lecture about the island was of great interest and of lasting memory. It was used, by the well-informed Prof. J. Z. Gilbert, to illustrate the formation and composition of a sand spit, which was a section of an extensive sand bar extending above water level. The bar had been built up, torn down, reformed, and shifted in position from time to time by the action of ocean shore currents and waves upon the coarser sediments brought to the shore line and dropped by both the Los Angeles and San Gabriel rivers. The spit and bar, with the overlying estuary mud, consisted of more than 50% water by volume, and so was unstable under the attack of tidal current and wave action.

The writer sees no reason to change the concept passed on to him at that time. The construction of the breakwater and the dredging of the harbor minimized the wave and current action and, with the filling in of the marshland between the island and the present location of the Pacific coast highway, the sand above and below the estuary and shore mud and clay settled to firmer bearing as the loosely held water was forced out of the pore space. At that time ground-water levels were well above high tide. Fresh water springs flowed into slough channels in the marsh at low tide. Some of these channels were dammed to provide for fresh water duck ponds, and there were numerous bodies of fresh water in slough channels cut off from tidewater.

With these facts in mind, it is difficult to agree with the authors that the filling and loading of that extensive marshland and island, more than 1 mile in average width, was not responsible for continued settlement and the more than 0.2 ft adopted for the period. The base of the cut and cover fill placed over garbage deposited along the shore of San Francisco Bay is continuing to settle with the compaction of the bay mud. Also, considering the extent and physical make-up of the sand, it is unlikely that all settlement, due to dewatering those sands to El. −75 for dry dock construction, ceased with the cessation of pumping at the end of the construction period. Even though water levels in wells were shown to recover, that necessarily is not proof that the pore space between sand grains is not, although slowly, contracting further as the lesser amounts of water held in them rise to the former level registered in the wells.

The void space that comprises the storage capacity of a ground-water reservoir contracts with the withdrawal of water or with the lessening of hydrostatic pressure. When the water column again reaches its former level at the higher part of the reservoir or basin, equal hydrostatic pressures are transmitted throughout; but the void space or storage capacity is never recovered, constituent grains never being unseated. This has been clearly demonstrated in the Santa Clara Valley of California through the observation of well water levels and discharge and artesian flow from wells during the period from 1933 to 1948 (24). The same thing can be assumed to be true in relation to oil sands. When the oil migrated to the sand beds it occupied voids filled with water in which the sediments were deposited. The oil and gases (being lighter) rose in the water, forcing the water down the dip of the strata, and were compressed by the pressure so developed in the formation. As the volume of gas, oil, and water was removed from the sands, the void space would contract. Just when the subsidence due to dewatering the top sand to El. −75 ended (if it has), and just when the subsidence due to voiding the oil sands of their fluid became effective, is a matter for question, and, in the opinion of the writer, indeterminate.

During 1920–1921 the writer made a study of the Southern California coastal plain, from Redondo Beach (Calif.) to beyond Long Beach, for the Shell Oil Company to locate and determine the character of geologic structures buried by the alluvium of the plain through the application of ground-water hydrology. The study consisted of a canvass of all water wells of the area; a compilation of available information as to log, depth, water yield, and past water levels; and the determination and correlation of the then existing elevation of well water levels through lines of levels run by accompanying survey parties. Several facts had a bearing on the subsidence that was then occurring with the lowering of ground-water levels and the restriction of the artesian flow area with the increase of pumpage. The bench marks of the areas between Signal Hill, Dominquez, and Palo Verdes Hills and along the coast south and east of Signal Hill were consistently so much lower than their published datum that it was obvious that all minus determinations could not be attributed to accumulative error. Owners of dug wells along some sloughs volunteered the information that the water from those wells had been potable but, since the breakwater was built and the harbor was dredged, the tides changed and came so far up the sloughs that they seeped into their wells and made the water so brackish that they had to be abandoned. Obviously this condition was due to subsidence of the slough channels, and to decline of ground-water levels. Well water levels dating from 1902 were obtained for comparison from United States Geological Survey (U.S.G.S.) publications (10)(25)(26) and from the U.S.G.S. office in Los Angeles, as well as from the owners themselves. These showed the cause; the level lines, the effect. The lowering of the bench marks was also corroborated by the office of the United States Coast and Geodetic Survey.

This paper, as well as one covering the same area and subject (27) is of value in "pointing up," by direct analogy with the exhaustion of the valuable mineral resource, petroleum, the fact that the even more valuable resource to the semiarid west—ground water—is being exhausted by the gradual elimination of ground storage capacity through the contraction of the underground void space under the weight of the overburden with the lowering of ground-water levels through overdraft. It has long been a matter of concern to the writer, and has become particularly acute during the severe winter drought of 1947–1948 in California and Nevada. The writer has recommended shutting down the pumping plants of the well field of the City of Palo Alto (Calif.) and purchasing the total domestic supply from San Francisco's Hetch Hetchy surface water supply in order to conserve what ground-water storage capacity remains. He has also opposed the drilling of new wells for clients in the Santa Clara and San Joaquin valleys of California, because there is now more well capacity in the basins concerned than there is ground-water capacity to satisfy it. Overdraft will reduce water levels, allow contraction of pore space, and thus reduce ground-water storage capacity.

In view of the facts, the writer is of the opinion that the accelerated subsidence of the Terminal Island area since 1941 is attributable, in the main, to the dewatering, through continued pumpage necessary for constructing the dry dock, of the sand and other surface deposits down to El. −75. The tests on the shale samples are interesting, yet they are in no way conclusive, nor do they change the writer's previously expressed opinion (28)(29) that the application of the theories of soil mechanics and the results of tests upon small samples are of value only as a check on judgment based on determinable facts and should not be used to evaluate cause and effect in numerical terms.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2339

RIVER INFILTRATION AS A SOURCE OF GROUND WATER SUPPLY

By Raphael G. Kazmann,[1] Assoc. M. ASCE

With Discussion by Messrs. C. V. Youngquist, and Raphael G. Kazmann.

Synopsis

The characteristics of a ground water supply based on the infiltration of water from a near-by stream are discussed. The effect of changes in river stage upon yield, as well as the effect of changes in river temperature on well field yields, drawdowns, and water temperatures, is revealed through the data presented. The absence of pathogenic bacteria from the ground water despite the contamination of the source of water is shown.

The paper describes a method of testing a well field of horizontal collectors to determine the firm yield of the field under the worst probable conditions of river stage and temperature.

The construction of a horizontal collector is described and the production record and operating characteristics of a field of seven such units are given in

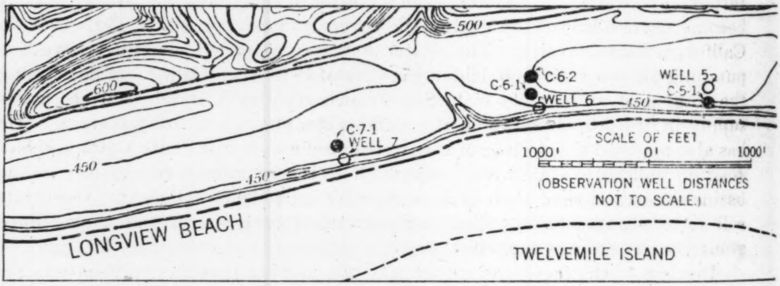

500
600
C-6-2
C-6-1
WELL 6
WELL 5
C-5-1
150
C-7-1
WELL 7
450
450
SCALE OF FEET
1000'
0'
1000'
(OBSERVATION WELL DISTANCES NOT TO SCALE.)
LONGVIEW BEACH
TWELVEMILE ISLAND

Fig. 1.—Well Field of the

Note.—Published in June, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Hydrologic Engr., Ranney Method Water Supplies, Inc., Columbus, Ohio.

detail. The paper is based on more than four years of careful observation of the water supply of the Indiana Ordnance Works near Charlestown, Ind., and was written while the writer worked for the United States Geological Survey.

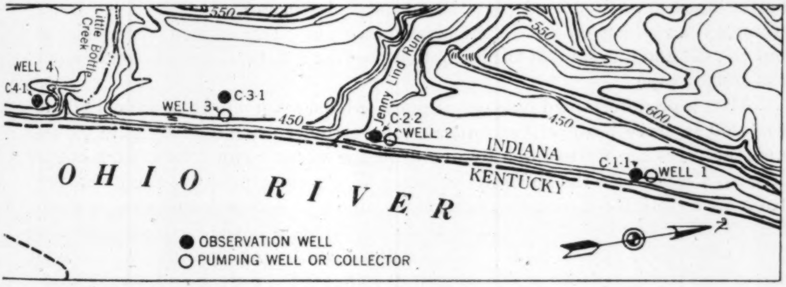
INTRODUCTION

An adequate supply of water is becoming more important as new industries develop which depend on large quantities of water for cooling purposes and in manufacturing processes. The producers of foods, synthetics of all kinds, and precision machinery are among the newer industrial water consumers. The problem of obtaining large quantities of satisfactory water at a minimum cost is a difficult one, even under the best of circumstances. If it is available, ground water is particularly desirable because of its cheapness and its relatively uniform quality.

This paper summarizes the data and conclusions obtained from a study of the ground water supply of the Indiana Ordnance Works. Since the well field is composed of seven horizontal collectors, a brief section is devoted to their construction. However, the greater part of the paper contains an analysis of data obtained during more than four years of continuous well field operation and includes the details of a test conducted to determine the capacity of the water production system.

LOCATION AND DESCRIPTION OF WELL FIELD

The well field of the Indiana Ordnance Works is situated on the Indiana shore of the Ohio River about 12 miles north of Louisville, Ky., between the river and the upland on a narrow flood-plain terrace composed primarily of gravel, although covered by a layer of silt and top soil. The geology is not significantly different from any one of a hundred terraces along the Ohio River. The log of observation well C-6-1, which is 30 ft from collector 6 (see Fig. 1) is typical:



INDIANA ORDNANCE WORKS

Depth (ft)	Description
0 to 18.5	Top soil
18.5 to 50	Gravel boulders and sand, mixed
50 to 65	Pea gravel
65 to 72	Sand and fine gravel
72 to 93	Fine gravel (screen set at 93 ft)

The bedrock floor, which is about 100 ft below the land surface at the river's edge, continues at that elevation inland from 200 ft to 500 ft. On the basis of test drilling inland there seems to be a sharp rise in the rock floor which flattens off about 30 ft or 40 ft below the land surface and meets limestone bluffs as much as 1,500 ft inland. On this basis alone the gravel area would seem to be a relatively large one. Actually, the width of the aquifer from which the collectors draw water ranges from about 200 ft to 500 ft, extending inland from the river's edge. Fig. 1 is a map of the Indiana Ordnance Works field which shows the collector locations. Production wells are plotted to scale but observation wells are spotted diagrammatically. The gravels extend into the river an undetermined distance, but gravel companies dredge gravel in the reach opposite the well field from about 500 ft to 1,500 ft offshore.

Construction of a Collector

A collector is a caisson in the lower end of which are one or more tiers of portholes. The portholes are spaced uniformly in each tier. After the caisson (usually 13 ft in inside diameter) reaches the requisite depth, a thick concrete plug is poured on the bottom to keep out water and extraneous material. After the plug has hardened, lengths of slotted well casing 8 in. in diameter are jacked hydraulically into the surrounding water bearing formation. The boring head of each slotted pipe lateral has holes in it enabling the returning water to carry fines with it to the central caisson. This flow of water washes out the silt and fine sand of the formation, leaving a natural gravel pack around the slotted pipe.

After a predetermined number of pipes, with a total length of from 1,000 ft to 1,200 ft, have been pushed out, all the portholes are closed by gate valves and the pipes are flushed one by one until the water coming from each one is free from sand. At the Indiana Ordnance Works the amount of sand removed from each collector during the pushing of pipe was between 50 cu yd and 100 cu yd. Fig. 2 shows a plan and section of collector 4 of the Indiana Ordnance Works which is a typical unit. The position of the laterals, relative to the Ohio River and to the bottom of the caisson, is indicated. Figs. 3 and 4 are photographs taken during the construction of similar collectors.

Collector Field Operation

The collector field consists of seven horizontal collectors spaced at intervals of from 1,300 ft to 2,000 ft within 100 ft of the river's edge at normal pool stage. In each unit two pumps were installed each capable of delivering 3,500 gal per min. Thus, the field had a pumping capacity of approximately 49,000 gal per min or about 70 mgd. The field was first pumped in April, 1941, and was in continuous operation until October, 1945.

From April, 1941, to April, 1945, inclusive, a period of forty-nine months of continuous operation, a total of 58.3 billion gal was pumped at an average rate of 39.7 mgd. The highest monthly pumping rate occurred in September, 1944, when an average of 51.2 mgd was pumped. The lowest monthly pumping rate during routine plant operation occurred in November, 1941, when 28.1 mgd were pumped. The days of greatest water production were September 15, 1942, when 55,975,000 gal were pumped and June 26, 1944, when 55,742,000 gal were pumped. No data are available for peak hourly water supply, but undoubtedly the field has been pumped at a rate of more than 60 mgd for short periods. The average daily water production for each month of plant operation is shown in Fig. 5.

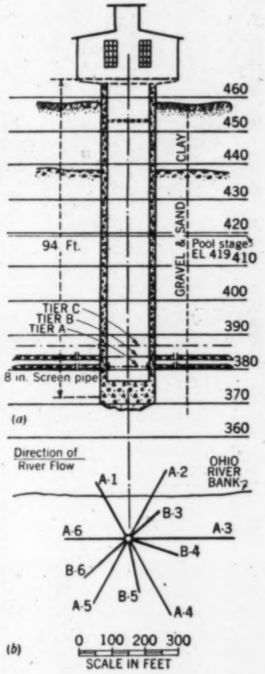


FIG. 2.—COLLECTOR NO. 4, FIG. 1

HYDROLOGY

It has been found that the principal source of water pumped by the collectors was water from the Ohio River which infiltrated to the aquifer. The source of the water can be demonstrated in a number of ways: By comparison of river and well hydrographs; by a study of river and collector field water temperatures; and by a study of the variations in water temperature which occurred in the several laterals of a single collector.

The relationship between water level, river stage, pumping rate, and the viscosity of water is indicated by a study of river and observation well hydrographs and pumpage.

For example, the Ohio River was approximately at pool stage, El. 420, throughout the months of January, 1944, and June, 1944. In January the average daily pumpage from the field was about 34 mgd; in June it was nearly 49 mgd. At collector 6 the average daily pumpage in January was slightly more than 5 million gal; in June it was slightly more than 9 million gal; yet, despite a constant pool stage and an increase in pumpage in the entire field, the average water level elevation in observation well C-6-2 (200 ft inland from collector 6) rose from 399.5 ft in January to 402 ft in June.

The only major variable factor not heretofore mentioned is temperature. In January the average collector water temperature was 59° F and in June it was 55° F. However, the river water temperature averaged 35° F in January and 78° in June. For rough computations, the average of the river and collector temperatures in January was 47° F; in June, 67° F. The kinematic viscosity of water changes from 1.38 to 1.01 centipoises in that range—a decline of more than 35%. This decline in water viscosity evidently accounts for an apparently contradictory condition: Same river level; more water pumped;

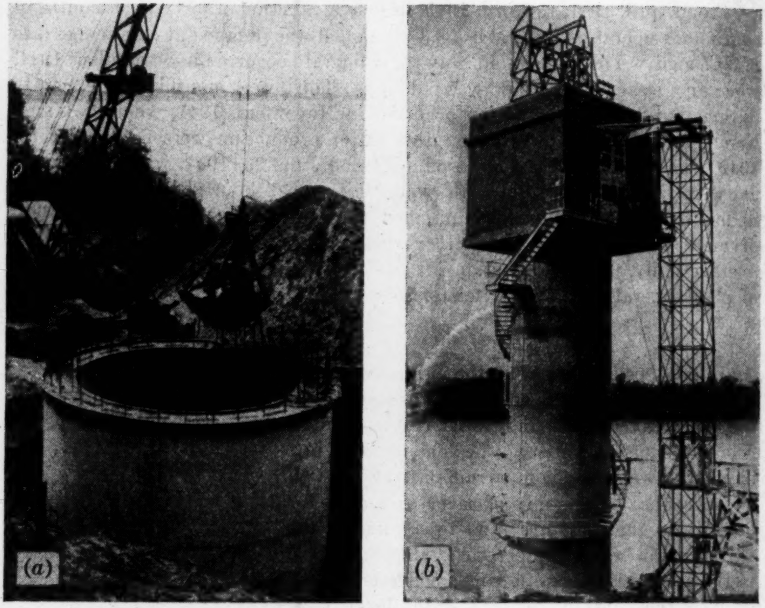


FIG. 3.—CONSTRUCTION VIEWS
(a) Sinking the Caisson
(b) Completed Collector Undergoing 3,000 Gal per Min Test

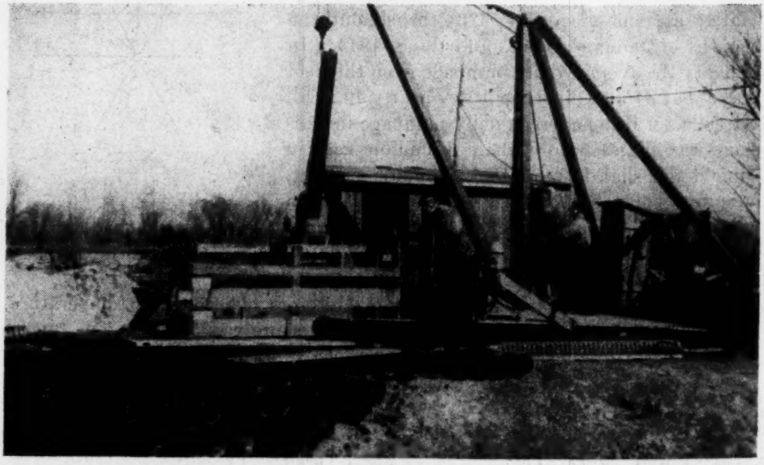


FIG. 4.—LOWERING SLOTTED PIPE INTO CAISSON FOR INSTALLATION AS RADIAL COLLECTORS

smaller drawdown in aquifer and pumped well. The subject is discussed more fully later in this paper.

The effect of a rise in the river stage was an immediate rise of water level in the pumped well and the observation well despite a constant pumping rate, as is illustrated by the rise in water level in well C–6–2 in April, 1945, paralleling a river rise despite a constant pumping rate at collector 6. The aquifer in the neighborhood of collector 6 was under water table conditions as long as the water level was below El. 420. Consequently, changes in water level represented a gain, or loss, in water storage.

Changes in the pumping rate at collector 6 also had an immediate effect on water levels in well C–6–2, regardless of the trend of water levels due to changes in the river stage. Detectable changes in water level in the vicinity of the collector were caused by minor variations in pumping rates.

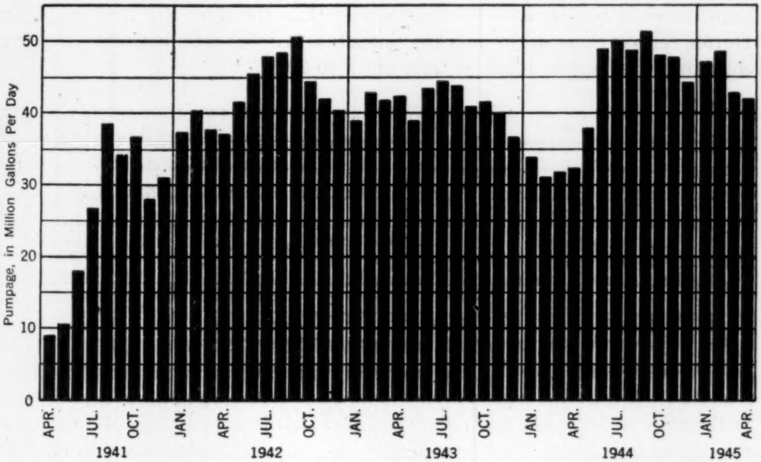


Fig. 5.—Average Daily Water Production from Well Field

River water temperatures and their effect on ground water temperatures form a very complex subject. Fig. 6 shows thermographs of the river water and of the water discharged by several of the collectors into one of the two main reservoirs of the plant. A study of the available data indicated that a seasonal trend in river temperatures was followed by a similar trend in collector water temperatures from eight to twelve weeks later. However, the period of lag has differed from collector to collector, being as little as six weeks in one and as much as fourteen weeks in another. The lag at any particular unit appeared to depend on the operation schedule and on the interference of near-by units and the geology of the aquifer, as well as on the abruptness of river temperature changes.

A study of the problem of temperature fluctuations in the aquifer shows that there were several factors involved: (1) The mixing of ground water with infiltrated river water when the field was first pumped; (2) the admixture of river waters of different temperatures while en route to the collector; (3) the heat storage of the aquifer and underlying rocks; and (4) the conduction of heat upward to the surface and laterally within the aquifer due to temperature gradients.

The water coming from the river entered the aquifer not only directly opposite the collector, but also upstream and downstream from it. Thus, different hydraulic gradients were established between the collector and the river depending upon the point of infiltration of the water. The nearer to the collector that infiltration occurred, the steeper the gradient and the greater the water velocity, all other things being equal. As a result, water that left the river a long time ago reached the collector by one path at the same time as water that left the river more recently reached the well along another path. Thus, an integrating process went on in the aquifer to produce water warmer than the river temperature lows and cooler than the river temperature highs. During certain times of the year the temperature differential between river and collector water was as much as from 18° F to 20° F.

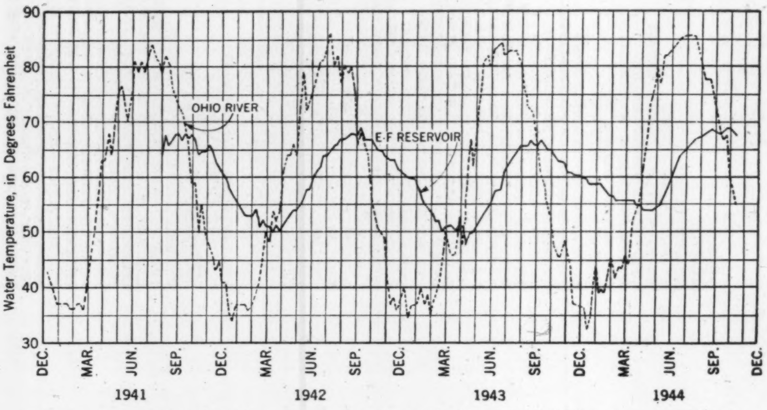


Fig. 6.—Thermographs of Ohio River Water and Pumped Water Entering a Reservoir

The foregoing description is, however, an oversimplification of the actual physical process involved. Superimposed on this idealized mechanism is the effect of temperature on the viscosity of the water and consequently on the manner of flow in the aquifer. The viscosity difference increases the percentage of warmer water entering the well as compared to the idealized case. In the summertime, when the river is warm, water coming from the reach opposite a well will have the advantage of decreased viscosity plus the steeper hydraulic gradient as compared to older (and colder) water already infiltrated from other reaches of the river. On the other hand, in the winter, when river temperatures are low, the viscosity factor will tend to increase the relative velocity of water infiltrated from the river at an earlier date and from greater distances as compared to the newer and colder water approaching the well along the direct path. This condition and others, such as heat flow within water and heat storage by the aquifer and its underlying bedrock, tend to make the exact mathematical analysis extremely involved for even the simplest example of flow to a well.

TEMPERATURE CHARACTERISTICS OF INFILTRATED WATER

The units at the Indiana Ordnance Works are composed of eight or ten laterals (actually horizontal wells) converging to a central caisson. In 1943, thermocouples were installed on eight of the laterals in collector 4 and arrangements were made to obtain periodic temperature readings. Fig. 7 gives typical thermographs.

When these thermographs and the layout of the laterals in Fig. 2 are studied, the relationship between the position of the lateral and the temperature variation becomes apparent. As might be expected, the laterals which project toward the river are more affected by river temperature changes than are those which project to the land side. A temperature differential of 16½° F existed on September 22, 1944, between laterals A-2 and A-5.

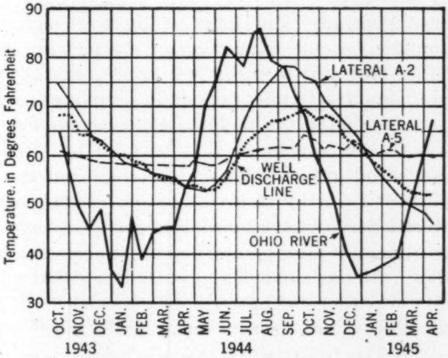


FIG. 7.—THERMOGRAPHS OF THE OHIO RIVER, DISCHARGE LINE FROM WELL 4, AND LATERALS A-2 AND A-5

One deduction to be made from this data is that, even under the same conditions of hydraulic gradient, the proportion of water entering the collectors from each lateral is constantly changing throughout the year in response to temperature changes in the aquifer. The reason for this is found in Darcy's equation for flow in porous media:

$$Q = \frac{P\,i\,A}{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which Q is the quantity of flow; P is the permeability of the materials composing the aquifer; i is the hydraulic gradient; A is the cross-sectional area perpendicular to the direction of flow; and μ is the viscosity of the fluid. Ignoring other factors, in the range of temperatures shown in Fig. 7 from 46° F to 78° F, the coefficient of viscosity ranges from 1.39 to 0.88 centipoises. In other words, the coefficient of viscosity of 78° F is only 63% of the coefficient at 46° F.

SANITARY QUALITY OF COLLECTOR WATER

The bacteriological quality of the raw water from each collector has been periodically tested. Although the 37° C plate count has almost always shown the presence of bacteria in the water obtained from the collectors, further tests with lactose broth incubated for 24 hours and 48 hours have rarely shown the presence of gas forming bacteria. For example, between August 1, 1944, and April 1, 1945, a typical eight-month period which included a long period of high water during which the river overflowed parts of the collector field area, a total of two hundred and twenty samples was tested. Of these, a total of eight samples (10-milliliter portions) formed gas in lactose broth and only two confirmed.

The collectors that produced the gas forming water samples were retested as soon as possible. No gas forming organisms were found. Had the aquifer been generally contaminated by the source of recharge, the retesting undoubtedly would have revealed gas forming organisms. After a study of the entire four-year record it was concluded that the contaminated samples were probably due to minor errors in sampling technique.

Considering the polluted character of the Ohio River next to the field, it was further concluded that harmful bacteria were removed or killed in the sands and gravels between the river and the collectors. How this is accomplished is worthy of investigation, because tests by the United States Public Health Service have shown that some gas forming bacteria survive the complete treatment of filter plants[2] making it necessary to resort to break-point chlorination. Among some of the possibilities that have been suggested are: Adsorption of pathogenic bacteria by the particles of silt, clay, sand, and gravel (note the use of diatomaceous earth in the latest techniques of filtration); death of pathogenic bacteria due to change in environment or lack of food; filtration by the fines at the bottom of the river; and elimination of pathogenic bacteria by action of such organisms as plankton or ground bacteria. However, until a systematic study is undertaken which leads to a workable hypothesis the facts are that coliform organisms rarely, if ever, were found in the water pumped by the collector field of the Indiana Ordnance Works.

WILL SILT CLOG THE INTAKE AREA?

In comparing the infiltration of river water to an aquifer and the passage of water through a slow or rapid sand filter, there appears to be only one major point of similarity—the fact that water must pass through a porous medium. In a filter all the available water must pass through, and deposit all its fines on, the surface of the filter. In the infiltration area of an aquifer the principal movement of water is along the river bottom in planes roughly perpendicular to the direction of infiltration. The fines, therefore, do not remain on the surface of the area as they would if raw river water were led to sand beds without prior sedimentation. The fines are continually removed by the river and replaced temporarily by material coming downstream—that is, the bed is constantly scoured and built up. Moreover, occasional periods of increased discharge put the fines back into suspension, leaving only the heavier sands and gravels on the river bottom. Evidently river infiltration to an adjacent aquifer should not be expected immediately upstream from a dam as the deposition of silt and fines will occur and preclude infiltration.

---

[2] "Studies of the Efficiency of Water Purification Processes," by H. W. Streeter, *Public Health Bulletin No. 172*, U. S. Public Health Service, Washington, D. C., May, 1927.

The reason sand beds in a filter plant must be scraped or backwashed is that a gelatinous layer of fine silt and organic matter forms on the filter bed surface. The efficiency of the filtration process probably depends on the existence of this layer. The filter plant operator wages a constant struggle to maintain a certain amount of the material on the sand bed, yet to prevent it from sealing the filter completely. Periodically this layer is removed by scraping or backwashing. As far as can be determined, no such continuous layer has ever formed on the surface of the infiltration area supplying the Indiana Ordnance Works well field.

There is no evidence that permanent silting occurred during the period of collector field operation. All the data indicate that the production capacity of the field remained unchanged during the entire period of operation.

Calibration of the Collector Field and Prediction of Firm Yield

The field had been operating successfully for about a year when, for various reasons, it was decided to make pumping tests to determine the magnitude of interference between the units and the probable minimum yield of the system when operating at full capacity.

Since the plant was in full production at the time, the testing had to be performed without interfering with plant operations and no water could be pumped to waste.

It was planned to operate the field, or most of the field, the same way save for one difference in pumping rate in one collector. Most of the time this was not feasible so all variations in plant water demand had to be met at the end of the field farthest from the point of principal interest. Thus, while tests were being run on unit 1, variations in plant load if any were to be taken by units 7 and 6, respectively.

Presumably, after allowing for river stage fluctuations and water temperatures, all differences in drawdowns in collectors 1, 2, 3, and possibly 4, would be due to the change in the pumping rate of collector 1.

A systematic scheme of pumping the field was prepared and the testing was started with the fullest cooperation of the War Department and the E. I. du Pont de Nemours Company, the plant operators. The tests lasted for nearly four months, during which time a number of pumping schedules were nullified because of operating demands. Eventually, however, seven reasonably satisfactory pumping schedules were carried through and temperature and water level data were obtained as needed.

From previous experience it had been found that about four days of pumping at a constant rate were sufficient to establish a steady flow of ground water in the aquifer, so each schedule was followed for four days. Water levels within each caisson could be determined by automatic air-line water level recorders. However, since there was an observation well within 30 ft of each caisson, water levels were also determined there by steel tape measurements. It was found that the drawdowns in these observation wells were substantially the same as those in the respective production wells. As it was easier to secure accurate measurements in the observation wells, and as these wells fluctuated less due to slight variations in pumpage, the water level elevations of the observation wells were used in the computations.

During the test period, collector water temperatures were changing from 60° F at the start of the period to 70° F at the end. In order that the data might be comparable, the drawdowns were corrected to a standard temperature of 68.5° F at the collector where the temperature was measured. The corrections were made on the basis that, in steady state flow through porous media, the drawdown is inversely proportional to the viscosity—that is, a well pumping 10 mgd at a temperature of 60° F with a drawdown of 10 ft would have had a drawdown of only 8.9 ft at the standard temperature, 68.5° F. To illustrate the method of operation and the type of data obtained, an operation schedule is summarized in Table 1. All the usable data obtained during the pumping tests are given in Table 2.

TABLE 1.—SAMPLE OPERATION SCHEDULE FOR TEST OF WELL FIELD AT INDIANA ORDNANCE WORKS

Date	COLLECTOR NO.						
	1	2	3	4	5	6	7
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(a) PUMPAGE, IN MILLION GALLONS PER DAY							
7-21-42	0.713	4.795	5.035	4.763	9.688	10.070	9.843
7-22-42	0.583	4.717	4.925	4.653	9.454	9.850	9.571
7-23-42	1.873	4.749	4.937	4.056	9.512	9.804	9.583
7-24-42	1.322	4.808	5.009	4.879	9.078	9.811	9.564
Total	4.491	19.069	19.906	18.351	37.732	39.535	38.561
Average	(1.10)	(4.75)	(5.00)	(4.60)	(9.45)	(9.90)	(9.65)
(b) COLLECTOR WATER TEMPERATURE, IN DEGREES FAHRENHEIT							
Average	65	65	62	63	61	62	60
(c) WATER LEVEL ELEVATIONS[a]							
7-21-42	412.69[b]	401.79[b]	400.76	401.06	395.53	398.25	406.58
7-22-42	414.09	402.81	400.91	401.11	395.48	399.56	406.51
7-23-42	413.31	403.33	400.93	402.06	396.40	399.43	406.24
7-24-42	414.61	403.76	401.12	401.11	395.35	399.64	406.39
Average	(414.00)	(403.30)	(400.93)	(401.34)	(395.68)	(399.47)	(406.43)
(d) DRAWDOWN,[c] IN FEET							
Observed	5.45	16.15	19.52	18.11	23.77	19.98	13.02
Corrected	5.22	15.45	17.90	16.84	21.50	18.34	11.58

[a] In observation wells. [b] Omitted from average because of antecedent conditions. [c] Mean river elevation: 419.45.

By a process of comparison the drawdown of each collector and its interference with all others, for a pumping rate of 10 mgd, was finally obtained. This was checked by recomputing the elevations at each collector taken from each pumping schedule using observed temperatures and river levels. These elevations were compared with the observed elevations; and, when a significant discrepancy appeared, the cause for the discrepancy was sought. Sometimes too much weight had been given to the poorer data. Minor corrections were then made on a trial basis. Finally, Table 3 was obtained, which summarizes the drawdowns caused at each production unit by any single collector pumping 10 mgd at a temperature of 68.5° F. Table 3 can be regarded as a mathematical model of the collector field in a state of steady flow.

TABLE 2.—CONSOLIDATED DATA SHEET FOR ALL USABLE PUMPING SCHEDULES

Schedule No.	COLLECTOR No.						
	1	2	3	4	5	6	7
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(a) AVERAGE PUMPAGE, IN MILLION GALLONS PER DAY							
1........	1.10	4.75	5.00	4.60	9.46	9.90	9.65
3........	6.80	9.25	4.80	6.75	9.40	9.90	0
3[a]........	9.25	4.10	4.75	5.35	9.40	9.80	4.10
4........	9.40	4.75	4.95	5.70	9.65	5.15	9.70
5........	9.20	4.70	4.75	4.95	5.05	9.80	9.50
7........	4.95	5.30	4.80	5.05	9.55	9.70	9.45
8A......	9.55	9.60	0.10	5.20	9.70	9.80	9.60
(b) AVERAGE DAILY DRAWDOWN, IN FEET[b]							
1........	5.22	15.45	17.90	16.84	21.50	18.34	11.58
3........	14.42	26.20	18.40	22.00	22.62	17.58	2.63
3[a]........	21.50	19.13	18.81	19.76	23.32	18.83	6.97
4........	21.78	19.70	18.60	19.76	23.31	13.88	11.84
5........	22.01	19.55	18.84	18.62	16.30	19.43	12.26
7........	14.58	17.48	19.28	20.11	24.28	20.34	12.75
8A......	24.36	29.35	11.04	19.24	24.58	20.85	14.27

[a] Variant. [b] Corrected to 68.5° F.

These tests furnish the factual basis for comparing the performance of the collector field with that at any future date. They will enable any theories as to apparently reduced collector field yields to be answered by a similar test. Any silting or reduced infiltration capacity of the aquifer would result in a reduction in water level elevations throughout the field as compared to the 1942 standard, under the same conditions of pumping, after corrections for temperature have been made.

TABLE 3.—MATHEMATICAL MODEL OF COLLECTOR FIELD, GIVING SPECIFIC DRAWDOWN, IN FEET PER 10 MGD, CAUSED BY PUMPING EACH COLLECTOR

Collector No.	COLLECTOR BEING PUMPED						
	1	2	3	4	5	6	7
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1.....	20.0	5.5	2.0	....	....	....	....
2.....	6.5	22.0	6.0	2.5	....	....	....
3.....	3.2	4.3	24.0	5.0	1.5	....	....
4.....	1.9	2.1	4.3	22.3	2.8	1.1	....
5.....	....	1.4	2.2	4.3	17.6	2.7	1.7
6.....	....	....	....	1.8	2.8	14.5	3.4
7.....	....	....	....	....	1.4	2.4	10.8

The test results, moreover, can be usefully applied in predicting minimum collector field yields and in making sure that the installed pump capacity in each unit matches its capacity.

The first steps in predicting minimum collector yields were: (1) To test the ability of the mathematical model to predict drawdowns on the basis of observed pumping rates, water temperatures, and river levels; and (2) to compare such predictions with the observed drawdowns. If these data were reasonably accurate, the model might then be used in predicting minimum yields; but, as minimum yields are to be expected during low pool stages in the winter and early spring, the mathematical model also had to be tested under these conditions.

Since the temperature distribution in the aquifer is so complex, it would be on the safe side to use the lowest measured water temperatures during any period. In the summertime the lowest temperatures in the aquifer are near the water collector; and, as long as river water temperatures are above collector water temperatures, this condition must exist. The original calibration tests were conducted under such conditions; hence, temperature corrections were based on the temperature of water discharged by the collectors.

However, in the winter when river temperatures are low, theoretically the temperature to use would probably be that existing somewhere between the river and the collectors. There appears to be no logical reason why the arithmetic average of the river and collector temperatures should be used when the temperature distribution in the aquifer is so complex. In any event, over a period of from thirty to sixty days of constant operation in the winter, the governing temperature would seem to approach the average river temperature during that period as a limit. A study of river temperature records since 1910 furnished by the Louisville Water Company shows a normal range in temperature for the winter period from 32° F to 43° F, with occasional freakish days above 43° F.

A complete record of pumpage at the Indiana Ordnance Works since April, 1941, has been available to the writer. It might, therefore, appear to be a simple matter to find a number of three-day or four-day periods of pumping at a constant rate which would serve to test the mathematical model. However, because of the man-power shortage and the confidence of the plant operators in the permanence of the supply, the systematic measurements in the observation wells which had been made at weekly intervals in 1941 and 1942 were decreased in frequency until, in 1945, only one measurement was made a month. The constantly changing water demand of the Ordnance Plant further tended to decrease the probability of the water level measurements occurring at the end of a period of constant pumping. However, a careful study of all the data revealed six periods between August, 1942, and December, 1944, during which the requisite conditions were approximately satisfied. Of these, three occurred in the summer and three in the winter.

Table 4 summarizes the water levels computed by the mathematical model, the observed water levels, and the difference between them.

All errors preceded by a minus (−) sign are on the unsafe side—that is, the observed water elevation was lower than the computed water elevation. Some of the errors result from major pumping variations, as on August 20, 1943, when the pumping rate at collector 1 changed considerably during the selected period. However, other large errors are not so readily explained and may result from temporary curtailment or increase in pumpage due to discharge-line pressure changes just prior to making the water level measurements.

A study of the data shows that the mathematical model of the ground water supply system works reasonably well when discharge temperatures are used in the summer and average river temperatures are used in the winter.

After the mathematical model was tested, the percentages of maximum errors on the unsafe side for each collector in numerical order were found to be 14.3, 5.8, 2.4, 3.1, 10.6, 16.2, and 14.0, respectively.

It is of interest to note that all the maximum observed errors on the unsafe side occurred during the period dated August 20, 1943, when the temperature factor favored large yields.

TABLE 4.—COMPARISON OF COMPUTED AND OBSERVED WATER LEVELS

Date	Water level elevation	COLLECTOR NO.							River stage
		1	2	3	4	5	6	7	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
8-21-42...	Observed...	395.0	399.5	400.0	401.0	403.2	398.8	406.0	419.7
	Computed..	398.0	399.4	399.7	401.3	403.0	400.0	405.4	....
	Error......	−3.0	0.1	−0.3	0.3	0.2	−1.2	0.6	....
8-20-43...	Observed...	399.3	399.2	399.1	400.7	402.0	396.5	406.0	419.9
	Computed..	402.4	400.4	399.6	401.3	403.9	400.3	407.8	....
	Error......	−3.1	−1.2	−0.5	−0.6	−1.9	−3.8	−1.8	....
12-24-43..	Observed...	398.3	395.4	393.1	393.1	395.3	393.9	400.8	419.8
	Computed..	398.2	392.9	391.2	393.1	394.2	393.4	400.7	....
	Error......	0.1	2.5	1.9	0.0	1.1	0.5	0.1	....
1-7-44....	Observed...	398.0	394.8	392.8	395.7	396.9	398.5	402.8	420.3
	Computed..	398.3	392.8	391.6	395.2	396.2	400.1	402.2	....
	Error......	−0.3	2.0	1.2	0.5	0.7	−1.6	0.6	....
7-21-44...	Observed...	397.3	400.2	403.3	401.0	398.0	397.4	406.2	420.0[a]
	Computed..	399.7	400.3	399.0	401.1	397.2	399.2	406.2	....
	Error......	−1.4	−0.1	4.3	−0.1	0.8	−1.8	0.0	....
12-16-44..	Observed...	393.2	396.1	395.2	397.2	390.7	389.4	401.4	420.3
	Computed..	390.7	392.0	391.5	393.1	391.2	392.0	400.0	....
	Error......	2.5	4.1	3.7	4.1	−0.5	−2.6	1.4	....

[a] Estimated.

The final step in determining the firm yield of the collector field was to use the construction data as a basis for computing yields. The maximum drawdown in each unit is governed by the elevation of the uppermost tier of installed laterals and the contractor's instructions stated that a minimum of 3 ft of water was to be maintained above the uppermost tier. These specifications were used as the basis for computations.

The firm yield of the field was found by trial, using the mathematical model corrected to a temperature of 35° F. This temperature is believed to be the average that can be expected to occur for fairly long periods during a normal winter. Pool elevation is taken as 419.0 ft in computing allowable drawdown. A significantly lower pool elevation can be expected only when wickets of the Louisville Dam are lowered to break up ice jams, a truly infrequent occurrence. The first step is to compute Table 5, assuming a pumpage of 10 mgd from each well. Using Cols. 9 and 10, Table 5, as a basis, a first approximation of firm water is obtained by multiplying the ratio between the available drawdown and the computed drawdown by 10 mgd. Data thus obtained are placed in the mathematical model and a new set of drawdowns is computed and similarly compared with the available drawdowns. This procedure is repeated and new values of pumpage are obtained for each collector until the computed drawdown and available drawdown coincide. After repeating this process three times, the firm yields of Table 6 were obtained based on the assumption that the whole field is pumped at the same time. Pump installation to at least the indicated capacity is required in each collector to obtain the maximum well field yield under adverse circumstances.

TABLE 5.—TRIAL SPECIFIC DRAWDOWNS, IN FEET, CAUSED BY PUMPING EACH COLLECTOR AT 10 MGD WHEN THE TEMPERATURE IS 35° F AND THE POOL STAGE IS 419.0

Drawdown at collector No. (1)	COLLECTOR BEING PUMPED							Computed drawdown (9)	Maximum drawdown available (10)
	1 (2)	2 (3)	3 (4)	4 (5)	5 (6)	6 (7)	7 (8)		
1.........	33.8	9.3	3.4	....	....	....	....	46.5	42.0
2.........	11.0	37.1	10.1	4.2	....	....	....	62.4	39.0
3.........	5.4	7.3	40.6	8.5	2.5	....	....	64.3	33.0
4.........	3.2	3.5	7.3	37.7	4.7	1.9	....	58.3	36.0
5.........	....	2.4	3.7	7.3	29.7	4.6	2.9	50.6	38.0
6.........	....	....	....	3.0	4.7	24.5	5.7	37.9	43.0
7.........	....	....	....	....	2.4	4.1	18.3	24.8	48.0

Additional data on river temperature probabilities and a considerable amount of statistical analysis will be required to establish design temperatures for collectors or wells obtaining water from river infiltration. Incidentally, this branch of hydrology may also be valuable to engineers interested in protecting levees from dangerous underseepage; a levee that may be safe from dangerous underseepage during a February or March flood might be unsafe from an equal river stage occurring in May, June, or July. The phenomena, it should be noted, are fundamentally identical.

TABLE 6.—FIRM YIELD WITH ENTIRE FIELD PUMPING

Collector No.	Yield (mgd)
1.........	10.0
2.........	5.8
3.........	4.3
4.........	5.9
5.........	6.8
6.........	10.5
7.........	22.5
Total....	65.8

CONCLUSIONS

Study of the water supply of the Indiana Ordnance Works has demonstrated that nearly all the water pumped has originated in the Ohio River and has infiltrated to the aquifer. Water collectors or wells deriving water from such a source will furnish water whose temperature variation follows the variation in temperature of the river water, but with a lag and reduction in amplitude fixed by the hydrology of the aquifer and the operation of the field.

At the Indiana Ordnance Works, the temperature of the water pumped from the collector field as a whole varied from 49° F to 70° F, although individual units furnished water whose temperature was as much as 2° F above or below this range. The range in temperature is determined by the temperature variations of the source of water, the distance to the recharge area, the landward extent and transmissibility of the aquifer, the rates and durations of discharge, the heat storage of the aquifer and movement of heat within it, and the water contribution from the land side of the aquifer.

Furthermore, the temperature of water discharged by each lateral of a unit is primarily determined by its relationship to the source of recharge. At the Indiana Ordnance Works, a temperature differential between 15° F and 20° F was found to exist during certain times of the year in water entering a single caisson from individual laterals.

Temperature of the water has a profound affect on the minimum quantity of water, or firm water, available from units whose maximum drawdown is fixed. The design temperature for computing the firm water approaches the lowest average river temperature prevailing over a one-month or a two-month period.

The firm yield of an existing collector field deriving its water from river infiltration can be determined by methods which have been described in this paper. Such tests are valuable in so far as they furnish the operator with a quantitative method to detect any deterioration in the collectors or the infiltration capacity of the aquifer. They are also useful in balancing installed pump capacity with collector productivity. The firm yield of a field must be known in order that additions to water supply capacity can be logically planned.

The bacteriological quality of the water discharged by the collectors at the Indiana Ordnance Works was extremely good. Gas forming organisms were rarely found in the water and it is believed that the gas formers as well as the few coliform organisms which were found by bacteriological tests were the results of careless sampling technique, despite the heavy pollution of the Ohio River. Tastes and odors were apparently absent from the water discharged by the collectors, although no systematic attempt was made to detect them. Natural purification of polluted water by aquifers undoubtedly merits more study than it has received in the past.

ACKNOWLEDGMENTS

Nearly all the data collected during the four-year period covered by this report were collected by the personnel of the Indiana Ordnance Works. That the thousands of measurements of water levels, pumpages, temperatures, and bacterial quality could be correlated at all is tribute enough to the industry and ability of these men and women.

The assistance of the War Department and the E. I. du Pont de Nemours Company is acknowledged with thanks. Special mention should be made of the capable and cooperative commanding officers of the Indiana Ordnance Works during the period: Brig.-Gen. R. E. Hardy and Col. Kelly H. Lewis, his

successor. Captains A. L. Ahlers, E. E. Keiser, and A. J. Klemka and their civilian aides, including John E. Ulrich and the late L. G. Couter, also rendered valuable assistance.

The plant managers of the Indiana Ordnance Works—J. F. Daly, R. F. Boltz, and M. L. Valentine—cooperated fully to facilitate obtaining the required information. Power superintendents F. M. Acker and C. C. Hagerty, as well as members of the power and water staff, including C. F. Parcels, G. R. Calhoun, S. G. Ferris, and C. M. Fair, were most obliging and helpful. Members of the Wilmington (Del.) office of the E. I. du Pont de Nemours Company, especially E. B. Showell and W. J. Sloan, were most cooperative in facilitating the study.

E. W. Silitch, M. ASCE, and A. D'Audiffret of the Ranney Method Water Supplies, Inc., rendered substantial assistance by furnishing engineering information and construction data as well as some of the photographs used in this article.

The writer desires to thank the various members of the Division of Ground Water of the United States Geological Survey for criticism and supervision as well as encouragement in conducting the study described in this paper. In particular, thanks are due to O. E. Meinzer, formerly geologist-in-charge of the Division of Ground Water, under whose general supervision the work was done; the late D. G. Thompson, geologist; and L. K. Wenzel and C. E. Jacob, Assoc. M. ASCE, hydraulic engineers, whose criticism has been invaluable.

Acknowledgment is also made to the Director, Geological Survey, United States Department of the Interior, for permission to publish the material contained in this paper.

DISCUSSION

C. V. Youngquist,[3] **M. ASCE.**—A timely and thorough analysis is made for the profession of a water-supply installation, utilizing as its source ground water, which is replenished principally by river infiltration. The writer would like to add some general statements on river infiltration with particular reference to experience in Ohio.

The use of ground water has more than doubled in the decade from 1937 to 1947 because of its increased demand by industry, municipalities, and agriculture. Ground water is desired because of its low temperature, constant chemical quality, clarity, and freedom from pollution. The increased demand has required a more thorough inventory of the ground water resources, particularly those that determine quantities available for further development. Under conditions of ground water occurrence in Ohio the source and rate of ground water recharge is all important if a permanent supply is to be developed.

Investigations of Ohio's ground water resources by the Ohio Water Resources Board in cooperation with the United States Geological Survey have established the fact that the greatest ground water pumpage in Ohio occurs in present stream valleys underlain by permeable glacial gravels. In these areas the principal source of ground water recharge is generally river infiltration, which occurs continuously through the stream bed and, in some instances, through permeable areas in the flood plains when the river is at flood stage.

River infiltration can be ascertained by the following methods: (1) Actual measurement of stream loss through infiltration by current meter in the vicinity of a producing well field, which is possible only when the ground water pumpage is large enough to be an appreciable factor of the stream flow (which, of course was not the case in the example described by the author); (2) correlation of river fluctuations with those of the water table in the pumped area; (3) observation of changes in the temperature of ground water due to infiltration; (4) observation of changes in the chemical quality of water due to infiltration; and (5) a pumping test conducted with properly arranged observation wells indicating a distortion in the shape of the cone of influence where river infiltration occurs. A plotting of drawdown against distance from the pumped well indicates steepening hydraulic gradients in a direction toward the stream from the pumping well as infiltration occurs.

All these methods have been used in Ohio ground water investigations. Although the investigations are not intensive or extensive enough, some quantitative data are available. For example, some direct measurements of river infiltration by current meter at Canton, Ohio, on the middle branch of Nimishillen Creek in the vicinity of the City of Canton's northeast well field, indicate infiltration rates of 3,600,000 gal a day per acre of stream bottom in certain reaches of channel at moderate stream flows. Additional data indicate presumptive evidence of rates of 5,000,000 gal a day per acre or 6,000,000 gal a day per acre at flood stages when water levels in the aquifer have been lowered sufficiently to receive the recharge.

---

[3] Chf. Engr., Ohio Water Resources Board, Dept. of Public Works, Columbus, Ohio.

These data are of increasing interest to engineers because they aid in the quantitative analysis of a projected ground water supply with more assurance than is generally believed. The stream-flow record, duration curve, flood-frequency analysis, and mass diagram are used in a manner similar to their use in the design of a surface reservoir. The dimensions and contour of the underground reservoir are ascertained as for a surface reservoir. Additional factors, particularly the physical properties of an aquifer (such as transmissibility and specific yield), must be considered in designing for the utilization of a ground water reservoir which is recharged by infiltration, but in general they are determinable to a fair degree of accuracy.

The underground reservoir permits withdrawals for periods of time at rates greater than the infiltration rate, just as a surface reservoir permits withdrawals greater than the run of the river. This is possible in an infiltration supply with ground water storage because infiltration rates increase with increased river stage. Often a ground water reservoir underlying a stream receives its major recharge during flood stages, in much the same way as a surface reservoir.

One advantage of a ground water reservoir replenished by river infiltration is that useful and valuable land is not removed from productivity by inundation. This is especially important in Ohio where most of the land is valuable and is needed for a growing population.

It is hoped that engineers will awaken to the tremendous possibilities for the utilization of ground water, especially infiltrated supplies. Then, engineers will not allow ground water to remain, as Malcolm Pirnie, M. ASCE, has recently stated, "a neglected natural resource."

Raphael G. Kazmann,[4] Assoc. M. ASCE.—At the time the water supply of the Indiana Ordnance Works was conceived, information was available concerning only one similarly planned major infiltrated water supply in the United States—the Des Moines, Iowa, filtration gallery. Consequently, Mr. Youngquist's discussion is, in a sense, a measure of the advance in ground water engineering during the seven years since 1941, and in particular during the five years since 1943. In the course of developing from a qualitative, descriptive science, ground water hydrology has reached the stage of as much a quantitative science as surface water hydrology, and probably is more susceptible of yielding reliable quantitative answers than is applied mechanics.

In the field of water supply, ground water hydrology recognizes, and has reliable quantitative answers for, the following categories of problems when encountered in aquifers composed of unconsolidated materials:

1. An underground water supply based on infiltration—
   - (*a*) From rainfall on the immediate area where the wells are located;
   - (*b*) From rainfall on a distant area where the aquifer outcrops; and
   - (*c*) From rainfall and from a river or lake touching a distant outcrop area.

---

[4] Chf. Hydrologic Engr., Ranney Method Water Supplies, Inc., Columbus, Ohio.

2. An underground water supply based on infiltration from a near-by source such as—

- (*a*) A permanent river or lake that can readily supply all desired quantities of water if the aquifer will transmit them (for example, the Mississippi, Missouri, and Ohio rivers and their larger tributaries or, say, Lake Michigan);
- (*b*) An intermittent stream crossing deep buried valleys, such as Nimishillen Creek in the Canton (Ohio) area;
- (*c*) A "flashy" stream to furnish a variable-quantity infiltration supply which will supplement existing surface water sources during dry weather and furnish large supplies during floods, when the turbidity of surface water forms a major treatment problem for water works operators;
- (*d*) A permanent or intermittent stream with sufficient subsurface storage to furnish water with a small, predetermined temperature fluctuation, as compared to the large temperature variation of the surface water which is supplying the aquifer; and
- (*e*) A permanent or intermittent stream with sufficient subsurface storage to furnish water whose mineral quality will fluctuate only slightly as compared to the variations in quality encountered in the surface source, in order to minimize water treatment problems.

These problems are not listed in order of importance or magnitude. The Indiana Ordnance Works is classified in category 2(*a*) as is the supply of the City of Manitowoc, Wis. The use of such a stream as that listed under category 2(*b*) as a source necessitates knowledge of flow-duration curves, the dimensions and contour of the underground reservoir, and the storage coefficient of the reservoir to determine the yield, as Mr. Youngquist has shown. In addition, the physical characteristics of the water already in storage should be determined, as well as the approximate quantity of areal rainfall reaching the underground reservoir, although this is usually a relatively small quantity. The recently completed stream infiltration supplies of the cities of Anderson, Ind., and Canton are classified in this category. In category 2(*d*) both the quantity and the approximate temperature variations of the water can be predicted.

Of course, there are other specialized water supply problems for which reliable quantitative answers can be obtained, but the bulk of all underground water supply problems are included in categories 1 and 2.

The reverse side of the water supply problem has not been touched, of course, in either the original paper or the discussion—that is, the problem of subsurface drainage and the removal of unwanted water from specific areas. The same quantitative ground water procedures are applicable to dewatering problems as to problems of underground water supply.

One paragraph of Mr. Youngquist's discussion is of special interest to hydrologic engineers because it summarizes, perhaps too briefly, the use of underground storage in an infiltration system. This paragraph compares the magnitude of withdrawals from an aquifer with withdrawals from a surface reservoir.

The function of the natural, underground reservoir is the same as that of a surface reservoir—to supply peak loads and to satisfy a predetermined minimum water demand during periods of low flow in the stream. The rate of infiltration to the aquifer (the equivalent of flow rate into a surface reservoir) is dependent on (1) the quantity of water flowing in the stream at any time; (2) the velocity of the stream; and (3) the stage of the stream relative to the general water level in the aquifer.

With the increase of each of these major factors, the rate of infiltration increases. If the subsurface storage is more or less depleted, as it will be at the end of a dry summer, the rate of infiltration during floods will be a maximum and will undoubtedly exceed the indicated 5 mgd or 6 mgd per acre of stream bed proposed by Mr. Youngquist. The bulk of recharge to an underground reservoir occurs during floods for precisely the same reason that surface water reservoirs are filled by floods: Most of the water is there at that time.

Summary.—Underground reservoirs do not remove productive land from cultivation and, equally important, their storage capacity never decreases because they do not silt up. The planners of new water supplies should not overlook the available underground water in an area. Such supplies can be relied on to furnish perennial, and predetermined, quantities of water.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5. 1852
TRANSACTIONS

Paper No. 2340

NEW FORMULAS FOR STRESSES IN CONCRETE PAVEMENTS OF AIRFIELDS

By H. M. Westergaard,[1] M. ASCE

With Discussion by Messrs. Robert Horonjeff, Evan P. Bone, and H. M. Westergaard.

Synopsis

The stresses investigated here are caused by loads. The load is a pressure transmitted through the oblong "footprint" of a tire of a landing gear. Three positions of this load are considered: The first is at a considerable distance from any edge or joint, in the interior of the area of a panel of the pavement; the second position is next to an edge or next to a joint that has no capacity for load transfer, but is at an appreciable distance from any corner of the panel; the third position is next to a joint that permits some transfer of load to the adjacent panel, but is at an appreciable distance from any corner of the panel. Formulas are given for stresses and deflections caused by such loads. The significant stresses are tensile stresses at the bottom of the slab. A fourth position of the load, at a corner of a panel, by which significant tensile stresses are produced at the top of the slab, is also important, but is not considered herein.

The boundary of the footprint of the tire is usually a curve that lies between an ellipse and a circumscribed rectangle, but for present purposes this curve can be replaced in most cases by an equivalent slightly more oblong ellipse, and the load can be considered to be distributed uniformly over the area of this ellipse. Specific formulas are derived under the assumption of such a load, but general formulas are also stated which apply for any shape of the footprint (including the case of dual tires).

The basic formulas for stresses under loads in the interior of the area of a panel have been published previously, but the remaining formulas are the result of new derivations.

The order of presentation is: Statement of assumptions; statement of the formulas; numerical examples; derivations; comments on evidence from tests; and conclusions.

Note.—Published in May, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Gordon McKay Prof. of Civ. Eng., Graduate School of Eng., Harvard Univ., Cambridge, Mass.

NOTATION

The following notation is used:

P = load transmitted through a tire to a panel of the pavement; if a part of the total load transmitted through the tire is on the other side of a joint or outside the edge, only the part transmitted to the panel under consideration is included in P;

a, b = semiaxes of an ellipse representing the footprint of a tire; if the load is next to an edge or joint, a is the semiaxis parallel to the edge or joint; either a or b may be the greater semiaxis, depending on whether the joint is longitudinal or transverse;

x, y = horizontal rectangular coordinates; if the load is next to an edge or joint, the axis of x is along the line of the edge or joint; if the footprint of the tire is represented by an ellipse, the axis of x is in the direction of the semiaxis a;

r, θ = corresponding polar coordinates;

z = deflection;

z_e = deflection at an edge or at a joint that has no capacity for load transfer;

z_j, z'_j = deflections at the line of a joint on the two sides of the joint when the joint has some capacity for load transfer;

σ_x, σ_y = tensile stresses at the bottom of the slab in the directions of x and y in the interior of the area of the panel;

σ_e = tensile stress at the bottom of the slab along an edge or along a joint that has no capacity for load transfer;

σ_j, σ'_j = tensile stresses along a joint at the bottom of the slabs on the two sides of the joint when the joint has some capacity for load transfer;

h = thickness of the slab, assumed constant;

E = modulus of elasticity of the concrete in the slab, assumed constant;

μ = Poisson's ratio of the concrete;

k = modulus of subgrade reaction, assumed to be a constant, measurable in pounds per cubic inch, and defining the reaction under the slab per unit of area as $k\,z$;

l = radius of relative stiffness, defined as:

$$l^4 = \frac{E\,h^3}{12\,(1 - \mu^2)\,k} \qquad (1)$$

Natural logarithms are used unless otherwise stated.

ASSUMPTIONS

The assumptions usually made in the study of slabs on a subgrade are: (1) Within the range of action the slab is elastic with single constant values of the modulus of elasticity E and Poisson's ratio μ; (2) the thickness, h, of the slab is constant; and (3) the reaction of the base on the slab is a vertical pressure, equal, per unit of area, to a constant k times the deflection z, the base being uniform in character and everywhere in contact with the slab. The constant k is called the modulus of subgrade reaction and is measurable in pounds per cubic inch. It is kept in mind that the third assumption is an empirical makeshift, which however has been found in the past to give usable results.

It is assumed furthermore—and this can be assumed without hesitation—that the average of the width and length of the footprint of the tire is greater than the thickness of the slab in all significant cases. Under this assumption, the ordinary theory of bending of plates or slabs can be used without the special corrections that would be needed if the loaded area were very small, as it sometimes must be considered to be in the corresponding problems of concrete highways.

Each panel of pavement is assumed to be fairly large and as yet uncracked. The load is assumed in no case to be close to more than one edge or joint; other edges or joints are assumed to be far enough away to have only a negligible influence on the stresses directly under the load.

STATEMENT OF THE FORMULAS

A. Stresses and Deflections in the Interior of the Area of a Panel.—

Case 1.—The load is applied in the interior of the area of the panel at a considerable distance from any edge or joint. The total load P is distributed uniformly over the area of the ellipse—

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The principal tensile stresses σ_x and σ_y, in the directions of x and y, at the bottom of the slab under the center of the load may be computed either by the formula—

$$\left.\begin{matrix}\sigma_x \\ \sigma_y\end{matrix}\right\} = \frac{3\,P}{8\,\pi\,h^2}\left[(1+\mu)\log\frac{E\,h^3}{k\left(\dfrac{a+b}{2}\right)^4} \mp 2\,(1-\mu)\,\frac{a-b}{a+b}\right] \ldots\ldots (3)$$

or by the equivalent formula—

$$\left.\begin{matrix}\sigma_x \\ \sigma_y\end{matrix}\right\} = \frac{P}{h^2}\left(0.275\,(1+\mu)\log_{10}\frac{E\,h^3}{k\left(\dfrac{a+b}{2}\right)^4} \mp 0.239\,(1-\mu)\,\frac{a-b}{a+b}\right) \ldots (4)$$

The deflection at any point x,y within or near the loaded area may be computed either by the formula—

$$z = \frac{P}{8\,k\,l^2}\left(1 - \frac{a^2+b^2+4\,x^2+4\,y^2}{16\,\pi\,l^2}\log\frac{E\,h^3}{k\left(\dfrac{a+b}{2}\right)^4} - \frac{a^2+4\,a\,b+b^2}{16\,\pi\,l^2} + \frac{(a-b)\,(x^2-y^2)}{2\,\pi\,l^2\,(a+b)}\right) \ldots (5)$$

(in which l is defined by Eq. 1) or by the equivalent formula—

$$
\begin{aligned}
z = {} & \frac{P}{4}\sqrt{\frac{3\,(1-\mu^2)}{E\,h^3\,k}} \\
& - \frac{0.275\,(1-\mu^2)\,P}{E\,h^3}\left[\frac{a^2+b^2}{4} + x^2 + y^2\right]\log_{10}\frac{E\,h^3}{k\left(\dfrac{a+b}{2}\right)^4} \\
& - \frac{0.239\,(1-\mu^2)\,P}{E\,h^3}\left[\frac{a^2+4\,a\,b+b^2}{8} + \frac{a-b}{a+b}\,(-x^2+y^2)\right] \quad \ldots . (6)
\end{aligned}
$$

Case 2.—The load again is applied in the interior of the area of the panel at a considerable distance from any edge or joint, but the load now has a more general type of distribution. The total load P is distributed uniformly over any area A having both axes of x and y as axes of symmetry. Such an area may be the footprint of either a single or a dual tire.

The principal stresses σ_x and σ_y at the bottom of the slab under the center of the load may be computed by the following formula:

$$
\left.\begin{matrix}\sigma_x \\ \sigma_y\end{matrix}\right\} = \frac{3\,P}{2\,\pi\,h^2}\left[(1+\mu)\,(K+0.1159) \pm \frac{1-\mu}{2}\,S\right] \ldots\ldots\ldots (7)
$$

in which K and S are "area coefficients" defined by the equations:[2]

$$
K = -\frac{1}{A}\int dA \log\frac{r}{l} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)
$$

and

$$
S = -\frac{1}{A}\int dA \cos 2\,\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)
$$

The integrations in Eqs. 8 and 9 are extended over the entire loaded area, and l is defined by Eq. 1.

The deflection at any point x,y within or near the loaded area may be computed by the formula:

$$
\begin{aligned}
z = \frac{P}{8\,k\,l^2} - \frac{P}{8\,\pi\,k\,l^4}\Bigg[& \frac{1}{A}\int r^2\,dA \log\frac{l}{r} + \frac{1.1159}{A}\int r^2\,dA \\
& + (K+0.1159)\,(x^2+y^2) + \tfrac{1}{2}\,S\,(x^2-y^2)\Bigg] \ldots (10)
\end{aligned}
$$

B. Stresses and Deflections at an Edge or at a Joint That Has No Capacity for Load Transfer.—

Case 3.—The load is next to an edge or next to a joint that has no capacity for load transfer. The axis of x is along the edge or joint. The total load P is distributed uniformly over the area of an ellipse which is tangent to the edge or joint and has the equation:

$$
\frac{x^2}{a^2} + \frac{(y-b)^2}{b^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)
$$

---

[2] "Stress Concentrations in Plates Loaded Over Small Areas," by H. M. Westergaard, *Transactions*, ASCE, Vol. 108, 1943, pp. 831–856, especially p. 835, Eqs. 21 and 22.

The tensile stress at the bottom along the edge or joint directly under the point of tangency of the ellipse, that is, at $x = y = 0$, may be computed either by the formula—

$$\sigma_e = \frac{3\,(1 + \mu)\,P}{\pi\,(3 + \mu)\,h^2} \left\{ \log \frac{E\,h^3}{100\,k \left(\frac{a + b}{2} \right)^4} + 1.84 - \frac{4}{3}\,\mu + (1 + \mu)\,\frac{a - b}{a + b} + 2\,(1 - \mu)\,\frac{a\,b}{(a + b)^2} + 1.18\,(1 + 2\,\mu)\,\frac{b}{l} \right\} \quad \ldots.(12)$$

or by the equivalent formula—

$$\sigma_e = \frac{2.2\,(1 + \mu)\,P}{(3 + \mu)\,h^2} \log_{10} \frac{E\,h^3}{100\,k \left(\frac{a + b}{2} \right)^4} + \frac{3\,(1 + \mu)\,P}{\pi\,(3 + \mu)\,h^2} \left[1.84 - \frac{4}{3}\,\mu + (1 + \mu)\,\frac{a - b}{a + b} + 2\,(1 - \mu)\,\frac{a\,b}{(a + b)^2} + 1.18\,(1 + 2\,\mu)\,\frac{b}{l} \right] \quad ..(13)$$

in which l, as elsewhere, is the radius of relative stiffness defined by Eq. 1.

The deflection at any point on the axis of symmetry perpendicular to the edge or joint at a moderate distance y from the edge or joint (or at the edge or joint itself) may be computed by the approximate formula:

$$z = \frac{P\,\sqrt{2 + 1.2\,\mu}}{\sqrt{E\,h^3\,k}} \left[1 - (0.76 + 0.4\,\mu)\,\frac{b}{l} \right] \left[1 - (0.76 + 0.4\,\mu)\,\frac{y}{l} \right] \quad ..(14)$$

The character of Eq. 14 as an approximate formula is indicated by the omission of quadratic terms in y, or in a and b. These quadratic terms are relatively unimportant when both b and y have moderate values.

Case 4.—As in Case 3, the load is again next to an edge or next to a joint that has no capacity for load transfer, but the total load P is now distributed uniformly over the area of one half of an ellipse. The ellipse has the principal radii a and b, and the diameter $2\,a$ is along the edge or joint. This condition may occur when one half of the footprint of the tire is outside the panel under consideration; P is then one half of the total pressure transmitted through the tire.

The tensile stress at the bottom of the slab along the edge or joint under the center of the ellipse may be computed by the following formula:

$$\sigma_e = \frac{2.2\,(1 + \mu)\,P}{(3 + \mu)\,h^2} \log_{10} \frac{E\,h^3}{100\,k \left(\frac{a + b}{2} \right)^4} + \frac{3\,(1 + \mu)\,P}{\pi\,(3 + \mu)\,h^2} \left[3.84 - \frac{4}{3}\,\mu - (1 - \mu)\,\frac{a - b}{a + b} + 0.5\,(1 + 2\,\mu)\,\frac{b}{l} \right] \quad ..(15)$$

The deflection at the center of the ellipse or at any other point on the axis of symmetry at a moderate distance y from the edge or joint may be computed by the approximate formula:

$$z = \frac{P\sqrt{2 + 1.2\,\mu}}{\sqrt{E\,h^3\,k}}\left[1 - (0.323 + 0.17\,\mu)\frac{b}{l}\right]\left[1 - (0.76 + 0.4\,\mu)\frac{y}{l}\right]\ldots(16)$$

Case 5.—As in Cases 3 and 4, the load is next to an edge or next to a joint that has no capacity for load transfer, but, as in Case 2, the load has a more general type of distribution. The total load P is distributed uniformly over any area A having the axis of y as an axis of symmetry. The axis of x is along the edge or joint.

The tensile stress at the bottom of the slab along the edge or joint at the point $x = y = 0$ may be computed by the formula:

$$\sigma_e = \frac{3\,(1 + \mu)\,P}{\pi\,(3 + \mu)\,h^2}\left[4\,K - 0.28 - \frac{4}{3}\mu - \mu^2 + (1 - \mu)\,S + 1.18\,(1 + 2\,\mu)\frac{\bar{y}}{l}\right]\ldots(17)$$

in which $\bar{y}$ is the distance from the edge or joint to the center of gravity of the load, l is the radius of relative stiffness defined by Eq. 1, and K and S are the area coefficients defined by Eqs. 8 and 9. In computing K and S by these formulas it should be noted that in this case, unlike Case 2, the axis of x, being along the edge or joint, is not an axis of symmetry.

The deflection at any point on the axis of symmetry perpendicular to the edge or joint at a moderate distance y from the edge or joint (or at the edge or joint itself) may be computed by the approximate formula:

$$z = \frac{P\sqrt{2 + 1.2\,\mu}}{\sqrt{E\,h^3\,k}}\left[1 - (0.76 + 0.4\,\mu)\frac{\bar{y}}{l}\right]\left[1 - (0.76 + 0.4\,\mu)\frac{y}{l}\right]\ldots(18)$$

C. Stresses and Deflections at a Joint That Has Some Capacity for Load Transfer.—

Case 6.—The load is at a joint and may be partly on one side and partly on the other side of the joint. The transfer devices are either continuous along the joint or are closely spaced. The joint is capable of transferring load with an efficiency j which may have any value in the range $0 < j \leqq 1$. The extreme value $j = 1$ represents 100% efficiency, which means that the deflections of the two adjacent slabs are certain to be equal at the line of the joint. The other extreme $j = 0$ would mean no transfer of load as in the preceding Cases 3 to 5. The meaning given here to any intermediate value of j is expressed by a formula which is recognized to be approximate only, but which nevertheless will serve the present purpose of estimating the relief of stress by load transfer; the formula is:

$$z_j - z'_j = (1 - j)\,(z_e - z'_e)\ldots\ldots\ldots\ldots(19)$$

in which z_j and z'_j are the deflections of the adjacent slabs at any point at the joint, while z_e and z'_e are the corresponding deflections that would occur at the same place if the joint had no capacity for load transfer. Eq. 19 must be considered as approximate only, because j is assumed to be constant over some distance.

Under these assumptions the tensile stresses σ_j and σ'_j at the bottom of the two slabs along the joint may be computed by the formulas:

$$\sigma_j = (1 - \tfrac{1}{2} j)\, \sigma_e + \tfrac{1}{2} j\, \sigma'_e \qquad (20)$$

and

$$\sigma'_j = \tfrac{1}{2} j\, \sigma_e + (1 - \tfrac{1}{2} j)\, \sigma'_e \qquad (21)$$

in which σ_e and σ'_e are the corresponding stresses that would be produced if the joint had no capacity for load transfer. That is, depending on the distribution of the load, σ_e and σ'_e may be defined by such formulas as Eqs. 13 15, or 17.

Similarly, the deflections of the two slabs at the joint may be computed by the formulas:

$$z_j = (1 - \tfrac{1}{2} j)\, z_e + \tfrac{1}{2} j\, z'_e \qquad (22)$$

and

$$z'_j = \tfrac{1}{2} j\, z_e + (1 - \tfrac{1}{2} j)\, z'_e \qquad (23)$$

It is observed that Eqs. 22 and 23 obey Eq. 19.

NUMERICAL EXAMPLES

The following values are selected as common to all the numerical examples:

(*a*) The total load transmitted through one tire is 50,000 lb. The footprint of the tire is taken as an ellipse with an average diameter $a + b = 30$ in. Either a or b is 20 in.

(*b*) The thickness of the pavement is $h = 10$ in.

(*c*) The modulus of elasticity and Poisson's ratio of the concrete are $E = 4{,}000{,}000$ lb-in.$^{-2}$ and $\mu = 0.15$. The modulus of subgrade reaction is $k = 200$ lb-in.$^{-3}$

With these values one finds the following constants which recur in the computations; the first being the radius of relative stiffness defined by Eq. 1: $l = 36.14$ in.; $l^2 = 1{,}305.8$ in.^{2}; and

$$\log_{10} \frac{E\, h^3}{100\, k \left(\dfrac{a + b}{2}\right)^4} = 0.5967 \qquad (24)$$

Example 1.—The load is in the interior of the area of the panel. The semiaxes of the loaded ellipse are $a = 20$ in. and $b = 10$ in.

The stresses σ_x and σ_y under the center of the load in the directions of the longer and shorter axis, respectively, are computed by Eq. 4 as

$$\left.\begin{matrix}\sigma_x \\ \sigma_y\end{matrix}\right\} = (410.6 \mp 33.9)\ \text{lb-in.}^{-2} = \left\{\begin{matrix}376.7\ \text{lb-in.}^{-2} \\ 444.5\ \text{lb-in.}^{-2}\end{matrix}\right. \qquad (25)$$

The three terms in Eq. 6 and the resultant deflection under the center of the load at $x = y = 0$ are stated as follows:

$$z = (0.02393 - 0.00109 - 0.00047)\text{ in.} = 0.02237\text{ in.} \tag{26}$$

The same results are obtained from Eq. 5.

At the end of the major axes, at $x = a$, $y = 0$, the value in Eq. 26 is changed, according to Eq. 6, by multiplying the second term by $\frac{21}{5}$ and the third by $\frac{7}{39}$. This gives the deflection at that point as $z = 0.01927$ in.

Example 2.—The load is next to a longitudinal joint that has no capacity for load transfer. The loaded elliptic area is tangent to the joint. The semiaxes are $a = 20$ in. and $b = 10$ in.

The stress at the joint is computed by Eq. 13. The two terms and the resulting value are as follows:

$$\sigma_e = (239.6 + 501.2)\text{ lb-in.}^{-2} = 740.8\text{ lb-in.}^{-2} \tag{27}$$

Eq. 14 gives the following values of the deflections z_e at the joint and z under the center of the load:

$$z_e = 0.0638\text{ in.} \qquad \text{and} \qquad z = 0.0493\text{ in.} \tag{28}$$

Example 3.—The load is next to a transverse joint that has no capacity for load transfer. The loaded elliptic area is tangent to the joint. The semiaxes are $a = 10$ in. and $b = 20$ in. The values in Eqs. 27 and 28 are changed by the exchange of the values of a and b to

$$\sigma_e = 663.8\text{ lb-in.}^{-2}, \quad z_e = 0.045\text{ in.}, \quad \text{and} \quad z = 0.025\text{ in.} \tag{29}$$

but the last of these values should be considered only as a crude approximation because the distance of 20 in. from the joint to the center of the load is rather large for this computation.

Example 4.—The load stradles a longitudinal joint, with half on each side; $a = 20$ in. and $b = 10$ in. The load on one side of the joint is 25,000 lb. The efficiency of the joint does not enter into the computation. Eqs. 15 and 16 give the stress and the deflection at the joint as

$$\sigma_e = 428.0\text{ lb-in.}^{-2} \qquad \text{and} \qquad z_e = 0.0373\text{ in.} \tag{30}$$

It is noted that the stress is a little less than σ_y in Example 1, in which the load is in the interior of the area, but the deflection is conspicuously greater.

Example 5.—The load is next to a longitudinal joint which has an efficiency $j = 0.8$ (or 80%) for load transfer. The loaded elliptic area is tangent to the joint. The semiaxes are $a = 20$ in. and $b = 10$ in. With σ_e taken from Eq. 27, Eqs. 20 and 21 give the following stresses on the two sides of the joint:

$$\sigma_j = 740.8\text{ lb-in.}^{-2}\,(1 - \tfrac{1}{2}\,0.8) = 444.5\text{ lb-in.}^{-2} \quad \text{and} \quad \sigma'_j = 296.3\text{ lb-in.}^{-2} \tag{31}$$

Similarly Eqs. 22, 23, and 28 give the deflections on the two sides of the joint:

$$z_j = 0.0383\text{ in.} \qquad \text{and} \qquad z'_j = 0.0255\text{ in.} \tag{32}$$

It is observed that σ_j in Eqs. 31 has the same value as σ_y in Eq. 25. This indicates that, under the circumstances, a joint efficiency of 80% or more would be desirable.

Example 6.—The load is next to a transverse joint which has an efficiency $j = 0.66$ for load transfer. The loaded elliptic area is tangent to the joint. The semiaxes are $a = 10$ in. and $b = 20$ in. Eqs. 20, 22, and 29 give:

$$\sigma_j = 444.7 \text{ lb-in.}^{-2} \quad \text{and} \quad z_j = 0.030 \text{ in.} \dots\dots\dots\dots (33)$$

DERIVATIONS

A. Stresses and Deflections in the Interior of the Area of a Panel.—

Eqs. 7, 8, and 9, which define the stresses when the loaded area has any shape, are taken from a former[3] paper by the writer, in which the use of area coefficients such as K and S in Eqs. 8 and 9, and place coefficients such as the constant 0.1159 in Eq. 7 were introduced.[3] Eq. 3 defines the stresses in the interior of the area when the loaded area is the ellipse in Eq. 2. It is obtained from Eq. 7 by substituting l from Eq. 1, omitting within the brackets a small positive term containing μ^2 and a nearly balancing negative term. By substituting the following values[4] for K and S:

$$K = \log \frac{2\,l}{a + b} + \tfrac{1}{2} \quad \text{and} \quad S = -\frac{a - b}{a + b} \dots\dots\dots (34)$$

Eq. 4 is a transcription of Eq. 3. With $a = b$ (that is, when the ellipse is a circle), Eq. 4 agrees with formulas published earlier.[5,6]

In a paper published in 1940,[7] the writer showed that even with a fairly large circular footprint of the tire the correction that would have to be added to the stress in the original formulas because of the large radius would be small. For example, with $a = b = 15$ in., but with other values as in the present numerical Example 1, the added stress would be only 4.6 lb-in.$^{-2}$. Such a small correction may well be ignored, and the correction of this type has been ignored in Eqs. 3, 4, and 7.

The 1940 paper[8] also contains a formula for the deflection at the center of a loaded circular area. That formula agrees with, and thus confirms, Eqs. 5 and 6 when these equations are applied with $a = b$ and $x = y = 0$; but Eqs. 5 and 6 apply when the loaded area is an ellipse. Eq. 5 is derived from Eq. 10 which applies when the loaded area is of a general type, but symmetrical about both axes of x and y. In performing this derivation the first integral in Eq. 10 can be obtained in a finite form; the second integral is merely the polar moment of inertia of the area; and K and S are given in Eqs. 34.

It remains to show the validity of Eq. 10. The terms containing x^2 and y^2 are verified by noting that they define the same curvatures as do the stresses in Eq. 7. On account of the symmetry of the load there can be no linear terms in x and y. The remaining terms in Eq. 10 are correct if the deflection due to a concentrated load P at the distance r from the point of application of P can be stated by the formula:

---

[3] "Stress Concentrations in Plates Loaded Over Small Areas," by H. M. Westergaard, *Transactions*, ASCE, Vol. 108, 1943, p. 834 et seq., especially Eqs. 15, 21, 22, and 51.

[4] *Ibid.*, pp. 840 and 841, Eqs. 51, 65, and 66.

[5] "Stresses in Concrete Pavements Computed by Theoretical Analysis," by H. M. Westergaard, *Public Roads*, April, 1926, pp. 25–35, especially Eq. 10.

[6] "Analytical Tools for Judging Results of Structural Tests of Concrete Pavements," by H. M. Westergaard, *ibid.*, December, 1933, pp. 185–188, especially Eq. 4.

[7] "Stresses in Concrete Runways of Airports," by H. M. Westergaard, *Proceedings*, 19th Annual Meeting, Highway Research Board, December, 1939 (published in 1940), pp. 197–202, especially Eqs. 6, 7, and 50.

[8] *Ibid.*, p. 199, Eq. 11.

$$z = \frac{P}{8\,k\,l^2} - \frac{P\,r^2}{8\,\pi\,k\,l^4}\left(\log\frac{l}{r} + 1.1159\right) \qquad (35)$$

Eq. 35 is obtained from the exact solution,[9,10] by omitting terms containing r^4 and higher powers of r. This concludes the verification of Eqs. 5, 6, and 10.

B. Stresses and Deflections at an Edge or at a Joint That Has No Capacity for Load Transfer.—

The basis of Eqs. 17 and 18, which apply when the load is next to an edge and is distributed over a not too large area of any shape, is two formulas which are taken from a paper published in 1923.[11] The first defines the deflection z produced at the point x,y by a load P at $x = y = 0$ when the axis of x is along an edge and the slab covers the entire region $y > 0$. The second formula, which is used as an auxiliary equation, defines in a related form the deflection z_i, produced at the point x,y with $y \geqq 0$ by a load P at $x = y = 0$ when the slab covers the entire xy-plane. The formulas are:

$$z = \frac{2\,P}{\pi\,k\,l^2}\int_0^\infty \frac{\gamma\cos\frac{\alpha\,x}{l}\left[\cos\frac{\beta\,y}{l} + (1-\mu)\,\alpha^2\sin\frac{\beta\,y}{l}\right]e^{-\gamma y/l}\,d\alpha}{1 + 4\,(1-\mu)\,\alpha^2\,\gamma^2 - (1-\mu)^2\,\alpha^4} \qquad (36)$$

and

$$z_i = \frac{P}{2\,\pi\,k\,l^2}\int_0^\infty \frac{\cos\frac{\alpha\,x}{l}\left[\beta\cos\frac{\beta\,y}{l} + \gamma\sin\frac{\beta\,y}{l}\right]e^{-\gamma y/l}\,d\alpha}{\beta^2 + \gamma^2} \qquad (37)$$

in which α, β, and γ are three positive values obeying the relations:

$$\alpha^2 + \beta^2 = \gamma^2 \qquad \text{and} \qquad 2\,\beta\,\gamma = 1 \qquad (38)$$

Eqs. 36 and 37 can be verified by showing that they both satisfy the governing differential equation—

$$l^4\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 z + z = 0 \qquad (39)$$

both give values converging toward zero as the distance goes to infinity; and both account for the load and satisfy the appropriate conditions at $y = 0$.

According to the Maxwell theorem of reciprocal deflections, z in Eq. 36 may also be interpreted as the deflection produced at the origin of the coordinates by a load P at point x,y. It follows that the diagram of deflections produced at the edge, at $y = 0$, by a load P at point $0,y$ may be obtained as the diagram of deflections produced at the distance y from the edge by a load P at the origin. It follows from this that the curvature $-\frac{\partial^2 z}{\partial x^2}$ produced at the origin by a load

---

[9] "Stress Concentrations in Plates Loaded Over Small Areas," by H. M. Westergaard, *Transactions*, ASCE, Vol. 108, 1943, p. 839, Eq. 48.

[10] "Stresses in Concrete Runways of Airports," by H. M. Westergaard, *Proceedings*, 19th Annual Meeting, Highway Research Board, December, 1939 (published in 1940), p. 197, Eqs. 30 and 34.

[11] "Om Beregning af Plader paa elastisk Underlag med særligt Henblik paa Spørgsmaalet om Spændinger i Betonveje," by H. M. Westergaard, *Ingeniøren*, Vol. 32, 1923, pp. 513–524, especially Eqs. 47, 30, and 34, and also Eqs. 48 and 33.

P at point x,y may be obtained as the curvature $-\frac{\partial^2 z}{\partial x^2}$ produced at point x,y by a load P at the origin of the coordinates.

The curvature in the direction of x at the origin of the coordinates defines the tensile stress along the bottom of the edge at that point as:

$$\sigma_e = -\frac{E\,h}{2}\frac{\partial^2 z}{\partial x^2} \quad \text{(40)}$$

According to the dual interpretation of the curvatures, if the load P is at point x,y, the function z in Eq. 40 is also the function z defined by Eq. 36. By use of Eq. 1 this gives the stress at the edge due to P at point x,y as:

$$\sigma_e = \frac{12\,(1-\mu^2)\,P}{\pi\,h^2}\int_0^\infty \frac{\alpha^2\,\gamma\cos\frac{\alpha\,x}{l}\left[\cos\frac{\beta\,y}{l} + (1-\mu)\,\alpha^2\sin\frac{\beta\,y}{l}\right]e^{-\gamma y/l}\,d\alpha}{1 + 4\,(1-\mu)\,\alpha^2\,\gamma^2 - (1-\mu)^2\,\alpha^4} \quad \text{(41)}$$

The function σ_e in Eq. 41 has a singularity at $x = y = 0$, with an infinite value at that point. The difficulty involved in this is overcome by dividing σ_e into two component parts, writing

$$\sigma_e = \bar{\sigma} + \sigma_c \quad \text{(42)}$$

and choosing the part $\bar{\sigma}$ so that it will have the same type of singularity as σ_e but will be of a simpler form. This leaves σ_c as a smooth function with a finite value at the origin of the coordinates.

The choice of the function $\bar{\sigma}$ is aided by noting the type of the singularity that the elastic surface of any plate must have at the point of application of a concentrated force applied at a free edge. A. Nádai[12] has defined that type. In this way one arrives at the following possible form of $\bar{\sigma}$:

$$\bar{\sigma} = -\frac{12\,(1+\mu)\,k\,l^4}{(3+\mu)\,h^2}\left[(3-\mu)\frac{\partial^2 z_i}{\partial x^2} + (1+\mu)\frac{\partial^2 z_i}{\partial y^2}\right] \quad \text{(43)}$$

in which z_i is the deflection produced at point x,y by a load P at the origin of the coordinates in a slab that covers the entire x,y-plane. The function z_i can be defined by Eq. 37; but in the vicinity of the origin of the coordinates the two second derivatives in Eq. 43 may also be expressed in the manner of Eq. 7 by the formula:[13]

$$\left.\begin{array}{r} -\dfrac{\partial^2 z}{\partial x^2} \\[2ex] -\dfrac{\partial^2 z}{\partial y^2} \end{array}\right\} = \frac{P}{4\,\pi\,k\,l^4}\left(\log\frac{l}{r} + 0.1159 \mp \tfrac{1}{2}\cos 2\,\theta\right) \quad \text{(44)}$$

This gives

$$\bar{\sigma} = \frac{12\,(1+\mu)\,P}{\pi\,(3+\mu)\,h^2}\left(\log\frac{l}{r} + 0.1159 - \frac{1-\mu}{4}\cos 2\,\theta\right) \quad \text{(45)}$$

The remaining part of σ_e is obtained as

$$\sigma_c = \sigma_e - \bar{\sigma} \quad \text{(46)}$$

---

[12] "Die elastischen Platten," by A. Nádai, Julius Springer, Berlin, 1925, p. 204.

[13] "Stress Concentrations in Plates Loaded Over Small Areas," by H. M. Westergaard, *Transactions*, ASCE, Vol. 108, 1943, p. 835 et seq., Eqs. 17, 21, 22, and 51.

with σ_e as in Eq. 41 and with $\bar{\sigma}$ again defined by Eq. 43, but this time with z_i substituted from Eq. 37, which thereby serves its purpose as an auxiliary formula. This gives:

$$\sigma_c = -\frac{24\,(1+\mu)\,P}{\pi\,(3+\mu)\,h^2}\int_0^\infty \frac{\cos\dfrac{\alpha\,x}{l}\,[1+(1-\mu)^2\,\alpha^2\,\beta^2]\,\beta\cos\dfrac{\beta\,y}{l}\,e^{-\gamma y/l}\,d\alpha}{1+4\,(1-\mu)\,\alpha^2\,\gamma^2-(1-\mu)^2\,\alpha^4}$$

$$-\frac{12\,(1-\mu^2)\,P}{\pi\,(3+\mu)\,h^2}\int_0^\infty \frac{\cos\dfrac{\alpha\,x}{l}\,[1+4\,(1-\mu)\,\alpha^2\,\beta^2]\,\gamma\sin\dfrac{\beta\,y}{l}\,e^{-\gamma y/l}\,d\alpha}{1+4\,(1-\mu)\,\alpha^2\,\gamma^2-(1-\mu)^2\,\alpha^4}$$

$$+\frac{6\,(1+\mu)\,(3-\mu)\,P}{\pi\,(3+\mu)\,h^2}\int_0^\infty \frac{\cos\dfrac{\alpha\,x}{l}\left[2\,\beta^3\cos\dfrac{\beta\,y}{l}+\beta\sin\dfrac{\beta\,y}{l}\right]e^{-\gamma y/l}\,d\alpha}{\beta^2+\gamma^2}\;\ldots(47)$$

in which, as before, α, β, and γ are defined by Eqs. 38.

As was anticipated, σ_c in Eq. 47 is in fact a smooth function without infinite values in the half plane $y \geqq 0$. In the region of small values of x and y the function σ_c may be written with adequate accuracy as a linear function of y in the form:

$$\sigma_c = [\sigma_c]_{x=y=0} + y\left[\frac{\partial\sigma_c}{\partial y}\right]_{x=y=0}\;\ldots\ldots\ldots\ldots(48)$$

In evaluating the constant term in Eq. 48 it is noted that:

$$\int_0^\infty \frac{\beta^3\,d\alpha}{\beta^2+\gamma^2} = \tfrac{1}{4}\;\ldots\ldots\ldots\ldots(49)$$

This leads to the following form of Eq. 48:

$$\sigma_c = \frac{12\,(1+\mu)\,P}{\pi\,(3+\mu)\,h^2}\left(\frac{3-\mu}{4} - B_1 + B_2\,\frac{y}{l}\right)\;\ldots\ldots\ldots\ldots(50)$$

in which B_1 and B_2 are dimensionless constants defined by the formulas:

$$B_1 = 2\int_0^\infty \frac{[1+(1-\mu)^2\,\alpha^2\,\beta^2]\,\beta\,d\alpha}{1+4\,(1-\mu)\,\alpha^2\,\gamma^2-(1-\mu)^2\,\alpha^4}\;\ldots\ldots\ldots\ldots(51)$$

and

$$B_2 = \tfrac{1}{2}\int_0^\infty \frac{[1+\mu-2\,(1-\mu)^2\,\alpha^2\,\beta^2]\,d\alpha}{1+4\,(1-\mu)\,\alpha^2\,\gamma^2-(1-\mu)^2\,\alpha^4}\;\ldots\ldots\ldots\ldots(52)$$

When the load P is distributed over an area A instead of being concentrated at the point x,y, Eq. 45 is replaced by

$$\bar{\sigma} = \frac{12\,(1+\mu)\,P}{\pi\,(3+\mu)\,h^2}\left(K + 0.1159 + \frac{1-\mu}{4}\,S\right)\;\ldots\ldots\ldots\ldots(53)$$

in which K and S are the place coefficients defined by Eqs. 8 and 9; and Eq. 50 is modified by replacing y by the distance $\bar{y}$ from the edge to the center of gravity of the load. Then Eqs. 42, 50, and 53 lead to the following formula for the tensile stress along the bottom of the edge at the point $x = y = 0$ due to the actual load:

$$\sigma_e = \frac{12\,(1+\mu)\,P}{\pi\,(3+\mu)\,h^2}\left[K + 0.8659 - \frac{\mu}{4} - B_1 + \frac{1-\mu}{4}S + B_2\frac{\bar{y}}{l}\right] \ldots (54)$$

in which B_1 and B_2 are defined by Eqs. 51 and 52.

Like the part σ_e of the stress, the deflection z_e at the edge at $x = y = 0$ can be expressed with adequate accuracy as a linear function of the distance $\bar{y}$ from the edge, provided that the loaded area is not too great. Thus one may write

$$z_e = [z]_{x=y=0} + \bar{y}\left[\frac{\partial z}{\partial y}\right]_{x=y=0} \ldots\ldots\ldots\ldots\ldots\ldots (55)$$

or, in terms of two dimensionless constants B_3 and B_4,

$$z_e = \frac{P}{k\,l^2}\left[B_3 - B_4\frac{\bar{y}}{l}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (56)$$

The values of B_3 and B_4 are determined through Eqs. 36 and 55 and are found to be

$$B_3 = \frac{1}{M\sqrt{8}}\left(\frac{M+1-\mu}{\sqrt{2-\mu+M}} + \frac{M-1+\mu}{\sqrt{2-\mu-M}}\right) \ldots\ldots\ldots (57)$$

in which the quantity M is defined as

$$M = \sqrt{(1-\mu)^2 + \mu^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (58)$$

and

$$B_4 = \frac{1}{\pi}\int_0^\infty \frac{[(1+\mu)\,\alpha^2 + 2\,\beta^2]\,d\alpha}{1 + 4\,(1-\mu)\,\alpha^2\,\gamma^2 - (1-\mu)^2\,\alpha^4} \ldots\ldots\ldots (59)$$

in which, as before, α, β, and γ are positive values defined by Eqs. 38.

Numerical values of the constants B_1, B_2, B_3, and B_4, as defined by Eqs. 51, 52, 57, and 59, and computed by members of the staff of the David Taylor Model Basin, United States Navy Department, at Carderock, Md., are given in Table 1. Numerical studies of these values have led to the following formulas through which the four constants are determined with quite satisfactory accuracy within the range of μ, and are given in forms that lend themselves to the final formulas:

TABLE 1.—VALUES OF THE CONSTANTS B_1, B_2, B_3, AND B_4 IN EQS. 54 AND 56 FOR DIFFERENT VALUES OF POISSON'S RATIO μ

μ	B_1	B_2	B_3	B_4
0 . . .	0.9352	0.2952	0.4082	0.3118
0.15. . .	0.9544	0.3822	0.4314	0.3510
0.2 . . .	0.9627	0.4131	0.4406	0.3715
0.3 . . .	0.9830	0.4786	0.4620	0.4106

$$4\,(B_1 - 0.8659) = 0.28 + \tfrac{1}{3}\,\mu + \mu^2 \ldots\ldots\ldots\ldots\ldots (60)$$

$$4\,B_2 = 1.18\,(1 + 2\,\mu) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (61)$$

$$B_3\sqrt{12\,(1-\mu^2)} = \sqrt{2 + 1.2\,\mu} \ldots\ldots\ldots\ldots\ldots (62)$$

$$B_4 = (0.76 + 0.4\,\mu)\,B_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (63)$$

Substitution from Eqs. 60 and 61 in Eq. 54 leads directly to Eq. 17 which defines σ_e for any shape of the loaded area. Eqs. 12, 13, and 15 are specific forms of Eq. 17, which are derived by use of Eq. 1, by replacing $\log(1 - \mu^2)$ by $-\mu^2$, and by substituting appropriate values of K and S; namely, for Eq. 15 the values in Eqs. 34, and for Eqs. 12 and 13 the following values which apply at the ends of the axis $2\,b$ of the elliptic area:[4]

$$K = \log\frac{2\,l}{a+b} + \frac{a-b}{2\,(a+b)} \qquad \text{and} \qquad S = -\frac{a-b}{a+b} + \frac{2\,a\,b}{(a+b)^2} \,\ldots\ldots(64)$$

It should be mentioned that Eq. 15, which applies when the load is distributed over one half of an ellipse, does not become identical, when $a = b$, to the formula—given elsewhere[4]—for the case of a load distributed over a semicircle.[14,15] The stresses defined by Eq. 15 are greater than those defined by the previous formula.[14,15]

The value of z defined by Eq. 18 when $y = 0$ is derived from Eq. 56 by substitution from Eqs. 62 and 63 and by use of Eq. 1. The second quantity in brackets in Eq. 18, which extends the applicability of the formula to points near the edge, is written so as to make the complete formula conform to the Maxwell theorem of reciprocal deflections. Eqs. 14 and 16 are merely specific forms of Eq. 18.

This completes the verification of the formulas for stresses and deflections at an edge.

C. Stresses and Deflections at a Joint That Has Some Capacity for Load Transfer.—

Assume that the load P is entirely on side A of the joint. For the purpose of the analysis this load can be resolved into two component parts: Component I consists of $\frac{1}{2}\,P$ on side A and $\frac{1}{2}\,P$ placed symmetrically on side B; component II consists of $\frac{1}{2}\,P$ on side A and $-\frac{1}{2}\,P$ on side B. Component I produces the deflections $\frac{1}{2}\,z_e$ on each side. Component II produces equal and opposite deflections on the two sides. According to Eq. 19 these deflections are $\pm\frac{1}{2}\,(1 - j)\,z_e$. This leads to the resultant deflections defined by Eqs. 22 and 23 with $z'_e = 0$. The complete Eqs. 22 and 23 are obtained by superposition of the effects of loads on both sides. Eqs. 20 and 21 which define the stresses are consistent with Eqs. 22 and 23.

COMMENTS ON THE EVIDENCE OBTAINED FROM TESTS AND EXPERIENCE

Since the 1920's experimental evidence has accumulated supporting the claim of usability of the formulas published previously for stresses in concrete highways under loads distributed over a small circular area. The Public Roads Administration—formerly The Bureau of Public Roads—has conducted many of the tests. A comprehensive report in five parts published in 1935, 1936, and 1943 by L. W. Teller and Earl C. Sutherland,[16] Members, ASCE, of

---

[14] "Stresses in Concrete Pavements Computed by Theoretical Analysis," by H. M. Westergaard, *Public Roads*, April, 1926, p. 32, Eq. 12.

[15] "Analytical Tools for Judging Results of Structural Tests of Concrete Pavements," by H. M. Westergaard, *ibid.*, December, 1933, p. 185, Eq. 5.

[16] "The Structural Design of Concrete Pavements," by L. W. Teller and Earl C. Sutherland, *ibid.*, October, November, and December, 1935; September and October, 1936; and April–May–June, 1943.

the Division of Tests, of the Public Roads Administration, should be mentioned especially. The last part of this report contains extensive and careful comparisons of tests and theory. These include a very satisfactory verification of the relation defined by Eq. 7 between the stresses in the directions of the long and short axes of an oblong footprint of a tire.[17] The new formulas that have been published are derived from the same basic theory as the formulas published previously, and may thus be considered to be supported by the same experimental evidence.

The reasonableness (in the light of general experience) of the sequence of numerical values of stresses and deflections obtained in the six numerical examples may be considered as further evidence in favor of the formulas.

CONCLUSIONS

The formulas that have been presented lend themselves to three purposes: The interpretation of experimental evidence; the interpretation of practical experience with pavements in service; and the forming of judgments from which policies of design may be adopted to meet new conditions such as new types and new magnitudes of the loads.

In relation to the complexity of the problem, which involves a number of significant variables, the formulas are simple to use.

ACKNOWLEDGMENTS

The manuscript from which this paper was prepared was the result of studies made by the writer while serving on active duty at the Bureau of Yards and Docks, United States Navy Department, as Commander, Civil Engineer Corps, United States Naval Reserve. Throughout the studies their use by the Board of Consultants on Airfield Pavements, Office of the Chief of Engineers, United States War Department, was kept in mind. The opinions contained in this paper are the private ones of the writer and are not to be construed as official, or as reflecting the views of the Navy Department or of the Naval Service at large.

The writer is especially indebted to the Staff of the Mathematical Section, Technical Service, of the Navy's David Taylor Model Basin, at Carderock, for supplying the numerical computations of the constants in Table 1.

---

[17] *Transactions*, ASCE, Vol. 108, 1943, pp. 864–868.

DISCUSSION

ROBERT HORONJEFF,[18] JUN. ASCE.—The comments presented by the writer are not to be construed as a criticism of the author's presentation but are offered principally to illustrate, to the practicing engineer, the effect in varying the modulus of subgrade reaction, *k*. The paper presented by Professor Westergaard is an important contribution to the science of airfield pavement design.

Normally, the values of Poisson's ratio, μ, and the modulus of elasticity, E, are assumed, but the modulus of subgrade reaction, k, is measured in the field by the so-called "field bearing test." This test normally consists of loading a round 30-in. plate on the subgrade with a total load equivalent to 10 lb per sq in. (7,070 lb) and of recording the deformation in inches with this load. The load is applied almost instantaneously (10 sec) and is held until practically complete deformation has taken place. The modulus of soil reaction $k = \frac{10}{\Delta}$, in which Δ represents the total movement in inches during the application of the load.

In comparison with other routine tests on soils, the field bearing test is expensive. It requires that the load be applied by an hydraulic jack working against a jacking frame. The jacking frame usually consists of two trucks, an anchored steel frame, or any other device, as long as the reactions are at least 8 ft from the bearing plate. The purpose of this discussion is to show that refinements in measuring the modulus of subgrade reaction, k, are unwarranted and that, in most cases, k can be assumed without testing by comparison of the bearing value of the soil in question with similar soils which have already been tested.

To illustrate this point, two curves are presented. Fig. 1(*a*) represents the maximum tensile stresses obtained by substitution in Eq. 4, Case 1, in which the load is applied to the interior of the area of the panel at a considerable distance from any edge or joint. The wheel load, P, used in the example amounts to 60 kips. Poisson's ratio, μ, was assumed to be 0.20, and the modulus of elasticity, E, for concrete, 4,000,000 lb per sq in. The tire imprint area was assumed as an ellipse, the semiaxis, a, being equal to 17 in., and the semiaxis, b, equal to 10 in. As an example, assume that the allowable stress in the concrete is equal to 550 lb per sq in., and that the modulus of subgrade reaction, k, is 100. From the curves in Fig. 1(*a*), the required thickness of concrete is 11 in. If the value of k is 300, the required thickness is 10 in. In other words, for a relatively large difference in the k-value, the required thickness of concrete varies only by 1 in. Fig. 1(*b*) presents the stresses computed from substitution in Eq. 13, Case 3, in which the load is next to an edge or a joint that has no capacity for load transfer. The values of P, μ, E, a, and b are the same as in Case 1, Fig. 1(*a*). Similar comparisons for $k = 100$ and $k = 300$ reveal, again, that large variations of k do not result in large variations in concrete thickness.

---

[18] Chf., Civ. Works Branch, Corps of Engrs., San Francisco, Calif.; Lecturer in Civ. Eng., Univ. of California, Berkeley, Calif.

Engineers who deal with airport design are well aware that a soil whose *k*-value is 100 is considered a relatively poor subgrade, whereas a soil with a *k*-value of 300 is considered a good subgrade. In other words, the *k*-values presented in Fig. 1 represent a range of subgrades from poor to good. The *k*-value for gravel and gravelly soils varies from 200 to 500 whereas it may be as low as 50 for fine-grain soils having high compressibility. Several publications are available[19] which show the approximate range of *k*-values for soil groups classified by either the Public Roads Administration soils classification or the Casagrande soils classification.[20]

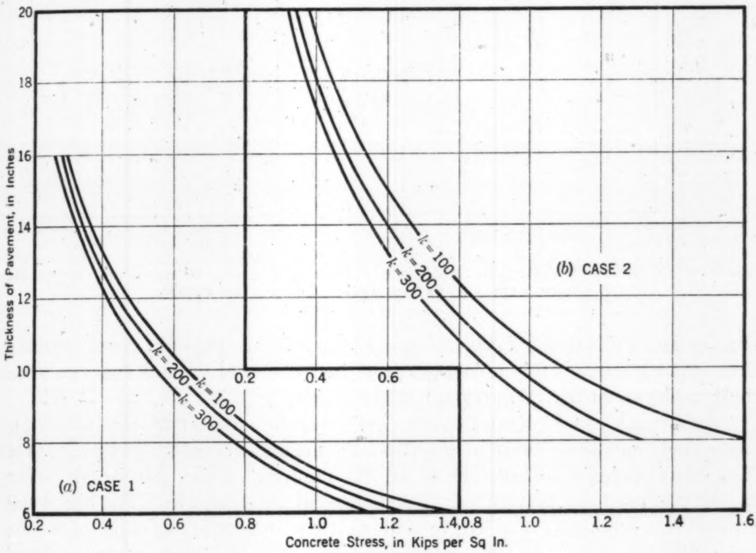


FIG. 1.—SIGNIFICANCE OF RELATIVE *k*-VALUES

This discussion and an examination of Fig. 1 reveal that the establishment of the *k*-value to the nearest 50 units is sufficiently accurate for most design problems, and that the project costs would not be adversely affected. Another conclusion that can be reached is that, for the average case, the engineer will find it more economical to increase the thickness of the pavement, rather than to improve the *k*-value of the subgrade by the addition of a better quality of material.

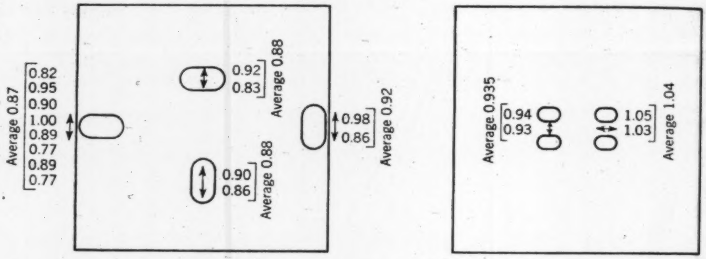
EVAN P. BONE,[21] ESQ.—The comprehensive formulas given in this paper prove to be very useful in the new design problems of pavements for heavy aircraft. It is possible to predict stresses under conditions not heretofore determinable from purely theoretical analysis. The design curves for rigid pavements given in an engineering manual[22] are based on Eq. 13, inasmuch as it has been determined that edge stress resulting from edge loading is critical for airfield pavements. Eq. 17, which is more general, has been found particularly applicable in the latest design problems of multiple-wheel landing gears.

---

[19] "Concrete Pavement Design," Portland Cement Assn., 1946.

[20] "Classification and Identification of Soils," by Arthur Casagrande, *Proceedings*, ASCE, June, 1947, p. 783.

[21] Engr., Rigid Pavement Laboratory, Corps of Engrs., Dept. of the Army, Mariemont, Ohio.



(*a*) SINGLE FOOTPRINT (*b*) DUAL FOOTPRINT

Fig. 2.—Plan of Slab Showing Ratio of Model Experimental Stress to Westergaard Theoretical Stress, with Arrows Indicating Directions of Stress

By the use of this formula it is now possible to compute the edge stress resulting from the superimposition of stresses from two wheels of various spacing, when both are symmetrical with respect to the y-axis.

As a rough check of the formulas used, experimental strain measurements were made on a scale model at the Rigid Pavement Laboratory of the Corps of Engineers under the direction of R. R. Philippe, Assoc. M. ASCE. The model consisted of a plaster slab on a sponge rubber subgrade. The horizontal dimensions, including the length and width of footprint, spacing of wheels (if multiple wheels), and the "radius of relative stiffness" were all proportioned to scale (approximately 1:20) corresponding to a chosen prototype. The scale value of the "radius of relative stiffness" was secured by use of Eq. 1, in which the terms E, μ, and k were taken as constants of the materials of which the model was constructed. The desired scale value of l was obtained by solving for h, which, for the plaster material used, equaled $\frac{1}{4}$ in. The subgrade modulus, k, of the sponge rubber was evaluated by measuring the volume displacement under a given load. The strains were measured by electrical gages of 1/8-in. active length.

Fig. 2 gives the ratios of the experimental stresses to the theoretical ones. Several items from which the average values were determined refer to different conditions and locations of the respective types of loading, as, for instance, the edge stresses at all four edges of the same model slab. It is observed that the average experimental stresses are lower than the corresponding theoretical stresses. This may be due to experimental error alone, but there are indications that the difference was influenced by the fact that the sponge rubber subgrade acted more as a semi-infinite body than as a liquid support, which defines k.

---

[22] "Ad Interim Engineering Manual for War Department Construction," Corps of Engrs., Office of the Chf. of Engrs., July, 1946, Pt. XII, Chapter 3, p. 16 (published by the War Dept., Washington, D. C.).

Fig. 3 shows a comparison of experimental and theoretical edge stress as the load moves across a joint onto an adjacent panel, the joint having no capacity for load transfer. These experimental stresses are consistently lower than the theoretical ones; yet the similarity of the shapes of the curves and locations of the peak stresses are significant to a distance approximately $2\,l$ from the edge of the slab.

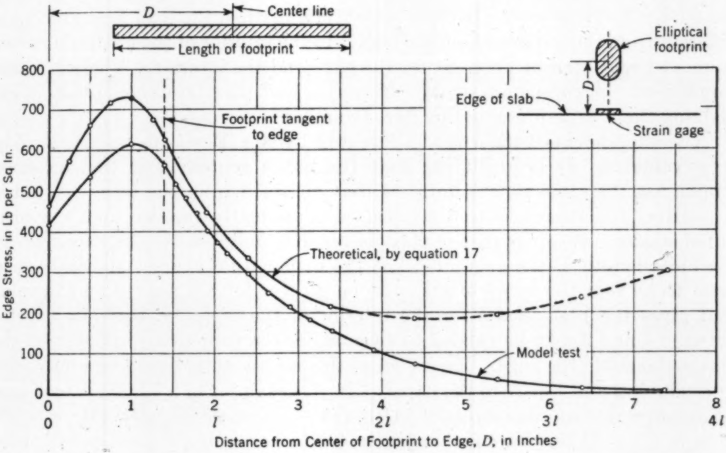


FIG. 3.—COMPARISON OF MODEL TEST SLAB STRESSES WITH WESTERGAARD THEORETICAL STRESSES ($P = 40.42$ LB; $h = 0.265$ IN.; $a = 0.845$ IN.; $b = 1.410$ IN.; $E = 2 \times 10^6$; $\mu = 0.2$; $k = 190$; AND $l = 2.0$ IN.)

Although the model measurements were not performed with a high degree of precision, they may serve to carry convincing verification of the validity of the theoretical analysis.

H. M. WESTERGAARD,[23] M. ASCE.—The curves presented by Mr. Horonjeff serve the useful purpose of showing the numerical meaning of the formulas when applied to a range of practical cases. These curves give renewed emphasis to the fact that the computed values of the stresses are not very sensitive even to fairly great variations of the modulus k of subgrade reaction. The particular test to determine k mentioned by Mr. Horonjeff deserves attention because it is fairly simple, can be made in advance, and has been found to give usable values of k within some range of applications. Outside this range, for example, in the case of very thick pavements, it is probable that values of k obtained directly from this test will require some coefficient of correction.

The fact is that—as has often been stated—the use of a definite value of k is only an empirical makeshift justified by usable results. The modulus k is not a simple mechanical property of the subgrade. The reaction of the subgrade at any one place depends not only on the deflection at that place but on the general pattern of deflections within the surrounding area; and the value of *k* that serves best under given circumstances depends on this pattern as well as on the mechanical properties of the subgrade itself. Therefore *k* becomes a property of the subgrade and the pavement and the character and position of the load. Thus, since the pattern of deflections when the load is at an edge is different from the pattern when the load is in the interior of the area, the proper value of *k* may well be different for the edge and the interior. These remarks supplement the comments by Mr. Horonjeff and may be of some help in explaining the results presented by Mr. Bone in his discussion.

---

[23] Gordon McKay Prof. of Civ. Eng., Graduate School of Eng., Harvard Univ., Cambridge, Mass.

The experiments with a model reported by Mr. Bone are unquestionably very valuable. It is gratifying that the curve representing the measured strains has the same general form as that obtained from the formulas, but it s puzzling that the measured strains are consistently smaller than the computed strains. A minor part of this discrepancy may possibly be explained by the finite length of the gage line, but the explanation nearest at hand would seem to be that the value of *k* should be greater for the edge than the value that gives the best results for the interior of the area. A correction factor between 1 and 2 may be needed to account for the supporting influence of the material outside the edge. To see what this means numerically one may refer to Example 2 and Eq. 27 in the paper. In this example of a 50,000-lb load on a 10-in. pavement an edge stress of 741 lb per sq in. was computed with $k = 200$ lb per cu in. If this *k* were changed to 400 lb per cu in., the computed stress would be reduced to 626 lb per sq in. This reduction is of the order of magnitude indicated by Mr. Bone's results.

Another possibility involves consideration of the energy of shearing stresses near the edge on vertical sections perpendicular to the edge. In the theory this energy was assumed to be relatively negligible. If it is found not to be negligible, this energy may account for a part of the discrepancy. A study of this influence would be difficult, but might prove to be worth undertaking.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2341

BEAM DEFLECTIONS BY SECOND AND THIRD MOMENTS

By Hsu Shih-Chang,[1] Esq.

With Discussion by Messrs. A. Floris, Robert B. B. Moorman, Frank J. McCormick, William A. Conwell, and Hsu Shih-Chang.

Synopsis

The various methods for computing beam deflections may be classified into two main groups—the double-integration method and the moment-area method. In both methods, beam deflection is considered a function of bending moment; and bending moment, in turn, is a function of the force system acting on the beam. In other words, deflection is computed, not directly from the forces acting on the beam, but rather indirectly through the medium of bending moment. Such indirect approach naturally makes the computations lengthy and complicated, and the aim of this paper is to suggest a more direct solution.

Introduction

With the conception of second and third moments of forces and couples, formulas are derived expressing slope deviation as a function of the second moment of forces and couples, and deflection as a function of the second and third moments of forces and couples. Although the fundamental differential equation of beam deflection—

$$\frac{d^2y}{dx^2} = \frac{M}{E\,I} \qquad (1)$$

—must be used in the derivation of such formulas, the final results are free from expressions of bending moment and they have the unique features of simplicity and lucidity.

The method can also be applied to a beam of variable moment of inertia provided that it is segmentally constant and provided that the secondary effects of axial thrust or tension, as well as the effect of the deflection due to shear, are not considered.

Notation.—The letter symbols in this paper are defined where they first appear, in the text or by illustrations, and are assembled alphabetically in the Appendix for convenience of reference.

Note.—Published in March, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Graduate, College of Eng., National Chung Cheng Univ., Nanchang, Kiangsi, China.

Second and Third Moments of Forces and Couples

Definition of Second and Third Moments.—Unless otherwise modified for a given case, this paper is based on the following assumptions: All forces are perpendicular to the reference line (ON, Figs. 1 and 2), all couples are considered to act at some point on the reference line, and the moment center is always taken on the reference line.

The second moment M'' and the third moment M''' of any force P_A, with respect to any point N taken as the moment center, are defined as the force P_A multiplied by the square and cube, respectively, of the perpendicular distance from the force to point N. Thus (see Fig. 1):

$$M_N = P_A\, a_A; \qquad M''_N = P_A\, a^2_A; \text{ and } M'''_N = P_A\, a^3_A \ldots\ldots (2)$$

in which the point force P_A is considered positive when acting upward and the moment arm a_A (Fig. 1) is positive when point A is on the left of point N.

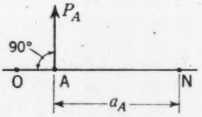


Fig. 1

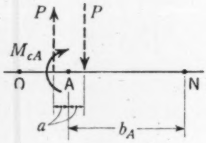


Fig. 2

Fig. 2 shows a couple M_{cA} acting at point A on the reference line ON. This couple can be replaced by two equal and opposite forces:

$$P = \frac{M_{cA}}{2\,a} \ldots\ldots (3)$$

in which the moment arm a is infinitely small. For this case the second moment is written:

$$M''_N = P\,(b_A + a)^2 - P\,(b_A - a)^2 = 2\,(2\,P\,a)\,b_A \ldots\ldots\ldots\ldots (4)$$

In other words,

$$M''_N = 2\,M_{cA}\,b_A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5a)$$

and the third moment, similarly, is written:

$$M'''_N = P\,(b_A + a)^3 - P\,(b_A - a)^3 = 3\,M_{cA}\,b^2_A \ldots\ldots\ldots (5b)$$

In Eqs. 5 the couple M_c is positive when it acts in a clockwise direction; and the reference distance, or "moment arm" b_A, is positive when point A is on the left of point N. If the couple M_c acts at point N:

$$M_N = M_c;\; M''_N = 0; \text{ and } M'''_N = 0 \ldots\ldots (6)$$

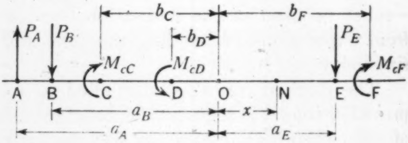


Fig. 3

Characteristics of Second and Third Moments.—Consider the balanced force system shown in Fig. 3, with points O and N, in turn, taken as the moment center. From the conditions of equilibrium:

$$\sum P = P_A - P_B - P_E = 0 \ldots\ldots\ldots\ldots\ldots\ldots (7a)$$

$$M_O = P_A\,a_A - P_B\,a_B + P_E\,a_E + M_{cC} - M_{cD} + M_{cF} = 0 \ldots\ldots (7b)$$

and

$$M_N = P_A\,(a_A + x) - P_B\,(a_B + x) + P_E\,(a_E - x) + M_{eC} - M_{eD} + M_{eF} = 0 \quad \text{(7c)}$$

Taking algebraic signs into consideration, Eqs. 7 can be written in more general terms as follows:

$$\sum_1^n P_n = 0 \quad \text{(8a)}$$

$$M_O = \sum_1^n (P_n\,a_n) + \sum_1^m M_{em} = 0 \quad \text{(8b)}$$

and

$$M_N = \sum_1^n P_n(a_n + x) + \sum_1^m M_{em} = 0 \quad \text{(8c)}$$

The second moments of the balanced force system (Fig. 3) with respect to points O and N, respectively, are:

$$M''_O = \sum_1^n (P_n\,a^2_n) + 2\sum_1^m (M_{em}\,b_m) \quad \text{(9a)}$$

and

$$M''_N = \sum_1^n P_n(a_n + x)^2 + 2\sum_1^m M_{em}\,(b_m + x) = \sum_1^n (P_n\,a^2_n) + 2\sum_1^m (M_{em}\,b_m) + x^2\sum_1^n P_n + 2\,x\left[\sum_1^n (P_n\,a_n) + \sum_1^m M_{em}\right] \quad \text{(9b)}$$

Substituting Eqs. 8 in Eqs. 9:

$$M''_N = \sum_1^n (P_n\,a^2_n) + 2\sum_1^m (M_{em}\,b_m) = M''_O \quad \text{((10a)}$$

or

$$M'' = \text{a constant} \quad \text{(10b)}$$

In Eq. 10*b* the subscript for M'' is unnecessary since the location of the moment center is immaterial. Eq. 10*b* defines the first characteristic of the second and third moments, for forces and couples; thus:

> Theorem I. The second moment of any balanced force system with respect to any point on the reference line is a constant.

Similar to Eqs. 9, the third moments of the balanced force system (Fig. 3) with respect to points O and N, respectively, are:

$$M'''_O = \sum_1^n (P_n\,a^3_n) + 3\sum_1^m (M_{em}\,b^2_m) \quad \text{(11a)}$$

$$M'''_N = \sum_1^n P_n(a_n + x)^3 + 3\sum_1^m M_{cm}\,(b_m + x)^2 = \sum_1^n (P_n\,a^3{}_n)$$
$$+ 3\sum_1^m (M_{cm}\,b^2{}_m) + 3\,x\left[\sum_1^n (P_n\,a^2{}_n) + 2\sum_1^m (M_{cm}\,b_m)\right]$$
$$+ 3\,x^2\left[\sum_1^n (P_n\,a_n) + \sum_1^m M_{cm}\right] + x^3\sum_1^n P_n \,.. \text{(11}b\text{)}$$

Substituting Eqs. 8 and 10 in Eqs. 11:

$$M'''_N = M'''_O + 3\,M''_O\,x = M'''_O + 3\,M''\,x \ldots\ldots\ldots\ldots (12)$$

Eq. 12 defines the second characteristic of second and third moments:

> Theorem II. The third moment of a balanced force system with respect to any point N on the reference line is equal to the third moment of the force system with respect to any other point O on the reference line, plus three times the distance ON multiplied by the second moment of the force system (with respect to any point on the reference line). The value of ON is positive when point N is on the right side of point O.

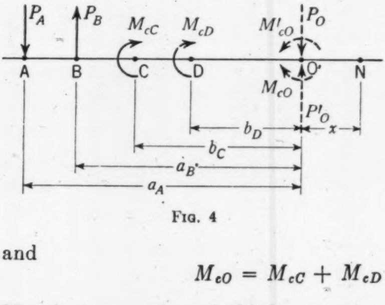


Fig. 4

The next step is to consider the unbalanced force system shown in Fig. 4, consisting of the parallel forces P_A and P_B and couples M_{cC} and M_{cD}. Let the resultant of the unbalanced force system be represented by a force P_O and a couple M_{cO}, both acting at point O; thus:

$$P_O = P_A - P_B \ldots.. (13a)$$

and

$$M_{cO} = M_{cC} + M_{cD} - P_A\,a_A + P_B\,a_B \ldots\ldots\ldots\ldots (13b)$$

The first, second, and third moments of the given unbalanced force system remain unchanged if two equal and opposite forces (P_O and P'_O) and two equal and opposite couples (M_{cO} and M'_{cO}) are introduced, all acting at point O, as shown by dotted lines in Fig. 4. The given unbalanced force system (P_A, P_B, M_{cC}, and M_{cD}), together with its equilibrant (the force P'_O and the couple M'_{cO}) forms a balanced force system. Thus, with any point N on the reference line taken as the moment center: M''_N of the given unbalanced force system (P_A, P_B, M_{cC}, and M_{cD}) is equal to the second moment M''_N of the balanced force system (P_A, P_B, M_{cC}, M_{cD}, P'_O, and M'_{cO}) plus the second moment M''_N of the resultant of the unbalanced force system (P_O and M_{cO}). Since the second moment of a balanced force system is constant (see theorem I and Eq. 10*b*), the second moment M''_N of the balanced force system (P_A, P_B, M_{cC}, M_{cD}, P'_O, and M'_{cO}) is equal to the second moment M''_O of the same balanced force system, which is equal to the second moment M''_O of the given unbalanced force system plus the second moment M''_O of the equilibrant of the given unbalanced force system (P'_O and M'_{cO}).

From Eq. 6, the second moment M''_O of the equilibrant (P'_O and M'_{eO}) vanishes. Hence:

$$M''_N = M''_O + M''_{O,N} \dots\dots\dots\dots\dots\dots (14)$$

Eq. 14 defines the third characteristic of second and third moments:

Theorem III. The second moment (M''_N) of the unbalanced force system with respect to any point N on the reference line is equal to the second moment of the unbalanced force system (M''_O) with respect to any other point O, plus the second moment of the resultant of the unbalanced force system ($M''_{O,N}$) at point O (P_O and M_{eO}) with respect to point N.

Similarly, it can be shown that:

$$M'''_N = M'''_O + 3\, M''_O\, x + M'''_{O,N} \dots\dots\dots\dots (15)$$

which defines theorem IV, as follows:

Theorem IV. The third moment (M'''_N) of the given unbalanced force system (P_A, P_B, M_{eC}, and M_{eD}) with respect to any point N on the reference line is equal to the third moment (M'''_O) of the unbalanced force system with respect to any other point O, also on the reference line, plus three times the product of x and the second moment (M''_O) of the unbalanced force system, with respect to point O; plus the third moment ($M'''_{O,N}$) of the resultant of the unbalanced force system at point O (P_O and M_{eO}) with respect to point N.

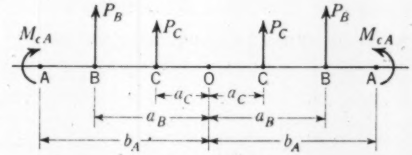


FIG. 5

Finally, for a force system symmetrical about a point O (for example, as shown in Fig. 5), the third moment with respect to point O is:

$$M'''_O = 3\, M_{eA}\, b^2_A + 3\,(-\,M_{eA})\,(-\,b_A)^2 + P_B\, a^3_B + P_B\,(-\,a_B)^3$$
$$+\, P_C\, a^3_C + P_C\,(-\,a_C)^3 = 0 \dots (16a)$$

or

$$M'''_O = 0 \dots\dots\dots\dots\dots\dots (16b)$$

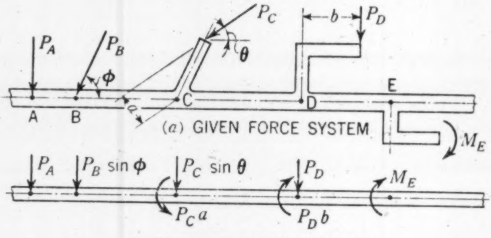
Eq. 16*b* defines the fifth characteristic:

Theorem V. The third moment of any symmetrical force system with respect to the point of symmetry is equal to zero.

SECOND AND THIRD MOMENTS OF FORCE SYSTEMS ACTING ON STRAIGHT BEAMS

A straight beam may be subjected to any system of loading; but, if the secondary effect of axial thrust or tension is neglected, only the couples and transverse loads applied on the beam axis cause flexure; any forces other than these need not be considered. The following typical cases of loading are particularly important in engineering problems.

Case 1. Concentrated Forces and Couples Acting Indirectly on Beam Axis.—For normal forces and couples acting directly on beam axis, use Eqs. 2 to 5, taking the beam axis as the reference line. For oblique or indirect loads, replace the given force system by an equivalent force system acting directly on the beam axis, as shown in Fig. 6.


P_A
P_B
ϕ
P_C
θ
b
P_D
a
A
B
C
D
E
M_E
P_A
$P_B \sin\phi$
$P_C \sin\theta$
P_D
M_E
$P_C a$
$P_D b$

(a) GIVEN FORCE SYSTEM

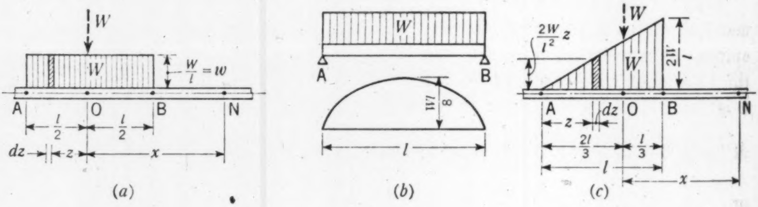
(b) EQUIVALENT FORCE SYSTEM

FIG. 6

Case 2. Uniformly Distributed Load Normal to Beam Axis.—By direct integration (see Fig. 7(*a*)):

$$M''_O = \int_{-l/2}^{l/2} \left(- \frac{W}{l}\, dz \right) z^2 = - \frac{W\, l^2}{12} \ldots\ldots\ldots\ldots (17a)$$

which is numerically equal to the area of the simple beam moment diagram,


W
W
$\frac{W}{l} = w$
A
O
B
N
$\frac{l}{2}$
$\frac{l}{2}$
dz
z
x
W
A
B
$\frac{Wl}{8}$
l
W
$\frac{2W}{l^2} z$
W
$\frac{2W}{l}$
A
dz
O
B
N
z
$\frac{2l}{3}$
$\frac{l}{3}$
l
x

(*a*) (*b*) (*c*)

FIG. 7

as shown in Fig. 7(*b*). Since the load is symmetrical about point O (Fig. 7(*a*)), Eq. 16*b* yields:

$$M'''_O = 0 \ldots\ldots\ldots\ldots (17b)$$

By Eq. 14:

$$M''_N = M''_O + M''_{O,N} = - \frac{W\, l^2}{12} - W\, x^2 \ldots\ldots\ldots (18a)$$

By Eq. 15:

$$M'''_N = M'''_O + 3\, M''_O\, x + M'''_{O,N} = - \frac{W\, l^2}{4}\, x - W\, x^3 \ldots (18b)$$

If $x = l/2$:

$$M''_B = - \frac{W\, l^2}{3}; \text{ and } M'''_B = - \frac{W\, l^3}{4} \ldots\ldots\ldots\ldots (19)$$

If $x = -\, l/2$:

$$M''_A = - \frac{W\, l^2}{3}; \text{ and } M'''_A = + \frac{W\, l^3}{4} \ldots\ldots\ldots\ldots (20)$$

Case 3. Triangular Load Normal to Beam Axis.—By integration, referring to Fig. 7(*c*):

$$M''_O = \int_0^l \left(- \frac{2\,W}{l^2}\, z\, dz \right) (\tfrac{2}{3}\, l - z)^2 = - \frac{W\, l^2}{18} \qquad \text{(21}a\text{)}$$

and

$$M'''_O = \int_0^l \left(- \frac{2\,W}{l^2}\, z\, dz \right) (\tfrac{2}{3}\, l - z)^3 = - \frac{W\, l^3}{135} \qquad \text{(21}b\text{)}$$

By Eqs. 14 and 15:

$$M''_N = M''_O + M''_{O,N} = - \frac{W\, l^2}{18} - W\, x^2 \qquad \text{(22}a\text{)}$$

and

$$M'''_N = M'''_O + 3\, M''_O\, x + M'''_{O,N} = - \frac{W\, l^3}{135} - \frac{W\, l^2}{6}\, x - W\, x^3 \qquad \text{(22}b\text{)}$$

For points A and B, assume that $x = -\tfrac{2}{3}\, l$ and $x = +\tfrac{1}{3}\, l$, respectively; thus:

$$M''_A = - \frac{W\, l^2}{2}\,; \text{ and } M'''_A = \tfrac{2}{5}\, W\, l^3 \qquad \text{(23}a\text{)}$$

and

$$M''_B = - \frac{W\, l^2}{6}\,; \text{ and } M'''_B = - \frac{W\, l^3}{10} \qquad \text{(23}b\text{)}$$

Second and third moments for other loading conditions may be computed by integration and substitution into Eqs. 14 and 15.

METHOD OF SECOND AND THIRD MOMENTS FOR COMPUTING BEAM DEFLECTIONS

Defining the x-axis as the undeflected position of the beam axis (which is the "reference line" defined in the preceding section) and the y-axis as the plane normal to the x-axis, with positive directions as shown in Fig. 8, let: δ_A be the vertical deflection at any point A, positive when upward; δ_{BA} be equal to $\delta_B - \delta_A$ and equal to the vertical deflection of point B relative to that of point A; y_{BA} be the vertical deflection of point B measured from the tangent to the elastic curve of the deflected beam axis at point A (positive when upward); θ_A be the angular deflection at point A, which can be taken as equal to $\frac{d\delta}{dx}$ at point A, assuming the same sign as the latter; and θ_{BA} be equal to $\theta_B - \theta_A$ and equal to the angular deflection at point B relative to that at point A. The sign conventions are illustrated in Fig. 8. It follows that:

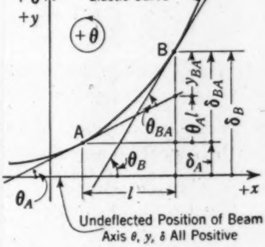


FIG. 8

$$\theta_{BA} = - \theta_{AB}; \text{ and } \delta_{BA} = - \delta_{AB} \qquad \text{(24)}$$

Fig. 9 shows a segment AB of a straight beam with constant flexural rigidity, subject to end shears V_A and V_B; end moments M_A and M_B; transverse loads P_a, P_c, $\cdots$, P_n; and couples M_b and M_d, $\cdots$, M_m.

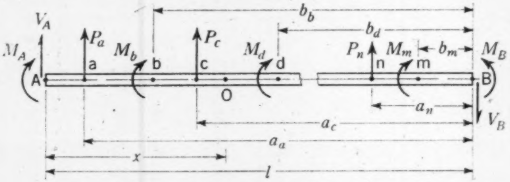


Fig. 9

The well-known differential equation of the elastic curve of the deflected beam axis is

$$\frac{d^2y}{dx^2} = \frac{M}{E\,I} \quad \text{(25)}$$

in which

$$M = M_A + V_A\,x; \quad \text{for} \quad 0 \leqq x \leqq l - a_a \quad \text{(26a)}$$

$$M = M_A + V_A\,x + P_a\,(x - l + a_a); \quad \text{for} \quad l - a_a \leqq x \leqq l - b_b \quad \text{(26b)}$$

$$M = M_A + M_b + V_A\,x + P_a\,(x - l + a_a); \quad \text{for} \quad l - b_b \leqq x \leqq l - a_c \quad \text{(26c)}$$

. . . .

$$M = M_A + \sum_1^m M_m + V_A\,x + \sum_1^n P_n\,(x - l + a_n); \quad \text{for} \quad l - b_m \leqq x \leqq l \quad \text{(26d)}$$

By integration, the angular deflection of point B relative to point A is:

$$\theta_{BA} = \frac{1}{E\,I}\int_0^l M\,dx = \frac{1}{E\,I}\left\{\left(M_A\int_0^l dx + \sum_1^m M_m \int_{l-b_m}^l dx\right) + \left[V_A\int_0^l x\,dx + \sum_1^n P_n \int_{l-a_n}^l (x - l + a_n)\,dx\right]\right\} = \frac{1}{2\,E\,I}\,M''_{AB,\,B} \quad \text{(27)}$$

Since the force system shown in Fig. 9 is in equilibrium and the second moment (with respect to any point on the beam axis) is, therefore, a constant:

$$\theta_{BA} = \frac{1}{2\,E\,I}\,M''_{AB} \quad \text{(28a)}$$

in which M''_{AB} is the second moment of the force system acting on the segment AB, including end moments and end shears; and point B is on the right side of point A. By Eq. 24,

$$\theta_{AB} = -\,\theta_{BA} = -\,\frac{M''_{AB}}{2\,E\,I} \quad \text{(28b)}$$

Thus, the more general expression may be written as

$$\theta_{BA} = \pm \frac{M''_{AB}}{2\,E\,I} \quad \text{(29a)}$$

which takes the positive sign when point B is on the right side of point A.

Referring to Fig. 8:

$$\theta_B = \theta_A + \theta_{BA} = \theta_A \pm \frac{1}{2\,E\,I} M''_{AB} \quad \text{(29b)}$$

which takes the positive sign when point B is on the right side of point A.

Eqs. 29 give the first fundamental theorem of the method of second and third moments, for beam deflections:

> Theorem VI. The angle θ_{BA} measured from the tangent to the elastic curve at any point A to the tangent at any other point B on the right side of point A, in a segment AB of a straight beam of constant flexural rigidity, is equal to the second moment of the force system acting on the segment AB, including end moments and end shears of the segment, taken with respect to any point on the undeflected beam axis, divided by twice the flexural rigidity of the beam (that is, by 2 *E I*).

The deflection of point B from the tangent at point A (Figs. 8 and 9) is:

$$y_{BA} = \frac{1}{E\,I}\int_0^l \int_0^x M\,dx\,dx = \frac{1}{E\,I}\left\{\left(M_A \int_0^l \int_0^x dx\,dx + \sum_1^m M_m \int_{l-b_m}^l \int_{l-b_m}^x dx\,dx\right) + \left[V_A \int_0^l \int_0^x x\,dx\,dx + \sum_1^n P_n \int_{l-a_n}^l \int_{l-a_n}^x (x - l + a_n)\,dx\,dx\right]\right\} = \frac{1}{6\,E\,I} M'''_{AB,\,B} \quad \text{(30)}$$

in which M'''_{AB} is the third moment of the force system acting on the segment AB, including end moments and end shears of the segment, with respect to point B, the latter point (B) being on the right side of point A. The more general form may be written as

$$y_{BA} = \pm \frac{M'''_{AB,\,B}}{6\,E\,I} \quad \text{(31a)}$$

which takes the positive sign when point B is on the right side of point A.

Referring again to Fig. 8, the deflection of point B is:

$$\delta_B = \delta_A + \theta_A\,l + y_{BA} = \delta_A + \theta_A\,l \pm \frac{M'''_{AB,\,B}}{6\,E\,I} \quad \text{(31b)}$$

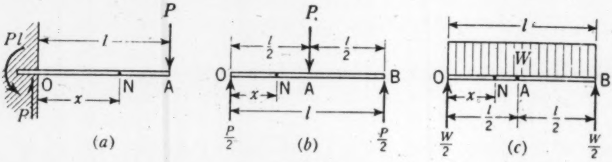
In Eq. 31*b* the positive sign is used when point B is on the right side of point A.

Eqs. 31 give the second fundamental theorem of the method of second and third moments, for beam deflections:

> Theorem VII. The deflection of any point B, measured from the tangent to the elastic curve at point A, of a segment (AB) of a straight beam of constant flexural rigidity, is equal to the third moment of the force system

acting on the segment AB, including end moments and end shears of the segment, taken with respect to point B, divided by six times the flexural rigidity; that is, by $6\,E\,I$.

Theorems VI and VII are similar in nature to theorems I and II of the moment-area method, and lead to identical results. The following examples illustrate the application of theorems VI and VII.



(a) (b) (c)

Fig. 10

Example 1. Cantilever Beam Loaded at Free End.—For the case shown in Fig. 10(*a*), $\theta_O = 0$ and $\delta_O = 0$. From theorem VI or Eq. 29*b*,

$$\theta_N = \frac{1}{2\,E\,I}\,M''_{ON} = \frac{1}{2\,E\,I}\,(P\,x^2 - 2\,P\,l\,x) \dots\dots (32a)$$

and, from theorem VII or Eq. 31*b*,

$$\delta_N = \frac{1}{6\,E\,I}\,M'''_{ON,N} = \frac{1}{6\,E\,I}\,(P\,x^3 - 3\,P\,l\,x^2) \dots\dots (32b)$$

At the free end, A (see Fig. 10(*a*)):

$$\theta_A = -\frac{P\,l^2}{2\,E\,I}\,;\ \text{and}\ \delta_A = -\frac{P\,l^3}{3\,E\,I} \dots\dots (33)$$

Example 2. Simple Beam with a Concentrated Load at Center.—For the case shown in Fig. 10(*b*), as in Fig. 10(*a*), $\theta_A = 0$ and $\delta_O = 0$. Again, from theorem VI or Eq. 29*b*,

$$\theta_A = \theta_O + \frac{1}{2\,E\,I}\,M''_{OA} = \theta_O + \frac{1}{2\,E\,I}\left(\frac{P}{2}\right)\left(\frac{l}{2}\right)^2 = 0 \dots\dots (34a)$$

and

$$\theta_O = -\frac{P\,l^2}{16\,E\,I} \dots\dots (34b)$$

Thus, for $x \leqq l/2$, by Eqs. 29*b* and 31*b*:

$$\theta_N = \theta_O + \frac{1}{2\,E\,I}\,M''_{ON} = \frac{P}{16\,E\,I}\,(4\,x^2 - l^2) \dots\dots (35a)$$

and

$$\delta_N = \theta_O\,x + \frac{1}{6\,E\,I}\,M'''_{ON,N} = \frac{1}{48\,E\,I}\,P\,x\,(4\,x^2 - 3\,l^2) \dots\dots (35b)$$

At the center A, $x = \frac{1}{2}\,l$:

$$\delta_A = -\frac{1}{48\,E\,I}\,P\,l^3 \dots\dots (35c)$$

Example 3. Simple Beam Subject to Uniformly Distributed Load.—By Eq. 29*b* with $\theta_A = 0$ and $\delta_O = 0$ (see Fig. 10(*c*)):

$$\theta_A = \theta_O + \frac{1}{2\,E\,I}\,M''_{OA} = \theta_O + \frac{1}{2\,E\,I}\left[\frac{W}{2}\left(\frac{l}{2}\right)^2 - \int_0^{l/2}\left(\frac{W}{l}\right)x^2\,dx\right]$$

$$= \theta_O + \frac{W\,l^2}{24\,E\,I} = 0 \ldots\ldots (36a)$$

and

$$\theta_O = -\,\frac{W\,l^2}{24\,E\,I} \ldots\ldots (36b)$$

By Eqs. 29*b* and 31*b*:

$$\theta_N = \theta_O + \frac{1}{2\,E\,I}\,M''_{ON} = -\,\frac{W\,l^2}{24\,E\,I} + \frac{1}{2\,E\,I}\left[\frac{W}{2}\,x^2 - \int_0^x\left(\frac{W}{l}\right)x^2\,dx\right]$$

$$= \frac{W}{24\,E\,I}\left(-\,l^2 + 6\,x^2 - 4\,\frac{x^3}{l}\right) \ldots\ldots (37a)$$

and

$$\delta_N = \delta_O + \theta_O\,x + \frac{1}{6\,E\,I}\,M'''_{ON,N} = -\,\frac{1}{24\,E\,I}\,W\,l^2\,x$$

$$+ \frac{1}{6\,E\,I}\left(\frac{W}{2}\,x^3 - \int_0^x \frac{W}{l}\,x^3\,dx\right) = \frac{1}{24\,E\,I}\,W\left(-\,l^2\,x + 2\,x^3 - \frac{x^4}{l}\right) \ldots (37b)$$

For $x = l/2$:

$$\delta_A = -\,\frac{5\,W\,l^3}{384\,E\,I} \ldots\ldots (38)$$

The computation of Eqs. 36 to 38 may be facilitated by using Eqs. 19, substituting $\frac{W}{l}\,x$ for W, and x for l. The second and third moments, referred to point N as moment center, due to uniform load only, are, respectively:

$$M'' = -\,\frac{W\,x^3}{3\,l}\,;\ \text{and}\ M''' = -\,\frac{W\,x^4}{4\,l} \ldots\ldots (39)$$

From which:

$$M''_{ON} = M''_{ON,N} = \frac{W}{2}\,x^2 - \frac{W\,x^3}{3\,l} \ldots (40a)$$

and

$$M'''_{ON,N} = \frac{W}{2}\,x^3 - \frac{W\,x^4}{4\,l} \ldots\ldots (40b)$$

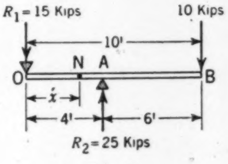


FIG. 11

Example 4. Overhanging Beam Loaded at Free End.—By Eq. 31*b*, with $\delta_O = 0$ and $\delta_A = 0$ (see Fig. 11):

$$\delta_A = \delta_O + \theta_O \times 4 + \frac{1}{6\,E\,I}\,M'''_{OA,A} = \theta_O \times 4 + \frac{1}{6\,E\,I}\left(-\,15 \times 4^3\right) = 0 \ldots (41a)$$

and

$$\theta_O = \frac{40}{E\,I} \qquad \text{(41}b\text{)}$$

For $x \leqq 4$ ft, Eqs. 29*b* and 31*b* yield:

$$\theta_N = \theta_O + \frac{1}{2\,E\,I} M''_{ON} = \frac{40}{E\,I} + \frac{1}{2\,E\,I}(-\,15)\,x^2 = \frac{1}{E\,I}(40 - 7.5\,x^2) \qquad \text{(42}a\text{)}$$

and

$$\delta_N = \delta_O + \theta_O\,x + \frac{1}{6\,E\,I} M'''_{ON,\,N} = \frac{1}{E\,I}(40\,x - 2.5\,x^3) \qquad \text{(42}b\text{)}$$

Similarly, for $x \geqq 4$ ft:

$$\theta_N = \theta_O + \frac{1}{2\,E\,I} M''_{ON} = \frac{1}{E\,I}(240 - 100\,x + 5\,x^2) \qquad \text{(43}a\text{)}$$

and

$$\delta_N = \delta_O + \theta_O\,x + \frac{1}{6\,E\,I} M'''_{ON,\,N}$$

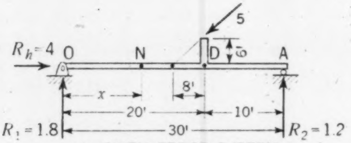
$$= \frac{1}{6\,E\,I}(10\,x^3 - 300\,x^2 + 1{,}440\,x - 1{,}600) \qquad \text{(43}b\text{)}$$

At the free end:

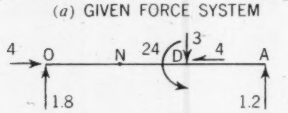
$$\theta_B = -\,\frac{260}{E\,I}\,;\ \text{and}\ \delta_B = -\,\frac{1{,}200}{E\,I} \qquad \text{(44)}$$

in which E is in kips per square foot, and I is in feet^4.

Example 5. Simple Beam Subject to Indirect Loading.—The external load on the beam in Fig. 12(*a*) is replaced by the equivalent force system as shown in Fig. 12(*b*). By Eq. 31*b*, with $\delta_O = 0$ and $\delta_A = 0$:


5
$R_h = 4$
O
N
D
6'
A
x
8'
20'
10'
30'
$R_1 = 1.8$
$R_2 = 1.2$

(*a*) GIVEN FORCE SYSTEM


4
O
N
24
D
3
4
A
1.8
1.2

(*b*) EQUIVALENT FORCE SYSTEM

FIG. 12

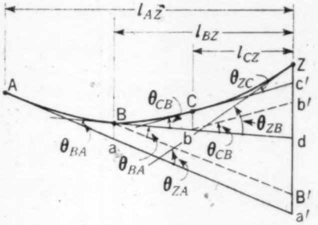

l_{AZ}
l_{BZ}
l_{CZ}
Z
c'
b'
d
B'
a'
A
B
C
a
b
θ_{ZC}
θ_{CB}
θ_{ZB}
θ_{BA}
θ_{BA}
θ_{CB}
θ_{ZA}

FIG. 13

$$\delta_A = \delta_O + \theta_O \times 30 + \frac{1}{6\,E\,I} M'''_{OA,\,A}$$

$$= 30\,\theta_O + \frac{1}{6\,E\,I}(1.8 \times 30^3 - 3 \times 10^3 - 3 \times 24 \times 10^2) = 0 \qquad \text{(45}a\text{)}$$

and

$$\theta_O = -\,\frac{640}{3\,E\,I} \qquad \text{(45}b\text{)}$$

for $x \leqq 20$ ft, by Eqs. 29*b* and 31*b*:

$$\theta_N = \theta_O + \frac{1}{2\,E\,I}\,M''_{ON} = -\,\frac{1{,}280}{6\,E\,I} + \frac{1}{2\,E\,I} \times 1.8\,x^2$$

$$= \frac{1}{6\,E\,I}\,(-\,1{,}280 + 5.4\,x^2) \ldots\ldots (46a)$$

and

$$\delta_N = \delta_O + \theta_O\,x + \frac{1}{6\,E\,I}\,M'''_{ON,N} = \frac{x}{6\,E\,I}\,(-\,1{,}280 + 1.8\,x^2) \ldots (46b)$$

For $x \geqq 20$ ft:

$$\theta_N = \theta_O + \frac{M''_{ON}}{2\,E\,I} = -\,\frac{1}{E\,I}\,(0.6\,x^2 - 36\,x + 333.3) \ldots\ldots\ldots (47a)$$

and

$$\delta_N = \delta_O + \theta_O\,x + \frac{1}{6\,E\,I}\,M'''_{ON,N}$$

$$= -\,\frac{1}{6\,E\,I}\,(1.2\,x^3 - 108\,x^2 + 2{,}000\,x + 4{,}800) \ldots\ldots (47b)$$

STRAIGHT BEAMS OF SEGMENTALLY CONSTANT FLEXURAL RIGIDITY

Fig. 13 shows the elastic curve of a beam whose flexural rigidity is $(E\,I)_{AB}$ for segment AB, $(E\,I)_{BC}$ for segment BC, and $(E\,I)_{CZ}$ for segment CZ. By Eq. 29*a*:

$$\theta_{ZA} = \theta_{BA} + \theta_{CB} + \theta_{ZC} = \frac{1}{2(E\,I)_{AB}}\,M''_{AB}$$

$$+ \frac{1}{2(E\,I)_{BC}}\,M''_{BC} + \frac{1}{2(E\,I)_{CZ}}\,M''_{CZ} \ldots\ldots (48a)$$

The general form, therefore, is:

$$\theta_{ZA} = \sum \frac{M''_{AB}}{2(E\,I)_{AB}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48b)$$

Substituting, $M''_{BC} = M''_{AC} - M''_{AB}$; and $M''_{CZ} = M''_{AZ} - M''_{AC}$. In Eq. 48*b*:

$$\theta_{ZA} = \frac{1}{2(E\,I)_{AB}}\,M''_{AB} + \frac{1}{2(E\,I)_{BC}}\,(M''_{AC} - M''_{AB})$$

$$+ \frac{1}{2(E\,I)_{CZ}}\,(M''_{AZ} - M''_{AC}) = \left[\frac{1}{2(E\,I)_{AB}} - \frac{1}{2(E\,I)_{BC}}\right] M''_{AB}$$

$$+ \left[\frac{1}{2(E\,I)_{BC}} - \frac{1}{2(E\,I)_{CZ}}\right] M''_{AC} + \frac{1}{2(E\,I)_{CZ}}\,M''_{AZ} = \Delta\!\left(\frac{1}{2\,E\,I}\right)_B M''_{AB}$$

$$+ \Delta\!\left(\frac{1}{2\,E\,I}\right)_C M''_{AC} + \Delta\!\left(\frac{1}{2\,E\,I}\right)_Z M''_{AZ} = \sum \Delta\!\left(\frac{1}{2\,E\,I}\right)_B M''_{AB} \ldots (48c)$$

in which

$$\Delta\!\left(\frac{1}{2\,E\,I}\right)_B = \frac{1}{2(E\,I)_{AB}} - \frac{1}{2(E\,I)_{BC}} \ldots\ldots\ldots\ldots\ldots (49a)$$

$$\Delta\!\left(\frac{1}{2\,E\,I}\right)_C = \frac{1}{2(E\,I)_{BC}} - \frac{1}{2(E\,I)_{CZ}} \ldots\ldots\ldots\ldots\ldots (49b)$$

and

$$\Delta\left(\frac{1}{2\,E\,I}\right)_Z = \frac{1}{2(E\,I)_{CZ}} - 0 = \frac{1}{2(E\,I)_{CZ}} \quad \text{(49}c\text{)}$$

Eq. 48*b* or Eq. 48*c* may be used, but the latter is the simpler.

Referring to Fig. 13, Eqs. 29*a* and 31*a*:

$$\begin{aligned} y_{ZA} &= \overline{\mathrm{a'Z}} = \overline{\mathrm{a'B'}} + \overline{\mathrm{B'd}} + \overline{\mathrm{db'}} + \overline{\mathrm{b'c'}} + \overline{\mathrm{c'Z}} = y_{BA} + \theta_{BA}\, l_{BZ} \\ &+ \theta_{CB}\, l_{CZ} + y_{CB} + y_{ZC} = \frac{1}{6(E\,I)_{AB}} M'''_{AB,B} + \frac{1}{2(E\,I)_{AB}} M''_{AB}\, l_{BZ} \\ &+ \frac{1}{2(E\,I)_{BC}} M''_{BC}\, l_{CZ} + \frac{1}{6(E\,I)_{BC}} M'''_{BC,C} + \frac{1}{6(E\,I)_{CZ}} M'''_{CZ,Z} \\ &= \frac{1}{6(E\,I)_{AB}} (M'''_{AB,B} + 3\,M''_{AB}\, l_{BZ}) + \frac{1}{6(E\,I)_{BC}} (M'''_{BC,C} + 3\,M''_{BC}\, l_{CZ}) \\ &\qquad + \frac{1}{6(E\,I)_{CZ}} M'''_{CZ,Z} \quad \text{(50)} \end{aligned}$$

By the second characteristic of second and third moments, Eq. 12 yields:

$$M'''_{AB,Z} = M'''_{AB,B} + 3\,M''_{AB}\, l_{BZ} \quad \text{(51}a\text{)}$$

and

$$M'''_{BC,Z} = M'''_{BC,C} + 3\,M''_{BC}\, l_{CZ} \quad \text{(51}b\text{)}$$

Substituting Eqs. 51 in Eq. 50:

$$y_{ZA} = \frac{1}{6(E\,I)_{AB}} M'''_{AB,Z} + \frac{1}{6(E\,I)_{BC}} M'''_{BC,Z} + \frac{1}{6(E\,I)_{CZ}} M'''_{CZ,Z} \quad \text{(52}a\text{)}$$

Therefore, the general formula is:

$$y_{ZA} = \sum \frac{1}{6(E\,I)_{AB}} M'''_{AB,Z} \quad \text{(52}b\text{)}$$

Substituting—

$$M'''_{BC,Z} = M'''_{AC,Z} - M'''_{AB,Z} \quad \text{(53}a\text{)}$$

and

$$M'''_{CZ,Z} = M'''_{AZ,Z} - M'''_{AC,Z} \quad \text{(53}b\text{)}$$

—in Eq. 52*b*:

$$\begin{aligned} y_{ZA} &= \frac{1}{6(E\,I)_{AB}} M'''_{AB,Z} + \frac{1}{6(E\,I)_{BC}} (M'''_{AC,Z} - M'''_{AB,Z}) \\ &\quad + \frac{1}{6(E\,I)_{CZ}} (M'''_{AZ,Z} - M'''_{AC,Z}) = \left[\frac{1}{6(E\,I)_{AB}} \right. \\ &\quad \left. - \frac{1}{6(E\,I)_{BC}}\right] M'''_{AB,Z} + \left[\frac{1}{6(E\,I)_{BC}} - \frac{1}{6(E\,I)_{CZ}}\right] M'''_{AC,Z} \\ &\quad + \frac{1}{6(E\,I)_{CZ}} M'''_{AZ,Z} = \Delta\left(\frac{1}{6\,E\,I}\right)_B M'''_{AB,Z} \\ &\quad + \Delta\left(\frac{1}{6\,E\,I}\right)_C M'''_{AC,Z} + \Delta\left(\frac{1}{6\,E\,I}\right)_Z M'''_{AZ,Z} \\ &\qquad = \sum \Delta\left(\frac{1}{6\,E\,I}\right)_B M'''_{AB,Z} \quad \text{(54)} \end{aligned}$$

in which (compared with Eqs. 49):

$$\Delta\left(\frac{1}{6\,E\,I}\right)_B = \frac{1}{6(E\,I)_{AB}} - \frac{1}{6(E\,I)_{BC}} \quad \text{(55a)}$$

$$\Delta\left(\frac{1}{6\,E\,I}\right)_C = \frac{1}{6(E\,I)_{BC}} - \frac{1}{6(E\,I)_{CZ}} \quad \text{(55b)}$$

and

$$\Delta\left(\frac{1}{6\,E\,I}\right)_Z = \frac{1}{6(E\,I)_{CZ}} \quad \text{(55c)}$$

For beams of variable flexural rigidity, Eqs. 48c and 54 become:

$$\theta_{ZA} = \int_{x=0}^{x=l} M''_{Ax}\, d\left[\frac{1}{2(E\,I)_x}\right] \quad \text{(56a)}$$

and

$$y_{ZA} = \int_{x=0}^{x=l} M'''_{Ax,x}\, d\left[\frac{1}{6(E\,I)_x}\right] \quad \text{(56b)}$$

in which $(E\,I)_x$, M''_{Ax}, and $M'''_{Ax,x}$ are functions of x. Solutions of problems by Eqs. 56 are naturally somewhat complicated. The following examples illustrate the application of Eqs. 48, 52, and 54.

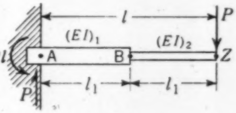


FIG. 14

Example 6. Cantilever Beam with Segmentally Constant Flexural Rigidity.—By Eqs. 48c and 54 (see Fig. 14):

$$\theta_Z = \Delta\left(\frac{1}{2\,E\,I}\right)_B M''_{AB} + \frac{1}{2(E\,I)_2} M''_{AZ} = \left[\frac{1}{2(E\,I)_1} - \frac{1}{2(E\,I)_2}\right](P\,l^2{}_1 - 2\,P\,l \times l_1) + \frac{1}{2(E\,I)_2}(P\,l^2 - 2\,P\,l \times l)$$

$$= -\frac{P\,l^2}{8}\left[\frac{3}{(E\,I)_1} - \frac{1}{(E\,I)_2}\right] \quad \text{(57a)}$$

and

$$\delta_Z = \Delta\left(\frac{1}{6\,E\,I}\right)_B M'''_{AB,Z} + \frac{1}{6(E\,I)_2} M'''_{AZ,Z} = \left[\frac{1}{6(E\,I)_1} - \frac{1}{6(E\,I)_2}\right] \times [P\,l^3{}_1 - 3\,P\,l \times l^2{}_1 + 3\,(P\,l^2{}_1 - 2\,P\,l \times l_1)\,l_1]$$

$$+ \frac{1}{6(E\,I)_2}(P\,l^3 - 3\,P\,l \times l^2) = -\frac{P\,l^3}{24}\left[\frac{7}{(E\,I)_1} + \frac{1}{(E\,I)_2}\right] \quad \text{(57b)}$$

Example 7. Simply Supported Beam with Segmentally Constant Flexural Rigidity and a Load at the Center.—Referring to Fig. 15, Eq. 52b yields:

$$\delta_A = y_{ZA} = \frac{1}{6(E\,I)_1} M'''_{AB,Z} + \frac{1}{6(E\,I)_0} M'''_{BZ,Z}$$

$$= \frac{P\,l^3}{384}\left[\frac{7}{(E\,I)_1} + \frac{1}{(E\,I)_0}\right] \quad \text{(58)}$$

STATICALLY INDETERMINATE BEAMS AND FRAMES

The following examples illustrate the use of the method of second and third moments to analyze statically indeterminate structures.

Example 8. Beam Fixed at One End and Simply Supported at the Other End, With a Concentrated Load at the Center.—Referring to Fig. 16(*a*), with $\theta_O = 0$ and $\delta_A = 0$, Eq. 31*a* yields:

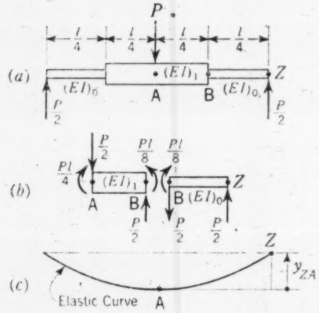

P
l/4
l/4
l/4
l/4
(a)
(EI)_1
(EI)_0
A
B
(EI)_0
Z
P/2
P/2
(b)
P/2
Pl/8
Pl/8
Pl/4
(EI)_1
A
B
B (EI)_0
Z
P/2
P/2
P/2
(c)
Z
y_ZA
Elastic Curve
A

FIG. 15

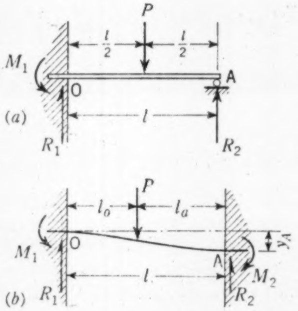

P
l/2
l/2
M_1
O
A
(a)
l
R_1
R_2
P
l_o
l_a
y_A
M_1
O
A
M_2
l
(b)
R_1
R_2

FIG. 16

$$\delta_A = y_{AO} = \frac{M'''_{OA,A}}{6\,E\,I} = 0 \qquad (59)$$

According to Eq. 12:

$$M'''_{OA,A} = M'''_{OA,O} + 3\,M''_{OA}\,l \qquad (60a)$$

and Eq. 10*b*,

$$M''_{OA} = M''_{OA,O} \qquad (60b)$$

Combining Eqs. 60:

$$M'''_{OA,A} = M'''_{OA,O} + 3\,M''_{OA,O}\,l = P\left(\frac{l}{2}\right)^3 - R_2\,l^3 + 3\left[-P\left(\frac{l}{2}\right)^2 + R_2\,l^2\right]l = 0 \qquad (61)$$

From which:

$$R_2 = \tfrac{5}{16}\,P; \quad R_1 = \tfrac{11}{16}\,P; \text{ and } M_1 = \tfrac{3}{16}\,P\,l \qquad (62)$$

Example 9. Beam Fixed at Both Ends But Displaced Vertically through a Distance y, with a Concentrated Load at Center.—Referring to Fig. 16(*b*), with $\theta_{AO} = 0$ and $M''_{OA,A} = 0$:

$$R_1\,l^2 - 2\,M_1\,l - P\,l_a^2 = 0 \qquad (63a)$$

Since $\delta_{AO} = -\,y_A$ and $M'''_{OA,A} = -\,6\,E\,I\,y_A$:

$$R_1\,l^3 - 3\,M_1\,l^2 - P\,l_a^3 = -\,6\,E\,I\,y_A \qquad (63b)$$

Combining Eqs. 63:

$$M_1 = \frac{1}{l^2}\left[P\, l^2_a\,(l - l_a) + 6\,E\,I\,y_A\right] \ldots\ldots\ldots\ldots (64a)$$

and

$$R_1 = \frac{1}{l^3}\,(3\,P\,l^2_a\,l - 2\,P\,l^3_a + 12\,E\,I\,y_A) \ldots\ldots\ldots\ldots (64b)$$

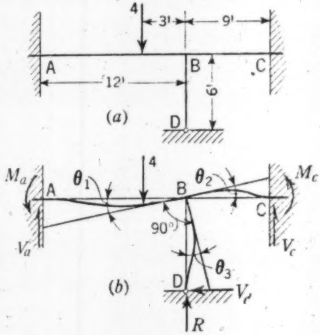

4
3'
9'
A
B
C
12'
6'
(a)
D
4
M_a
A
θ_1
B
θ_2
M_c
C
90°
V_a
V_c
θ_3
(b)
D
V_d
R

FIG. 17

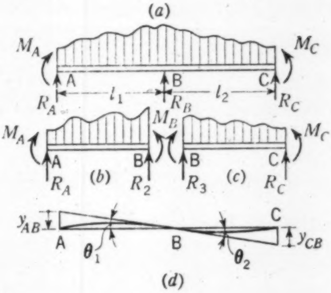

(a)
M_A
M_C
A
B
C
R_A
l_1
R_B
l_2
R_C
M_A
M_B
M_C
A
B
B
C
R_A
(b)
R_2
R_3
(c)
R_C
y_{AB}
C
A
B
θ_1
θ_2
y_{CB}
(d)

FIG. 18

Example 10. Continuous Frame From the Static Condition of Equilibrium (See Fig. 17).—With $M_B = 0$:

$$12\,V_a - M_a - 12 + 6\,V_d - M_c + 9\,V_c = 0 \ldots\ldots\ldots\ldots (65)$$

From Eqs. 29*a* and 31*a*:

$$\theta_1 = \frac{1}{2\,E\,I}\,M''_{AB,\,B} = \frac{1}{2\,E\,I}\,(V_a \times 12^2 - 2\,M_a \times 12 - 4 \times 3^2)$$

$$= \frac{1}{2\,E\,I}\,(144\,V_a - 24\,M_a - 36) \ldots (66a)$$

$$\theta_2 \doteq \frac{1}{2\,E\,I}\,M''_{BC,\,B} = \frac{1}{2\,E\,I}\,(18\,M_c - 81\,V_c) \ldots\ldots\ldots (66b)$$

and

$$\theta_3 = \frac{y_{DB}}{6} = \frac{1}{36\,E\,I}\,M'''_{BD,\,D} = \frac{1}{36\,E\,I}\,(V_d \times 6^3 - 3 \times V_d$$

$$\times\, 6^2 \times 6) = -\,\frac{1}{2\,E\,I}\,24\,V_d \ldots (66c)$$

When $\theta_1 = -\,\theta_2$, equating Eqs. 66*a* and 66*b* yields $144\,V_a - 24\,M_a - 36 = -\,18\,M_c + 81\,V_c$, which reduces to:

$$8\,M_a - 48\,V_a - 6\,M_c + 27\,V_c = -\,12 \ldots\ldots\ldots\ldots (67a)$$

When $\theta_2 = \theta_3$, equating Eqs. 66*b* and 66*c* yields $18\,M_c - 81\,V_c = -\,24\,V_d$, which reduces to:

$$6\,M_c - 27\,V_c + 8\,V_d = 0 \ldots\ldots\ldots\ldots\ldots\ldots (67b)$$

Therefore, since $y_{BA} = \frac{1}{6\,E\,I} M'''_{AB,\,B} = 0$; and $M'''_{AB,\,B} = V_a \times 12^3 - 3M_a \times 12^2 - 4 \times 3^3 = 0$; or

$$V_a = \tfrac{1}{16} + \frac{M_a}{4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (68a)$$

Similarly, $y_{BC} = \frac{1}{6\,E\,I} M'''_{BC,\,B} = 0$; and $M'''_{BC,\,B} = V_c \times 9^3 - 3\,M_c \times 9^2 = 0.$

Therefore,

$$V_c = \frac{M_c}{3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (68b)$$

Substituting Eqs. 68 into Eq. 65:

$$8\,M_a + 24\,V_d + 8\,M_c = 45 \ldots\ldots\ldots\ldots\ldots\ldots (69a)$$

substituting Eqs. 68 into Eq. 67*a*:

$$4\,M_a - 3\,M_c = 9 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (69b)$$

and, substituting Eq. 68*b* into Eq. 67*b*:

$$8\,V_d - 3\,M_c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (69c)$$

Solving Eqs. 69 simultaneously: $M_c = 1.174$ kip-ft; $M_a = 3.13$ kip-ft; and $V_d = 0.44$ kips. From Eqs. 68 and from static conditions of equilibrium: $V_c = 0.391$ kips; $V_a = 0.844$ kips; and $R = 2.766$ kips.

Example 11. Derivation of the Three-Moment Equation.—Fig. 18 shows a continuous beam of two spans subject to any transverse load. For simplicity, assume that the supports A, B, and C are unyielding, so that $\theta_1 = -\theta_2$. By geometry, $\frac{y_{AB}}{l_1} = -\frac{y_{CB}}{l_2}$; or $\frac{-M'''_{AB,\,A}}{6\,E\,I\,l_1} = -\frac{+M'''_{BC,\,C}}{6\,E\,I\,l_2}$. Simplifying:

$$\frac{1}{l_1} M'''_{AB,\,A} = \frac{1}{l_2} M'''_{BC,\,C} \ldots\ldots\ldots\ldots\ldots\ldots (70a)$$

Eq. 70*a* may be written in the following form:

$$\frac{1}{l_1}(-R_2\,l^3{}_1 - 3\,M_B\,l^2{}_1 + M'''_{\overline{AB},\,A}) = \frac{1}{l_2}(R_3\,l^3{}_2 + 3\,M_B\,l^2{}_2 + M'''_{\overline{BC},\,C}) \ldots (70b)$$

in which $M'''_{\overline{AB},\,A}$ represents the third moment, with respect to point A, of the external loads only, on the span AB, end shears and end moments being excluded; and $M'''_{\overline{BC},\,C}$ represents the third moment, with respect to point C, of the external loads only, on the span BC. Similarly, $M_{\overline{AB},\,A}$ and $M''_{\overline{AB},\,A}$ represent the first and second moments, respectively, with respect to point A, of the external loads only, on span AB. For span AB (Fig. 18(*b*)): $M_{AB,\,A} = M_A - M_B - R_2\,l_1 + M_{\overline{AB},\,A} = 0$; and

$$R_2 = \frac{1}{l_1}(M_A - M_B + M_{\overline{AB},\,A}) \ldots\ldots\ldots\ldots\ldots (71a)$$

For span BC (Fig. 18(c)): $M_{BC,C} = M_B - M_C + R_3 l_2 + M_{\overline{CB},C} = 0$; and

$$R_3 = \frac{1}{l_2}(M_C - M_B - M_{\overline{BC},C}) \ldots\ldots\ldots\ldots\ldots\ldots (71b)$$

Substituting Eqs. 71 in Eq. 70*b*, the three-moment equation is derived, thus:

$$l_1 M_A + 2(l_1 + l_2) M_B + l_2 M_C = \left(\frac{M'''_{\overline{AB},A}}{l_1} - \frac{M'''_{\overline{BC},C}}{l_2}\right) - (l_1 M_{\overline{AB},A} - l_2 M_{\overline{BC},C}) \ldots\ldots (72)$$

Example 12. Continuous Beam.—The problem in Fig. 19 may be solved with the three-moment equation. By Eq. 72, $2(5 + 10) M_B = \left[\frac{500 \times 3^3}{5} - \frac{(-12{,}000) \times 10^3}{4 \times 10}\right] - [5 \times 500 \times 3 - 10 \; (-12{,}000) \times 5] = -304{,}800$ lb-ft; and $M_B = -10{,}160$ lb-ft. For span AB (Fig. 19(c)): $M_{BC,B} = -R_3 \times 10 + 12{,}000 \times 5 - 10{,}160 = 0$. Consequently, $R_3 = 4{,}984$ lb; and $R_2 = 9{,}348$ lb.

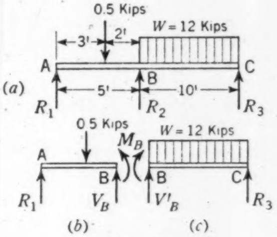


FIG. 19

ACKNOWLEDGMENTS

The writer is indebted to Prof. Fang-Yin Tsai, formerly dean of the College of Engineering, and Prof. T. M. Yu, formerly of the department of civil engineering, both of the National Chung Cheng University, Nanchang, Kiangsi, China, for their invaluable help in the preparation of the paper.

APPENDIX. NOTATION

The following letter symbols, used in the paper and in its discussions, conform essentially to American Standard Letter Symbols for Mechanics, Structural Engineering and Testing Materials (ASA—Z10a—1932), prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1932:

a = lever arm of a point force;
b = reference distance to the center of a couple;
E = modulus of elasticity;
I = moment of inertia;
l = span length, or (where stated) equivalent length of a uniform load;
M = moment:
  M'' = second moment;
  M''' = third moment;
  M_c = a couple, or torque;
m = a variable number;

n = a variable number;
P = point force or concentrated load, with appropriate subscripts to denote the point of application;
R = resultant force;
V = vertical shear, or total shear force;
W = total distributed weight, or load;
w = distributed load per unit distance;
x = variable distances parallel to the x-axis;
y = variable distances and deflections parallel to the y-axis;
z = variable distances along a beam, to define the location of an increment dz of distributed load;
δ = vertical deflections (see also y); and
θ = angular deflection.

DISCUSSION

A. FLORIS,[2] ESQ.—By means of his second-moment and third-moment concepts, the author derives formulas for the determination of slopes and deflections, which are used afterward in the analysis of hyperstatic structures. It is stated in the "Synopsis" of this paper that "The various methods for computing beam deflections may be classified into two main groups—the double-integration method and the moment-area method."

The application of the author's second and third moments to statical problems is facilitated by the aid of seven theorems. From these theorems the last two are admittedly similar in nature to those of the moment-area method. The labor involved in these computations is by no means less than the work required by other methods.

In enumerating the various methods for determining slopes and deflections, the author overlooks the principle of virtual displacements and the Castigliano theorems related to it. By the use of this general principle, and with the Müller-Breslau formulas for quick integration involving algebra only,[3] the most rapid results can be obtained in a direct manner.

The slopes and deflections (and hence the hyperstatic quantities) can be determined also by means of a fourth-order differential equation of the elastic curve.[4,5] By using Laplace transforms the tedious process of determining the four constants of integration is eliminated, and the differential equation is transformed into an algebraic equation.[6] Even without the use of the Laplace transformation the problem can be arranged in a manner suitable for practical solution.[7]

The second moment as defined by the author has a physical meaning that he has not utilized. If W is taken as the mass of the system, the second moment is only the moment of inertia of this mass with respect to an axis normal to the center line of the beam. Using the author's notation, M''_O is then the moment of inertia of the system with respect to the normal through point O, the center of gravity of the mass, M''_N is the moment of inertia of the mass with respect to the normal through point N, etc. Some of these values for masses of various shapes are given in reference books.[8]

If the mass is replaced by the loading area, the moment of inertia for area is obtained, which is more familiar to the structural engineer. Conversely, the moments of inertia for mass are derived from the moments of inertia for areas. Mass here is taken as equivalent to load. Of all the moments of inertia, only M''_O needs to be determined by integration. The moments with respect to axes A, B, and N can be found by means of the well-known formula for moments of inertia at points other than the center of gravity. From the values of moments of inertia for few basic loadings, the values of the moment of inertia for a variety of loadings can be obtained by superposition. Table 1 is a sample of second moments for three loadings given by the author.

---

[2] Asst. to Structural Engr., Pacific Elec. Ry. Co., Los Angeles, Calif.

[3] "Graphische Statik der Baukonstruktionen," by H. F. B. Müller-Breslau, Alfred Kröner, Stuttgart, Vol. III, Pt. II, 1908, p. 90.

[4] "Applied Elasticity," by John Prescott, Dover Publications, New York, N. Y., 1946, p. 52.

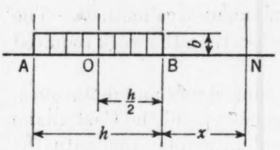
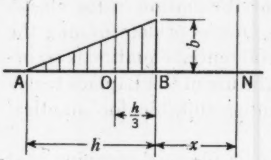
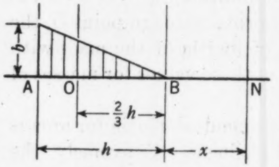
[5] "An Introduction to the Theory of Elasticity for Engineers and Physicists," by R. V. Southwell, Clarendon Press, Oxford, 1936, p. 172.

[6] "Applied Mathematics for Engineers and Physicists," by Louis A. Pipes, McGraw-Hill Book Co., Inc., New York and London, 1946, p. 217.

[7] "On the Theory of the Elastic Curve of the Deflected Beam," by P. C. Khoudiakoff, *Bulletin*, Polytechnic Soc., Imperial Technical School, Moscow, 1909, p. 213 (in Russian).

[8] "Hütte, des Ingenieurs Taschenbuch," Wilhelm Ernst & Son, Berlin, Pt. I, 1908, p. 219.

TABLE 1.—SECOND MOMENTS OR MOMENTS OF INERTIA

Loading	Mass	Area	Axis	Notation	Moment of Inertia: Mass	Moment of Inertia: Area
(a) RECTANGULAR						
	m	$b\,h$	A	I_a	$\frac{m\,h^2}{3}$	$\frac{b\,h^3}{3}$
			O	I_o	$\frac{m\,h^2}{12}$	$\frac{b\,h^3}{12}$
			B	I_b	$\frac{m\,h^2}{3}$	$\frac{b\,h^3}{3}$
			N	I_n	$\frac{m\,h^2}{12}+m\left(\frac{h}{2}+x\right)^2$	$\frac{b\,h^3}{12}+b\,h\left(\frac{h}{2}+x\right)^2$
(b) TRIANGULAR						
	m	$\frac{b\,h}{2}$	A	I_a	$\frac{m\,h^2}{2}$	$\frac{b\,h^3}{4}$
			O	I_o	$\frac{m\,h^2}{18}$	$\frac{b\,h^3}{36}$
			B	I_b	$\frac{m\,h^2}{6}$	$\frac{b\,h^3}{12}$
			N	I_n	$\frac{m\,h^2}{18}+m\left(\frac{h}{3}+x\right)^2$	$\frac{b\,h^3}{36}+\frac{b\,h}{2}\left(\frac{h}{3}+x\right)^2$
(c) TRIANGULAR						
	m	$\frac{b\,h}{2}$	A	I_a	$\frac{m\,h^2}{6}$	$\frac{b\,h^3}{12}$
			O	I_o	$\frac{m\,h^2}{18}$	$\frac{b\,h^3}{36}$
			B	I_b	$\frac{m\,h^2}{2}$	$\frac{b\,h^3}{4}$
			N	I_n	$\frac{m\,h^2}{18}+m\left(\frac{2}{3}h+x\right)^2$	$\frac{b\,h^3}{36}+\frac{b\,h}{2}\left(\frac{2}{3}h+x\right)^2$

Considerable originality is displayed by Mr. Hsu and his interesting paper deserves careful study.

ROBERT B. B. MOORMAN,[9] M. ASCE.—Although the author is to be commended for his efforts to simplify computations for the deflections of beams, the writer feels that the simplification, by introducing second and third moments of the loads, is manifest only in some special cases and cannot be said to be general. The author contends that the use of second and third moments of the loads to compute deflections is a more direct method than that of considering the deflection to be a function of the bending moment. With this the writer cannot agree. Actually the angle changes that occur along the axis of a structure are used in computing the deflections. For beams, columns, or curved members the usual procedure is to assume the angle change equal to the bending moment divided by $E\,I$.

---

[9] Prof., Civ. Eng., Univ. of Missouri, Columbia, Mo.

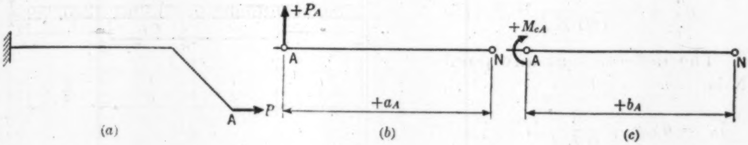


Fig. 20

A direct method of computing deflections should not be limited to straight beams but should also be applicable to a simple arrangement as shown in Fig. 20(*a*). Point O of theorem III should be specified as the point at which the resultant is located on the reference line and not as "any other point O." The same is probably true for theorem IV.

TABLE 2.—Summary of Moment Formulas

Loading	First moment	Second moment	Third moment
Load (see Fig. 20 (*b*)).....	$M_N = P_A\,a_A$	$M''_N = P_A\,a^2{}_A$	$M'''_N = P_A\,a^3{}_A$
Couple (see Fig. 20 (*c*))...	$M_N = M_{cA}$	$M''_N = 2\,M_{cA}\,b_A$	$M'''_N = 3\,M_{cA}\,b^2{}_A$
Balanced force system....	....	$M'' = \text{constant}$	$M'''_N = M'''_O + 3\,M''\,x$
Unbalanced force system (point O at resultant)...	....	$M''_N = M''_O + M''_{O.N}$	$M'''_N = M'''_O + 3\,M''_O\,x + M'''_{O.N}$
Force system symmetrical about point O.........	....	....	$M'''_O = 0$

The usefulness of the paper would probably be enhanced if the information were summarized somewhat as demonstrated in Table 2. Likewise it would be helpful to have the information contained in the section entitled "Second and Third Moments of Force Systems Acting on Straight Beams" extended and tabulated.

That the proposed method can be applied expeditiously to certain deflection problems is illustrated by the following example. Let it be required to determine the equation for the deflection and the equation for the slope at any point for the beam shown in Fig. 21. The deflection at end B is zero, or

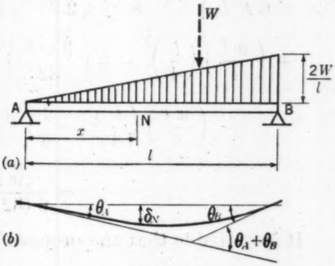


Fig. 21

$$\delta_B = \theta_A\,l + \frac{1}{6\,E\,I}\,M'''_{AB,\,B} = 0 \ldots\ldots\ldots\ldots\ldots (73)$$

Substituting for $M'''_{AB,B}$ in Eq. 73:

$$\theta_A\, l + \frac{1}{6\,E\,I}\left(-\frac{W\,l^3}{10} + \frac{W\,l^3}{3}\right) = 0 \ldots\ldots\ldots\ldots\ldots (74)$$

from which

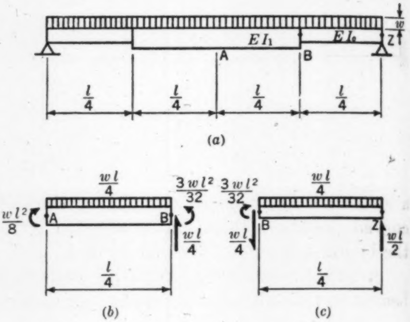
$$\theta_A = -\frac{7}{180\,E\,I}\,W\,l^2 . (75)$$

The deflection at any point N is

$$\delta_N = \theta_A\, x + \frac{1}{6\,E\,I}\,M'''_{AN,N}$$

$$= -\frac{7}{180\,E\,I}\,W\,l^2\,x + \frac{1}{6\,E\,I}\left(\frac{W}{3}\,x^3 - \frac{W\,x^2}{l^2} \times \frac{x^3}{10}\right)$$

$$= \frac{W\,x}{180\,E\,I\,l^2}\,(-3\,x^4 + 10\,x^2\,l^2 - 7\,l^4) \ldots (76)$$



(a)

(b)

(c)

FIG. 22

The slope at any point N is

$$\theta_N = \theta_A + \frac{1}{2\,E\,I}\,M''_{AN} = -\frac{7}{180\,E\,I}\,W\,l^2 + \frac{1}{2\,E\,I}\left(\frac{W}{3}\,x^2 - \frac{W\,x^2}{l^2} \times \frac{x^2}{6}\right)$$

$$= \frac{W}{180\,E\,I\,l^2}\,(-15\,x^4 + 30\,x^2\,l^2 - 7\,l^4) \ldots\ldots\ldots\ldots\ldots (77)$$

To illustrate a case in which the computations for deflection are about the same whether or not the proposed method or the moment-area method is used, consider the beam shown in Fig. 22. Let it be required to determine the deflection at point A, which may be expressed as

$$\delta_A = y_{ZA} = \frac{1}{6\,E\,I_1}\,M'''_{AB,Z} + \frac{1}{6\,E\,I_0}\,M'''_{BZ,Z}$$

$$= \frac{1}{6\,E\,I_1}\left[3\left(\frac{w\,l^2}{8}\right)\left(\frac{l}{2}\right)^2 - \frac{1}{4}\left(\frac{w\,l}{4}\right)\left(\frac{l}{4}\right)^2\left(\frac{3\,l}{8}\right) - \left(\frac{w\,l}{4}\right)\left(\frac{3\,l}{8}\right)^3 + \left(\frac{w\,l}{4}\right)\left(\frac{l}{4}\right)^3 - 3\left(\frac{3\,w\,l^2}{32}\right)\left(\frac{l}{4}\right)^2\right] + \frac{1}{6\,E\,I_0}\left[3\left(\frac{3\,w\,l^2}{32}\right)\left(\frac{l}{4}\right)^2 - \left(\frac{w\,l}{4}\right)\left(\frac{l}{4}\right)^3 - \frac{1}{4}\left(\frac{w\,l}{4}\right)\left(\frac{l}{4}\right)^2\left(\frac{l}{8}\right) - \left(\frac{w\,l}{4}\right)\left(\frac{l}{8}\right)^3\right]$$

$$= \frac{w\,l^4}{6\,(32)^2\,E}\left(\frac{67}{I_1} + \frac{13}{I_0}\right) \ldots\ldots\ldots\ldots\ldots\ldots (78)$$

It is probable that the proposed method will be found useful in simplifying those analyses[10] which involve the moment of the $\frac{M}{E\,I}$-diagram about the support. In a case of this kind the loads can be used in place of the bending moments.

---

[10] "Continuous Frame Analysis by Elastic Support Action," by J. Charles Rathbun and C. W. Cunningham, *Transactions*, ASCE, Vol. 113, 1948, p. 583, Eqs. 16.

FRANK J. MCCORMICK,[11] Assoc. M. ASCE.—A study of this paper reveals that, as defined by the author, the second moment (M'') is equal to twice the area of the bending moment diagram, and the third moment (M''') is equal to six times the moment of that area with respect to one end.

This ingenious substitution of these analytical expressions for the geometrical concepts used in the moment-area and conjugate beam methods is not likely to seem advantageous to most readers, for the invention of these latter methods was an attempt to avoid the mathematical abstractions of the double-integration process in favor of the concreteness of geometrical or physical concepts. In fact, the advantage of being able to visualize the operations involved has led to the invention of many similar substitution methods. Among these are the elastic weight, moment distribution, and column analogy methods in structural analysis, and the membrane, electrical, and hydrodynamical analogies in the field of elasticity. All these represent an effort to simplify the problem, either by replacing the quantities involved by more familiar ones, or by replacing relatively abstract mathematical quantities by analogous concrete physical ones.

Where the conjugate beam method refers the student back to the familiar concepts of shear and bending moment, the method proposed by the author requires the student to learn the new concepts of second and third moments of forces and couples. This could be defended if by so doing less actual labor need be applied for a solution. However, the author has not shown this to be true in the examples presented.

WILLIAM A. CONWELL,[12] M. ASCE.—The scholarly presentation of this paper and its adherence to fundamentals in the approach to the solution of deflection problems make it a definite contribution to engineering literature.

In the "Synopsis," the author states "* * * the aim of this paper is to suggest a more direct solution" and further indicates that it is proposed to solve deflection problems without use of the bending moment. It should be noted that, although the author achieves the elimination of the bending moment, he does not do it without introducing another moment concept—namely, that of second and third moments. There is some question as to whether this substitution is an improvement. If the sign convention is remembered, the second and third moments of a force are readily understood. However, the second and third moments of a moment are not so clearly visualized, inasmuch as they involve a coefficient which has no physical significance and so introduces an element of artificiality. Likewise, the expressions for angle changes and deflections (Eqs. 27 and 30) involve divisors which have no direct physical significance and must, therefore, be remembered if used. Furthermore, if the design computations for the structure are available, the bending moments on which to base deflection calculations are also at hand. In most instances, it would be necessary to compute second and third moments. Still further, the bending moment is intimately associated with the geometry of deflections and thus is an invaluable aid in the visualization of structural action.

---

[11] Associate Prof. of Applied Mechanics, Kansas State College, Manhattan, Kans.

[12] Gen. Engr., Structural Eng. and Design Dept., Duquesne Light Co., Pittsburgh, Pa.

Since the method of this paper has the effect of separating the mind of the designer from the action of the structure, it should offer something to balance this disadvantage. In the use of the column analogy,[13] for instance, the designer's mind is taken up with the computation of stresses in a fictitious column cross section whose stresses are analogous to the bending moments in the structure being analyzed. In the process of computing column stresses, the designer is not thinking in terms of the action of the structure, but at the conclusion of this work he has a good basis for visualizing it. The only excuse for interposing the column analogy in the analysis is that it produces results more quickly in some cases than would otherwise be obtained and, thus, conserves the time of the designer. Whether the same can be stated of the method under discussion will be determined as it is used in competition with other methods.

Practically the entire portent of this paper is carried in theorems VI and VII. Theorems I to V, inclusive, are almost exclusively devoted to definitions. This raises the question as to whether greater emphasis should not be given to theorems VI and VII than is accorded them in the paper at the end of the series of definitive theorems.

From the beginning of his studies in strength of materials, the engineer has been aware that angle changes vary as $\frac{M\,a}{E\,I}$ and deflections vary as $\frac{M\,a^2}{E\,I}$. Accordingly, since M is generally equal to $\Sigma P\,a$, he concluded at an early point in his progress that angle changes vary as $\Sigma\frac{P\,a^2}{E\,I}$ and deflections vary as $\Sigma\frac{P\,a^3}{E\,I}$. It has remained, however, for the author of this paper to employ this general conclusion in such a way as to produce quantitative results. It is, therefore, possible that the method of second and third moments, when used in conjunction with methods more generally employed, will prove to be a valuable tool in determining deflections.

HSU SHIH-CHANG,[14] ESQ.—The purpose of this paper was to simplify the procedure for computing beam deflections, thus eliminating the unnecessary computations involved in other methods. For example, the equation for the deflection at any point N of a cantilever beam that supports a single force P at the free end (Fig. 23(a)) is determined by area moments in three steps as follows:

(1) Construct the bending moment diagram shown in Fig. 23(b);

(2) Compute the area of the bending moment diagram between points A and N—

$$A_m = -P\,(l - x)\,x - \tfrac{1}{2}\,(P\,x)\,x \ldots\ldots\ldots\ldots\ldots (79)$$

---

[13] "The Column Analogy," by Hardy Cross, *Bulletin No. 215*, Eng. Experiment Station, Univ. of Illinois, Urbana, 1930.

[14] Asst., Dept. of Mech. and Elec. Eng., National Chung Cheng Univ., Nanchang, Kiangsi, China.

and

(3) Compute the moment of that area about point N—

$$Q_n = -[P(l-x)\,x]\,\frac{x}{2} - (\tfrac{1}{2}P x^2)\left(\frac{2x}{3}\right) = -\tfrac{1}{6}P x^2(3l - x) \quad \text{..(80)}$$

Eq. 80 is to be divided by $E I$ to obtain the equation of deflection:

$$\delta_N = -\frac{P x^2}{6 E I}(3l - x) \quad \text{.......(81)}$$

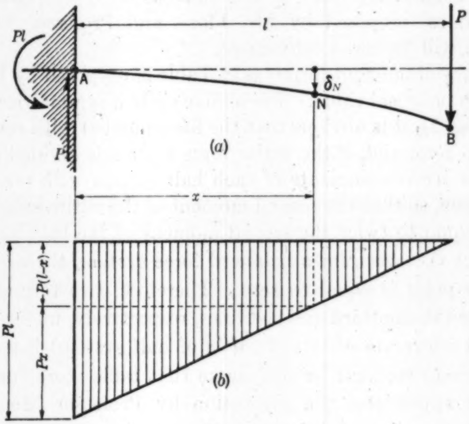


FIG. 23

By the proposed method, it is necessary only to compute the third moment of the reaction P and the couple $P l$ (both at built-in end A) with respect to point N; thus:

$$P x^3 - 3Plx^2 = -Px^2(3l - x) \quad \text{.......(82)}$$

Eq. 82 is divided by $6 E I$ to obtain the equation of deflection—

$$\delta_N = -\frac{P x^2}{6 E I}(3l - x) \quad \text{.......(83)}$$

Although the results are the same, the proposed method simplifies the procedure of computation considerably.

Since the moment center of the second and third moments of any force system, either balanced or unbalanced, may be chosen at any point on the reference line according to the characteristics of second and third moments, the computer can select some convenient points as moment centers to make the calculation simple and easy for any given case. Fig. 28, described subsequently, is an illustration of this procedure. Also, using the proposed method, the designer can formulate the general solutions of angle changes, deflections, and statically indeterminate problems, for a given support condition of beam carrying any loading, expressed as functions of the second and third moments of forces and couples. For any given loading condition, the desired results are obtained quickly from the formulas, which are convenient for practical use. Tables 4 and 5 (explained in detail subsequently) are prepared for this purpose.

Since the bending moment on which the other methods depend is very useful in other fields of structural engineering and is more generally employed, and since the concept of second and third moments, as adapted to the problem of this paper, is new, it is natural that it would be difficult for it to replace other methods immediately. As the concept of the second and third moments becomes more familiar, however, and as its usefulness becomes recognized in other fields of engineering (such as the analysis of hyperstatic structures and continuous frames, proposed by Mr. Floris and Professor Moorman), it is possible that it will be found convenient.

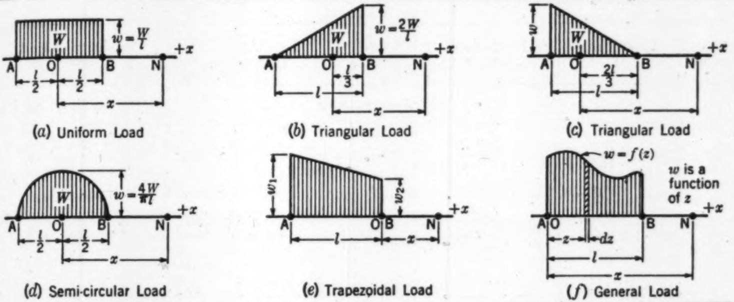
The summary of moment formulas in Table 2, proposed by Professor Moorman, is of much practical value. For a force system symmetrical about point O (referring to Fig. 5), it is obvious that the first moment with respect to point O is also equal to zero; and, if the entire force system is divided into two halves at point O, the second moments of each half system with respect to point O must be the same, so that the second moment of the entire system with respect to point O is equal to twice the second moment of the half force system with respect to point O. Also, for a balanced force system, the first moment with respect to any point is equal to zero. Therefore, two formulas $M = 0$ and $M_O = 0$ inserted in the third and last lines, respectively, under "first moment" in Table 2 and a formula $M''_O = 2$ (M''_O of half system) inserted in the last line under "second moment" would make that table more complete.

The writer appreciates the suggestion by Professor Moorman that the information of second and third moments of loadings be summarized, and has prepared Table 3. If Eq. 54, theorem II, and Table 3 are applied to the example in Fig. 22, the solution can be simplified as follows: Referring to Fig. 24, the deflection at point A may be expressed as

$$\begin{aligned}
\delta_A = y_{ZA} &= \Delta\left(\frac{1}{6\,E\,I}\right)_B M'''_{AB,Z} + \frac{1}{6\,E\,I_0} M'''_{AZ,Z} \\
&= \left(\frac{1}{6\,E\,I_1} - \frac{1}{6\,E\,I_0}\right)\left(M'''_{AB,B} + 3\,M''_{AB}\,\frac{l}{4}\right) \\
&\quad + \frac{1}{6\,E\,I_0} M'''_{AZ,Z} \\
&= \left(\frac{1}{6\,E\,I_1} - \frac{1}{6\,E\,I_0}\right)\left\{3\left(\frac{w\,l^2}{8}\right)\left(\frac{l}{4}\right)^2 - \tfrac{1}{4}\,w\left(\frac{l}{4}\right)^4\right. \\
&\quad \left. + 3\left[2\left(\frac{w\,l^2}{8}\right)\left(\frac{l}{4}\right) - \tfrac{1}{3}\,w\left(\frac{l}{4}\right)^3\right]\left(\frac{l}{4}\right)\right\} \\
&\quad + \frac{1}{6\,E\,I_0}\left[3\left(\frac{w\,l^2}{8}\right)\left(\frac{l}{2}\right)^2 - \tfrac{1}{4}\,w\left(\frac{l}{2}\right)^4\right] \\
&= \frac{w\,l^4}{6(32)^2\,E}\left(\frac{67}{I_1} + \frac{13}{I_0}\right) \quad \text{.......... (84)}
\end{aligned}$$

TABLE 3.—SECOND AND THIRD MOMENTS AT LOADINGS

(Expressed in Terms of Total Load W and Load Intensity w)



(a) Uniform Load (b) Triangular Load (c) Triangular Load

(d) Semi-circular Load (e) Trapezoidal Load (f) General Load

SECOND MOMENT			THIRD MOMENT		
Notation	Total load,[a] W	Intensity of load, w	Notation	Total load, W	Intensity of load, w
		(a) UNIFORM LOAD			
M''_A	$-\frac{Wl^2}{3}$	$-\frac{wl^3}{3}$	M'''_A	$+\frac{Wl^3}{4}$	$+\frac{wl^4}{4}$
M''_O	$-\frac{Wl^2}{12}$	$-\frac{wl^3}{12}$	M'''_O	0	0
M''_B	$-\frac{Wl^2}{3}$	$-\frac{wl^3}{3}$	M'''_B	$-\frac{Wl^3}{4}$	$-\frac{wl^4}{4}$
M''_N	$-\frac{Wl^2}{12} - Wx^2$	$-\frac{wl^3}{12} - wlx^2$	M'''_N	$-\frac{Wl^2}{4}x - Wx^3$	$-\frac{wl^3}{4}x - wlx^3$
		(b) TRIANGULAR LOAD			
M''_A	$-\frac{Wl^2}{2}$	$-\frac{wl^3}{4}$	M'''_A	$+\frac{2Wl^3}{5}$	$+\frac{wl^4}{5}$
M''_O	$-\frac{Wl^2}{18}$	$-\frac{wl^3}{36}$	M'''_O	$-\frac{Wl^3}{135}$	$-\frac{wl^4}{270}$
M''_B	$-\frac{Wl^2}{6}$	$-\frac{wl^3}{12}$	M'''_B	$-\frac{Wl^3}{10}$	$-\frac{wl^4}{20}$
M''_N	$-\frac{Wl^2}{18} - Wx^2$	$-\frac{wl^3}{36} - \frac{wl}{2}x^2$	M'''_N	$-\frac{Wl^3}{135} - \frac{Wl^2}{6}x - Wx^3$	$-\frac{wl^4}{270} - \frac{wl^3}{12}x - \frac{wl}{2}x^3$
		(c) TRIANGULAR LOAD			
M''_A	$-\frac{Wl^2}{6}$	$-\frac{wl^3}{12}$	M'''_A	$+\frac{Wl^3}{10}$	$+\frac{wl^4}{20}$
M''_O	$-\frac{Wl^2}{18}$	$-\frac{wl^3}{36}$	M'''_O	$+\frac{Wl^3}{135}$	$+\frac{wl^4}{270}$
M''_B	$-\frac{Wl^2}{2}$	$-\frac{wl^3}{4}$	M'''_B	$-\frac{2Wl^3}{5}$	$-\frac{wl^4}{5}$
M''_N	$-\frac{Wl^2}{18} - Wx^2$	$-\frac{wl^3}{36} - \frac{wl}{2}x^2$	M'''_N	$+\frac{Wl^3}{135} - \frac{Wl^2}{6}x - Wx^3$	$+\frac{wl^4}{270} - \frac{wl^3}{12}x - \frac{wl}{2}x^3$

TABLE 3.—(*Continued*)

SECOND MOMENT			THIRD MOMENT		
Notation	Total load,[a] W	Intensity of load, w	Notation	Total load, W	Intensity of load, w
		(*d*) SEMICIRCULAR LOAD			
M''_A	$-\frac{5\,W\,l^2}{16}$	$-\frac{5\,w\,\pi\,l^3}{64}$	M'''_A	$+\frac{7\,W\,l^3}{32}$	$+\frac{7\,w\,\pi\,l^4}{128}$
M''_O	$-\frac{W\,l^2}{16}$	$-\frac{w\,\pi\,l^3}{64}$	M'''_O	0	0
M''_B	$-\frac{5\,W\cdot l^2}{16}$	$-\frac{5\,w\,\pi\,l^3}{64}$	M'''_B	$-\frac{7\,W\,l^3}{32}$	$-\frac{7\,w\,\pi\,l^4}{128}$
M''_N	$-\frac{W\,l^2}{16}-W\,x^2$	$-\frac{w\,\pi\,l^3}{64}-\frac{w\,\pi\,l}{4}\,x^2$	M'''_N	$-\frac{3\,W\,l^2}{16}\,x-W\,x^3$	$-\frac{3\,w\,\pi\,l^3}{64}\,x-\frac{w\,\pi\,l}{4}\,x^3$
		(*e*) TRAPEZOIDAL LOAD			
M''_A		$-\frac{(w_1+3\,w_2)\,l^3}{12}$	M'''_A		$+\frac{(w_1+4\,w_2)\,l^4}{20}$
M''_B		$-\frac{(3\,w_1+w_2)\,l^3}{12}$	M'''_B		$-\frac{(4\,w_1+w_2)\,l^4}{20}$
M''_N		$-\frac{(3\,w_1+w_2)\,l^3}{12}-\frac{(2\,w_1+w_2)\,l^2}{3}\,x-\frac{(w_1+w_2)\,l}{2}\,x^2$	M'''_N		$-\frac{(4\,w_1+w_2)\,l^4}{20}-\frac{(3\,w_1+w_2)\,l^3}{4}\,x-\frac{(2\,w_1+w_2)\,l^2}{2}\,x^2-\frac{(w_1+w_2)\,l}{2}\,x^3$
		(*f*) GENERAL LOAD			
M''_A		$-\int_0^l f(z)\,z^2\,dz$	M'''_A		$+\int_0^l f(z)\,z^3\,dz$
M''_B		$-\int_0^l f(z)\,(l-z)^2\,dz$	M'''_B		$-\int_0^l f(z)\,(l-z)^3\,dz$
M''_N		$-\int_0^l f(z)\,(x-z)^2\,dz$	M'''_N		$-\int_0^l f(z)\,(x-z)^3\,dz$

[a] Formulas for total load W are omitted in Tables 3(*e*) and 3(*f*).

The advantage of applying Eq. 54 may be further shown by another numerical example. Let it be required to determine the equation for the deflection at any point for the beam shown in Fig. 25. Applying Table 3(*e*) and computing the second and third moments,

$$M'''_{AN,N} = 135\,x^3 - 3 \times 1{,}350\,x^2 - \frac{(4 \times 9 + 9 - 0.3\,x)\,x^4}{20}$$

$$= x^2\,(-\,4{,}050 + 135\,x - 2.25\,x^2 + 0.015\,x^3)\,..(85)$$

Substituting $x = 10$ and $x = 20$, respectively, into Eq. 85, $M'''_{AB,B} = -\,291{,}000$; and $M'''_{AC,C} = -\,852{,}000$; also

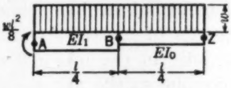
$$M''_{AB} = 135 \times 10^2 - 2 \times 1{,}350 \times 10 - \frac{(3 \times 9 + 9 - 0.3 \times 10)\,10^3}{12} = -\,16{,}250$$

and

$$M''_{AC} = 135 \times 20^2 - 2 \times 1{,}350 \times 20 - \frac{(3 \times 9 + 9 - 0.3 \times 20)\, 20^3}{12} = -20{,}000$$

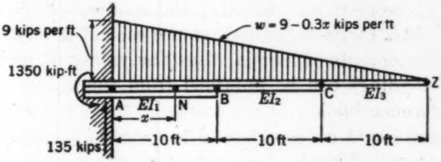
Then, applying Eq. 54, the deflection at any point N is: For $0 \leqq x \leqq 10$ ft—

$$\delta_N = \frac{1}{6\,E\,I_1} M'''_{AN,N} = \frac{x^2}{6\,E\,I_1} (-4{,}050 + 135\,x - 2.25\,x^2 + 0.015\,x^3) \quad (86)$$



$\frac{wl^2}{8}$
A
EI_1
B
Z
EI_0
w
$\frac{l}{4}$
$\frac{l}{4}$

FIG. 24



9 kips per ft
$w = 9 - 0.3x$ kips per ft
1350 kip-ft
A
EI_1
N
B
EI_2
C
EI_3
Z
x
135 kips
10 ft
10 ft
10 ft

FIG. 25

and, for $10 \text{ ft} \leqq x \leqq 20$ ft—

$$\begin{aligned}
\delta_N &= \Delta \left(\frac{1}{6\,E\,I} \right)_B M'''_{AB,N} + \frac{1}{6\,E\,I_2} M'''_{AN,N} \\
&= \left(\frac{1}{6\,E\,I_1} - \frac{1}{6\,E\,I_2} \right) [M'''_{AB,B} + 3\,M''_{AB}\,(x - 10)] \\
&\qquad + \frac{1}{6\,E\,I_2} M'''_{AN,N} \\
&= \left(\frac{1}{6\,E\,I_1} - \frac{1}{6\,E\,I_2} \right) [-291{,}000 + 3\,(-16{,}250)\,(x - 10)] \\
&\qquad + \frac{x^2}{6\,E\,I_2} (-4{,}050 + 135\,x - 2.25\,x^2 + 0.015\,x^3) \\
&= \frac{125}{E\,I_1} (262 - 65\,x) + \frac{1}{6\,E\,I_2} (-196{,}500 + 48{,}750\,x - 4{,}050\,x^2 \\
&\qquad + 135\,x^3 - 2.25\,x^4 + 0.015\,x^5) \quad (87)
\end{aligned}$$

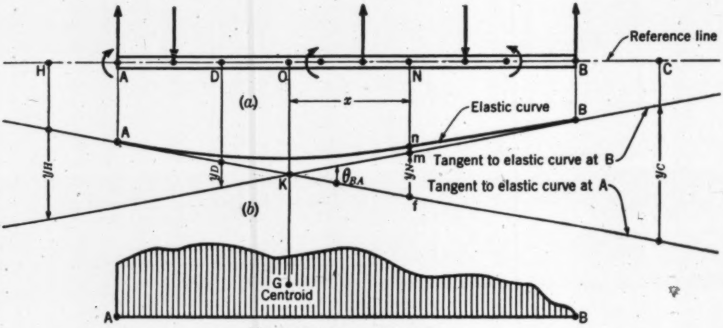
Similarly, for $20 \text{ ft} \leqq x \leqq 30$ ft:

$$\begin{aligned}
\delta_N &= \Delta \left(\frac{1}{6\,E\,I} \right)_B M'''_{AB,N} + \Delta \left(\frac{1}{6\,E\,I} \right)_C M'''_{AC,N} + \frac{1}{6\,E\,I_3} M'''_{AN,N} \\
&= \left(\frac{1}{6\,E\,I_1} - \frac{1}{6\,E\,I_2} \right) [-291{,}000 + 3\,(-16{,}250)\,(x - 10)] \\
&\qquad + \left(\frac{1}{6\,E\,I_2} - \frac{1}{6\,E\,I_3} \right) [-852{,}000 + 3\,(-20{,}000)\,(x - 20)] \\
&\qquad + \frac{x^2}{6\,E\,I_3} (-4{,}050 + 135\,x - 2.25\,x^2 + 0.015\,x^3) \\
&= \frac{125}{E\,I_1} (262 - 65\,x) + \frac{125}{E\,I_2} (202 - 15\,x) \\
&\qquad + \frac{1}{6\,E\,I_3} (-348{,}000 + 60{,}000\,x - 4{,}050\,x^2 \\
&\qquad + 135\,x^3 - 2.25\,x^4 + 0.015\,x^5) \quad (88)
\end{aligned}$$

For the arrangement shown in Fig. 20(*a*), the beam may be cut into two parts at the hip, each part being considered as a straight beam, and solved separately by the proposed method. The writer cannot agree with Professor Moorman's proposal to modify point O of theorem III. By statics, the resultant of an unbalanced force system can be represented by a single force and a couple both acting at any point on the reference line. Therefore, point O of theorem III is an independent point on the reference line similar to that of theorem II, and can be selected at will. In applying theorem III, point O is chosen before computing the resultant at point O.

Mr. Floris suggests that the second moments of loadings are equivalent to the moments of inertia; thus the values of second moments of various shapes of loadings can be obtained from the values of moments of inertia given in reference books. Unfortunately there is no characteristic in the moment of inertia that corresponds to the second moment of a couple, so the analogy of moment of inertia to second moment is only partial. Although the values of second moments of loadings can be taken from reference books by the analogy of moment of inertia to second moment, it is difficult to find a similar analogy to the more important term—third moments. Therefore, it is as well to obtain values of both second and third moments from a separate table, such as Table 3.

The second and third moments of a balanced force system with respect to any point on the reference line have some geometrical significance. Fig. 26(*a*) shows a balanced force system acting on the straight beam AB with the undeflected position of the beam axis as the reference line. Fig. 26(*b*) shows the elastic curve of beam AB, where the two tangents to the elastic curve at ends A and B intersect at point K. Applying theorems I and VI,



(*c*) Bending-moment Diagram

Fig. 26

$$M''_{AB} = 2\,E\,I\,\theta_{BA} = \text{a constant} \ldots\ldots\ldots\ldots\ldots (89)$$

Take any point N on the reference line, and let y_N be the intercept of the vertical line through point N between the two tangents to the elastic curve at points A and B, as shown in Figs. 26(*a*) and 26(*b*). Ordinate y_N will be positive when measured upward from the tangent at point A to the tangent at point B.

Applying theorem VII, Eq. 31*a* gives

$$y_N = \overline{\mathrm{fm}} = \overline{\mathrm{fn}} - \overline{\mathrm{mn}} = y_{NA} - y_{NB} = \frac{M'''_{AN,N}}{6\,E\,I} - \left(- \frac{M'''_{NB,N}}{6\,E\,I} \right)$$

$$= \frac{1}{6\,E\,I}\left(M'''_{AN,N} + M'''_{NB,N}\right) = \frac{1}{6\,E\,I} M'''_{AB,N} \quad \text{(90)}$$

or

$$M'''_{AB,N} = 6\,E\,I\,y_N \quad \text{(91)}$$

Similarly, $M'''_{AB,C} = 6\,E\,I\,y_C$; $M'''_{AB,D} = 6\,E\,I\,y_D$; and $M'''_{AB,H} = 6\,E\,I\,y_H$ —in which the values of $M'''_{AB,D}$ and $M'''_{AB,H}$ must be negative, because y_D and y_H are measured downward. Since $y_O = 0$ at the intersection K, it is obvious that

$$M'''_{AB,O} = 0 \quad \text{(92)}$$

Eqs. 89, 91, and 92 define the geometrical significance of the second and third moments of a balanced force system, acting on a straight beam AB (point B being on the right side of point A), as follows:

(1) The second moment, with respect to any point on the reference line, is equal to the angle between the two tangents to the elastic curve at points A and B, measured from the tangent at point A to the tangent at point B, multiplied by 2 *E I*. Therefore, the second moment is a constant.

(2) The third moment, with respect to any point N on the reference line, is equal to the intercept of the vertical line through point N between the two tangents at points A and B, measured from the tangent at point A to the tangent at point B, multiplied by 6 *E I*.

(3) The third moment, with respect to point O, which is on the same vertical line as the intersection K of the two tangents at points A and B, is equal to zero.

Comparing the elastic curve, Fig. 26(*b*), with the corresponding bending moment diagram, Fig. 26(*c*), since the intersection K and the centroid G of the area of the bending moment diagram occupy the same vertical line,[15] it is interesting to note that the three points—point O which is taken as the moment center of the balanced force system to obtain zero third moment, the intersection K of the two tangents to the elastic curve at points A and B, and the centroid G of the area of the bending moment diagram—are all on the same vertical line. To determine the position of point O, point K, or point G requires the application of theorem II, Eq. 12; thus:

$$M'''_{AB,N} = M'''_{AB,O} + 3\,M''_{AB}\,x \quad \text{(93)}$$

in which x (equal to the distance ON) is positive when point N is on the right side of point O. From Eq. 92, since $M'''_{AB,O} = 0$, Eq. 93 reduces to

$$M'''_{AB,N} = 3\,M''_{AB}\,x \quad \text{(94a)}$$

or

$$x = \frac{M'''_{AB,N}}{3\,M''_{AB}} \quad \text{(94b)}$$

Eq. 94*b* gives the distance from any point N to point O. The position of point O, once determined, may be selected as the origin. Then, from Eq. 94*a*, the third moment of the balanced force system, with respect to any point N on the reference line, can be computed easily as the distance x multiplied by a constant $3\ M''_{AB}$.

---

[15] "Improved Method of Finding Beam Deflections," by Ralph W. Stewart, *Civil Engineering*, February, 1934, p. 88, Fig. 1.

The derivation of Eqs. 52 can be simplified, as follows: Referring to Fig. 27 (which is similar to Fig. 13), Eq. 91 yields

$$y_{ZA} = \overline{\mathrm{aZ}} = \overline{\mathrm{ab}} + \overline{\mathrm{bc}} + \overline{\mathrm{cZ}} = \frac{1}{6\ (E\ I)_{AB}} M'''_{AB,Z} + \frac{1}{6\ (E\ I)_{BC}} M'''_{BC,Z} + \frac{1}{6\ (E\ I)_{CZ}} M'''_{CZ,Z} \ldots (95a)$$

Therefore, the general formula is

$$y_{ZA} = \sum \frac{1}{6\ (E\ I)_{AB}} M'''_{AB,Z} \ldots\ldots\ldots\ldots\ldots\ldots (95b)$$

For the beam with both ends built in and on the same level (in which case the two tangents to the elastic curve at the ends of the beam coincide with each other), Eqs. 89 and 91 make evident the fact that the first moment, second moment, and third moment of the balanced force system acting on the beam, with respect to any point on the reference line, are all equal to zero. Since the moment center of the first, second, and third moments may be taken at any point on the reference line, it is a matter of technique to choose the convenient moment center on the reference line for a given case in order to make the computation as simple and easy as possible. For example, let it be required to determine the two moments M_1 and M_2 at the ends of the beam shown in Fig. 28. The four equations for solving four unknowns M_1, M_2, R_1, and R_2 are

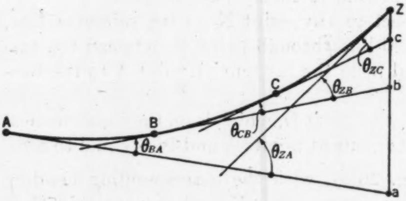


FIG. 27

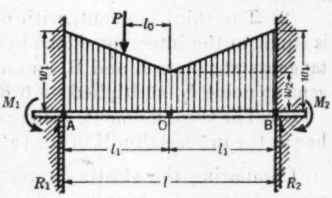


FIG. 28

$$R_1 + R_2 = P + l_1\ (w_1 + w_2) \ldots\ldots\ldots\ldots\ldots (96a)$$

$$M_{AB} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (96b)$$

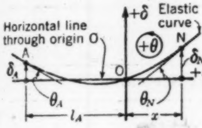
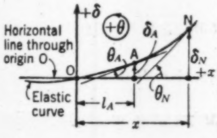
$$M''_{AB} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (96c)$$

and

$$M'''_{AB} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (96d)$$

in which the subscripts following the subscripts AB in symbols M_{AB}, M''_{AB}, and M'''_{AB} are all unnecessary since the location of the moment center is immaterial. From inspection of Fig. 28, the two trapezoidal loads are symmetrical about point O. Therefore, the first and third moments of the two trapezoidal loads with respect to point O are both equal to zero, and the second moment of the two trapezoidal loads with respect to point O is equal to twice the second moment of one trapezoidal load only with respect to point O. Also, point O is in the center of beam AB. Therefore, the most convenient point of moment center for determining M_1 and M_2 is point O. By Eqs. 96*b* and 96*d*,

TABLE 4.—Formulas of Angle Changes and Deflections[a]

Diagram	Boundary condition	Angle change θ_N of any point N	Deflection δ_N of any point N
		(*a*) Point A on Right Side of Origin O	
 +δ Elastic curve Horizontal line through origin O +θ N A δ_A O δ_N +x θ_A θ_N l_A x	θ_A (given).	$\frac{1}{2EI}(-M''_{OA} \pm M''_{ON}) + \theta_A$	$\frac{1}{6EI}(-3M''_{OA} \pm M'''_{ON,N}) + \theta_A x$
	$\theta_A = 0$....	$\frac{1}{2EI}(-M''_{OA} \pm M''_{ON})$	$\frac{1}{6EI}(-3M''_{OA}x \pm M'''_{ON,N})$
	δ_A (given).	$\frac{1}{6EIl_A}(-M'''_{OA,A} \pm 3l_A M''_{ON}) + \frac{\delta_A}{l_A}$	$\frac{1}{6EIl_A}(-M'''_{OA,A}x \pm l_A M'''_{ON,N}) + \frac{\delta_A}{l_A}x$
	$\delta_A = 0$....	$\frac{1}{6EIl_A}(-M'''_{OA,A} \pm 3l_A M''_{ON})$	$\frac{1}{6EIl_A}(-M'''_{OA,A}x \pm l_A M'''_{ON,N})$
		(*b*) Point A on Left Side of Origin O	
 +δ +θ δ_A N Horizontal line through origin O θ_A A δ_N O +x θ_N Elastic curve l_A x	θ_A (given).	$\frac{1}{2EI}(M''_{OA} \pm M''_{ON}) + \theta_A$	$\frac{1}{6EI}(3M''_{OA}x \pm M'''_{ON,N}) + \theta_A x$
	$\theta_A = 0$	$\frac{1}{2EI}(M''_{OA} \pm M''_{ON})$	$\frac{1}{6EI}(3M''_{OA}x \pm M'''_{ON,N})$
	δ_A (given).	$\frac{1}{6EIl_A}(-M'''_{OA,A} \pm 3l_A M''_{ON}) - \frac{\delta_A}{l_A}$	$\frac{1}{6EIl_A}(-M'''_{OA,A}x \pm l_A M'''_{ON,N}) - \frac{\delta_A}{l_A}x$
	$\delta_A = 0$....	$\frac{1}{6EIl_A}(-M'''_{OA,A} \pm 3l_A M''_{ON})$	$\frac{1}{6EIl_A}(-M'''_{OA,A}x \pm l_A M'''_{ON,N})$

[a] The values of x and the "±" signs before M''_{ON} and $M'''_{ON,N}$ all take positive sign when point N is on the right side of origin O.

$$M_{AB,O} = R_1 l_1 - M_1 - P l_0 - R_2 l_1 + M_2 = 0 \ldots\ldots\ldots (97a)$$

and

$$M'''_{AB,O} = R_1 l^3_1 - 3 M_1 l^2_1 - P l^3_0 - R_2 l^3_1 + 3 M_2 l^2_1 = 0 \ldots (97b)$$

Eliminating the two unknowns R_1 and R_2, both of which occur in Eqs. 97,

$$M_1 - M_2 = \frac{P\, l_0\, (l^2_1 - l^2_0)}{2\, l^2_1} \qquad \text{(98)}$$

From Eq. 96*c* and Table 3(*e*),

$$M''_{AB,O} = (R_1 + R_2)\, l^2_1 - 2\, M_1\, l_1 - 2\, M_2\, l_1 - P\, l^2_0 - 2 \times \frac{(3\, w_1 + w_2)\, l^3_1}{12} = 0 \qquad \text{(99)}$$

TABLE 5.—GENERAL SOLUTIONS OF STATICALLY INDETERMINATE PROBLEMS

(Assume the Supports to Be Unyielding)

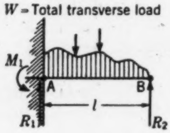
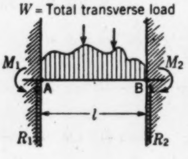
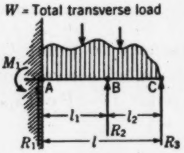
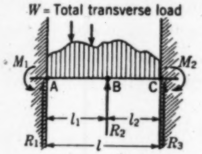
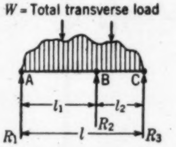
Support condition	Reactions and Statically Indeterminate Couples	
	Unknown	Solution
(a) Beam with One Fixed End and the Other End Freely Supported		
	M_1....	$\frac{1}{2\, l^2}\,(M'''_{\overline{AB},B} - l^2\, M_{\overline{AB},B})$
	R_1.....	$\frac{1}{2\, l^3}\,(M'''_{\overline{AB},B} - 3\, l^2\, M_{\overline{AB},B})$
	R_2.....	$-\frac{1}{2\, l^3}\,(M'''_{\overline{AB},B} - 3\, l^2\, M_{\overline{AB},B}) + W$
(b) Beam with Both Ends Fixed		
 W = Total transverse load M1 M2 A B l R1 R2	M_1....	$\frac{1}{l^2}\,(M'''_{\overline{AB},B} - l\, M''_{\overline{AB},B})$
	R_1.....	$\frac{1}{l^3}\,(2\, M'''_{\overline{AB},B} - 3\, l\, M''_{\overline{AB},B})$
	M_2....	$-\frac{1}{l^2}\,(M'''_{\overline{AB},B} - 2\, l\, M''_{\overline{AB},B}) - M_{\overline{AB},B}$
	R_2.....	$-\frac{1}{l^3}\,(2\, M'''_{\overline{AB},B} - 3\, l\, M''_{\overline{AB},B}) + W$
(c) Beam on Three Supports with One End Fixed		
 W = Total transverse load M1 A B C l1 l2 R2 l R1 R3	M_1....	$\frac{1}{l^2_1\, l_2\, (3\, l + l_2)}\,[l\,(l + l_2)\, M'''_{\overline{AB},B} - l^2_1\, M'''_{\overline{AC},C} + l^2_1\, l^2_2\, M_{\overline{AC},C}]$
	R_1.....	$\frac{1}{l^3_1\, l_2\, (3\, l + l_2)}\,[(3\, l^2 - l^2_2)\, M'''_{\overline{AB},B}$ $- 3\, l^2_1\, M'''_{\overline{AC},C} + 3\, l^2_1\, l^2_2\, M_{\overline{AC},C}]$
	R_2.....	$-\frac{1}{l^3_1\, l^2_2\, (3\, l + l_2)}\,[2\, l^3\, M'''_{\overline{AB},B}$ $- l^2_1\, (2\, l + l_2)\, M'''_{\overline{AC},C} + 3\, l^2_1\, l_2\, l^2\, M_{\overline{AC},C}]$
	R_3.....	$\frac{1}{l_1\, l^2_2\, (3\, l + l_2)}\,[(2\, l + l_2)\, M'''_{\overline{AB},B} - 2\, l_1\, M'''_{\overline{AC},C}$ $+ 3\, l_1\, l_2\, (l + l_2)\, M_{\overline{AC},C}] + W$

TABLE 5.—(*Continued*)

Support condition	Reactions and Statically Indeterminate Couples	
	Unknown	Solution
(*d*) Beam on Three Supports with Both Ends Fixed		
 W = Total transverse load M1 M2 A B C l1 l2 R2 R1 l R3	M_1....	$\frac{1}{2\,l^2_1\,l_2\,l}\,(l^2\,M'''\overline{AB},{}_B - l^2_1\,M'''\overline{AC},{}_C + l^2_1\,l_2\,M''\overline{AC},{}_C)$
	R_1.....	$\frac{1}{2\,l^3_1\,l_2\,l}\,[l\,(l + 2\,l_1)\,M'''\overline{AB},{}_B - 3\,l^2_1\,M'''\overline{AC},{}_C + 3\,l^2_1\,l_2\,M''\overline{AC},{}_C]$
	R_2.....	$-\frac{1}{2\,l^3_1\,l^3_2}\,[l^3\,M'''\overline{AB},{}_B - l^2_1\,(l + 2\,l_2)\,M'''\overline{AC},{}_C + 3\,l^2_1\,l_2\,l\,M''\overline{AC},{}_C]$
	M_2....	$\frac{1}{2\,l_1\,l^2_2\,l}\,[l^2\,M'''\overline{AB},{}_B - l_1\,(l + l_2)\,M'''\overline{AC},{}_C$ $+ l_1\,l_2\,(3\,l + l_2)\,M''\overline{AC},{}_C] - M\overline{AC},{}_C$
	R_3.....	$\frac{1}{2\,l_1\,l^3_2\,l}\,[l\,(l + 2\,l_2)\,M'''\overline{AB},{}_B - l_1\,(l + 3\,l_2)\,M'''\overline{AC},{}_C$ $+ 3\,l_1\,l_2\,(l + l_2)\,M''\overline{AC},{}_C] + W$
(*e*) Beam on Three Supports		
 W = Total transverse load A B C l1 l2 R2 R1 l R3	R_1.....	$\frac{1}{2\,l^2_1\,l_2\,l}\,(l\,M'''\overline{AB},{}_B - l_1\,M'''\overline{AC},{}_C + l_1\,l^2_2\,M\overline{AC},{}_C)$
	R_2.....	$-\frac{1}{2\,l^2_1\,l^2_2}\,[l\,M'''\overline{AB},{}_B - l_1\,M'''\overline{AC},{}_C + l_1\,l_2\,(l + l_1)\,M\overline{AC},{}_C]$
	R_3.....	$\frac{1}{2\,l_1\,l^2_2\,l}\,[l\,M'''\overline{AB},{}_B - l_1\,M'''\overline{AC},{}_C + l_1\,l_2\,(2\,l + l_2)\,M\overline{AC},{}_C] + W$

Substituting the value of $(R_1 + R_2)$ from Eq. 96*a* into Eq. 99 and simplifying,

$$M_1 + M_2 = \frac{6\,P\,(l^2_1 - l^2_0) + l^3_1\,(3\,w_1 + 5\,w_2)}{12\,l_1} \quad \text{.........}(100)$$

Solving Eqs. 98 and 100 simultaneously,

$$M_1 = \frac{1}{24\,l^2_1}\,[6\,P\,(l_1 + l_0)^2\,(l_1 - l_0) + l^4_1\,(3\,w_1 + 5\,w_2)] \quad \text{....}(101a)$$

and

$$M_2 = \frac{1}{24\,l^2_1}\,[6\,P\,(l_1 + l_0)\,(l_1 - l_0)^2 + l^4_1\,(3\,w_1 + 5\,w_2)] \quad \text{....}(101b)$$

Table 4 gives the formulas for angle changes and deflections in a beam carrying any loading under various given conditions. In applying this table, the origin O can be selected at any point on the elastic curve. The angle change or deflection at point A is given by problem or may be found from the support condition of beam. The angle change and deflection at any point are obtained directly from the formulas. All the formulas in Table 4 can be derived by Eqs. 29 and 31. Since the derivations are somewhat complicated, they are omitted from this discussion.

Table 5 gives the general solutions of statically indeterminate problems for some customary support conditions. The beam must be straight and must possess a constant flexural rigidity. The supports are assumed to be unyielding. The loading on the beam may be any transverse loads and any couples (each of the couples acting at a point on the beam axis, like that shown in Fig. 6(*b*)), where the couples are not shown in the table. Symbol W denotes the total transverse load, the reactions at supports being excluded; and $M_{\overline{AB},B}$, $M''_{\overline{AB},B}$, and $M'''_{\overline{AB},B}$ represent the first, second, and third moments, respectively, with respect to point B, of the external loading only, on span AB—end shears and end moments being excluded. The reactions at supports are assumed positive when they act upward. The statically indeterminate couples are assumed positive when their directions of rotation are as indicated in the table, respectively. The solutions in each case of Table 5 are obtained by solving simultaneous equations. Their derivations are cumbersome and lengthy, and are omitted.

Tables 4 and 5 are advantageous for practical use. For example, let it be required to determine the deflection at the free end D for the beam shown in Fig. 29(*a*). Cut the beam at point C into two parts as shown in Figs. 29(*b*) and 29(*c*). Consider the end shear of 2 kips and the end moment of 8 kip-ft at end C of beam ABC in Fig. 29(*b*) as the external force and couple, respectively. From the solution for R_3 in Table 5(*c*), substitute 8 ft for l_1, 5 ft for l_2, 13 ft for l, 12 kips for W, and zero for $M'''_{\overline{AB},B}$; thus:

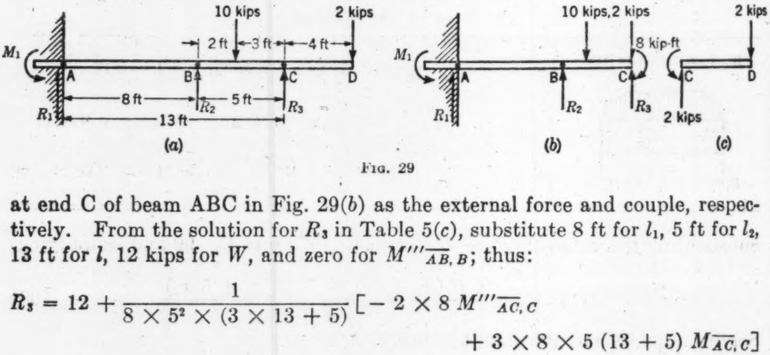


FIG. 29

$$R_3 = 12 + \frac{1}{8 \times 5^2 \times (3 \times 13 + 5)} [- 2 \times 8 \, M'''_{\overline{AC},C} + 3 \times 8 \times 5 \, (13 + 5) \, M_{\overline{AC},C}]$$

$$R_3 = 12 + \frac{1}{8{,}800} [- 16 \, (- 10 \times 3^3) + 2{,}160 \, (- 10 \times 3 + 8)] = 7.09 \text{ kips}$$

In Fig. 29, point C is selected as the origin. From the formula of deflection in Table 4(*b*) for the boundary condition $\delta_A = 0$, substitute C for O, B for A, D for N, 5 ft for l_A, and 4 ft for x; thus:

$$\delta_D = \frac{1}{6 \, E \, I \times 5} (- 4 \, M'''_{CB,B} + 5 \, M'''_{CD,D})$$

$$\delta_D = \frac{1}{30 \, E \, I} \{- 4 \, [10 \times 2^3 - (7.09 - 2) \, 5^3 + 3 \times 8 \times 5^2] + 5 \, (2 \times 4^3 - 3 \times 8 \times 4^2)\} = - \frac{291}{6 \, E \, I} \text{ ft}$$

in which E is in kips per square foot, and I is in feet^4.

The writer is grateful to Mr. Floris, Professor Moorman, Professor McCormick, and Mr. Conwell for their discussions of this paper.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

TRANSACTIONS

Paper No. 2342

DEVELOPMENT AND HYDRAULIC DESIGN, SAINT ANTHONY FALLS STILLING BASIN

By Fred W. Blaisdell,[1] Assoc. M. ASCE

With Discussion by Messrs. Donald E. Blotcky, M. M. Culp, Paul Baumann, A. J. Peterka, Louis M. Laushey, J. H. Douma, Robert F. Ewald, and Fred W. Blaisdell.

Synopsis

Tests made to develop rules for the design of a stilling basin to dissipate hydrodynamic energy are described in this paper. The energy dissipator has been named the SAF stilling basin, SAF (denoting "Saint Anthony Falls") being a coined word used to differentiate this design from other stilling basin designs. In size, this stilling basin is the smallest known to the writer. The size is reduced through the use of baffles and sills within the stilling basin to assist in the dissipation of the energy of water flowing at high velocities.

The paper begins with a discussion of the reasons for making the tests reported and a brief discussion of previous work on energy dissipators. The scope of the test program is then discussed as are the apparatus and test methods used for each of the three groups or series of tests into which the study is arbitrarily divided. Each element involved in the design of the stilling basin is discussed separately in Section 6; under subheadings each of the various steps leading to the determination of the design equations is discussed; and the pertinent test data are summarized in tables and illustrations. The aim throughout has been to present the data in sufficient detail to permit an independent analysis by the reader so that he can arrive at his own conclusions and discuss the paper intelligently. The results are summarized for convenience of reference so that, after evaluating the design, the reader can obtain all the equations and information essential to the hydraulic design of the SAF stilling basin without "thumbing" through the entire paper.

Note.—Published in February, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] **Project Supervisor, U. S. Dept. of Agriculture, SCS, St. Anthony Falls Hydr. Laboratory, Minneapolis, Minn.**

1. Introduction

Energy dissipators in some form have been made a part of many hydraulic structures throughout the world when the destructive energy in flowing water must be brought under control. Most of these energy dissipators have performed their intended function adequately—particularly those that have received the benefit of model tests. However, some of them have not performed their intended function, with the result that they have been destroyed or the structure of which they were a part has been damaged.

Except in unusual circumstances, only the larger structures are the subjects of individual model studies. However, they ordinarily pay dividends in improved performance and in reduced construction, maintenance, and possible replacement costs. In general, the structures built by the Soil Conservation Service (SCS), United States Department of Agriculture, are of such a size that few of them can justify, economically, the individual model studies that have proved so profitable for the larger structures. However, it becomes economical to make model studies if their costs can be spread over a large number of small structures, as is the case with the SCS where a large number of structures are required to further its soil and water conservation program. For use with certain of these structures it was essential that an efficient and economical stilling basin be developed and that design rules be formulated so that future stilling basins could be designed without recourse to further model studies. For this reason a study of methods of dissipating energy was instigated by the SCS in cooperation with the Minnesota Agricultural Experiment Station at the St. Anthony Falls Hydraulic Laboratory in Minneapolis, Minn.

2. Previous Work

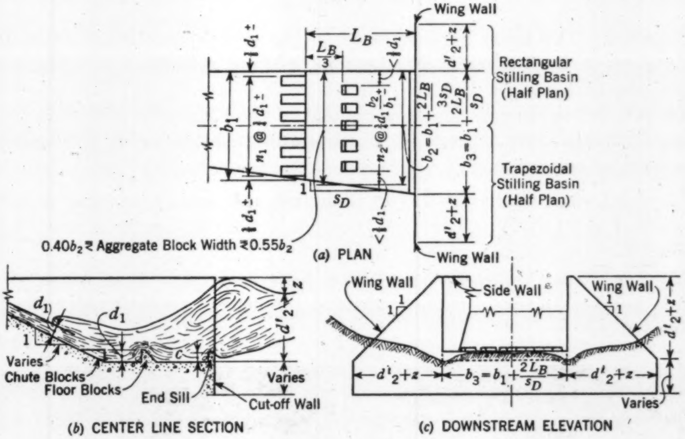
The more important references consulted during the study upon which this paper is based are listed in the Bibliography (see Appendix I). The energy dissipators described in the references cited may be divided into three groups: Those that could be eliminated from further consideration as not being adaptable to the problem at hand (outlets for flumes or chutes and culverts), those that required critical study to evaluate them, and those that required tests to determine their potentialities.

No mention will be made of the first group. Of those that were classified in the second group, the simple stilling basin studied by C. Maxwell Stanley, M. ASCE (1),[2] is outstanding. This energy dissipator consists of a pool in which the energy in the high-velocity flow is dissipated. The pool is frequently formed by a sill or, in the case of high overflow dams, by a secondary low dam. Necessarily, the simple stilling basin is larger than the SAF basin described in this paper for comparable stilling action, since its design does not include the chute and floor blocks which dissipate much of the energy in the SAF basin. The elements and dimensions of the SAF stilling basin are defined in Fig. 1. The tests made to determine the best proportions of each element will be discussed separately in Section 6.

---

[2] Numerals in parentheses, thus: (1), refer to corresponding items in the Bibliography (Appendix I).

Exploratory tests were made on three types of stilling basin that were classified in the third group. The nappe-cutting energy dissipator studied at the Royal Technical University, in Stockholm, Sweden, is described by Wolmar K. A. Fellenius (2*a*). This stilling basin was tested to check the claims made for it. The few tests at Minneapolis did not show any appreciable increase in the energy dissipation when a nappe cutter was used over the energy dissipation when no nappe cutter was used. As a result, no further consideration was given to this type.


Wing Wall
L_B
$\frac{L_B}{3}$
$\frac{1}{2}d_1\pm$
$<\frac{1}{2}d_1$
b_1
$n_1 @ \frac{1}{2}d_1\pm$
$n_2 @ \frac{3}{4}d_1\frac{b_2}{b_1}\pm$
$b_2=b_1+\frac{2L_B}{3s_D}$
$b_3=b_1+\frac{2L_B}{s_D}$
d'_2+z
1
s_D
Rectangular Stilling Basin (Half Plan)
Trapezoidal Stilling Basin (Half Plan)
$0.40b_2 \lesseqgtr$ Aggregate Block Width $\lesseqgtr 0.55b_2$
d_1
d'_2
z
c
Varies
Chute Blocks
Floor Blocks
End Sill
Cut-off Wall
Side Wall
d'_2+z
$b_3=b_1+\frac{2L_B}{s_D}$
Varies

(*a*) PLAN

(*b*) CENTER LINE SECTION

(*c*) DOWNSTREAM ELEVATION

FIG. 1.—PROPORTIONS OF THE SAF STILLING BASIN

Armin Schoklitsch (3*a*) has published results of experiments performed on an energy dissipator that is similar to Mr. Stanley's simple stilling basin except that a sill or bucket is used at the basin entrance. This sill causes the water to enter the stilling pool above its floor so that energy dissipation takes place both above and below the jet. A test with the sill removed, making the stilling basin similar to that tested by Mr. Stanley, showed that the use of the sill greatly increased the efficiency of the stilling basin as an energy dissipator. Mr. Schoklitsch has published charts giving the dimensions and performance of this stilling basin for different discharges and heights of dam. The tests on the Schoklitsch energy dissipator at Minneapolis were made after the SAF stilling basin studies were well advanced. At that time the SAF stilling basin was approximately one half of the size of the Schoklitsch dissipator that gave comparable results. Further experimentation on the Schoklitsch energy dissipator was abandoned in favor of the SAF stilling basin study.

Jacob E. Warnock, M. ASCE (4*a*), has described a rectangular stilling basin developed by the Bureau of Reclamation, United States Department of the Interior (USBR). Exploratory tests indicated that this basin was particularly efficient and that its size could be reduced considerably. As a result, additional study was given to a stilling basin of this type, and from this study generalized design rules were formulated for proportioning the SAF stilling basin. Two general views of a stilling basin model are shown in Fig. 2.

3. Test Program

Notation.—The letter symbols used in the paper are defined where they first appear, in the text or by illustrations, and are assembled for convenience of reference in Appendix II.

The studies leading to the development of the SAF stilling basin are conveniently divided into three parts which are listed below in the chronological order of their performance:

(*1*) *The Culvert Outlet Series.*—In these tests the basin dimensions were first determined tentatively. The tests covered a narrow range of the Froude number, **F**.

(*2*) *The Flume Outlet Series.*—These tests covered a wide range of the Froude number and were made to expand the field of application of the SAF stilling basin.

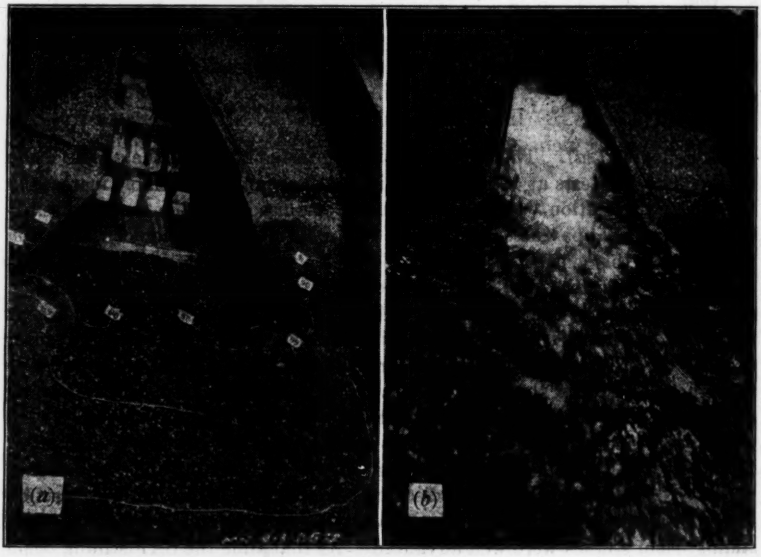


Fig. 2.—Model of SAF Stilling Basin

(*a*) Scour After Running Model 20 Hours; Basin Floor Elevation 88.0; Maximum Scour Elevation 87.4
(*b*) Design Discharge, Tailwater Elevation 93.0

(*3*) *The Turbine Room Series.*—The previous series of tests had shown the end sill height to be a function of the Reynolds number, **R**. The turbine room tests were made to extend the range of the Reynolds number, since its highest value for the culvert and flume outlet series was less than would be obtained for most prototype structures, and to check the equation for the end sill height. The tests also covered a wide range of the Froude number.

The SAF stilling basin design was developed and verified as a result of two hundred seventy-four tests. The number of tests in each series and the range of the variables are given in Table 1. In Table 1, Q is the discharge; V_1 and d_1 are the velocity and depth, respectively, at the entrance to the stilling basin; and d_2 is the theoretical depth after the hydraulic jump. The Froude number is defined by the equation:

TABLE 1.—Range of Test Variables

Series	No. of tests	Q (ft^3/sec)		V_1 (ft/sec)		d_1 (ft)		d_2 (ft)		**F**		**R** × 10^{-3}	
		From	To	From	To	From	To	From	To	From	To	From	To
Culvert outlet...	100	0.09	0.4	2.9	12	0.04	0.17	0.17	0.8	3	57	12.7	54
Flume outlet....	108	0.04	0.8	2.8	22	0.05	0.15	0.13	1.8	5	200	14.2	237
Turbine room...	66	0.40	21	9.7	44	0.03	1.27	0.49	5.5	7	288	40.6	2,100
Total...........	274	0.04	21	2.8	44	0.03	1.27	0.13	5.5	3	288	12.7	2,100

$$\mathbf{F} = \frac{V^2_1}{g\, d_1} \qquad (1a)$$

in which g is the acceleration due to the effect of gravity. The Reynolds number is defined by the equation:

$$\mathbf{R} = \frac{V_1\, d_1}{\nu} \qquad (1b)$$

in which ν is the kinematic viscosity.

4. Apparatus and Procedure

Both the culvert and flume outlet series of tests were made in a channel 8 ft long by 18 in. wide by 24 in. deep. Plate glass formed one side of the channel. Water was supplied to the models through a 4-in. pipe line and the rate of flow was controlled by a gate valve. The rate of flow through the model was determined by a calibrated 1-ft type HS-flume (5). Discharge measurements accurate to 1% were readily obtainable.

The channel in which the turbine room series of tests was made is shown in Fig. 3. A glass panel was placed in one side of this channel at the place where the stilling basin models were to be located. The rate of flow to the models was controlled by a gate valve in the 12-in. supply line. Three methods of measuring the flow were used. A calibrated 1.5-ft type H-flume (5) was used for discharges up to 5 cu ft per sec. For discharges greater than 5 cu ft per sec, the pressure drop across an 18-in. by 12-in. reducing flange was calibrated and used to determine the discharge until its calibration became invalid, probably because of rusting at the piezometers. A ½-in. Prandtl (6) pitot tube in the approach to the stilling basin model was then used for measuring the higher discharges.

Previous experiments on outlets (7) have shown that it is impossible to follow, visually, the progress of the erosion under the jet and "white water" in and beyond the end of the stilling basin. Therefore, all models were split along their longitudinal center lines and one half of them were pressed against the glass side of the experimental channel. Check tests (8) showed that, within the limits of experimental error, the results obtained on the half model agreed with the full model. As a result all the tests reported in this paper were made on "half models" in which a glass plate was placed on the longitudinal center line.

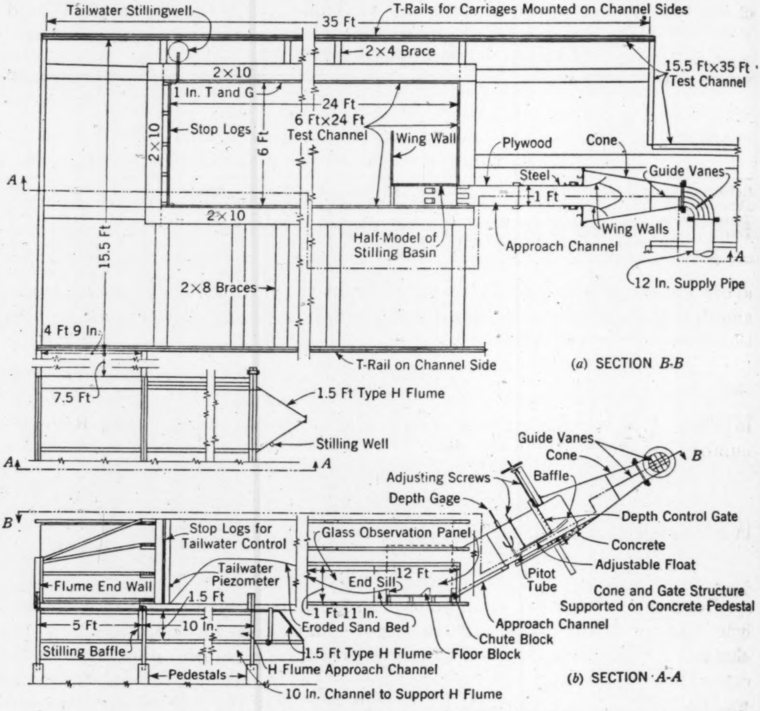

Tailwater Stillingwell
35 Ft
T-Rails for Carriages Mounted on Channel Sides
2×4 Brace
15.5 Ft×35 Ft Test Channel
2×10
1 In. T and G
24 Ft
6 Ft×24 Ft Test Channel
Stop Logs
6 Ft
Wing Wall
Plywood
Cone
Steel
1 Ft
Guide Vanes
A
Wing Walls
Half-Model of Stilling Basin
Approach Channel
12 In. Supply Pipe
15.5 Ft
2×8 Braces
4 Ft 9 In.
T-Rail on Channel Side
(a) SECTION B-B
7.5 Ft
1.5 Ft Type H Flume
Stilling Well
Guide Vanes
Cone
B
Adjusting Screws
Baffle
Depth Gage
Depth Control Gate
Stop Logs for Tailwater Control
Glass Observation Panel
Concrete
Adjustable Float
Tailwater Piezometer
12 Ft
Pitot Tube
Flume End Wall
End Sill
Cone and Gate Structure Supported on Concrete Pedestal
1.5 Ft
1 Ft 11 In.
5 Ft
10 In.
Eroded Sand Bed
Approach Channel
Chute Block
Stilling Baffle
1.5 Ft Type H Flume
Floor Block
Pedestals
H Flume Approach Channel
(b) SECTION A-A
10 In. Channel to Support H Flume

Fig. 3.—Test Apparatus, Turbine Room Series

For convenience, the approach to the stilling basin models for the culvert outlet series was considered to be a 3-ft-square culvert built to a scale of 1 in. equals 1 ft. A half section of the culvert barrel, 1.5 in. by 3 in. in section and 7.5 ft long, was placed on a slope of 1% and connected to the 4-in. supply line. An open channel transition section was placed between the culvert and the stilling basin to spread out the water and drop it to the stilling basin floor level. The floor of the transition was given a parabolic shape. The width of the downstream end of the transition was varied to give the desired depth of flow at the stilling basin entrance. In designing the transition, the velocities in the culvert barrel and along the transition were assumed to be identical.

For the flume outlet series, a 3-in.-square pipe, 7.5 ft long, was connected to the 4-in. supply line and a rectangular nozzle 3 in. wide with an adjustable lid was placed on a slope of 1 on 2 (a flatter slope was used for a few tests) and was used to control the depth of approach at the stilling basin entrance. The open channel between the nozzle and the stilling basin was 3 in. wide and its slope was 1 on 2.

The approach to the stilling basin model for the turbine room series is shown in Fig. 3. The gate for controlling the depth of flow at the entrance to the model is at the downstream end of the cone connected to the 12-in. supply line. As it left the gate, the stream was ordinarily well distributed across the channel leading to the model stilling basin. However, at the larger gate openings and at the larger discharges, the disturbances existing in the cone persisted and the depth distribution across the channel section was not satisfactory. Accordingly, an adjustable steel "float" was placed downstream from the gate to insure satisfactory flow conditions at the entrance to the stilling basin. This "float" was used for all tests in this series.

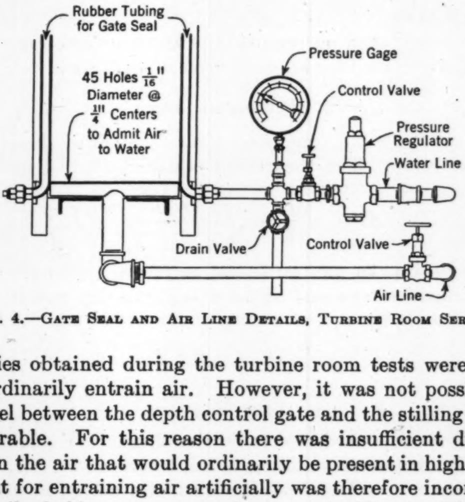

Rubber Tubing for Gate Seal
45 Holes $\frac{1}{16}$" Diameter @ $\frac{1}{4}$" Centers to Admit Air to Water
Pressure Gage
Control Valve
Pressure Regulator
Water Line
Drain Valve
Control Valve
Air Line

FIG. 4.—GATE SEAL AND AIR LINE DETAILS, TURBINE ROOM SERIES

The velocities obtained during the turbine room tests were such that the water would ordinarily entrain air. However, it was not possible to provide as long a channel between the depth control gate and the stilling basin as would have been desirable. For this reason there was insufficient distance for the water to entrain the air that would ordinarily be present in high-velocity flows. An arrangement for entraining air artificially was therefore incorporated in the design. A detail of this arrangement is shown in Fig. 4. The air flow was measured by a square-edged orifice using equations and coefficients published by H. S. Bean, E. Buckingham, and P. S. Murphy (9).

Stop logs were used in both test channels for tailwater control. Water levels and stream-bed contours were measured with the aid of point gages mounted on carriages in such a way that levels anywhere in the test channel could be obtained readily. These gages were also used in setting the models in their correct positions.

All models were of the "movable bed" type; the stream bed downstream from each stilling basin was formed in commercial concrete sand. The water leaving the model stilling basin was permitted to erode the stream bed until the scour-hole dimensions had become stabilized. This action required about 30 min for the culvert and flume outlet series and 2 hours for the turbine room series. The size of the scour hole was taken as an indication of the effectiveness of the stilling basin (1*a*)(8).

The method of testing the models was as follows: The stilling basin was installed and the stream bed was filled with sand to an elevation higher than the anticipated elevation of the eroded stream bed. The stream bed was then flooded so that the initial rush of water through the stilling basin would not erode the stream bed excessively. The gate valve in the supply line was then opened to give the desired discharge and the tailwater level was adjusted to the correct depth. The discharge, tailwater depth, and water-surface profiles were recorded near the end of each test run. The stream bed was drained at the conclusion of each test. For the culvert and flume outlet series, the center-line profile of the eroded stream bed was recorded on data sheets. For the turbine room series, a contour map of the eroded stream bed was also obtained. Elevations of the stream bed at selected points near the stilling basin were also obtained and recorded.

Photographs were taken during many of the tests (see Fig. 2(*b*)). Motion pictures were also obtained for selected typical conditions.

5. HYDRAULIC JUMP

All the SAF stilling basin dimensions are related either directly or indirectly to the hydraulic jump. The theoretical equation for the hydraulic jump is

$$d_2 = -\frac{d_1}{2} + \sqrt{\frac{2\,V^2_1\,d_1}{g} + \frac{d^2_1}{4}} \qquad \text{(2a)}$$

The derivation of Eq. 2*a* can be found in most books on hydraulics (10*a*). Numerous experiments have proved the validity of this equation which can be simplified to

$$d_2 = \frac{d_1}{2}\left(-1 + \sqrt{8\,\mathrm{F} + 1}\right) \qquad \text{(2b)}$$

6. TEST RESULTS

(*a*) *Length of Basin.*—As noted in Section 2, the SAF stilling basin was developed from the rectangular stilling basin described by Mr. Warnock (4*a*). The single rectangular stilling basin tested had a design length L_B of 3.75 d_2 although Mr. Warnock used 3.00 d_2. This basin was designed for a flow of Q = 275 cu ft per sec and a depth d_1 = 1.0 ft. The dimensions of this stilling basin (shown in Fig. 5) are in terms of the prototype, assuming the linear scale ratio to be 12 to 1. The results of the tests on this stilling basin under approximate design flow conditions are presented in Fig. 6(*a*) and Table 2, test C3. The profile shown in Fig. 6(*a*) is taken along the channel center line. Note that the toe of the roller is not at the upstream end of the basin. This test indicated that the stilling basin was much longer than was necessary to secure adequate energy dissipation, although it was entirely satisfactory from a purely hydraulic point of view.

TABLE 2.—SUMMARY OF DATA FOR TESTS TO DETERMINE LENGTH OF BASIN

(Linear Scale Ratio, $L_r = 12$)

Test No.	Flow, Q (ft³/sec) (1)	d_2 (ft) (2)	d'_2 (ft) (3)	L_B (ft) (4)	c (ft) (5)	F (6)	$\frac{d'_2}{d_2}$ (7)	$\frac{L_B}{d_2}$ (8)	$\frac{c}{d_2}$ (9)	Scour[a] (ft) (10)	Basin[b] (ft) (11)	z_s (ft) (12)	$\frac{z_s}{d_2}$ (13)	Performance[c] (14)
(a) CULVERT OUTLET SERIES C (d_1 = 1 FT AND b_1 = 9 FT EXCEPT: IN TEST C75 d_1 = 2 FT AND b_1 = 4.5 FT; AND IN TEST C76 d_1 = 0.5 FT AND b_1 = 18 FT)														
C63..	92	2.10	2.28	6.56	0.292	3.25	1.08	3.12	0.139	−1.0	−1.0	0.5	0.24	S
C64..	91	2.07	2.33	4.38	0.292	3.18	1.13	2.11	0.141	−1.0	−1.0	0.3	0.14	S
C56..	141	3.44	3.15	12.75	0.500	7.63	0.92	3.71	0.145	−2.4	−2.0	0.9	0.26	F
C57..	140	3.41	3.15	8.50	0.500	7.52	0.92	2.49	0.147	−2.1	−2.0	0.9	0.26	S−
C59..	140	3.41	3.20	9.88	0.500	7.52	0.94	2.90	0.147	−2.2	−2.0	0.9	0.26	F+
C60..	142	3.46	3.20	7.00	0.500	7.73	0.92	2.02	0.145	−2.3	−2.0	0.5	0.14	F
C75..	210	7.28	5.99	15.33	1.000	8.45	0.82	2.11	0.137	−4.8	−4.9	1.2	0.16	S
C71..	198	5.00	4.10	10.00	0.833	15.00	0.82	2.00	0.167	−3.7	−3.2	1.1	0.22	F
C72..	200	5.06	4.10	8.33	0.833	15.33	0.81	1.64	0.164	−3.7	−3.2	2.1	0.41	F
C76..	198	3.64	3.12	5.00	0.500	30.1	0.86	1.37	0.138	−2.7	−2.1	0.0	0.00	F
C3...	296	7.71	6.21	26.25	1.250	33.6	0.81	3.40	0.162	−5.2	−5.0	0.9	0.12	F+
C11..	288	7.49	6.15	20.00	1.250	31.8	0.82	2.67	0.167	−5.1	−5.0	0.3	0.04	S
C12..	283	7.35	6.21	17.50	1.000	30.7	0.84	2.38	0.136	−4.8	−5.0	0.1	0.01	S
C16..	282	7.32	6.09	14.00	1.000	30.5	0.83	1.91	0.136	−4.3	−5.0	1.1	0.15	G+
C19..	287	7.46	6.26	12.00	1.000	31.6	0.84	1.61	0.134	−4.4	−5.0	0.9	0.12	G
C22..	281	7.30	6.08	12.00	1.000	30.3	0.83	1.64	0.137	−4.2	−5.0	0.9	0.12	G
C85..	273	7.08	6.03	10.92	1.000	28.6	0.85	1.54	0.141	−4.1	−5.0	0.7	0.10	G
C28..	276	7.16	6.07	8.75	1.000	29.2	0.85	1.22	0.140	−4.5	−5.0	0.9	0.13	G−
C32..	275	7.13	6.08	7.00	1.000	29.0	0.85	0.98	0.140	−5.0	−5.0	2.0	0.28	F
C37..	278	7.20	6.06	5.00	1.000	29.6	0.84	0.69	0.139	−6.1	−5.0	2.9	0.40	P
C65..	386	10.19	9.80	10.00	1.417	57.0	0.96	0.98	0.139	−8.0	−8.5	1.4	0.14	G
C66..	375	9.90	9.83	8.33	1.417	53.9	0.99	0.84	0.143	−8.2	−8.5	1.9	0.19	G−
C67..	383	10.11	9.83	7.14	1.417	56.2	0.97	0.71	0.140	−8.5	−8.5	2.6	0.26	S
C68..	378	9.96	9.83	6.00	1.417	54.7	0.99	0.60	0.142	−8.7	−8.5	2.2	0.22	S−
C69..	379	10.00	9.83	5.00	1.417	55.0	0.98	0.50	0.142	−9.2	−8.5	2.8	0.28	F
C70..	378	9.96	9.83	4.00	1.417	54.7	0.99	0.40	0.142	−9.0	−8.5	1.9	0.19	F
(b) FLUME OUTLET SERIES F (d_1 = 1.20 FT AND b = 6 FT EXCEPT: IN TESTS F52 AND F57 d_1 = 1.03 FT)														
F52..	200	7.62	6.47	14.02	1.021	31	0.85	1.84	0.134	+0.8	0.0	1.0	0.13	G
F57..	200	7.62	6.47	12.06	1.021	31	0.85	1.58	0.134	+0.5	0.0	1.2	0.16	G
F51..	284	10.15	8.63	14.21	0.762	40	0.85	1.40	0.075	+0.4	0.0	1.6	0.16	G
F55..	284	10.15	8.63	12.18	0.762	40	0.85	1.20	0.075	+0.7	0.0	1.9	0.19	G
F49..	375	13.61	11.45	16.33	1.021	70	0.84	1.20	0.075	+0.7	0.0	4.1	0.30	S−
F53..	375	13.61	11.45	13.61	1.021	70	0.84	1.00	0.075	+0.5	0.0	2.9	0.21	S−
F44..	375	13.61	11.41	10.88	1.021	70	0.84	0.80	0.075	+0.6	0.0	3.5	0.26	F
F48..	448	16.38	13.92	16.38	1.228	100	0.85	1.00	0.075	+0.9	0.0	4.2	0.26	S+
F54..	448	16.38	13.92	13.10	1.228	100	0.85	0.80	0.075	+1.0	0.0	4.7	0.29	S
F45..	448	16.38	13.92	9.83	1.228	100	0.85	0.60	0.075	+1.0	0.0	5.1	0.31	F
F46..	548	20.18	17.16	24.22	1.488	150	0.85	1.20	0.074	+1.0	0.0	3.9	0.19	G
F47..	548	20.18	17.16	20.18	1.488	150	0.85	1.00	0.074	+1.1	0.0	4.1	0.20	G
F50..	548	20.18	17.16	16.15	1.488	150	0.85	0.80	0.074	+1.1	0.0	4.7	0.23	G
F56..	548	20.18	17.16	12.11	1.488	150	0.85	0.60	0.074	+1.2	0.0	5.3	0.26	G

[a] Elevation of maximum depth of scour, in feet, on the channel center line near the end of the stilling basin.

[b] Elevation of the stilling basin floor.

[c] Symbols denoting criteria for stilling basin performance, as follows:

G = good; maximum depth of scour only slightly below top of end sill and downstream bed level gradually rising as in Fig. 6(*c*).

S = satisfactory; no scour deeper than floor of stilling basin with the downstream bed level gradually rising.

F = fair; maximum depth of scour slightly below floor of stilling basin, or secondary scour holes, as in Figs. 6(*a*), 6(*f*), and 8(*d*) (structure not in danger of being undermined).

P = poor; maximum depth of scour considerably below floor of stilling basin and a large scour hole as in Fig. 6(*b*) (structure in danger of being undermined).

The next step in the investigation of length was to shorten the basin sufficiently to bring the toe of the roller back to the upstream end. The results of this shortening are presented in Fig. 6(*b*) and test C12, Table 2. The performance of the basin was not affected by this length reduction and, surprisingly, the maximum depth of the scour hole at the end of the stilling basin was reduced by 0.4 ft. Because these findings were encouraging, the basin length was reduced progressively with the results shown in Fig. 6 and Table 2.

The maximum depth of the scour hole decreased with a decrease in basin length until a length of 1.5 d_2 was reached. Further reduction in the basin length resulted in an increased depth of scour. However, the maximum depth of scour did not reach the elevation of the stilling basin floor until the length had been reduced to 0.98 d_2.

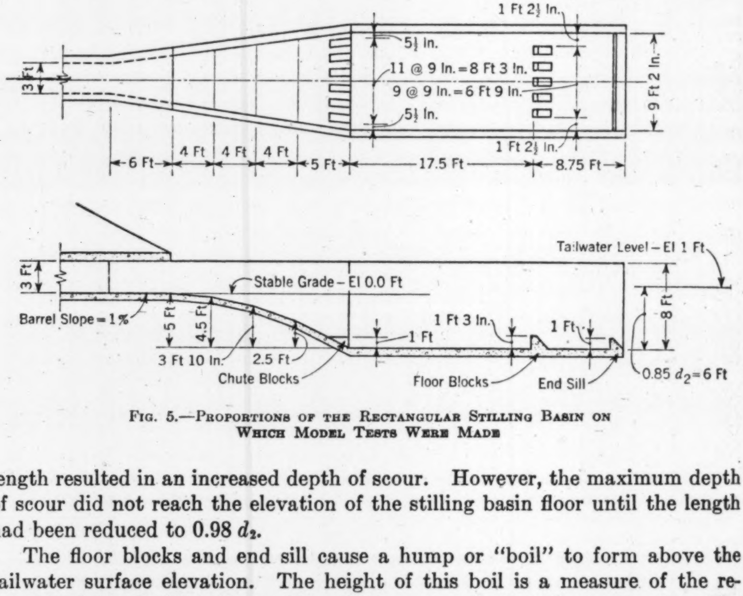

1 Ft 2½ In.
5½ In.
11 @ 9 In. = 8 Ft 3 In.
9 @ 9 In. = 6 Ft 9 In.
5½ In.
1 Ft 2½ In.
9 Ft 2 In.
3 Ft
6 Ft
4 Ft
4 Ft
4 Ft
5 Ft
17.5 Ft
8.75 Ft
Tailwater Level – El 1 Ft
3 Ft
Stable Grade – El 0.0 Ft
8 Ft
Barrel Slope = 1%
5 Ft
4.5 Ft
1 Ft 3 In.
1 Ft
1 Ft
3 Ft 10 In.
2.5 Ft
Chute Blocks
Floor Blocks
End Sill
0.85 d_2 = 6 Ft

FIG. 5.—PROPORTIONS OF THE RECTANGULAR STILLING BASIN ON WHICH MODEL TESTS WERE MADE

The floor blocks and end sill cause a hump or "boil" to form above the tailwater surface elevation. The height of this boil is a measure of the required height of stilling basin side wall if overtopping is to be prevented. The height of boil, determined for each test run, remained constant at about 1 ft above the tailwater surface elevation until the basin length was reduced to 0.98 d_2. For this length of basin the boil height increased to more than 2 ft. For $L_B = 0.69\, d_2$ the boil height reached 3 ft above the tailwater elevation. This increase in boil height required no higher side walls since the maximum boil height occurred downstream from the end of the stilling basin. However, the large size of the scour hole for $L_B = 0.98\, d_2$ was an indication that the energy dissipation was poor. A study of the data obtained from this series of tests resulted in the decision tentatively to select 1.25 d_2 as the best length of stilling basin.

After this length was selected, the study of basin lengths was deferred until the dimensions and spacing of the floor blocks and chute blocks, end sill height, and basin shape had been investigated and tentative conclusions had been derived from the investigation. When the study of basin length was resumed, tests were planned to broaden the scope of the previous work. The principal dimensions of these basins and a summary of the results are given in Table 2.

As a result of these tests it was found, unfortunately, that the length of the stilling basin for discharges other than 275 cu ft per sec could not be predicted assuming that $L_B = 1.25\, d_2$. The stilling basins for smaller discharges were too short and those for higher discharges were longer than necessary. To determine the best length of basin for each discharge, L_B was decreased from run to run until the capability of the stilling basin to dissipate the energy became unsatisfactory.

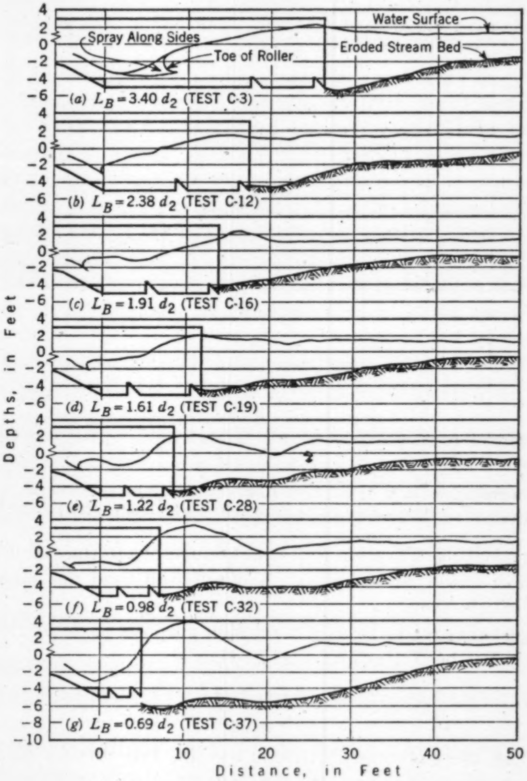


FIG. 6.—EFFECT OF LENGTH OF RECTANGULAR STILLING BASIN ON CENTER-LINE BED AND WATER-SURFACE PROFILES (SEE TABLE 2)

After the data obtained for each test were studied, the composite performance of each length of stilling basin was placed in one of the four categories listed in Table 2, Col. 14, and the performances were plotted in Fig. 7. For satisfactory performance the length of the stilling basin in terms of d_2 is shown to be considerably greater for the lower values of **F** than it is for the higher values of **F**. However, the data should be sufficient to permit a satisfactory determination of the basin length for all values of **F** between 3 and 150. In this connection it should be noted that a rating of "fair" was given to performances resulting in erosion that is not dangerous to the stability of an outlet when a short cutoff wall is used. A cutoff wall need not be used when the rating is "satisfactory."

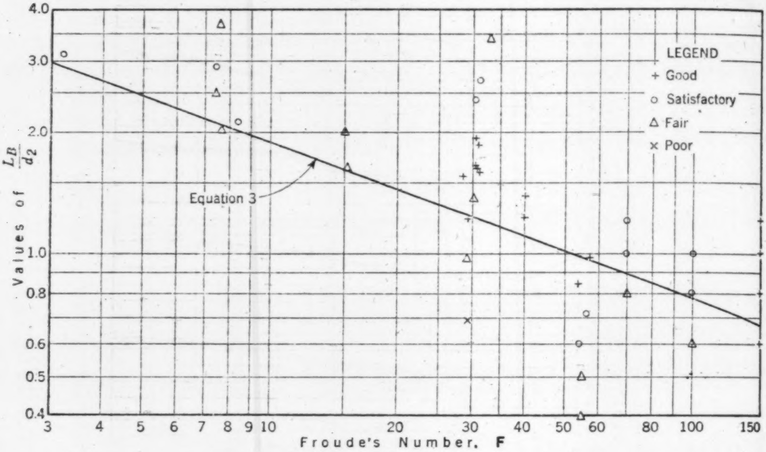


FIG. 7.—DIMENSIONLESS PLOT OF STILLING BASIN PERFORMANCE AS AFFECTED BY BASIN LENGTH
(Correction: Add a point at $\mathbf{F} = 3.18$, $L_B/d_2 = 2.11$)

A suggested curve to be used for design purposes in determining the length of the basin has been drawn in Fig. 7—that is:

$$\frac{L_B}{d_2} = \frac{4.5}{\mathbf{F}^{0.38}} \quad \text{(3)}$$

Of course, there is some latitude in locating this curve. Eq. 3 is suggested as being conservative but not to the extent that the material in the outlet is wasted. Eq. 3 was developed for a range of the Froude number from 3 to 150 but was later used to design experimental stilling basins for the turbine room series having values of **F** as high as 288. Although no test in the turbine room series was designed specifically to check the equation for the length of the stilling basin, the results of the turbine room tests indicate that the original design equation is entirely satisfactory and that no change in either its form or constants is indicated.

(*b*) *Chute Blocks.*—The chute blocks at the entrance to the stilling basin serve to increase the effective depth of the entering stream, break it up into a number of small jets, and help to create the turbulence required for effective energy dissipation.

The height of the chute blocks was originally assumed to be d_1 and their width and spacing 0.75 d_1 at the downstream end. Their tops were made horizontal and their sides vertical. In plan they decreased in width with distance upstream to take into account the decrease in transition width (see Fig. 5). No change was made in the chute blocks until the basin dimensions had been tentatively selected and the floor blocks had been tentatively located. A test was then made on a solid chute block such as is used in the Schoklitsch energy dissipator (3*a*). Tests made on this change, when compared to tests made on the original chute block design, showed that there was less energy dissipated, that flow conditions in the channel downstream from the stilling basin were not as good, and that the height of the boil was greater. The chute blocks were next made 1.75 d_1 high at their downstream end and the top was sloped 1 on 3 so that the jet leaving the top of the chute blocks was directed at the base of the floor blocks. The result of this change was an increase in the depth of erosion near the end of the stilling basin over that obtained for the chute blocks originally used.

Chute blocks with the original dimensions were replaced on the transition. However, they were rectangular in plan instead of following the arrangement shown in Fig. 5; and their sides were placed parallel to the transition center line. Little difference was found between the rectangular and trapezoidal chute blocks—the advantage, if any, accruing to the rectangular blocks. The chute blocks used in all subsequent tests had a height of d_1 and a width and spacing of approximately 0.75 d_1. Throughout the tests made from Froude numbers of 3 to 288, no reason was found for making a change in these dimensions.

In making the subsequent tests, sometimes a block and sometimes a space were placed next to the side wall. Apparently, it makes no difference in the performance of the stilling basin whether a chute block or a space is next to the side wall as long as the blocks are symmetrical about the center line of the outlet.

(*c*) *Floor Blocks.*—The function of the floor blocks is to create the turbulence through which energy is dissipated. Energy is also removed from the water by impact against these blocks.

Longitudinal Position.—No particular attention was given to the location of the floor blocks during the first length-of-basin tests. After a length equal to 1.25 d_2 had been tentatively selected as the best, the longitudinal spacing of the floor blocks was investigated. The results of these tests are presented in Table 3(*a*) and Fig. 8. When the floor blocks were placed at a distance of $L_B/5$ from the upstream end of the basin (Table 3(*a*), test C41, and Fig. 8(*a*)), there was insufficient distance between them and the chute blocks to permit the turbulence caused by the chute blocks to become fully effective in dissipating energy. This condition is shown by the greater depth of erosion and slightly higher boil, for example, than in the case of test C28 in which the distance to the floor blocks is $L_B/3.2$ (Table 3(*a*), test C28, and Fig. 8(*b*)). On the other hand, when the distance to the floor blocks was 2 $L_B/3$ (Table 3(*a*), test C40, and Fig. 8(*d*)), there was insufficient distance between the floor blocks and the end sill to permit the turbulence caused by the floor blocks to become fully effective in dissipating the energy. This condition is also shown by the greater depth and volume of erosion than for runs having a somewhat lesser distance to the floor blocks. The boil was higher than for any other run of this group because the floor blocks and end sill seemed to act together in deflecting the stream upward. The maximum depths of erosion for the two extreme floor block locations were approximately the same, indicating that it is as bad to have too little distance between the chute blocks and the floor blocks as it is to have the floor blocks and end sill too close together.

TABLE 3.—SUMMARY OF DATA FOR TESTS MADE ON FLOOR BLOCKS

(Linear Scale Ratio, $L_r = 12$)

Test No.	Flow, Q (ft³/sec) (1)	d_2 (ft) (2)	d'_2 (ft) (3)	F (4)	$\frac{d'_2}{d_2}$ (5)	$\frac{L_B}{d_2}$ (6)	$\frac{c}{d_2}$ (7)	Scour[a] (ft) (8)	z_a (ft) (9)	$\frac{z_a}{d_2}$ (10)	Miscellaneous (11)
(a) LONGITUDINAL POSITION ($b_1 = 9$ FT; $d_1 = 1$ FT; $L_B = 8.75$ FT; $c = 1$ FT; AND BASIN FLOOR ELEVATION = − 5 FT)											
C41...	276	7.17	6.09	29.3	0.85	1.22	0.14	−4.9	1.1	0.15	$L_B/5$[b]
C28...	276	7.17	6.07	29.3	0.85	1.22	0.14	−4.5	0.9	0.13	$L_B/3.2$[b]
C39...	272	7.04	6.09	28.3	0.87	1.24	0.14	−4.7	1.9	0.27	$L_B/2$[b]
C40...	282	7.32	6.09	30.4	0.83	1.19	0.14	−5.0	2.1	0.29	$2\ L_B/3$[b]
(b) HEIGHT ($b_1 = 9$ FT; $d_1 = 1$ FT; $L_B = 8.75$ FT; $c = 0.75$ FT; AND BASIN FLOOR ELEVATION = − 5 FT)											
....	275	7.14	6.09	29.1	0.85	1.23	0.14	−5.0	1.9	0.27	$3\ d_1/2$[c]
C43...	276	7.17	6.09	29.3	0.85	1.22	0.14	−4.5	1.3	0.18	d_1[c]
C44...	274	7.09	6.07	28.7	0.86	1.23	0.14	−5.0	1.9	0.27	$3\ d_1/4$[c]
(c) ARRANGEMENT ($b_1 = 9$ FT; $d_1 = 1$ FT; $L_B = 8.75$ FT; $c = 1$ FT; AND BASIN FLOOR ELEVATION = − 5 FT)											
C42...	281	7.29	6.10	30.2	0.84	1.20	0.14	−4.9	1.7	0.23	....[d]
(d) AGGREGATE WIDTH ($b_1 = 4.5$ FT; $d_1 = 2$ FT; $L_B = 15.33$ FT; $c = 1$ FT; AND BASIN FLOOR ELEVATION = − 4.9 FT)											
C73...	202	6.96	6.03	7.8	0.87	2.20	0.14	−4.0	2.0	0.29	67[e]
C74...	199	6.86	5.99	7.6	0.87	2.23	0.15	−5.0	1.4	0.20	56[e]
C75...	210	7.26	5.99	8.4	0.83	2.11	0.14	−4.8	1.2	0.17	44[e]

[a] Elevation of maximum depth of scour, in feet, on the channel center line near the end of the stilling basin. [b] Distance from upstream end of stilling basin to floor blocks. [c] Height of floor blocks. [d] Floor blocks are in line with chute blocks. Compare with test C28 in which floor blocks are in line with spaces between chute blocks. [e] Percentage of stilling basin width occupied by floor blocks.

Both the depth of erosion and the scour pattern are nearly identical when the floor blocks are located $L_B/3.2$ and $L_B/2$ from the upstream end of the basin (Table 3(*a*), tests C28 and C39, and Figs. 8(*b*) and 8(*c*)), and it is impossible to state definitely which spacing is the better from the standpoint of scour. However, the boil is 1 ft higher for test C39 and therefore the longitudinal floor block spacing of $L_B/2$ was eliminated from further consideration.

The floor blocks were placed $L_B/3$ from the upstream end of the stilling basin for all subsequent tests. Although the subsequent tests covered a range of the Froude number from 3 to 288, no reason was discovered for changing the longitudinal floor block location.

Height.—The height of the floor blocks for the first length-of-basin tests was assumed to be $5\ d_1/4$. During these tests the floor block height was reduced to d_1 with no apparent adverse effects. When the floor block height was investigated, a further reduction to $3\ d_1/4$ was made. The result of this test (Table 3(*b*)) was an increase in both the height of boil and depth of scour.

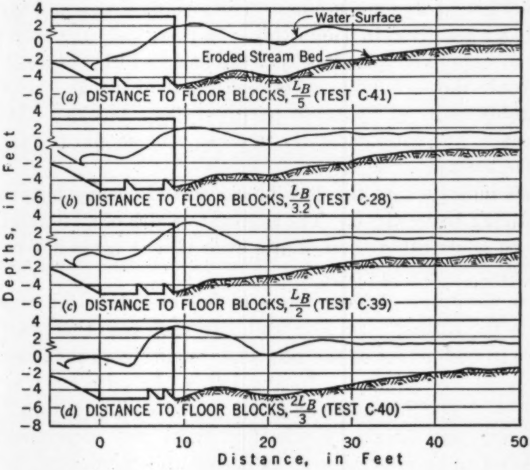


Fig. 8.—Effect of Floor Block Locations on Center-Line Bed and Water-Surface Profiles

The result of increasing the height of the floor blocks to $3\ d_1/2$ was the same—an increase in both the height of boil and depth of scour over that for a block height of d_1. Accordingly, a floor block height equal to d_1 was used for all subsequent tests.

Width and Spacing.—The width and spacing of the floor blocks was ordinarily made equal to those of the chute blocks. However, for the stilling basins that diverged in plan (see Fig. 1 and Section 6(*h*)), the floor block width and spacing were increased in proportion to the increase in stilling basin width at the floor block location.

Arrangement.—Ordinarily, the floor blocks were opposite the spaces between the chute blocks to intercept and break up the stream issuing through these spaces. To test the effectiveness of this arrangement, a single test was made with the floor blocks in line with the chute blocks. The results of this test (C42) are given in Table 3(*c*). They may be compared with the results of test C28, although the tabulated data do not give the complete story since a bad whirl developed along the side of the downstream channel for test C42 that caused considerable bed erosion and bank erosion. For all subsequent tests the floor blocks were alined with the spaces between the chute blocks.

Aggregate Floor Block Width.—Insufficient water can pass between the floor blocks if they occupy too much of the stilling basin width; they then act more like a sill than like individual blocks. This fact was demonstrated vividly when a test was made on a narrow stilling basin with a higher value of d_1 than had been used prior to that time. The floor blocks, when 3 $d_1/4$ wide, obstructed 67% of the stilling basin width (Table 3(*d*), test C73). The boil caused by these wide blocks was very high and overtopped the basin side wall. (In all previous tests, where the floor blocks had obstructed 43% of the basin width, the boil height was much less.) The results of two additional tests on this stilling basin are given in Table 3(*d*). In these tests the floor block width was reduced to occupy 56% and 44% of the basin width, respectively, so that more water could flow around and between the floor blocks. The boil height was considerably reduced by these narrower floor blocks. The scour was slightly increased, but not sufficiently to cause concern since it did not extend lower than the basin floor level.

These tests indicate that the floor blocks should occupy between 40% and 55% of the stilling basin width. The aggregate width of all floor blocks, therefore, should be held within these limits even if it is necessary to reduce the width of the floor blocks to do so.

Transverse Position.—A floor block of full or half width was sometimes placed next to the stilling basin side wall. In each case this practice resulted in an increase in boil height at the side wall, requiring an increase in side-wall height to prevent overtopping. To avoid this, the criterion was established that no floor block should be placed closer to the stilling basin side wall than 3 $d_1/8$. The side-wall height (see Section 6(*f*)) was determined with this criterion in mind and no deviation from it should be permitted.

(*d*) *End Sill.*—The end sill, at the downstream end of the stilling basin (see Fig. 1), deflects the bottom currents upward and away from the stream bed. In addition, a ground roller is generated under the deflected jet. This ground roller transports bed material upstream and deposits it at the end of the stilling basin.

Originally, before much thought had been given to the problem, the end sill height was assumed to be d_1. This assumption proved satisfactory for the first runs of the culvert outlet series in which case Q, d_1, and d_2 were constant. Toward the end of series C (culvert outlet tests) a few tests were made on basins designed for indifferent values of Q and d_2 but with the same value of d_1. During these tests it was discovered that the end sill caused a high boil when d_2 was low and the end sill blocked off a large part of the stilling basin exit area—that is, when the ratio of end sill height c to d_2 was high. The height of the end sill was therefore made proportional to d_2 rather than to d_1. A little thought shows this to be more logical. The function of the end sill is to deflect the bottom currents upward and in doing so, the entire stream leaving the stilling basin is deflected. To have the same degree of deflection for all depths of water at the end of the basin, the height of the end sill should vary with d_2. The satisfactory height of end sill used in the first few tests was (in terms of d_2) equal to $d_2/7$. Therefore, this end sill height was used with entirely satisfactory results until the tests of the flume outlet series, which were made at higher values of **F**, showed that further changes were required.

The design conditions and results of the flume outlet tests made to determine the best end sill height are summarized in Table 4. The tests were made by holding all variables constant for each Froude number with the exception of the height of end sill. Tests were made at that end sill height which, in the judgment of the observer, would give the best results. Additional tests were made using higher and lower end sills to check the assumed value of c.

TABLE 4.—SUMMARY OF DATA FOR TESTS TO DETERMINE HEIGHT OF END SILL; FLUME OUTLET SERIES F

($L_r = 12$; $b = 6$ Ft; and $d'_2/d_2 = 0.85$)

Test No.	c (ft) (1)	$\frac{c}{d_2}$ (2)	Scour[a] (ft) (3)	z_a (ft) (4)	$\frac{z_a}{d_2}$ (5)	Remarks (6)
(*a*) $Q = 100$ CU FT PER SEC; $d_1 = 0.6$ FT; $d_2 = 5.1$ FT; $d'_2 = 4.3$ FT; $L_B = 6.1$ FT; $L_B/d_2 = 1.20$; $\mathbf{F} = 40$; AND $\mathbf{R}_m \times 10^{-3} = 30.1$						
F72....	0.25	0.0490	−0.4	0.3	0.06	....
F70....	0.50	0.0980	+0.1	0.9	0.18	....
F71....	0.76	0.150	+0.5	1.0	0.20	Best height of end sill.
F73....	1.01	0.198	+1.0	1.7	0.33	....
(*b*) $Q = 637$ CU FT PER SEC; $d_1 = 1.8$ FT; $d_2 = 18.8$ FT; $d'_2 = 16.2$ FT; $L_B = 14.3$ FT; $L_B/d_2 = 0.76$; $\mathbf{F} = 60$; AND $\mathbf{R}_m \times 10^{-3} = 191.7$						
F81....	0.95	0.050	0.0	1.8	0.10	Best height of end sill.
F80....	1.91	0.102	+1.3	5.0	0.27	High boil.
F82....	2.86	0.152	+2.2	6.9	0.37	High boil.
(*c*) $Q = 374$ CU FT PER SEC; $d_1 = 1.2$ FT; $d_2 = 13.6$ FT; $d'_2 = 11.4$ FT; $L_B = 10.2$ FT; $L_B/d_2 = 0.75$; $\mathbf{F} = 70$; AND $\mathbf{R}_m \times 10^{-3} = 123.0$						
F60....	0.68	0.050	−0.5	2.3	0.17	....
F58....	1.02	0.075	+0.7	3.3	0.24	Best height of end sill.
F59....	1.36	0.100	+1.0	4.5	0.33	Second hole.
(*d*) $Q = 158$ CU FT PER SEC; $d_1 = 0.6$ FT; $d_2 = 8.2$ FT; $d'_2 = 7.0$ FT; $L_B = 6.1$ FT; $L_B/d_2 = 0.75$; $\mathbf{F} = 100$; AND $\mathbf{R}_m \times 10^{-3} = 47.7$						
F76....	0.41	0.050	−0.1	0.6	0.07	....
F75....	0.82	0.100	+0.4	1.9	0.23	Best height of end sill.
F74....	1.22	0.149	+1.0	3.2	0.39	Second hole.
(*e*) $Q = 448$ CU FT PER SEC; $d_1 = 1.2$ FT; $d_2 = 16.4$ FT; $d'_2 = 13.9$ FT; $L_B = 12.3$ FT; $L_B/d_2 = 0.75$; $\mathbf{F} = 100$; AND $\mathbf{R}_m \times 10^{-3} = 138.5$						
F66....	0.82	0.050	−0.1	1.6	0.10	Sill heights equally good; depth of scour greater for test F66 but boil height greater for test F65.
F65....	1.23	0.075	+1.0	3.6	0.22	
F64....	1.63	0.099	+1.1	5.1	0.31	Second hole.
(*f*) $Q = 549$ CU FT PER SEC; $d_1 = 1.2$ FT; $d_2 = 20.2$ FT; $d'_2 = 17.2$ FT; $L_B = 15.1$ FT; $L_B/d_2 = 0.75$; $\mathbf{F} = 150$; AND $\mathbf{R}_m \times 10^{-3} = 165.4$						
F62....	1.01	0.050	0.0	2.8	0.14	Height of end sill satisfactory.
F63....	1.49	0.074	+1.1	5.0	0.25	Best height of end sill.
F61....	2.02	0.100	+1.6	8.0	0.40	High boil.

[a] Elevation of maximum depth of scour, in feet, on the channel center line near the end of the stilling basin. The basin floor is at El. 0.

The plotted center-line profiles of the water surface and the eroded sand beds were studied to determine the best end sill height. The height judged best on this basis is noted in Table 4. Ordinarily, a low end sill did not deflect the jet upward sufficiently to prevent stream-bed erosion close to the outlet and to permit the formation of a ground roller under the jet. On the other hand, a high end sill caused the jet to shoot into the air. Thus, a higher side wall was required ordinarily and the landing stream eroded a large hole some distance downstream from the stilling basin. In Table 4, the terms "second hole" and "high boil" indicate the same condition; because of the test setup the second hole did not always appear and the term "high boil" is substituted as a synonomous term.

The flume outlet series was completed by making tests to check the over-all performance of the stilling basin. The results of these tests are summarized in Table 5. The end sill heights used in designing the stilling basins for these tests were found to be entirely satisfactory.

Considerable study was given to the problem of reducing the end sill height observations of the culvert and flume outlet series to a single curve before a successful solution was obtained. Plotting c/d_2 versus **F** did not reduce the data to such a form that a single curve could be used. Studies of this plot indicated that the variation was a function of the discharge per unit width of stilling basin. Plotting the product of c/d_2 and the discharge per unit width versus **F** produced a very satisfactory curve which was used in designing end sills for the tests listed in Table 5. The equation of this curve was not dimensionless, and it was therefore necessary to devise a method of making the equation generally applicable. Although two methods were used to accomplish this result, only the simpler and most straightforward method will be described. In the Reynolds number, defined in Eq. 1*b*, the term $V_1 d_1$ is the discharge per unit width. The kinematic viscosity, ν, can be considered constant for design purposes. For this condition the Reynolds number is proportional to the discharge per unit width. Since **R** is dimensionless, it does not have the undesirable dimensional properties of the equation first developed for the height of end sill. A consideration of the previously developed equation indicated that **F** had little influence on the determination of the end sill height. Therefore, the Froude number was tentatively eliminated from further consideration and c/d_2 was plotted against **R**. The curve drawn through these data was quite well defined and had the equation,

$$\frac{c}{d_2} = \frac{24}{\sqrt{\mathbf{R}}} \qquad (4)$$

Unfortunately, the experimental data did not cover the complete range of Reynolds numbers that may be expected in designing field structures—because the models were operated in accordance with the Froude model law (gravitational forces controlling) whereas the prototype-model relationship for the end sill height developed from the culvert and flume outlet series of tests requires that the model be operated in accordance with the Reynolds model law (viscous forces controlling). Since the two model laws require different and incompatible operating conditions, it is not possible to operate a model in such a way as to fulfil the requirements of both laws simultaneously.

The end sill heights computed from Eq. 4 are unbelievably low for the Reynolds numbers within the range of practical application. This condition led to the suspicion that the form of Eq. 4 might be incorrect, although nothing occurred during the culvert and flume outlet series to indicate that Eq. 4 could not be safely extrapolated, and showed the advisability of making additional tests at higher values of **R**. The turbine room series of tests was designed

TABLE 5.—SUMMARY OF DATA FOR TESTS TO CHECK HEIGHT OF END SILL; FLUME OUTLET SERIES F

(L_r = 12 and b = 6.0 Ft)

Test No.	Flow, Q (ft³/sec) (1)	d_1 (ft) (2)	d_2 (ft) (3)	d'_2 (ft) (4)	L_B (ft) (5)	c (ft) (6)	$\mathbf{R}_m \times 10^{-3}$ (7)	$\frac{d'_2}{d_2}$ (8)	$\frac{L_B}{d_2}$ (9)	$\frac{c}{d_2}$ (10)	Scour[a] (ft) (11)	z_a (ft) (12)	$\frac{z_a}{d_2}$ (13)
(*a*) FROUDE NUMBER F = 5													
F103...	36	0.6	1.6	1.7	4.0	0.38	14.1	1.06	2.50	0.237	0.0	0.7	0.44
F108...	101	1.2	3.2	3.4	7.9	0.46	39.9	1.06	2.47	0.141	+0.1	0.5	0.16
F105...	184	1.8	4.9	5.1	11.9	0.50	72.9	1.04	2.43	0.102	+0.2	0.8	0.16
(*b*) FROUDE NUMBER F = 10													
F107...	142	1.2	4.8	4.9	9.0	0.50	56.1	1.02	1.88	0.105	+0.1	0.5	0.10
F104...	260	1.8	7.2	7.3	13.5	0.55	103.2	1.02	1.88	0.077	+0.4	0.3	0.04
(*c*) FROUDE NUMBER F = 20													
F106...	201	1.2	7.0	6.5	10.1	0.58	79.5	0.93	1.44	0.082	0.0	0.5	0.07
(*d*) FROUDE NUMBER F = 30													
F89....	86	0.6	4.4	3.7	5.5	0.54	30.8	0.85	1.25	0.125	+0.3	1.2	0.27
F92....	246	1.2	8.7	7.4	11.0	0.64	93.3	0.85	1.26	0.073	0.0	1.1	0.10
F93....	246	1.2	8.7	7.4	11.0	0.64	97.3	0.85	1.26	0.073	+0.1	0.8	0.09
F91....	450	1.8	13.1	11.1	16.5	0.71	160.7	0.85	1.26	0.054	+0.5	0.9	0.07
(*e*) FROUDE NUMBER F = 60													
F87....	123	0.6	6.3	5.3	6.0	0.70	43.8	0.85	0.95	0.111	+0.2	1.7	0.27
F85....	347	1.2	12.5	10.7	11.9	0.82	124.1	0.85	0.95	0.065	0.0	1.3	0.10
F90....	637	1.8	18.8	16.0	17.9	0.91	227.7	0.85	0.95	0.048	−0.1	1.0	0.05
(*f*) FROUDE NUMBER F = 100													
F88....	159	0.6	8.2	7.0	6.4	0.90	56.7	0.85	0.78	0.110	0.0	2.0	0.24
F84....	448	1.2	16.4	13.9	12.8	1.06	160.7	0.85	0.78	0.065	+1.0	4.1	0.25
(*g*) FROUDE NUMBER F = 150													
F83....	549	1.2	20.2	17.2	13.5	1.36	196.4	0.85	0.67	0.067	+1.1	2.8	0.14

[a] Elevation of maximum depth of scour, in feet, on the channel center line near the end of the stilling basin. The basin floor is at El. 0.

TABLE 6.—SUMMARY OF DATA,

Test No.	Flow, Q (ft³/sec) (1)	V_1 (ft/sec) (2)	d_1 (ft) (3)	d_2 (ft) (4)	d'_2 (ft) (5)	L_B (ft) (6)	c (ft) (7)	$\mathbf{F}$ (8)	$\mathbf{R} \times 10^{-3}$ (9)	$\frac{d'_2}{d_2}$ (10)
T38......	0.408	13.6	0.03	0.572	0.419	0.348	0.067	191	41.2	0.732
T39......	0.399	13.3	0.03	0.560	0.473	0.348	0.039	183.1	40.6	0.845
T37......	0.408	10.2	0.04	0.489	0.427	0.413	0.058	80.8	43.4	0.874
T46......	1.05	21.0	0.05	1.146	0.958	0.602	0.076	274	98.6	0.835
T47......	1.06	20.7	0.111	1.142	0.945	0.602	0.076	120	....	0.829
T44......	1.06	17.7	0.06	1.050	0.883	0.672	0.069	161	99.2	0.840
T45......	1.02	16.9	0.102	1.005	0.864	0.672	0.069	87.0	....	0.860
T42......	1.016	14.5	0.07	0.924	0.774	0.736	0.064	93.7	104.2	0.838
T43......	1.010	14.5	0.094	0.919	0.775	0.736	0.064	69.2	....	0.844
T40......	0.970	9.7	0.10	0.716	0.629	0.900	0.054	29.2	98.6	0.878
T35......	2.02	25.2	0.08	1.740	1.433	0.961	0.100	247	212	0.823
T36......	1.99	24.9	0.08	1.715	1.432	0.961	0.150	240	214	0.835
T33......	2.00	20.0	0.10	1.525	1.327	1.098	0.191	124	204	0.870
T34......	1.98	19.8	0.10	1.510	1.300	1.098	0.100	121.7	201	0.861
T29......	2.01	16.8	0.12	1.386	1.167	1.223	0.191	72.6	191	0.841
T30......	1.99	16.6	0.12	1.373	1.173	1.223	0.100	71.2	192	0.854
T31......	1.99	16.6	0.12	1.373	1.182	1.223	0.070	71.2	197	0.861
T32......	2.00	16.7	0.144	1.380	1.187	1.223	0.070	60.0	....	0.860
T25......	1.95	13.0	0.15	1.182	1.051	1.388	0.290	35.0	206	0.889
T26......	2.00	13.3	0.15	1.212	1.063	1.388	0.191	36.8	208	0.876
T27......	2.01	13.4	0.15	1.221	1.075	1.388	0.100	37.2	200	0.881
T28......	1.99	13.3	0.15	1.207	1.075	1.388	0.062	36.4	197	0.890
T21......	1.94	9.7	0.20	0.985	1.017	1.622	0.290	14.6	198	1.032
T22......	2.02	10.1	0.20	1.030	1.049	1.622	0.191	15.8	209	1.018
T23......	2.01	10.0	0.20	1.020	1.032	1.622	0.100	15.5	214	1.011
T24......	2.01	10.0	0.20	1.020	1.040	1.622	0.062	15.5	220	1.019
T53......	4.00	33.3	0.12	2.82	2.397	1.472	0.197	288	426	0.850
T50......	4.00	25.0	0.16	2.410	2.053	1.754	0.169	121	404	0.852
T51......	4.00	25.0	0.16	2.410	2.070	1.754	0.169	121	407	0.859
T52......	4.00	25.0	0.16	2.410	2.070	1.754	0.169	121	417	0.859
T48......	4.50	22.5	0.20	2.405	1.820	1.986	0.148	78.5	435	0.757
T49......	4.50	22.2	0.223	2.398	1.924	1.986	0.148	68.7	....	0.803
T41......	4.00	16.0	0.25	1.871	1.597	2.271	0.131	31.8	408	0.853
T13......	7.00	38.9	0.18	4.023	3.408	2.187	0.112	261	730	0.847
T14......	7.00	38.9	0.18	4.023	2.656	2.187	0.192	261	707	0.660
T15......	7.00	38.9	0.18	4.023	3.005	2.187	0.192	261	715	0.747
T16......	7.00	38.9	0.18	4.023	3.033	2.187	0.305	261	677	0.754
T17......	7.00	38.9	0.18	4.023	3.430	2.187	0.305	261	690	0.853
T5.......	7.00	35.0	0.20	3.800	2.70	2.329	0.106	190	752	0.710
T6.......	7.00	35.0	0.20	3.800	2.715	2.329	0.190	190	750	0.715
T7.......	7.00	35.0	0.20	3.800	2.746	2.329	0.266	190	756	0.722
T8.......	7.00	35.0	0.20	3.800	3.19	2.329	0.266	190	750	0.839
T9.......	7.00	35.0	0.20	3.800	3.36	2.329	0.106	190	760	0.884
T1.......	7.00	28.0	0.25	3.359	2.856	2.656	0.099	97	686	0.850
T2.......	7.00	28.0	0.25	3.359	2.62	2.656	0.169	97	730	0.780
T3.......	7.00	28.0	0.25	3.359	2.65	2.656	0.235	97	745	0.789
T4.......	7.00	28.0	0.25	3.359	2.67	2.656	0.336	97	753	0.795
T18......	7.00	21.2	0.33	2.875	2.45	3.119	0.146	42.3	716	0.852
T19......	7.00	21.2	0.33	2.875	2.454	3.119	0.204	42.3	716	0.854
T20......	7.00	21.2	0.33	2.875	2.477	3.119	0.292	42.3	718	0.862
T10......	7.00	14.0	0.50	2.232	2.25	3.882	0.062	12.2	745	1.008
T11......	7.00	14.0	0.50	2.232	2.225	3.882	0.250	12.2	730	0.997
T12......	7.00	14.0	0.50	2.232	2.232	3.882	0.500	12.2	740	1.000
T62......	9.70	43.7	0.222	5.02	4.345	2.720	0.360	267	800	0.865
T63......	9.42	42.5	0.222	4.875	4.345	2.720	0.360	252	756	0.891
T61......	9.94	39.8	0.25	4.826	4.15	2.934	0.340	196	805	0.860
T60......	14.4	36.0	0.40	5.48	4.81	4.280	0.396	100.6	1,150	0.878
T59......	14.8	29.6	0.50	4.98	4.295	4.915	0.353	54.6	1,210	0.862
T58......	14.9	23.8	0.625	4.39	4.065	5.586	0.318	28.2	1,450	0.926
T54......	15.0	20.0	0.75	3.96	3.84	6.154	0.281	16.5	1,500	0.970
T57......	20.0	20.0	1.00	4.50	4.41	7.782	0.315	12.4	1,910	0.980
T56......	21.0	16.8	1.25	4.09	4.245	8.795	0.252	7.0	2,060	1.038
T55......	21.0	16.5	1.27	4.05	3.69	9.497	0.286	6.7	2,140	0.911

---

[a] Elevation of maximum depth of scour, in feet, near the end of the basin (Col. 12) and at the end of the wing caused bad erosion in back of wing wall; see duplicate run for normal conditions.

TURBINE ROOM SERIES T ($b = 1$ Ft)

$\frac{c}{d_1}$ (11)	SCOUR, IN FEET[a] Basin (12)	 Wing Wall (13)	z_m (ft) (14)	$\frac{z_m}{d_2}$ (15)	$\frac{d''_2}{d_2}$ (16)	Remarks (17)	Test No.
0.117	−0.21	−0.18	0.14	0.25	....	Conditions not as recommended[b]	T38
0.070	−0.03	−0.17	−0.01	−0.02	0.73	....	T39
0.118	−0.05	−0.08	0.11	0.22	0.79	Conditions not as recommended[b]	T37
0.066	−0.12	−0.48	0.32	0.28	0.63	....	T46
0.066	−0.01	−0.47	0.36	0.31	0.72	Air added to water, 1.24 cu ft per sec	T47
0.066	−0.11	−0.36	0.14	0.13	0.74	....	T44
0.069	−0.06	−0.27	0.17	0.17	0.69	Air added to water, 0.70 cu ft per sec	T45
0.069	−0.05	−0.24	0.18	0.20	0.82	....	T42
0.070	0.00	−0.19	0.15	0.16	....	Air added to water, 0.35 cu ft per sec	T43
0.075	0.00	−0.14	0.10	0.14	0.82	....	T40
0.057	−0.32	−0.37	0.20	0.12	....	....	T35
0.087	−0.23	−0.37	0.65	0.38	....	Conditions not as recommended[b]	T36
0.125	−0.14	−0.08	0.62	0.41	....	Conditions not as recommended[b]	T33
0.066	−0.19	−0.21	0.20	0.13	....	....	T34
0.138	−0.02	−0.19	0.43	0.31	....	Conditions not as recommended[b]	T29
0.073	−0.11	−0.21	0.11	0.08	....	....	T30
0.051	−0.14	−0.19	0.10	0.07	....	Conditions not as recommended[b]	T31
0.051	−0.06	−0.12	0.05	0.04	....	Air added to water, 0.40 cu ft per sec[b]	T32
0.245	0.25	−0.03	0.42	0.36	....	Conditions not as recommended[b]	T25
0.158	0.10	−0.02	0.35	0.29	....	Conditions not as recommended[b]	T26
0.082	0.02	−0.10	0.10	0.08	....	....	T27
0.051	−0.08	−0.02	0.07	0.06	....	Conditions not as recommended[b]	T28
0.294	0.14	0.24	0.24	0.24	....	Wing wall is too long[b]	T21
0.185	0.06	0.00	0.18	0.17	....	Conditions not as recommended[b]	T22
0.098	−0.13	0.00	0.12	0.12	....	Conditions not as recommended[b]	T23
0.061	−0.17	−0.03	0.17	0.17	....	....	T24
0.070	−0.10	0.26	0.29	0.10	0.50	Slope, top of wing wall, 1 on 1	T53
0.070	0.00	−0.31	0.33	0.14	0.64	....	T50
0.070	−0.02	0.27	0.36	0.15	....	Stepped wing wall	T51
0.070	0.00	0.23	0.28	0.12	....	Slope, top of wing wall, 1 on 1	T52
0.062	−0.23	−0.44	0.13	0.05	0.65	Conditions not as recommended[b]	T48
0.062	−0.24	−0.47	0.24	0.10	0.68	Air added to water, 0.46 cu ft per sec	T49
0.070	−0.05	−0.18	0.25	0.13	0.82	....	T41
0.0278	−0.45	−0.58	−0.34	−0.08	0.62	Conditions not as recommended[b]	T13
0.0477	−0.05	−0.21	0.07	0.02	....	Conditions not as recommended[b]	T14
0.0477	−0.09	−0.21	0.06	0.02	....	Conditions not as recommended[b]	T15
0.0758	0.18	0.23	0.68	0.17	....	Conditions not as recommended[b]	T16
0.0758	0.07	0.45	0.26	0.06	....	....	T17
0.0278	−0.50	−0.58	−0.19	−0.05	0.66	Conditions not as recommended[b]	T5
0.050	−0.23	−0.20	0.04	0.01	....	Conditions not as recommended[b]	T6
0.070	−0.10	−0.27	0.49	0.13	....	Conditions not as recommended[b]	T7
0.070	−0.09	−0.15	0.54	0.14	....	....	T8
0.0278	−0.51	−0.58	−0.33	−0.09	....	Conditions not as recommended[b]	T9
0.0295	−0.34	−0.28	−0.31	−0.09	0.66	Conditions not as recommended[b]	T1
0.050	−0.28	−0.23	−0.13	−0.04	....	Conditions not as recommended[b]	T2
0.070	−0.04	−0.10	0.45	0.13	....	Conditions not as recommended[b]	T3
0.100	0.18	....	0.84	0.25	....	Conditions not as recommended[b]	T4
0.051	−0.08	0.05	0.09	0.03	....	Conditions not as recommended[b]	T18
0.071	0.09	0.36	0.24	0.08	....	....	T19
0.102	0.02	0.30	0.48	0.17	....	Conditions not as recommended[b]	T20
0.0278	−0.08	0.15	0.45	0.20	0.89	Conditions not as recommended[b]	T10
0.112	0.14	0.30	0.376	0.17	....	Conditions not as recommended[b]	T11
0.224	0.27	0.35	0.50	0.22	....	Conditions not as recommended[b]	T12
0.072	−0.33	−0.02	0.61	0.12	0.55	Slope, top of wing wall, 1 on 1[c]	T62
0.074	−0.06	0.36	0.54	0.11	....	Slope, top of wing wall, 1 on 1	T63
0.070	0.03	0.50	0.57	0.12	0.57	Slope, top of wing wall, 1 on 1	T61
0.072	0.10	1.16	1.0	0.18	0.58	Slope, top of wing wall, 1 on 1	T60
0.071	0.05	1.04	0.67	0.13	0.68	Slope, top of wing wall, 1 on 1	T59
0.072	0.11	1.52	0.34	0.08	0.52	Slope, top of wing wall, 1 on 1	T58
0.071	−0.02	1.39	0.10	0.02	0.71	Slope, top of wing wall, 1 on 1	T54
0.070	0.04	1.78	0.89	0.20	....	Slope, top of wing wall, 1 on 1	T57
0.062	−0.12	1.90	0.84	0.21	....	Slope, top of wing wall, 1 on 1	T56
0.071	−0.02	2.21	0.41	0.10	....	Slope, top of wing wall, 1 on 1	T55

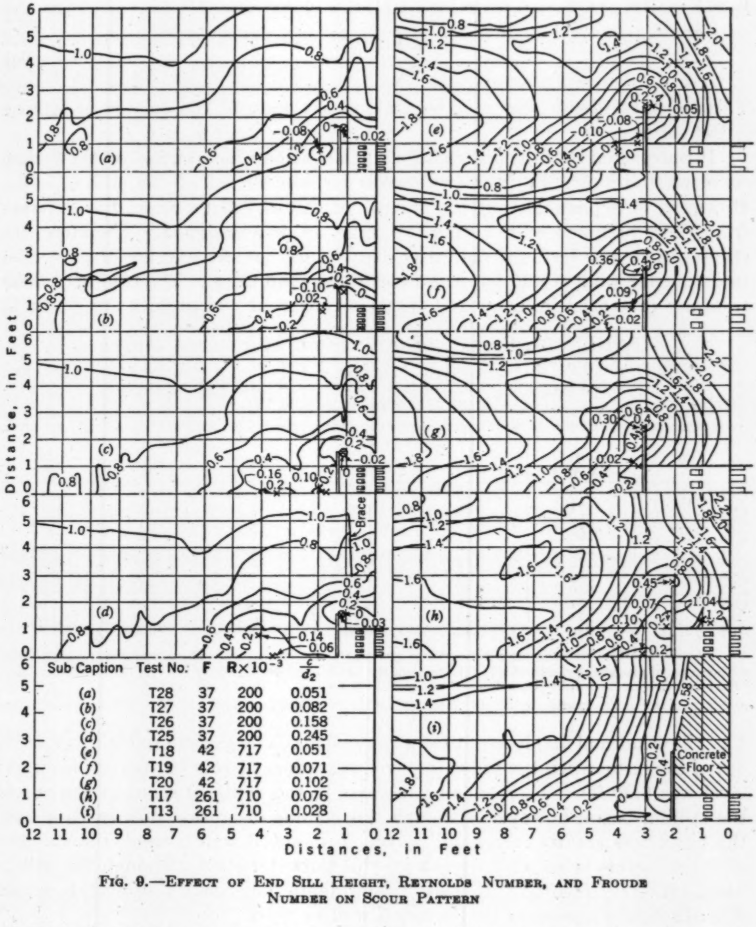
wall (Col. 13). [b] Basin dimensions or controllable flow conditions not as recommended in this paper. [c] Washout primarily to determine whether or not the end sill height equation originally proposed (Eq. 4) can be extrapolated beyond the range of the Reynolds numbers for which it was derived to those Reynolds numbers that are to be expected in full-size structures. These tests confirm the opinion that the extrapolated heights of end sill appear to be far too low. The results of the sixty-three tests made as part of the turbine room series are summarized in Table 6. The tests have been listed in order of increasing Reynolds numbers and grouped according to constant Froude numbers. The individual tests in each group have different amounts of air entrainment or different heights of end sill or different tailwater depths.

Typical scour patterns for several different heights of end sill are presented in Fig. 9. The height of end sill computed from Eq. 4 was used to obtain the results shown in Figs. 9(*a*), 9(*f*), and 9(*i*). The results of tests made with the end sill height recommended as a result of the turbine room series of tests are shown in Figs. 9(*b*), 9(*f*), and 9(*h*). The results presented in Figs. 9(*a*) to 9(*d*) and 9(*e*) to 9(*g*), inclusive, are for approximately the same Froude numbers (37 and 42, respectively), but at different Reynolds numbers; the tests in the latter group have a Reynolds number more than three and one-half times that of the first mentioned group (717,000 versus 200,000). The results shown in Figs. 9(*h*) and 9(*i*) are for a Froude number near the upper limit of what may be expected in a prototype structure (261) whereas the Reynolds number has a moderate value (710,000).

The 6-ft width of test channel was too narrow for many of the tests, with the result that certain features of the eroded sand bed are not indicative of what is to be expected in the prototype. These nontypical erosion patterns are shown in Figs. 9(*e*) to 9(*i*) near the downstream end and on the outside edge where a depression has been eroded along the side of the test channel. Typical erosion patterns are presented in Figs. 9(*a*) to 9(*d*). The tailwater depth for these latter tests was such that a stream bank could be maintained between the stream and the outside edge of the test channel. In the former tests the tailwater was so high that the stream bank could not be maintained and the disturbance caused by the water striking the vertical side of the channel resulted in the erosion of a depression along the channel wall.

Those tests of the turbine room series that have Reynolds numbers within the range covered by the tests of the culvert and flume outlet series check the previous conclusions. However, a study of the data obtained for Reynolds numbers beyond the range covered by the previous tests shows that the scour near the stilling basin is much less if an end sill height equal to 7% of the theoretical tailwater depth is used. Further study of the data shows that the height of the end sill relative to the tailwater depth (c/d_2) is independent of both the Froude and the Reynolds numbers. The data obtained during the flume and culvert outlet series were then re-analyzed and it was found that entirely satisfactory scour patterns were obtained for an end sill height equal to 7% of the theoretical tailwater depth. Admittedly, the end sill height as given by Eq. 4 resulted in slightly better scour patterns to $\mathbf{R} = 118{,}000$, where $c/d_2 = 0.07$, but the difference is so small as not to warrant the use of a separate equation for the lower Reynolds numbers. As a result of all tests made to determine the height of end sill, the recommended equation is

$$\frac{c}{d_2} = 0.07 \quad \text{(5)}$$



Sub Caption	Test No.	F	R×10⁻³	$\frac{c}{d_2}$
(a)	T28	37	200	0.051
(b)	T27	37	200	0.082
(c)	T26	37	200	0.158
(d)	T25	37	200	0.245
(e)	T18	42	717	0.051
(f)	T19	42	717	0.071
(g)	T20	42	717	0.102
(h)	T17	261	710	0.076
(i)	T13	261	710	0.028

FIG. 9.—EFFECT OF END SILL HEIGHT, REYNOLDS NUMBER, AND FROUDE NUMBER ON SCOUR PATTERN

End sill heights as given by both Eqs. 4 and 5 were used for a few tests until it became certain that Eq. 4 would be eliminated from the final list of recommended design equations and Eq. 5 would be substituted for it. Subsequent tests were made simply to verify Eq. 5.

(*e*) *Tailwater Depth.*—A certain depth of tailwater above the stilling basin floor is required to prevent the hydraulic jump from being washed out of the stilling basin. This depth can ordinarily be computed from Eqs. 2. However, the use of the floor blocks and of the end sill makes it possible to reduce the actual tailwater depth d'_2 below the theoretical tailwater depth d_2 given by Eqs. 2. Mr. Warnock (4*b*) suggests a 15% reduction, or

$$\frac{d'_2}{d_2} = 0.85 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6a)$$

This tailwater depth was used in making the tests of the culvert and flume outlet series.

It is expected that the design tailwater elevation will be such that the surface roller on the hydraulic jump will never be washed out of the stilling basin. However, it is possible that the actual conditions will be somewhat different from those assumed at the time the structure was designed. Therefore, the stilling basin was tested at tailwater depths both greater and less than normal to determine what would happen under these conditions. The results of these tests are presented in Table 7. Water-surface and bed profiles are shown in Fig. 10 for normal and lowered tailwater levels. The surface roller was washed out of the basin when the tailwater level was reduced to 5.00 ft (normal tailwater level was 6.00 ft). There was no increase in the depth of scour at the end of the stilling basin, however, although the volume of erosion downstream from the basin was greatly increased. Even if no cutoff wall is used, the stability of the structure is not endangered for this tailwater level, although the stilling basin action is impaired. If the tailwater depth falls below this level, however, it is absolutely necessary that a cutoff wall be used.

TABLE 7.—EFFECT OF ABNORMAL TAILWATER DEPTH

($L_r = 12$; $b_1 = 9.0$ Ft; $d_1 = 1.0$ Ft; d_2 (Average) $= 7.20$ Ft; $L_B = 10.92$ Ft; $c = 1.00$ Ft; F (Average) $= 29.5$; $L_B/d_2 = 1.52$; $c/d_2 = 0.14$; Stilling Basin Floor Elevation $= -5.00$ Ft)

Test No.	Flow, Q (ft^3/sec)	d'_2 (ft)	$\frac{d'_2}{d_2}$	Scour[a]	z_s (ft)	$\frac{z_s}{d_2}$
C92..........	277	11.49	1.60	−4.0	1.8	0.11
C91..........	274	9.56	1.33	−4.0	0.8	0.11
C90..........	277	8.54	1.19	−4.0	1.1	0.15
C83..........	286	7.71	1.07	−4.0	1.2	0.17
C84..........	280	7.08	0.98	−4.0	1.0	0.14
C85..........	272	6.03	0.84	−4.1	0.8	0.11
C86..........	280	5.00	0.69	−5.0	7.0[b]	0.97
C87..........	280	3.80	0.53	−7.0	7.1[b]	0.99
C88..........	277	2.77	0.38	−8.5	7.4[b]	1.03
C89..........	275	1.80	0.25	−10.1	7.9[b]	1.10
C98..........	284	−0.40	−0.06	−13.0	10.5[b]	1.46

[a] Elevation of maximum depth of scour, in feet, on the channel center line near the end of the stilling basin.
[b] Surface roller on hydraulic jump washed out of stilling basin.

The following brief explanation will define the term "air-jet" boundaries, in Fig. 10. The turbulence at the entrance to the stilling basin causes much air to be entrained at that point. In addition, the water in the channel approaching the stilling basin may contain entrained air. The vigorous turbulence in and beyond the stilling basin prevents the ready separation of this entrained air. Therefore, the presence of air in the water may be taken as an indication that the flow is very turbulent and, conversely, the absence of air may be assumed to indicate that the turbulence is insufficient to maintain the air in the "suspended" state. The lower boundary of the "air jet" shown in Fig. 10 is the lower limit of air entrainment and therefore indicates the lower limit of extreme turbulence as defined. The upper boundary of the "air jet" is taken as the lower boundary of the roller normally associated with the hydraulic jump. Much entrained air is also found in this roller.

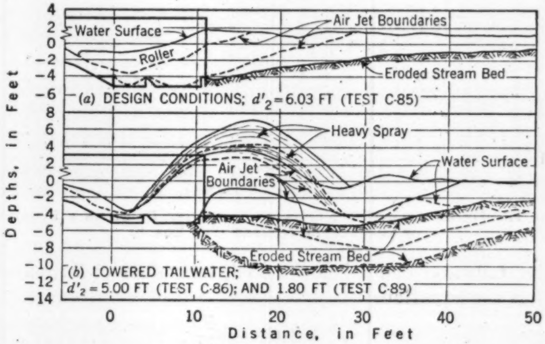


FIG. 10.—CENTER-LINE BED AND WATER-SURFACE PROFILES FOR NORMAL AND SUBNORMAL TAILWATER DEPTHS

Unfortunately, in making a check of the stilling basin design during the flume outlet series, it was found that the roller ordinarily present on the hydraulic jump was washed out of the stilling basin at values of the Froude number of less than about $\mathbf{F} = 30$. Several methods were available to overcome this difficulty: The tailwater depth could be increased, the end sill height could be increased, or the stilling basin could be lengthened. Increasing the end sill height increased the height of the boil caused by the end sill and increased the depth of scour near the end of the stilling basin. The increase required in the length of the stilling basin to insure its proper functioning was so great as to be uneconomical. The method finally adopted was to increase the tailwater depth. The equation for that part of the tailwater depth curve between $\mathbf{F} = 3$ and $\mathbf{F} = 30$ was therefore changed to

$$\frac{d'_2}{d_1} = 1.10 - \frac{\mathbf{F}}{120}, \qquad (6b)$$

to insure the proper functioning of the stilling basin at the lower Froude numbers. This curve and the data on which it is based are shown in Fig. 11.

For many runs of the turbine room series the tailwater depth was slowly lowered until the roller was washed out of the stilling basin. The object of this procedure was to determine the relative tailwater depth at which the basin would cease to function as was assumed in the design and to determine the safety factor for this part of the stilling basin. The results of these tests are presented in Table 6 and Fig. 11. They indicate that d'_2 can be reduced below $0.85\ d_2$ when the Froude number is high—that is, when $\mathbf{F} > 120$. A curve based on the turbine room tests has been added to Fig. 11 to permit a lowering of the relative tailwater depth below $0.85\ d_2$ for values of **F** greater than 120. This curve has the equation,

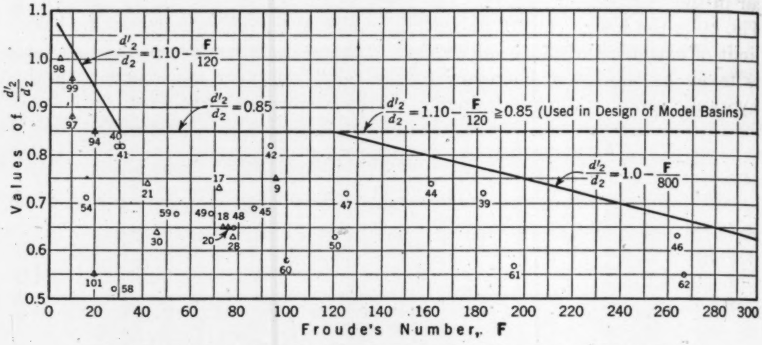


FIG. 11.—DIMENSIONLESS PLOT SHOWING TAILWATER DEPTH AS A FUNCTION OF THE FROUDE NUMBER (SEE EQS. 6)

$$\frac{d'_2}{d_2} = 1.00 - \frac{\mathbf{F}}{800} \qquad (6c)$$

The difference between the curve and the plotted points of Fig. 11 is the safety factor which insures that the stilling basin will perform as anticipated by the designer. The safety factor is small, but it is felt to be sufficient since a tailwater level somewhat lower than the design value will not endanger the basin, although the roller will be washed out of it. For this condition the floor blocks and end sill deflect the stream into the air and break it up as is shown by Fig. 10(*b*) where the tailwater depth is reduced to 5 ft (the design depth being 6 ft). This statement is inserted not as a justification for using a tailwater depth below that recommended but to show that an unanticipated lowering of the design tailwater depth or a discharge greater than that for which the outlet is designed will not endanger the outlet structure. (In Fig. 11, the plotted points are identified as follows: The triangles refer to the tests of the flume outlet series, the circles to the turbine room series, and the numbers to the test numbers.)

During the culvert outlet series a few tests were made in which the tailwater level was held at a constant elevation while the discharge was increased until the roller was washed out of the stilling basin. These tests were made to determine what would happen if the tailwater level were lower than was assumed by the designer or the actual discharge were greater than that assumed for design purposes. Conversely, the discharge was increased for several different tailwater elevations to wash the roller out of the stilling basin and then was decreased until the roller reappeared. The results of these tests indicate the maximum discharge at which the basin will continue to function properly for any tailwater depth. It is apparent from Fig. 12 that the stilling basin on which these tests were made would function properly for the design tailwater depth until a discharge equal to about 118% of the design value was reached. Similarly, for the design discharge, the basin would function properly until the tailwater depth was reduced to 90% of the design value or 76% of the theoretical value. When the tailwater depth was increased to 7.20 ft or 120% of the design (102% of the theoretical) value, the roller could not be washed out of the stilling basin even though the discharge was increased to 598 cu ft per sec or 217% of the design discharge.

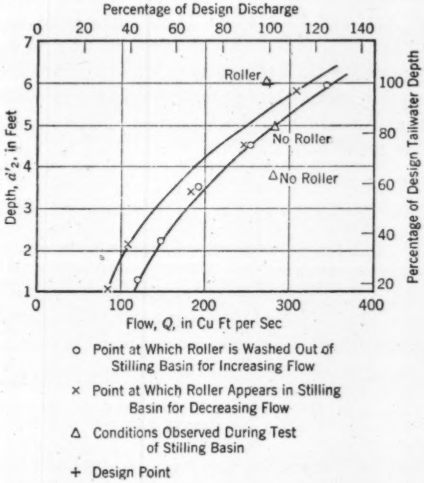


FIG. 12.—EFFECT OF VARIATIONS IN DISCHARGE AND TAILWATER DEPTH ON STILLING BASIN PERFORMANCE

(*f*) *Side-Wall Height.*—The flow in the stilling basin is very turbulent and the water surface is rough so that some freeboard above tailwater level is necessary if overtopping of the side walls is to be prevented. In addition to the surface roughness, the standing wave or boil, caused by the floor blocks and end sill, requires freeboard above the tailwater level. For Froude numbers less than about $\mathbf{F} = 20$ the crest of the standing wave is in the stilling basin whereas for higher Froude numbers the crest occurs downstream from the end of the basin and its full height need not be considered in designing the side-wall height.

Average profiles of the water surface in the stilling basin were obtained for all series, but the maximum height of splash was obtained only for the turbine room series. From the latter tests the height of the side wall is determined. The maximum height of the splash z_m in the stilling basin was divided by d_2. Both z_m and z_m/d_2 are shown in Table 6. Although there is considerable scatter to the data, z_m/d_2 is apparently independent of $\mathbf{F}$. The maximum value of z_m/d_2 for those tests of the turbine room series in which the recommended design conditions exist is 0.31; and the minimum, -0.02. The maximum value of z_m/d_2 occurs at the exit end of the stilling basin. A study of the data shows that, if the height of the stilling basin side walls z above the maximum tailwater level is given by the equation—

$$z = \frac{d_2}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

—the freeboard will be sufficient to keep the splash in the stilling basin. Because of the scatter in the data, the freeboard provided by this equation will be greater, in some cases, than is necessary to protect the structure adequately, but the safety factor is not excessive for the average case. Two values of z_m/d_2 were obtained from motion pictures made during the culvert and flume series of tests (series C and F). These observations substantiate the results of the turbine room tests (series T).

Frequently the elevations of the tailwater and stream bed at the time the stilling basin is constructed are much higher than those that are ultimately expected and for which the stilling basin is designed. In this case the height of the side wall must be based on the maximum expected tailwater level. If the side walls are constructed so low that water will pour over them into the stilling basin, the basin must handle, in effect, a quantity of water greater than that for which it was designed. The result is the large increase in the size of the scour hole shown in Fig. 13. This condition occurs because the "air jet," leaving the stilling basin is not deflected upward by the end sill as much as if no water entered the basin over the top of the side walls. Consequently, the side walls should always be high enough to extend above the maximum tailwater level.

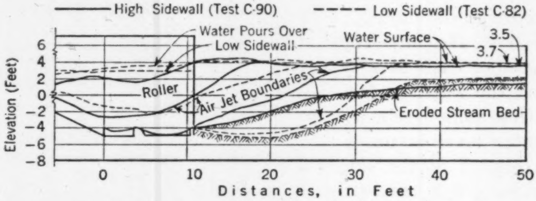
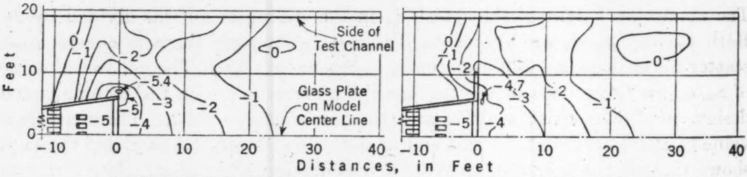


FIG. 13.—EFFECT OF WATER POURING OVER LOW SIDE WALLS INTO STILLING BASIN ON CENTER-LINE BED AND WATER-SURFACE PROFILES

(*g*) *Wing Walls.*—The SAF stilling basin was developed mainly for use in connection with earth dams. For this type of dam, wing walls are used at the end of the stilling basin to protect the toe.

Shape of Wing Walls.—The wing walls used in the past have been rectangular in downstream elevation. During the culvert outlet series (C) tests were made to determine the best length of this wing wall. During runs made with stilling basins having no wing walls it was observed that there was considerable scour at the toe of the dam and that the deepest scour occurred close to the end of the stilling basin side walls, as shown in Fig. 14(*a*). The principal



(*a*) NO WING WALL (TEST C-46) (*b*) WING WALL 4.5 FT. LONG (TEST C-47)

FIG. 14.—EFFECT OF RECTANGULAR WING WALL ON SCOUR PATTERN

cause of the scour is an eddy along the side of the channel which is set in motion and is driven by the water that flows along the center of the downstream channel after it leaves the stilling basin. The upstream end of this eddy causes the scour at the toe of the dam. The wing wall has a twofold function: It calms the water in back of it and prevents the eddy from extending as far upstream as it would if the wing wall were absent; and it lessens the concentration of flow at the end of the stilling basin. The use of a wing wall results in a decreased depth of scour at the end of the stilling basin (Fig. 14(*b*)), although it has little or no effect on the scour at the channel center line.

The results of the tests made to determine the best length of rectangular wing wall are presented in Table 8. Too short wing walls are ineffective in preventing scour and there is nothing to be gained by making the wing walls too long. As a result of these tests, it was suggested that the minimum length of wing wall be 40% of the stilling basin side-wall height. Lengths longer than this do not change the hydraulic performance significantly.

TABLE 8.—EFFECT OF WING-WALL LENGTH ON DEPTH OF SCOUR; SERIES C

($L_r = 12$; $b_1 = 9$ Ft; $Q = 275$ Cu Ft per Sec; $d_1 = 1$ Ft; $d_2 = 7.13$ Ft; $d'_2 = 6$ Ft (Which Equals El. + 1.0); $L_B = 8.75$ Ft; $c = 1$ Ft; $F = 29$; $d'_2/d_2 = 0.84$; $L_B/d_2 = 1.23$; $c/d_2 = 0.14$; and Stilling Basin Floor Elevation = − 5.0 Ft)

Test No.	Length[a] (%)	ELEVATION OF MAXIMUM DEPTH OF SCOUR, IN FEET		
		Maximum	Place where maximum occurred	Channel[b]
C46.....	0	−5.4	In line with the end of the stilling basin and 2.5 ft outside the side wall	−4.2
C48.....	32	−5.0	At end of wing wall	−4.3
C54.....	42	−4.5	In line with the stilling basin side wall and close to the end of the basin	−4.0
C47.....	47	−4.7		−4.0
C49.....	53	−4.5		−4.2

[a] Length of the wing wall expressed as a percentage of the side-wall height. [b] Depth of scour, in feet, on the channel center line near the end of the stilling basin.

This rectangular wing wall was not entirely satisfactory since the depth of scour at its end was sufficient to cause some concern. It was felt that the concentration of flow lines at the end of the wing wall caused this scour. A submerged extension of the wing wall to keep the concentrated streamlines above the stream bed was tried. The height of this extension above the stilling basin floor was 0.5 d'_2 and the length was 0.6 times the side-wall height. The results were excellent in that the depth of scour at the end of the wing wall and (what is just as important) the erosion in back of the wing wall were considerably decreased. The results of this test are shown in Fig. 15(*b*). The results of a comparable test made with the originally proposed wing-wall length are presented in Fig. 15(*a*) for purposes of comparison. Subsequent to this test an additional test was made in which the top of the wing wall was sloped. The slope used was 1 on 1 beginning at the top of the stilling basin side wall. The results of this test (see Fig. 15(*c*)) were such that this wing-wall shape was used for all subsequent tests and has been shown in Fig. 1 as the recommended wing-wall shape.

Location of Wing Walls.—In the past, the wing walls used on most outlets built by the SCS have been placed perpendicular to the center line of the outlet structure. As a result, this arrangement was used throughout the test program. However, after the SAF stilling basin design had been released, questions were raised as to whether the wing wall should be perpendicular, parallel, or at some intermediate angle to the basin center line. Tests were made to answer these questions.

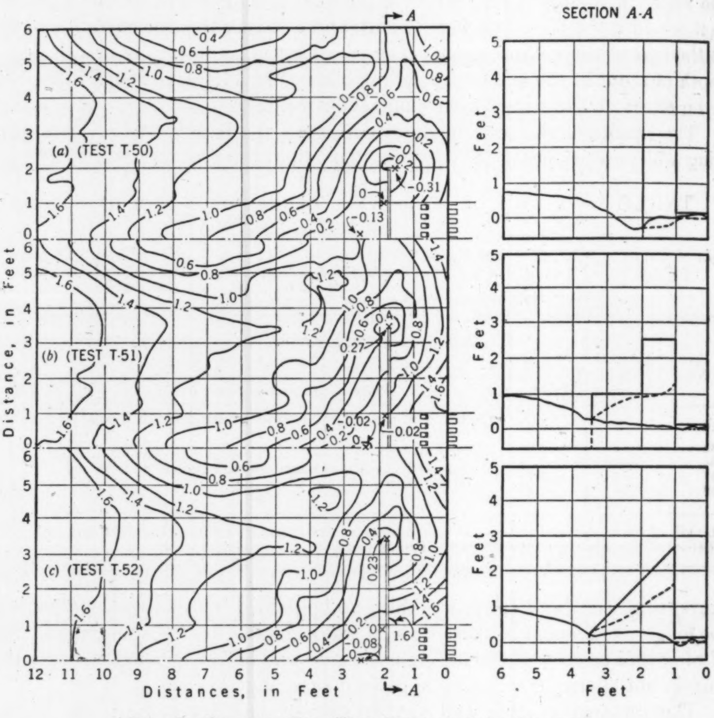
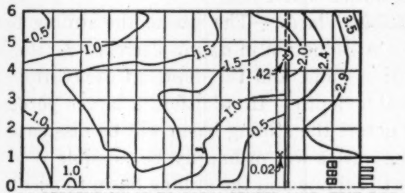


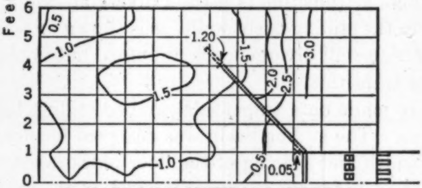
Fig. 15.—Effect of Wing-Wall Slope on Scour Pattern

The stilling basin used in this group of tests had the dimensions shown in Table 6 for test T63. The stilling basin model had deteriorated somewhat in the fifteen months since test T63 was made, and the turbine room test setup had been partly dismantled. As a result, no attempt was made to measure the discharge, although the same valve setting was used for each of the tests in this series. It is believed that the flow was somewhat greater than that for test T63. Less sand was also available for the movable bed than was desirable. Therefore, the test results are only comparable among themselves. Plan views of each of the stilling basins and wing-wall arrangements tested are shown in Fig. 16. The top of the wing wall was sloped at 1 on 1 in all cases.

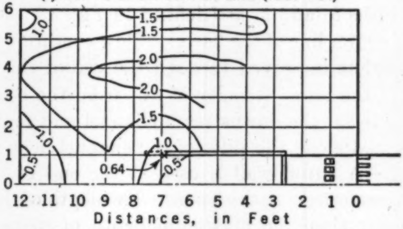
The results of the tests made on three different positions of the wing wall are given in Fig. 16. The boil height for test T66 was satisfactory ($z_m/d_2 = 0.10$) and the scour pattern (Fig. 16(*a*)) was similar to that for other tests made with the wing wall placed perpendicular to the basin center line. The maximum boil height for test T64 was 0.6 ft greater at the end of the stilling basin ($z_m/d_2 = 0.23$) than for test T66, but was still below the top of the basin side wall. The scour pattern was changed somewhat (Fig. 16(*b*)) and in general was improved. Erosion of the dam fill beside the stilling basin was decreased by a foot or more. The boil height for test T65 was the highest observed in this group of tests ($z_m/d_2 = 0.34$). This boil height occurred at the very end of the stilling basin. The high boil did, however, cause a deeper secondary scour hole than for the other two tests of this group. The scour pattern (Fig. 16(*c*)) for this test was also satisfactory. In no tests of this group did the scour descend to the elevation of the stilling basin floor, although the rise in the bed elevation along the channel center line downstream from the basin was less as the wing wall was brought nearer the outlet center line. For tests T64 and T65 there was an opening between the stilling basin wall and the wing wall that was not noticed until the tests had been completed. Water passed through this opening and caused an erroneous scour pattern at this point on the outside of the



(*a*) PERPENDICULAR TO BASIN CENTER LINE (TEST T-66)



(*b*) 45° TO BASIN CENTER LINE (TEST T-64)



(*c*) PARALLEL TO BASIN CENTER LINE (TEST T-65)

FIG. 16.—EFFECT OF TRIANGULAR WING-WALL POSITION ON SCOUR PATTERN

basin wall. Contours believed to be erroneous have been omitted from Figs. 16(*b*) and 16(*c*).

These tests indicate that the wing wall may be extended parallel to the basin center line if field conditions make it necessary to do so although the boil height is considerably increased. The wing wall perpendicular to the stilling basin center line produces entirely satisfactory flow and scour conditions, but the best over-all conditions were obtained when the wing wall was set at an angle of 45° to the outlet center line.

(*h*) *Shape of Stilling Basin.*—The size of the stilling basin varies with the initial flow depth. A reduction in d_1 will decrease d_2, the length of the basin, the height of the side walls, and the depth of excavation necessary to secure the required tailwater depth. In addition, a larger percentage of the energy in the water as it enters the stilling basin will be dissipated. A saving in the over-all cost of the outlet will ordinarily be possible if a transition is placed between the culvert or flume and the stilling basin to accomplish this reduction in d_1. In cases where a transition is used, a diverging transition side wall can be extended to form the stilling basin walls, as is shown on a half plan of Fig. 1. The limiting basin side-wall divergence is unknown but is probably as large as can be used on the transition.

A few tests were made on a trapezoidal-shaped stilling basin as part of the culvert outlet series. The stilling basin was designed for flow conditions at its entrance and the width and spacing of the floor blocks were multiplied by the ratio b_2/b_1 to compensate for the increase in width of the stilling basin at their location. (The widths b_1 and b_2 are defined in Fig. 1.) All blocks had their axes parallel to the center line of the basin. Flow conditions in the downstream channel were somewhat improved through the use of the trapezoidal stilling basin, as a result of the lower velocity of exit from the basin and because the stream spreads out to fill the downstream channel and reduces the size of the eddies that are formed along the sides of the channel near the stilling basin.

(*i*) *Cutoff Wall.*—A cutoff wall is used at the end of the stilling basin to prevent scour from undermining the basin. The depth of the cutoff wall, therefore, must be greater than the maximum depth of scour at the end of the stilling basin.

Serious erosion near the end of the stilling basin is prevented by the end sill which deflects the jet upward as it leaves the basin. A ground roller under the jet brings material upstream and further aids in preventing erosion. During the tests of the recommended design the scour sometimes reached an elevation slightly below the floor of the stilling basin but never reached a depth at the end of the basin greater than the thickness of a floor slab that might be used. Therefore, a cutoff wall of only nominal depth need be used at the end of the stilling basin.

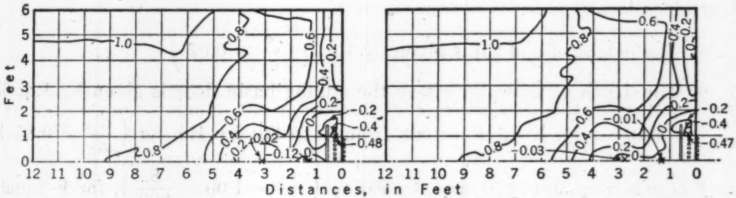
(*j*) *Effect of Air Entrainment.*—Air is ordinarily entrained by the water flowing in channels laid on a steep slope—resulting in a greatly increased depth of flow of the mixture. However, no air was naturally entrained during the model tests because of the low velocities or short length of test channel. Therefore, five runs were made in which from 10% to 117% of air was mixed with the water. The results of these tests and their parallel runs made without air entrainment are summarized in Table 6.

It was intended that the quantity of air mixed with water as it passed under the depth control gate (see Fig. 3) be computed from the equation:

$$Q_a = 0.2\, Q \sqrt{\mathbf{F}_{aw}} - Q \dots\dots\dots\dots\dots\dots\dots (8)$$

in which Q_a is the air flow at atmospheric pressure and Q is the water flow, both in cubic feet per second, and $\mathbf{F}_{aw}$ is the Froude number for the air-water mixture. Eq. 8 is an average of the equations derived by Warren DeLapp, Jun. ASCE (11*a*), from an analysis of the data published by L. Standish Hall, M. ASCE (11). However, an error was made in the solution of the equation and the actual quantities of air used are somewhat different from those intended. Nevertheless, the qualitative results give an indication of what may be expected when air is entrained with the water.

In designing the test basins, it was assumed that the velocity of the water would be unchanged by entrained air. The initial depth of flow was increased, therefore, to compensate for the larger total quantity of mixture flowing in order to keep the velocity constant. The result is a lowering of the Froude number for the flow of mixture below that obtained for water flow alone. The stilling basin was designed for the higher Froude number obtained when there was no air entrainment and the same basin was used for the test with air entrainment. In either case the tailwater depth is the same whether or not air entrainment occurs since the downstream velocities are low enough that the air separates from the water after leaving the stilling basin. No increase in side-wall height is necessary when the water transports entrained air. The results of a typical test are given in Fig. 17(*b*) where the percentage of air used was 117—that is, the air flow was greater than the water flow. The results presented in Fig. 17(*a*) were obtained on the same stilling basin when no air was entrained in the water.



(*a*) F = 264; R = 99000; $\frac{Q_a}{Q} = 0.00$ (TEST T-46) (*b*) F = 125; $\frac{Q_a}{Q} = 1.17$ (TEST T-47)

FIG. 17.—EFFECT OF ENTRAINED AIR ON SCOUR PATTERN

(Correction: In caption (*a*) R should be boldface, denoting the Reynolds number.)

These tests show that the effect of air entrainment can be neglected in the design of the SAF stilling basin. The resulting structure will safely handle any flow in which air is entrained.

7. SUMMARY

The following conclusions are reached as a result of the tests made to develop and verify the SAF stilling basin design.

1. The length of the stilling basin for Froude numbers between **F** equals 3 and **F** equals 300 is determined by Eq. 3 $\left(\frac{L_B}{d_2} = \frac{4.5}{\mathbf{F}^{0.38}}\right)$.

2. The height of the chute blocks and floor blocks is d_1 and the width and spacing are approximately $\frac{3}{4}\,d_1$.

3. The distance from the upstream end of the stilling basin to the floor blocks is $\frac{L_B}{3}$.

4. No floor block should be placed closer to the side wall than $\frac{3}{8}\,d_1$.

5. The floor blocks should be placed downstream from the openings between the chute blocks.

6. The floor blocks should occupy between 40% and 55% of the stilling basin width.

7. The widths and spacings of the floor blocks for diverging stilling basins should be increased in proportion to the increase in stilling basin width at the floor block location.

8. The height of end sill is given by Eq. 5 $\left(\frac{c}{d_2} = 0.07\right)$.

9. The depth of tailwater above the stilling basin floor is given by Eq. 6*b* $\left(\frac{d'_2}{d_2} = 1.10 - \frac{\mathbf{F}}{120}\right)$, for **F** equals from 3 to 30; by Eq. 6*a* $\left(\frac{d'_2}{d_2} = 0.85\right)$, for **F** equals from 30 to 120; and, by Eq. 6*c* $\left(\frac{d'_2}{d_2} = 1.00 - \frac{\mathbf{F}}{800}\right)$, for **F** equals from 120 to 300.

10. The height of the side wall above the maximum tailwater depth to be expected during the life of the structure is given by Eq. 7 $\left(z = \frac{d_2}{3}\right)$.

11. Wing walls should be equal in height to the stilling basin side walls. The top of the wing wall should have a slope of 1 on 1.

12. The wing wall should be placed at an angle of 45° to the outlet center line.

13. The stilling basin side walls may be parallel (rectangular stilling basin) or they may diverge as an extension of the transition side walls (trapezoidal stilling basin).

14. A cutoff wall of nominal depth should be used at the end of the stilling basin.

15. The effect of entrained air should be neglected in the design of the stilling basin.

8. REMARKS

The safety factor incorporated in the design of the SAF stilling basin is low but it is felt to be adequate. As more exact methods of analysis are applied to the design of a structure, it is possible to reduce the safety factor, with increased assurance that the structure will perform as intended by the designer. The SAF stilling basin is based on a large number of tests and knowledge of its performance is such that the safety factor can be safely reduced to a minimum. In addition, the tests show that, although a discharge somewhat higher than the design discharge might wash the roller out of the basin, this action would do little or no damage. A lowering of the tailwater somewhat below the design level would have the same effect and result.

Throughout the tests it was observed that the performance of the SAF stilling basin was excellent at discharges less than the design value. For the design conditions, the SAF stilling basin provides an economical method of dissipating energy and preventing dangerous and uncontrolled stream bed erosion. The hydraulic design of the SAF stilling basin is simple; only the depth and velocity of the flow approaching the basin are needed before entering a design chart (12) or solving the design equations.

9. ACKNOWLEDGMENTS

The work covered in this paper was performed at the St. Anthony Falls Hydraulic Laboratory of the University of Minnesota where the Office of Research of the SCS and the Minnesota Agricultural Experiment Station are cooperating in the solution of problems pertinent to the activities of the SCS. The work was done under the administrative direction of M. L. Nichols, chief, Office of Research, and under the technical direction of C. E. Ramser, M. ASCE, research specialist in hydrology. The tests were largely performed by Charles A. Donnelly, senior engineering aide, under the direction of the writer. Albert N. Huff was project supervisor at Minneapolis during these investigations. From him and from Donald A. Parsons much constructive criticism was forthcoming. The writer also wishes to acknowledge the most helpful, constructive criticism offered by the anonymous (to the writer), expert reviewers to whom the paper was submitted by the Society's Committee on Publications.

APPENDIX I. BIBLIOGRAPHY

---

(1) **"Study of Stilling-Basin Design,"** by C. Maxwell Stanley, *Transactions*, ASCE, Vol. 99, 1934, pp. 490–523. (*a*) p. 513.

(2) **"Hydraulic Laboratory Practice,"** edited by John R. Freeman, A.S.M.E., New York, N. Y., 1929. (*a*) p. 555.

(3) **"Hydraulic Structures,"** by Armin Schoklitsch, translated by Samuel Shulits, A.S.M.E., New York, N. Y., 1937, Vol. II. (*a*) p. 919.

(4) **"Spillways and Energy Dissipators,"** by Jacob E. Warnock, *Proceedings of Hydraulics Conference*, Studies in Engineering, *Bulletin No. 20*, Univ. of Iowa, Iowa City, 1940, p. 142. (*a*) p. 158. (*b*) p. 159.

(5) **"Devices for Measuring Rates and Amounts of Runoff,"** by L. L. Harrold and D. B. Krimgold, *Bulletin SCS-TP-51*, U.S.D.A., Washington, D. C., July, 1943 (multilithed).

(6) **"Applied Hydro- and Aeromechanics,"** by L. Prandtl and O. G. Tietjens, translated by J. P. Den Hartog, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1934. (*a*) p. 230.

(7) **"Tests of a Standard Culvert Outlet for Use with Drop Inlet Culverts,"** by Fred W. Blaisdell, SCS, U.S.D.A., Washington, D. C., October, 1941 (mimeographed).

(8) **"The Use of Sand Beds for Comparing Relative Stilling Basin Performance,"** by Fred W. Blaisdell, *Transactions*, Am. Geophysical Union, Vol. 23, 1942, Pt. II, pp. 633–638.

(9) **"Discharge Coefficients of Square-Edged Orifices for Measuring the Flow of Air,"** by H. S. Bean, E. Buckingham, and P. S. Murphy, *Journal of Research*, National Bureau of Standards, Vol. 2, March, 1929, p. 561.

(10) **"Handbook of Hydraulics,"** by H. W. King, McGraw-Hill Book Co., Inc., New York, N. Y., 2d Ed., 1929. (*a*) p. 334.

(11) **"Open Channel Flow at High Velocities,"** by L. Standish Hall, *Transactions*, ASCE, Vol. 108, 1943, p. 1394. (*a*) p. 1448.

(12) **"The SAF Stilling Basin,"** by Fred W. Blaisdell, SCS, U.S.D.A., in cooperation with the Minnesota Agricultural Experiment Station, Minneapolis, Minn., December, 1943 (mimeographed).

(13) **"Model Study of Green Mountain Dam Spillway,"** by J. H. Douma, *Civil Engineering*, March, 1940, pp. 153–156.

(14) **"Design of a Flume Outlet,"** by Albert N. Huff, SCS, U.S.D.A., Washington, D. C., October, 1941 (dittoed).

(15) **"The Hydraulic Jump in Terms of Dynamic Similarity,"** by B. A. Bakhmeteff and A. E. Matzke, *Transactions*, ASCE, Vol. 101, 1936, p. 630. (*a*) p. 676. (*b*) p. 643. (*c*) p. 660. (*d*) p. 643, Fig. 6, curve $\frac{L}{d_2}$. (*e*) p. 638, Fig. 3, ϵ_j. (*f*) p. 643, Fig. 6.

(16) **"The Study of Energy Dissipators with the Aid of a Small-Scale Model,"** by L. Escande, *Le Génie Civil*, December 16, 1939, pp. 429–433 (French) (translated by U. S. Army, Office of Chf. of Engrs., Washington, D. C.).

(17) **"Use of Models at Fort Peck Dam,"** by Gail A. Hathaway, *Civil Engineering*, November, 1936, pp. 741–745.

(18) **"Protection Against Scour Below Overfall Dams,"** by E. W. Lane and W. F. Bingham, *Engineering News-Record*, March 14, 1935, pp. 373–378.

(19) **"Model Tests Aid Calaveras Spillway Stilling Basin Reconstruction,"** by C. A. Lauenstein, *Western Construction News*, October, 1939, p. 327.

(20) **"Energy Loss at the Base of a Free Overfall,"** by Walter L. Moore, *Transactions*, ASCE, Vol. 108, 1943, pp. 1343–1392.

(21) **"Hydraulic Tests on the Spillway of the Madden Dam,"** by Richard R. Randolph, Jr., *ibid.*, Vol. 103, 1938, pp. 1080–1132.

(22) **"The Hydraulic Jump as a Means of Dissipating Energy,"** by Ross M. Riegel and John C. Beebe, *Technical Reports*, Miami Conservancy Dist., Pt. III, Dayton, Ohio, 1917, pp. 63–118.

(23) **"Jet Deflectors for High-Dam Outlet Conduits,"** by Harold A. Thomas and Wallis S. Hamilton, *Civil Engineering*, May, 1939, pp. 297–300.

(24) **"Experiments Aid in Design at Grand Coulee,"** by Jacob E. Warnock, *ibid.*, November, 1936, pp. 737–741.

(25) **"Scour-Control and Scour-Resistant Design for Hydraulic Structures,"** by Brooks T. Morris, *Transactions*, Am. Geophysical Union, Vol. 23, 1942, Pt. I, p. 60 (with bibliography).

(26) **"Kolkabwehr und Stauraumverlandung,"** by Armin Schoklitsch, Julius Springer, Berlin, 1935, pp. 78–173; "Prevention of Scour and Energy Dissipation," translated by Edward F. Wilsey, USBR, 1935.

(27) **"Hydraulics of Open Channels,"** by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., New York, N. Y., 1932, p. 241. (*a*) Chapter 18.

(28) **"Aerodynamics of the Airplane,"** by Clark B. Millikan, John Wiley & Sons, Inc., New York, N. Y., 1941, pp. 3–8.

(29) **"Principles of Naval Architecture,"** edited by Henry E. Rossell and Lawrence B. Chapman, Soc. of Naval Architects and Marine Engrs., Vol. 2, 1939, pp. 52–59.

(30) **"Fluid Mechanics for Hydraulic Engineers,"** by Hunter Rouse, Eng. Societies Monographs, 1938, Fig. 109. (*a*) p. 230.

(31) **"Engineering for Dams,"** by William P. Creager, Joel D. Justin, and Julian Hinds, John Wiley & Sons, Inc., New York, N. Y., 1944, Vol. 1, p. 84.

(32) **"Aerodynamics of the Perisphere and Trylon at World's Fair,"** by Alexander Klemin, Everett B. Schaefer, and J. G. Beerer, Jr., *Transactions*, ASCE, Vol. 104, 1939, p. 1449.

(33) **"Modern Conceptions of the Mechanics of Fluid Turbulence,"** by Hunter Rouse, *ibid.*, Vol. 102, 1937, p. 468.

(34) **"Baffle-Pier Experiments on Models of Pit River Dams,"** by I. C. Steele and R. A. Monroe, *Transactions*, ASCE, Vol. 93, 1929, pp. 451–546. (*a*) p. 518 (*b*) p. 499.

(35) **"Model and Prototype Studies on Unique Spillway: Part III of Symposium on Fontana Dam Spillway,"** by A. J. Peterka, *Civil Engineering*, June, 1946, p. 250.

(36) **"Determining the Energy Lost in the Hydraulic Jump,"** by J. C. Stevens, *Engineering News-Record*, June 4, 1925, p. 928–929.

(37) *Ibid.*, May 20, 1943, p. 58.

(38) **"Flow of Water in Steep Chutes with Special Reference to Self-Aeration,"** by R. Ehrenberger, translation by U. S. Bureau of Reclamation, Denver, Colo., 1926.

(39) **"Treatise on Hydraulics,"** by Mansfield Merriman, John Wiley & Sons, Inc., New York, N. Y., 1912, p. 350.

(40) *Encyclopædia Britannica*, New York, N. Y., 11th. Ed., 1910, Vol. XIV, pp. 76–77.

(41) **"Hydraulic Models,"** *Manual of Engineering Practice No.* 25, ASCE, 1942. (*a*) p. 14, Eq. 40.

(42) **"The High Velocity Flow of Water in a Small Rectangular Channel,"** by Warren W. DeLapp, thesis presented to the University of Minnesota, at Minneapolis, in June, 1947, in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Hydraulic Engineering.

(43) **"Flow through Diverging Open Channel Transitions at Supercritical Velocities,"** by Fred W. Blaisdell, SCS, U.S.D.A., June, 1947 (mimeographed).

APPENDIX II. NOTATION

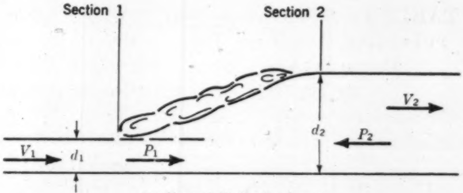
The following letter symbols, adopted for use in this paper and for the guidance of discussers, conform essentially to American Standard Letter Symbols for Hydraulics (ASA—Z10.2—1942), prepared by a Committee of the American Standards Association, with Society representation and approved by the Association in 1942. In addition, the symbols of linear concepts are defined by Fig. 1.

b = stilling basin width:
  b_1 = at the entrance;
  b_2 = at the floor block location;
  b_3 = at the exit;
c = height of end sill;
d = flow depth:
  d_1 = at the stilling basin entrance (lower conjugate depth for the hydraulic jump);
  d_2 = upper conjugate depth for the hydraulic jump (see Eq. 2*b*);
  d'_2 = tailwater depth, measured above the stilling basin floor;
  d''_2 = tailwater depth at which the hydraulic jump was washed out of the stilling basin;
$\mathbf{F}$ = the Froude number (Eq. 1*a*); $\mathbf{F}_{aw}$ = the Froude number when air is entrained;
g = acceleration due to the effect of gravity;
L_B = length of the stilling basin;
L_r = assumed linear scale ratio between prototype and model;
Q = rate of flow of water; Q_a = rate of flow of air;
$\mathbf{R}$ = the Reynolds number (Eq. 1*b*); $\mathbf{R}_m$ = the Reynolds number for the model;
V_1 = velocity of flow at stilling basin entrance;
z = height of stilling basin side walls above maximum tailwater level:
  z_a = average maximum boil height above tailwater level;
  z_m = maximum boil height in stilling basin above tailwater level; and
ν = kinematic viscosity.

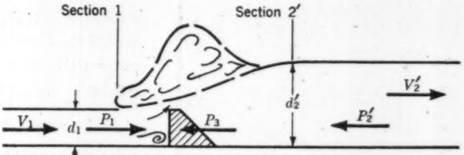
DISCUSSION

Donald E. Blotcky,[3] Assoc. M. ASCE.—Criteria for design of the Saint Anthony Falls (SAF) stilling basin (Minneapolis, Minn.) are based on scour conditions considered satisfactory in the movable bed model. Because of the limited knowledge of the mechanics of energy dissipation, empirical equations must be used. However, it is desirable to submit such equations to a rational analysis, thus striving toward the goal of explaining all hydraulic phenomena by rational methods.

It is the writer's belief that use of the theoretical equation for the hydraulic jump (Eqs. 2) is undesirable where formation of the jump is aided by blocks and sills.



(a) Normal Hydraulic Jump



(b) Hydraulic Jump Complicated by Floor Blocks

Fig. 18.—Velocity, Depth, and Pressure Relationships in Hydraulic Jumps

Energy dissipation can be divided into two categories as far as stilling basin action is concerned: First, that in which the energy is dissipated due to impact of water on water; and, second, that in which energy

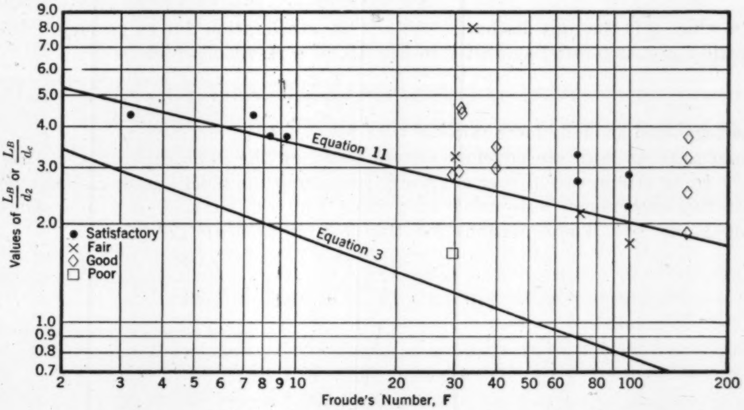


Fig. 19.—Dimensionless Plot of Stilling Basin Performance for Comparison of Eqs. 3 and 11

is dissipated due to impact of water on a submerged solid object and the impact of water on water.

---

[3] Office Engr., T. C. Forrest, Jr., Cons. Engr., Dallas, Tex.

TABLE 9.—Summary of Computations Based on Test Data as Applied to Eq. 10

Test No. (1)	L_B (ft) (2)	d_c (ft) (3)	$\frac{L_B}{d_c}$ (4)	F (5)
C57...	8.50	1.96	4.34	7.52
C75...	15.33	4.07	3.76	8.45
C76...	5.00	1.56	3.20	30.1
C3.....	26.25	3.23	8.13	33.6
C16...	14.00	3.12	4.51	30.5
C28...	8.75	3.08	2.84	29.2
C37...	5.00	3.09	1.62	29.6
F52....	14.02	3.25	4.33	31.0
F51....	14.21	4.10	3.47	40.0
F49....	16.33	4.95	3.30	70.0
F53....	13.61		2.75	
F44....	10.88		2.20	
F48....	16.38	5.67	2.89	100.0
F54....	13.10		2.30	
F45....	9.83		1.70	
F46....	24.22	6.40	3.79	150.0
F47....	20.18		3.20	
F50....	16.15		2.52	
F56....	12.11		1.90	

In the first case, exemplified by the hydraulic jump, the degree of energy loss is well known (27). In the analytical solution of the hydraulic jump (Fig. 18(*a*)), the phenomenon starts at the toe of the jump, at Section 1, and has a theoretical end of jump at Section 2, which is located at the point of maximum d_2. It should be noted that the jump acts from Section 1 to Section 2 without interference. Eqs. 2 relates the various parts of the jump as shown in Fig. 18(*a*). The second case (Fig. 18(*b*)) is usually a hydraulic jump with floor blocks added, and will hereinafter be called a "block jump" as distinct from a normal hydraulic jump. The addition of blocks, which cause additional unknown energy losses, invalidates Eqs. 2. In place of d_2, the writer suggests that d_c, the critical depth, be used in presenting the design equation for the length of basin. Since the following equalities are true,

$$f(d_c) = \psi(d_2) = \alpha(d_1) = \theta(\mathbf{F}) \quad \text{....(9)}$$

any one of the equal functions can be substituted and the result will be changed in form only. It would, perhaps, be even more desirable to present the equation for L_B as a function of **F** and d'_2. In Table 9, values of d_c are given for various runs as computed from

$$d_c = \sqrt[3]{q^2/g} \quad \text{..........(10)}$$

in which q is the unit discharge and g is the acceleration due to gravity. The ratio L_B/d_c is plotted against **F** in Fig. 19, as a result of which

$$L_B/d_c = \frac{6.2}{\mathbf{F}^{0.25}} \quad \text{..........(11)}$$

is suggested as being more rational than Eq. 3. It is, however, no more exact, inasmuch as both are envelope curves based on the same experimental data.

In an attempt to further analyze the action of the block jump, various runs of the experiment were used as a basis for computation of values shown in Table 10. The equations for the computations are (27*a*):

$$\epsilon_j = \frac{(d_2 - d_1)^3}{4\, d_2\, d_1} \quad \text{..........(12)}$$

$$\epsilon_1 = d_1 \left(1 + \frac{\mathbf{F}_1}{2} \right) \quad \text{..........(13)}$$

$$\epsilon'_2 = d'_2 + \frac{q^2}{2\, g\, (d'_2)^2} \quad \text{..........(14)}$$

$$\epsilon_2 = (\epsilon_1 - \epsilon_j) \quad \text{..........(15)}$$

$$\epsilon_{bj} = (\epsilon_1 - \epsilon'_2) \quad \text{..........(16)}$$

$$\epsilon_b = (\epsilon_{bj} - \epsilon_j) \quad \text{..........(17)}$$

In these equations, ϵ_j, ϵ_{bj}, and ϵ_b represent energy loss due to the hydraulic jump, the block jump, and the blocks alone, respectively; d'_2 is the depth of flow at section $2'$; ϵ_1, ϵ_2, and ϵ'_2 represent the specific energy at section 1, section 2, and section $2'$, respectively; and $\mathbf{F}_1$ is the Froude number at section 1.

TABLE 10.—SUMMARY OF BLOCK JUMP ANALYSIS

Test No. (1)	ϵ_j (2)	ϵ_1 (3)	ϵ'_2 (4)	ϵ_2 (5)	ϵ_b (6)	$\frac{\epsilon'_2}{\epsilon_1}$ (7)	$\frac{d'_2}{d_1}$ (8)	$\frac{\epsilon_b}{\epsilon_j}$ (9)	**F** (10)
C57	1.03	4.76	3.52	3.73	0.21	0.74	3.15	0.20	7.52
C75	2.53	10.44	6.93	7.91	0.98	0.66	3.00	0.39	8.45
C76	4.25	8.02	3.31	3.77	0.46	0.41	6.24	0.11	30.1
C3	9.80	17.80	6.64	8.00	1.36	0.37	6.21	0.14	33.6
C16	8.62	16.25	6.50	7.63	1.13	0.40	6.09	0.13	30.5
C28	8.14	15.60	6.47	7.46	0.99	0.41	6.07	0.12	29.2
C37	8.28	15.80	6.47	7.52	1.05	0.41	6.06	0.13	29.6
F52	9.12	17.00	6.88	7.88	1.00	0.40	6.28	0.11	31
F51	14.72	25.2	9.10	10.48	1.38	0.36	7.19	0.08	40
F49	29.26	43.2	11.91	13.94	2.03	0.28	9.54	0.07	70
F48	44.49	61.2	14.37	16.70	2.33	0.23	11.60	0.05	100
F46	69.16	91.2	17.60	22.10	4.50	0.19	14.30	0.06	150
T37	1.157	1.66	0.441	0.50	0.059	0.27	10.68	0.05	80.8
T42	2.405	3.340	0.801	0.93	0.129	0.24	11.06	0.05	93.7

The values in Col. 8, Table 10, when plotted (Fig. 20) against the corresponding Froude numbers, result in the equation

$$\frac{d'_2}{d_1} = 1.05\,\mathbf{F}^{0.522} \quad \ldots\ldots\ldots\ldots\ldots (18)$$

which is comparable to Eqs. 6 and is in a more desirable form.

In the hydraulic jump, $\mathbf{F} = 3$ is the theoretical transition point between the direct jump and the undulated jump. It has been noted (15*a*) that observed values of d_2 within the undulated jump zone are in excess of those computed by Eq. 2. It has also been stated (15*b*) that the actual transition point between the direct jump and the undulated jump is at about $\mathbf{F} = 4$. It is interesting to note that the undular jump also occurs in the block jump. This is evident because the ratio d'_2/d_2 is greater than 1 for small values of **F**, and has occurred for values of **F** as high as 15. For this reason Eq. 18 should not be used for values of **F** less than 15. Further experimentation is necessary to locate the probable range of values for the undulated jump in the block jump.

TABLE 10.—(*Continued*)

Test No. (1)	$\frac{d'_2}{d_1}$ (8)	**F** (10)
T39	15.8	183.1
T46	19.15	274
T44	14.7	161
T35	17.95	247
T53	19.95	288
T50	12.82	121
T17	19.05	261
T8	15.95	190
F103	2.83	5
F107	4.08	10
C63	2.28	3.25

Col. 6 of Table 10 gives the actual head loss caused by the blocks; Col. 9 gives the ratio of the efficiency of the blocks as compared to that of the hydraulic jump. It should be noted that the blocks are most effective, compared to the hydraulic jump, at low values of **F**.

Fig. 21 is a plot of the values of Cols. 7 and 8, Table 10, against **F**. Study of the manner in which the plotted points follow the curves for the hydraulic jump causes speculation as to the actual nature of the relationship between parts of the block jump. It is apparent that the ratios are comparable to those in a hydraulic jump and thus might be susceptible of an analytical solution based on change of momentum.

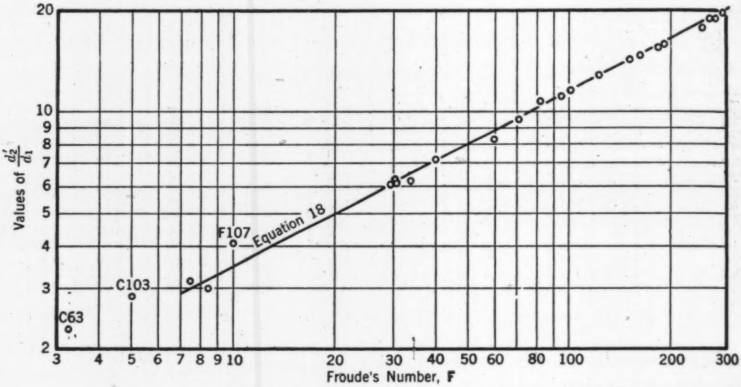


FIG. 20.—RELATION BETWEEN DEPTH RATIO AND FROUDE NUMBER FOR BLOCK JUMP

A schematic picture of the conditions obtaining in the block jump is given in Fig. 18(*b*). Applying the momentum theorem, P_3 is the only new force involved over that in the solution for the hydraulic jump. Change of momentum may then be expressed by

$$P'_2 + P_3 - P_1 = \frac{Q\,w}{g}\,(V_1 - V'_2) \ldots\ldots\ldots\ldots\ldots (19)$$

in which $P_1 = A_1\,w\,\bar{y}_1$; $P'_2 = A'_2\,w\,\bar{y}'_2$; A is the area at any section; w is the unit weight of water; $\bar{y}$ is the depth to the center of gravity at the section; and V'_2 is the velocity at section 2′. To determine P_3, recourse must be had to the analysis of forces acting on a flat plate perpendicular to flow and completely submerged in the fluid (see Fig. 22(*a*)). It can be shown by dimensional analysis (28)(29) that the force acting on the flat plate is

$$F = \theta(R)\,\rho\,V^2\,A_p \ldots\ldots\ldots\ldots\ldots\ldots\ldots (20a)$$

in which $\theta(R) = C_f\,\frac{1}{2}$; $\rho = \frac{w}{g}$; V is the velocity of approach; A_p is the area of the plate; and C_f is a coefficient of form.

Thus,

$$F = C_f\,w\,A_p\,\frac{V^2}{2\,g} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (20b)$$

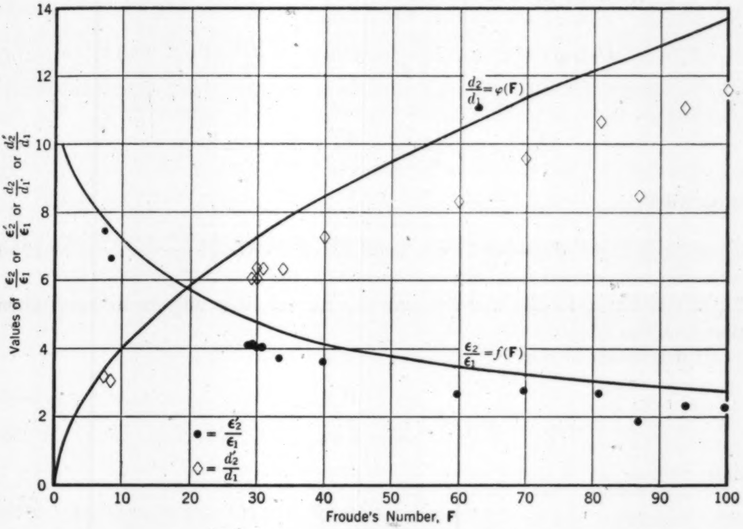
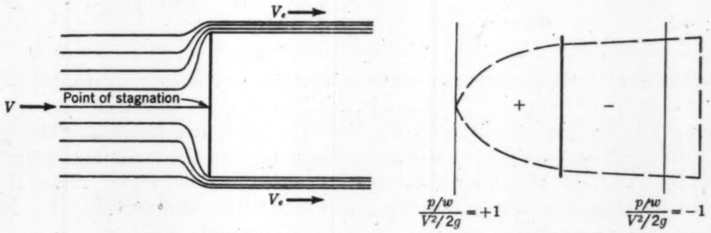
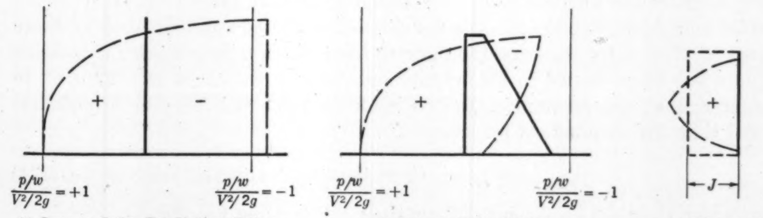


FIG. 21.—RELATION BETWEEN CALCULATED BLOCK JUMP ENERGIES AND DEPTHS AND THOSE OF NORMAL HYDRAULIC JUMP



V_e
V
Point of stagnation
V_e
+
−
$\frac{p/w}{V^2/2g} = +1$
$\frac{p/w}{V^2/2g} = -1$

(*a*) Velocities - Flat Plate (*b*) Pressure Bulbs - Flat Plate



+
−
$\frac{p/w}{V^2/2g} = +1$
$\frac{p/w}{V^2/2g} = -1$
+
−
$\frac{p/w}{V^2/2g} = +1$
$\frac{p/w}{V^2/2g} = -1$
+
J

(*c*) Pressure Bulbs - Flat Plate on Bottom (*d*) Pressure Bulbs - Blocks (*e*) Value of *J*

FIG. 22.—RELATIONSHIP OF QUANTITIES IN FLOW AROUND FLAT PLATES AND BLOCKS

If $F = P_3$ is substituted in Eq. 19,

$$A'_2 \, w \, \bar{y}'_2 + C_f \, w \, A_p \frac{V^2}{2 \, g} - A_1 \, w \, \bar{y}_1 = \frac{Q \, w}{g} (V_1 - V'_2) \ldots\ldots (21a)$$

or

$$\frac{(d'_2)^2}{2} + C_f \frac{A_p}{b} \frac{V^2}{2 \, g} - \frac{d^2_1}{2} = \frac{q}{g} (V_1 - V'_2) \ldots\ldots\ldots\ldots (21b)$$

from which

$$d'_2 = \sqrt{\frac{2 \, q}{g} (V_1 - V'_2) + d^2_1 - C_f \frac{A_p}{b} \frac{V^2}{g}} \ldots\ldots\ldots\ldots (21c)$$

Eq. 21*c* is dimensionally and dynamically correct; it is only in its application that troubles arise.

If it is assumed that

$$A_p = W \, d \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22a)$$

$$W = k \, b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22b)$$

and

$$V'_2 = q/d'_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22c)$$

in which W is the width of plate; d, the depth of plate; and k, the percentage of basin width covered by blocks; then Eq. 21*c* may be rewritten

$$d'_2 = \sqrt{\frac{2 \, q}{g}\left(V_1 - \frac{q}{d'_2} \right) + d^2_1 - C_f \frac{k \, V^2}{g} d} \ldots\ldots\ldots\ldots (23)$$

Eq. 23 is an implicit equation which cannot, by any simple means, be solved into an explicit function. For this reason Eq. 23 must be solved by trial and error. As an original value for d'_2, the value of d_2 can be used. The third solution will generally give the final value of d'_2. In the foregoing analysis it has been assumed that the resistance P_3 is caused by a flat plate. Since this is not true, it must be shown that the general conditions still prevail. Fig. 22(*b*) shows the approximate pressure bulbs on a flat plate under the conditions required by the analysis (30). By bringing the flat plate in contact with a solid boundary (30*a*) the pressure distribution is altered approximately as shown in Fig. 22(*c*). By changing the plate to the form of a floor block the pressure bulbs are again varied to the approximate shapes shown in Fig. 22(*d*). In analyzing C_f the assumption can be made that the skin friction is negligible and that, for all practical purposes,

$$C_f = C_d \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

in which C_d is the coefficient of form drag. Rewriting Eq. 20*b*,

$$\frac{F}{A_p} = p = C_d \, w \frac{V^2}{2 \, g} \ldots\ldots\ldots\ldots\ldots\ldots (25a)$$

or

$$\frac{p}{w} = C_d \frac{V^2}{2\,g} \qquad (25b)$$

It is noted that the coefficient C_d is actually a nondimensional number, shown to vary with **R**, which, when multiplied by the velocity head of stagnation, gives the pressure head caused by the stagnation of flow. Referring to Fig. 22(a), the pressure under ideal circumstances would be the pressure due to stagnation of flow and thus equal to $\frac{V^2}{2\,g}$ or $\frac{V^2_s}{2\,g}$. Since the streamlines of flow are approximately zero in width at the point of separation the velocity V_s must be greater than V, the velocity of approach. For this reason, the negative pressure bulb is greater than the positive pressure bulb; C_d can then be written as

$$C_d = J_+ + J_- \qquad (26)$$

in which J_+ is the coefficient in $J_+ \frac{V^2}{2\,g}$ to represent the plus pressure on the front of the plate and J_- is the coefficient in $J_- \frac{V^2}{2\,g}$ to represent the negative pressure on the back of the plate caused by the velocity V_s. The true physical nature of J is shown in Fig. 22(e), where it is shown to be the average height of the pressure bulb over the area of the plate.

In two-dimensional flow, J is one side of a plane rectangle; in three-dimensional flow it is the height of a rectangular solid. Thus, by measuring the pressure bulbs on the back and front of a block, the value of C_d is obtainable. Distinction should be made for the C_d of a single block and the C_d for a block group, since interference would cause a change in value. In studies pertaining to cavitation on baffle piers the coefficient J_- was determined to be equal to 0.68 for a single cubical pier (31).

As another approach to C_d, Eq. 23 can be rewritten as

$$C_d = \frac{\frac{2\,q}{g}\left(V_1 - \frac{q}{d'_2}\right) + d^2_1 - (d'_2)^2}{k\,d\,\frac{V^2}{g}} \qquad (27)$$

By assuming k equal to 0.5, d equal to d_1, and V equal to V_1, the value of C_d can be computed for various runs of the experiments. Included in the foregoing assumptions is the inherent fact that the chute blocks will not dissipate any appreciable amount of energy and thus are ignored. The computed values of C_d based on these assumptions are plotted against **R** in Fig. 23.

Up to about $\mathbf{R} = 10^6$, the value of C_d is practically constant, which indicates that C_d is independent of **R** for this range. However, one point (T58) with $\mathbf{R} = 1.45 \times 10^6$ and with $\mathbf{F} = 28.2$ is definitely below the values of the other runs. Two possibilities can explain the reason for this change. First, if the range of undulated jumps extends to as high a value of **F** as 28.2, then the observed d'_2 would be too high, and consequently the run could not be used in an equation based on change of momentum. Second, the value of $\mathbf{R}_{crit}$ has

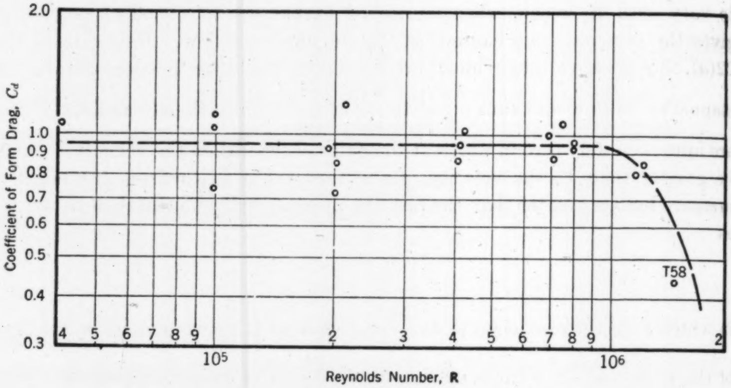


FIG. 23.—PLOT OF C_d VERSUS R FOR BLOCK JUMP

been reached, in which the boundary layer adjacent to the block has changed from laminar to turbulent flow. Whether a solid, streamlined as slightly as is the block, can exhibit this characteristic is as yet unknown. Other experimenters have made tests to determine C_d for a body similar to the floor block, but so far have given as a result only that C_d was constant and a value. No attempt was made to correlate the value of C_d to **R** and no mention is made of the range of **R** (32). It is entirely possible for the smooth blocks used in the experiments to exhibit this characteristic, but if so the value computed for C_d cannot be used in application to a prototype. In prototype flow it would be almost impossible for a laminar boundary layer to exist next to a surface as rough as a concrete block.

In a model controlled by the Froude number the value of **R** is different for the model and prototype. However, it has been stated that "* * * the Froude number refers to flow with a free surface, and actually has no relationship whatever with the inner mechanism of confined flow * * *" (33). Since the foregoing derivations are based on the premise that the blocks are completely submerged this quotation will hold and **R** will be the same, as regards C_d, for model and prototype. The confined flow is between a solid boundary and a streamline which has a pressure greater than zero and which is not affected by surface disturbances. If the submergence is not sufficient to satisfy these conditions the water would be deflected into the air and a free surface of flow would then affect the flow around the blocks.

The author has given aid to designers from two standpoints: (*a*) By presenting design equations the designer has at hand a reasonable method to use;

and (*b*) perhaps of more importance, the designer may use a standardized method so that prototype structures may be compared in action after construction.

M. M. Culp,[4] **Assoc. M. ASCE.**—Early gully control work involving the use of chutes and drop inlet spillways has demonstrated the need for some mechanical device to assist in the dissipation of hydrodynamic energy at the outlet end of such spillways. Erosion damage below several drop inlets and chutes led to field inspections and study of the problem. It seemed that the principal factors involved were:

1. Tailwater elevation;
2. Soil condition in the gully bottom and banks;
3. The stability of the channel grade below the spillway (which affects tailwater);
4. The magnitude, duration, and frequency of discharge through the spillway; and
5. Vegetative growth or other protective cover in the immediate vicinity of the outlet.

Where damage occurred it was obviously the result of eddies produced by a high velocity jet of water emerging from the spillway. Severe damage, requiring extensive repair, has occurred at the outlet of several spillways. For example, Fig. 24 shows the outlet of a twin 6-ft by 6-ft reinforced concrete drop inlet. The outlet apron had been undermined and so badly cracked that it had to be completely removed during repair operations.

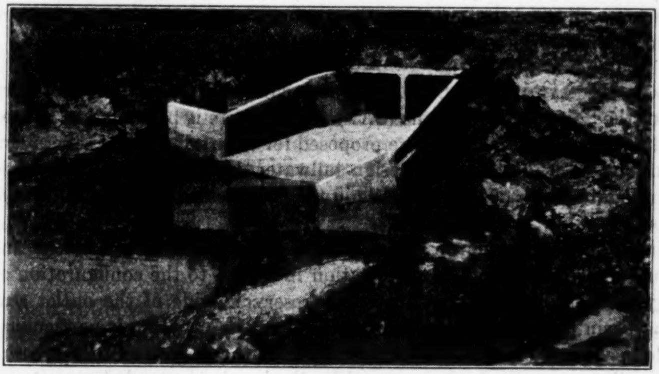


Fig. 24.—Failure of Apron on Twin 6-Ft by 6-Ft Drop Inlet Culvert Due to Undermining and Scouring

---

[4] Civ. Engr., Head, Design and Construction Section, Regional Eng. Div., SCS, U. S. Dept. of Agriculture, Milwaukee, Wis.

Several spillways constructed with Saint Anthony Falls (SAF) outlets during the past two years have not been subjected to a real test, as yet, and final conclusions cannot be drawn. However, several flows of less than design capacity have occurred, and the short experience to date indicates that the SAF outlet will function satisfactorily.

The designer of this type of outlet is confronted with the problem of constructing a stage-discharge curve for the natural channel below the spillway. Associated with this problem is the determination of the probable grade and cross section of a stable channel below the spillway. It is a well-recognized fact that natural channels tend to degrade when the bed load is removed by deposition in a reservoir. The problem is complicated further by variations in channel roughness with the stage, and by nonuniform flow conditions that may exist. The problem may be simplified, however, by the construction of a low drop spillway in the channel below the SAF outlet. Such a spillway provides a control from which backwater computations lead to reasonably correct tailwater elevations.

The writer sugggests that Mr. Blaisdell give consideration to the use of the word "chute," in lieu of "flume," to describe the high velocity open spillway shown in Fig. 3. This suggested nomenclature conforms to the definition given in the ASCE *Manual of Engineering Practice No. 11*, "Letter Symbols and Glossary for Hydraulics."

Mr. Blaisdell is to be complimented on a clear presentation of a long, tedious, and difficult program of hydraulic tests of a generalized nature. The writer is confident that the results of his work, as reported in the paper under discussion, will have continued and increasing use, especially in the field of upstream engineering.

Paul Baumann,[5] **M. ASCE.**—An abundance of information on the model tests of the Saint Anthony Falls (SAF) stilling basin is presented in this paper, and simple empirical formulas are proposed for determination of several critical dimensions. In addition, the design tailwater depth, d'_2, as compared to d_2, the conjugate depth of the hydraulic jump, is expressed in terms of the Froude number, **F**.

Since the primary purpose of a stilling basin is the dissipation of energy, it would have been helpful if information in regard to the configuration of the energy gradient between the upstream (reservoir) side of the outlet and the stilling basin proper had been presented. Such information is particularly desirable in view of the fact that the depth d_1 at the upper end of the basin and of the roller, respectively, is governed by the loss in energy between it and the reservoir surface (assuming the energy line to coincide with the reservoir surface for small velocity of approach). The coefficient ϕ to be applied to the factor $\sqrt{2\,g\,h}$ must be determined experimentally for each problem. Values of ϕ determined on models therefore furnish valuable information for those who must prepare stilling basin designs without the aid of model tests.

In general, it has been found that ϕ does not drop much below 0.9 for supercritical flow of water under rounded gates, although, for water discharging over ogee sections of considerable height, ϕ may drop to 0.5 and lower, depending on the smoothness of the surface, the curvature of the ogee section as well as that of the apron, and the transition between the crest and the stilling basin. Such a loss in head, and therefore in energy, is reflected in the depth d_1, which may be considerably in excess of the depth that would be established without such loss. The conjugate depth, d_2, derived from Eq. 2*a*, is necessarily affected correspondingly.

---

[5] Asst. Chf. Engr., Los Angeles County Flood Control Dist., Los Angeles, Calif.

It is an interesting fact in hydraulics that Eq. 2*a*, which is derived from the condition that the sum of the static and dynamic pressure above the jump equals the sum of the static and dynamic pressure below the jump, leads to a loss in energy without the application of any empirical factors. The validity of this equation has been confirmed on models as well as on prototypes. This equation may, with close approximation, be written

$$d_2 = 0.45 \left(v_1 \sqrt{\frac{d_1}{3.28}} - d_1 \right) \dots\dots\dots\dots\dots\dots (28)$$

In contradistinction, the equation for the depth d_1 must always include the empirical factor ϕ and must read

$$d^3_1 - d^2_1 H_0 + \frac{q^2}{\phi^2\, 2\, g} = 0 \dots\dots\dots\dots\dots\dots (29)$$

in which H_0 is the vertical distance from the floor of the stilling basin to the upstream energy line (reservoir surface); and q is the discharge per unit width of stilling basin. Since ϕ appears in the second power, it is readily conceivable that its influence on the value of d_1 is significant.

In discussing the paper, "Baffle-Pier Experiments on Models of Pit River Dams" (34), the writer (34*a*) attempted to determine the most efficient design depth (d'_2) based on the concept that the dissipation of energy (the transformation of kinetic into heat energy) is due to the roller and that the rise in temperature of the roller should be a maximum for the design depth d'_2. The equation for d'_2 so derived is, after simplification and correction,

$$(d'_2)^3 = \frac{q^2}{g} \left(2 - \frac{d'_2}{2\, H} \right) \dots\dots\dots\dots\dots\dots (30)$$

in which H is the vertical distance between the upstream energy line (reservoir surface) and the stilling basin water surface for design depth d'_2. In arriving at this equation, the empirical formula—

$$V_B = \alpha\, Q \sqrt{\frac{H}{g}} \dots\dots\dots\dots\dots\dots (31)$$

—was used for the volume of the roller. In Eq. 31, V_B is the volume of the roller; α is a coefficient; and Q is the total discharge ($= q\, b$).

The difference between the conjugate depth d_2 and the values for d'_2 resulting from Eq. 30 is demonstrated by Fig. 25, which indicates that d_2 is considerably in excess of d'_2. This is particularly noticeable with an increase in head. The values, however, would tend to become reconciled if d_2 were determined by the use of a proper coefficient ϕ with the value of d_1 as previously discussed.

To demonstrate the relation between values for d'_2 determined by Eq. 30 and by the author, respectively, Fig. 12, an important part of the paper, is

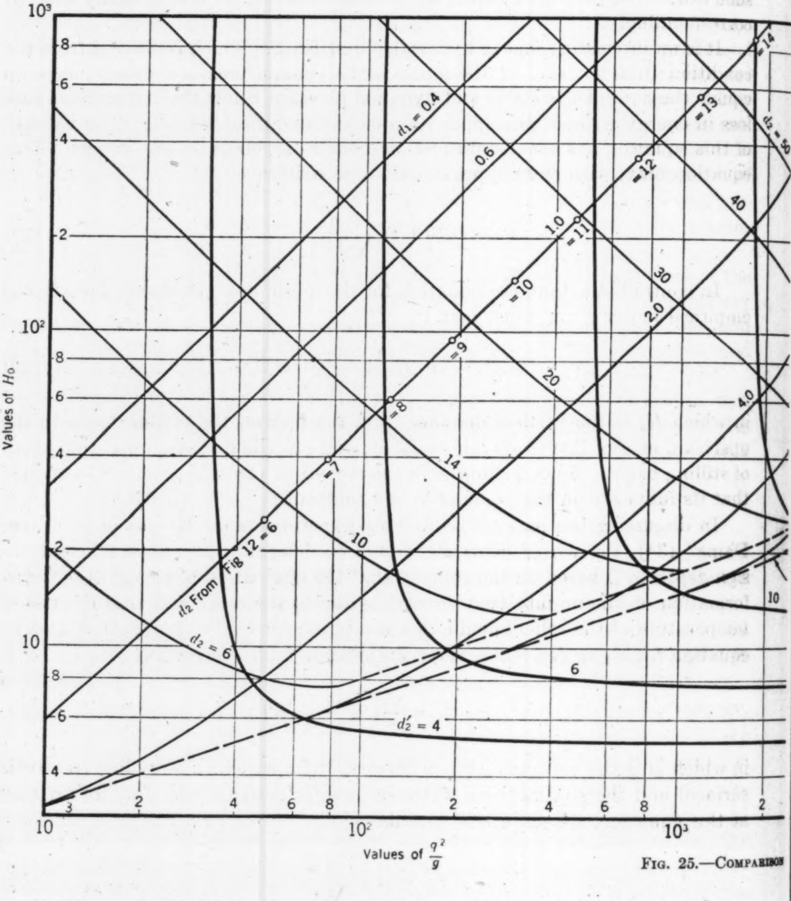


FIG. 25.—COMPARISON

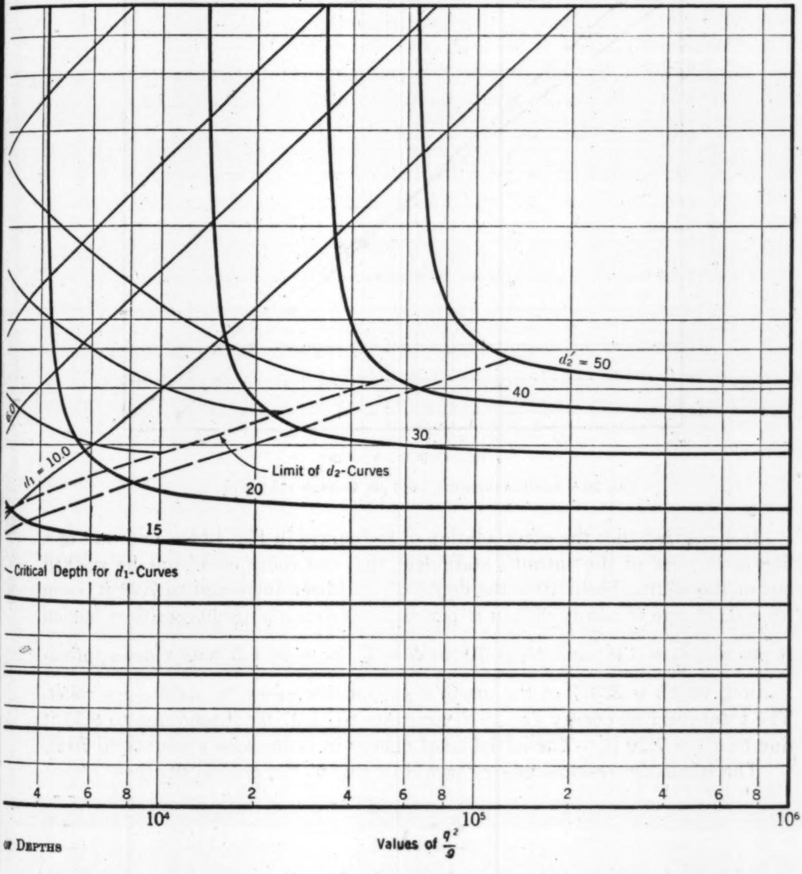
used. It was derived from the series of culvert outlet tests which are not shown in the paper.

For $d_1 = 1$ ft and $b_1 = 9$ ft, Eq. 29 may be rewritten

$$H_0 = 1 + \frac{q^2}{\phi^2 \, 2\, g} \quad \text{.......................(32)}$$

in which $q = \frac{Q}{b_1}$.

The curves shown in Fig. 12 plot as straight lines on semilogarithmic paper. They may, therefore, be extrapolated so as to cover depths d'_2 considerably in excess of 7 ft, as shown in Fig. 26. The two curves intersect and therefore have a common point, whose approximate coordinates are: $Q = 540$ cu ft per sec and $d'_2 = 7.9$ ft. The two curves would be expected to merge into one beyond this point. Hence, this point would coincide with a break in the curves. Only the curve through points at which the roller was washed out of the basin for increasing flow is used. Points for d'_2-values, as plotted in Fig. 25, naturally fall on the curve for $d_1 = 1$ ft for a unit value of ϕ. It may be noted that, despite the d'_2-values so determined being considerably at variance with the d'_2-values from Eq. 30 for small values of Q $\left(\text{that is, } \frac{q^2}{g} \text{ and } H_0\right)$, they ap-



of Depths

proach each other with an increase in Q and H_0, and coincide for $d'_2 = 13$ ft±. Below this point the model tests indicate that a greater design depth is necessary; above this point a design depth equal to or smaller than that resulting from Eq. 30 is required.

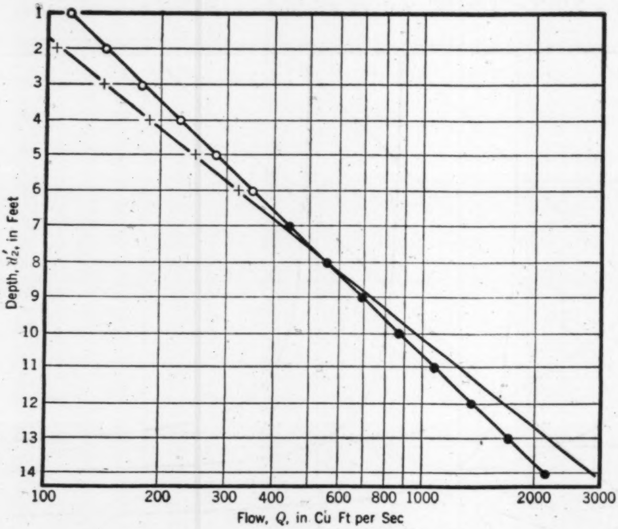


FIG. 26.—SEMILOGARITHMIC PLOT OF CURVES FROM FIG. 12

It is realized that the extrapolation of the curves in Fig. 12 is somewhat of a liberty in view of the author's statement that the roller could not be washed out of the stilling basin after the depth d'_2 had been increased to 7.20 ft, even for a discharge of nearly 600 cu ft per sec. For such a discharge ($Q = 598$ cu ft per sec; $\frac{q^2}{g} = 138$; and $H_0 = 70$ for $\phi = 1$), $d_2 = 16.1$ ft with close approximation, which is 223% of the depth d'_2 found necessary to maintain a roller. The total residual energy due to d_2 amounts to 16.37 ft; it amounts to 8.53 ft due to $d'_2 = 7.20$ ft. The initial total energy in both cases amounts to 70 ft.

The minimum residual energy would be due to the critical depth—

$$d_c = \sqrt[3]{\frac{q^2}{g}} \qquad (10)$$

—which establishes itself for $d'_2 = 2\,H$ in Eq. 30. However, the critical depth in the stilling basin is not compatible with its purpose, which is stilling action to minimize downstream scour.

A. J. PETERKA,[6] ASSOC. M. ASCE.—General studies of the type reported in this paper have been needed for some time. Designers in particular have seized every available scrap of information that would ease the problem of designing small stilling basins that will operate for the first time as prototype structures, or larger basins that will be submitted to a laboratory for refinement of features. Unfortunately, in most hydraulic laboratory investigations, it is not possible to concentrate on general solutions since funds are usually available only to solve the problems for a specific installation. Because of this situation only a limited amount of information is available for general solutions.

---

[6] Hydr. Engr., Bureau of Reclamation, U. S. Dept. of Interior, Denver, Colo.

The author is to be complimented for his approach to a problem that is difficult in many ways. His results, no doubt, will serve the purpose for which they were intended. However, there will be a tendency for designers to use this material for purposes beyond its range of reliability and the intent of this discussion is to indicate some of these limiting factors rather than to criticize, adversely, the excellent treatment given by the author.

The stilling basin operation, indicated by Fig. 6, appears to be greatly dependent on the chute blocks, baffle blocks, and end sill. It also appears that the configuration of the movable bed downstream from the apron had some effect in determining the length of the basin. Since at the start of each test the bed was molded above the elevation of the apron, there was probably some resistance offered to the flow that could favorably affect the performance of a short apron. Without a high bed, somewhat different action might take place, as the result of a different distribution of velocities.

The stilling basin profiles indicate that for L_B less than about 2 d_2 very little energy is dissipated on the apron. Thus, the basin becomes a distributor rather than a dissipator. The high-velocity flow entering the basin is deflected upward toward the surface, and the hydraulic jump acts merely to distribute the energy over a greater area with most of the dissipation occurring in the channel downstream from the apron. If the energy is not to be dissipated on the apron, the writer feels that it would be more economical to dispense with the apron accessories, consisting of chute blocks and baffle piers, and to adopt a curved deflector bucket to "throw" the water away from the structure. This method has been used successfully by others. A special application of this type of energy distributor was used successfully on the Fontana Dam spillway in the Tennessee Valley (35).

The advisability of relying primarily on such relatively fragile structures as chute blocks and baffle piers for satisfactory apron performance is questioned by the writer. If the prototype apron is submerged, as many of them are, underwater inspection and repair of blocks and piers are difficult. If the apron accessories should become damaged or lost, for any reason, the safety of the entire structure is endangered. As a precaution against undermining the apron, it might prove enlightening in future tests to determine the necessary depth of cutoff wall with some or all of the apron accessories removed. The resulting erosion pattern would also indicate the importance of keeping close watch on the condition of these same accessories if a deep cutoff wall is not used.

The main criterion for evaluating apron efficiency, during the author's tests, was the extent of the erosion in the model, and no mention of wave action below the apron is made. From Figs. 2(*a*) and 6, it appears that considerable wave action and resulting surface roughness occurred. The destructive effect of these waves is often as important as the scour of the channel bottom. If the basin discharged into an unlined canal or steep banked stream with soft or overhanging sides, some type of bank protection would be necessary. The prevalence of high surface velocities below the apron, if they existed, might also be of concern.

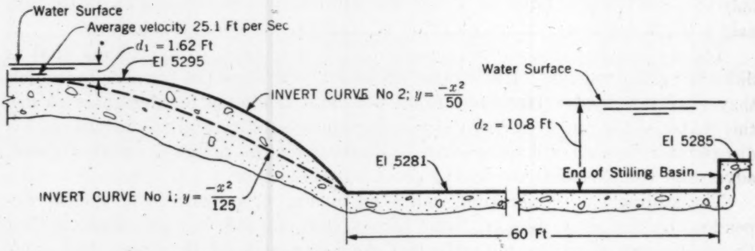

Water Surface
Average velocity 25.1 Ft per Sec
d_1 = 1.62 Ft
El 5295
INVERT CURVE No 2; $y = \frac{-x^2}{50}$
Water Surface
d_2 = 10.8 Ft
El 5285
End of Stilling Basin
El 5281
INVERT CURVE No 1; $y = \frac{-x^2}{125}$
60 Ft

FIG. 27.—STILLING BASIN PROFILE, HORSETOOTH OUTLET WORKS

No mention was made by the author of any changes in the transition parabola during the tests and it is assumed that the same parabola was used throughout. During hydraulic model studies for Horsetooth Dam outlet works, in Colorado (Fig. 27), made in the Hydraulic Laboratory of the Bureau of Reclamation under the writer's supervision, it was found that the shape of the parabola had considerable effect on the stability of the hydraulic jump. Steeper slopes at the point of entry into the basin resulted in improved basin efficiency, lower waves in the downstream canal, and more uniform velocity distribution at the end of the basin as shown in Fig. 28. Apron accessories were not used in this stilling basin.

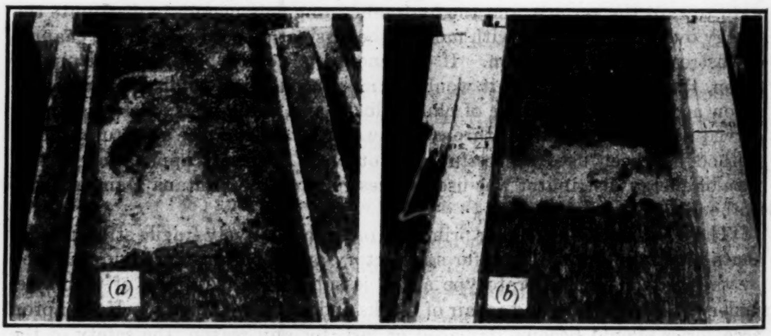


FIG. 28.—STILLING BASIN, HORSETOOTH OUTLET WORKS (SEE CURVES IN FIG. 27)

(a) Unsymmetrical Jump with Trajectory Curve No. 1
(b) Symmetrical Jump with Trajectory Curve No. 2

During tests, supervised by the writer, on the Enders Dam spillway, in Nebraska, which also makes use of a transition parabola (Fig. 29), the value of the proposed chute blocks, d_1 in height, was not readily apparent. Subsequent tests, and check tests, both with and without chute blocks indicated that, in this case at least, erosion was greater with the blocks in place. Removal of the blocks decreased the extent and depth of the erosion of the sand bed below the apron, as shown in Fig. 30. Furthermore, observations indicated no change in the appearance of the operation when the blocks were removed. Use of the blocks, however, did increase the amount that the tailwater elevation could be reduced without causing the jump to wash out of the apron. Finally, it was shown that the same range of useful tailwater could be produced by increasing the height of the dentated end sill from 9 ft to 12 ft, as shown in Fig. 29.

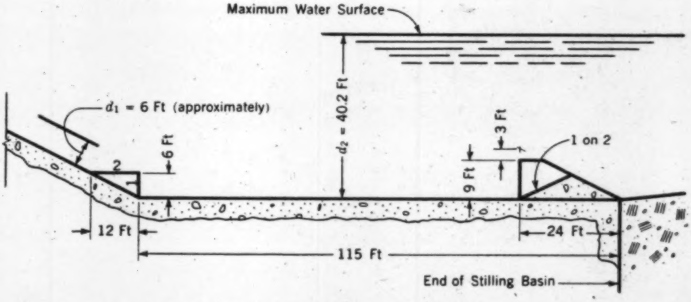


Fig. 29.—Stilling Basin Profile, Enders Spillway

The explanation of the action of these particular chute blocks, offered by the writer, is pure conjecture but may be of some value. Although the Enders spillway chute blocks were equal in height to d_1, as proposed by the author, they were not sufficiently tall. Because of the steep transition profile, the top length of the chute blocks was relatively short. Thus, the directional effect given to the water leaving the top surfaces was negligible, and the flow continued downward much as if the chute blocks were not in place. Had the stilling basin approach been flatter, the top length of the chute blocks would have been proportionately longer, giving greater directional effect to the water and inducing energy dissipating eddies to form. It appears that, if the transition profile is to be varied from one structure to another, the top length of the blocks is of greater importance than the block height.

The author's conclusions regarding the lateral spacing of the floor blocks are in good agreement with results of tests made by the writer on Kentucky Dam in the Tennessee Valley, and on some of the larger structures of the Bureau of Reclamation. A good over-all estimate for design purposes is 50%. Model tests of certain specific installations may show a need for increasing the block area more than 50% to provide flow stability on the apron, but this increase also produces a more concentrated boil. Reducing the block area substantially below 50% results in less turbulence but does not take full advantage of the possibilities.

The 45° wing wall shown by the author has also proved a satisfactory solution to a vexing problem. In most stilling basins the erosion below the downstream corners of the basin is somewhat greater than at any other location, because the downstream channel is wider than the stilling basin, creating a dead-water area at each side of the basin just beyond the end sill. The water leaving the end sill is directed upward, inducing a current from the dead-water area to

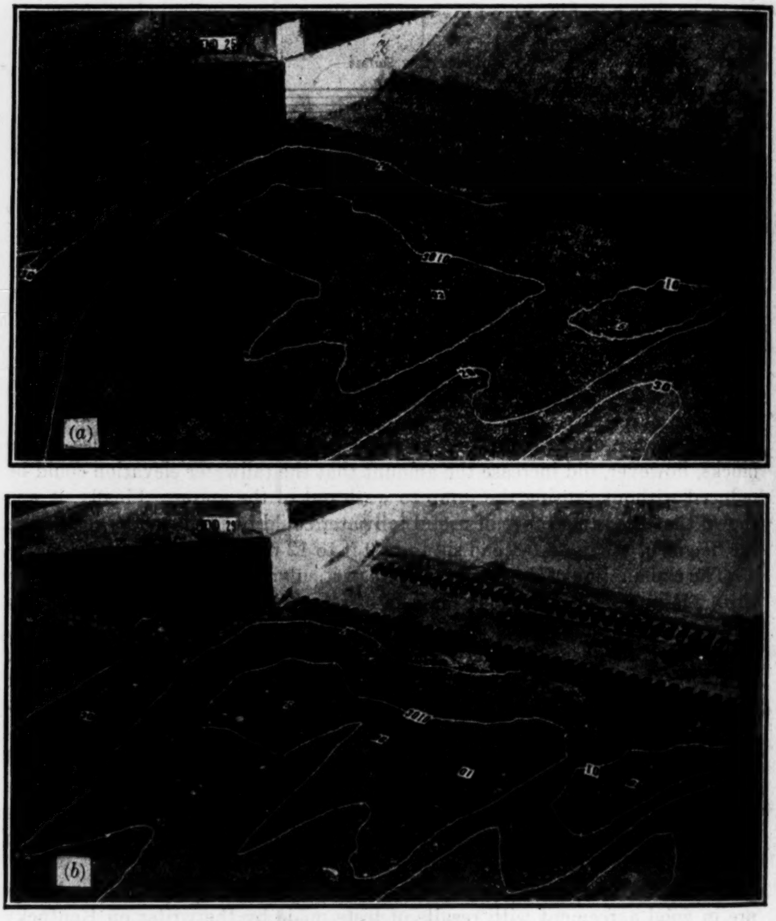


FIG. 30.—EFFECT OF CHUTE BLOCKS ON THE SCOURING BELOW THE SPILLWAY BASIN, ENDERS DAM (45° WING WALL)

(a) Blocks Removed
(b) Blocks Installed

flow laterally under the main flow as it leaves the end sill. This circulation results in excessive erosion at the apron corners even if surface velocities appear to be low. Addition of the 45° wing wall reduces the amount of circulation and the resulting erosion considerably, as shown by comparison of Figs. 30(*b*) and 31. If the dead-water area is large, or if it extends behind the wing wall, special treatment at the corners of the apron may be necessary.

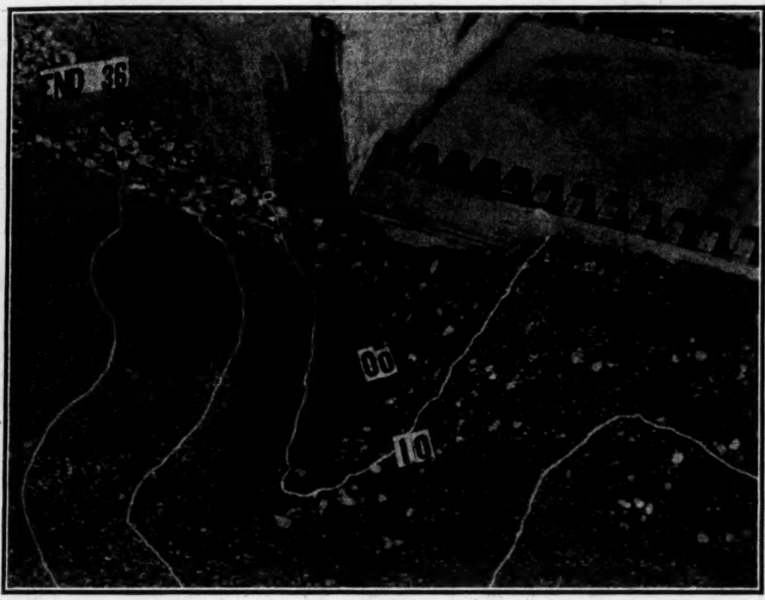


FIG. 31.—EROSION AT APRON CORNER WITH 90° WING WALL

Louis M. Laushey,[7] **Jun. ASCE.**—The results reported in this paper lead one step closer to the economical design of stilling basins. During 1946 and 1947 the writer conducted somewhat similar tests to determine the effects of a rectangular end sill on the hydraulic jump in a level, rectangular channel. Emphasis was concentrated on the tailwater depth required to form a jump, the proportioning of the height of sill, and the length of the basin required to accommodate the jump. It is interesting to compare the results of these tests with a rectangular end sill in a rectangular basin to those obtained on the Saint Anthony Falls (SAF) basin which uses both a sill and an array of blocks to dissipate the kinetic energy.

The tests on end sills of various heights were conducted in a level, steel flume 12 in. wide and 40 ft long. Water was circulated by a centrifugal pump, and was measured by a calibrated venturi meter. The depth of flow was con-

---

[7] Instr., Civ. Eng. Dept., Carnegie Inst. of Technology, Pittsburgh, Pa.

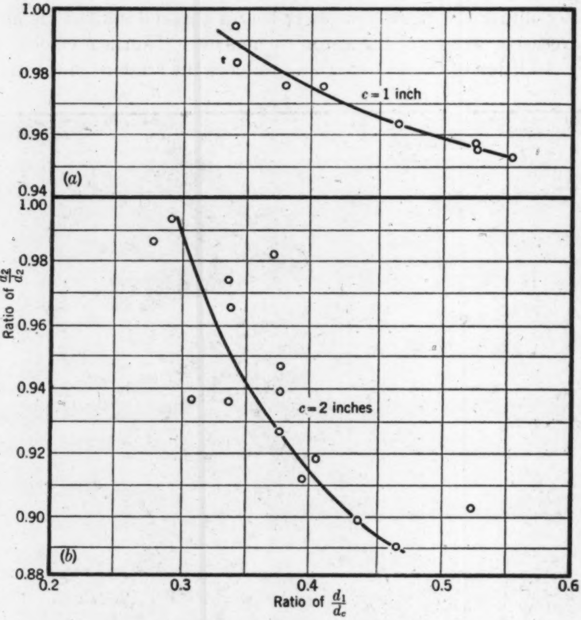


FIG. 32.—DIMENSIONLESS PLOT SHOWING TAILWATER DEPTH RATIOS AS A FUNCTION OF THE RATIO d_1/d_c FOR TYPICAL SILLS

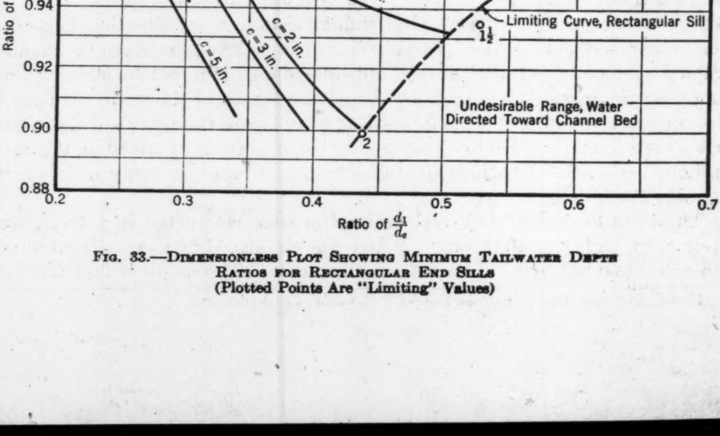


FIG. 33.—DIMENSIONLESS PLOT SHOWING MINIMUM TAILWATER DEPTH RATIOS FOR RECTANGULAR END SILLS (Plotted Points Are "Limiting" Values)

trolled at the upper end of the flume by raising or lowering a sluice gate hinged to a horizontal steel plate, 6 ft long, to guide the water parallel to the channel bottom. The jump was formed on the test section representing the apron, the downstream end of which was 8 ft from the outfall end of the flume. Sills were introduced at the end of the apron. Depths of flow were controlled by an adjustable tail gate at the end of the flume, and were measured in open manometers. Seven pressure openings were drilled at 2-ft intervals along the center line of the channel; four openings were on the apron, and three were downstream. The sills were made of $\frac{1}{4}$-in. steel plate, 12 in. long, with heights varying from $\frac{1}{2}$ in. to 5 in. The maximum dimensions of the flume in the reach below the jump were 12 in. wide, 20 in. deep, and 8 ft long.

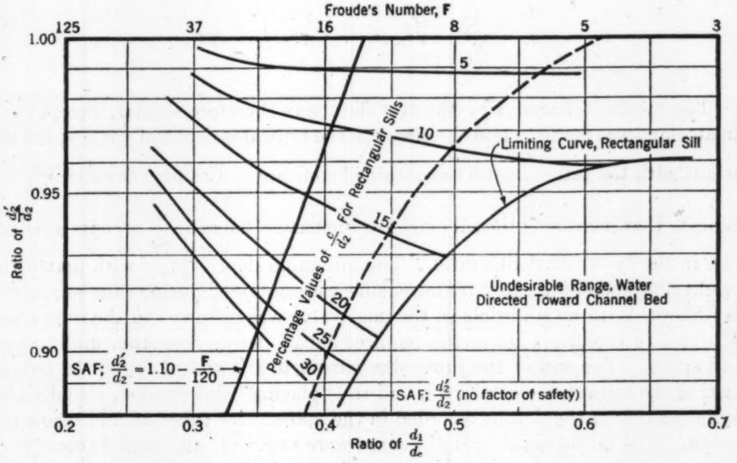


FIG. 34.—COMPARISON OF TAILWATER DEPTH RATIOS FOR THE SAF BASIN AND THE END SILL BASIN

Comparison of Tailwater Depths Required.—Results of the tests to determine the tailwater depth required for a jump in a basin with an end sill are shown in Figs. 32, 33, and 34. The curves in Fig. 32 are typical curves for sills 1 in. and 2 in. high, the points having been plotted to show the precision attained. The ordinates, values of $\frac{d'_2}{d_2}$, are the same ratios as those used in the SAF analysis, but the dimensionless ratio $\frac{d_1}{d_c}$ was chosen for the abscissa. This is the ratio of initial depth to critical depth before the jump, and it is helpful in visualizing the flow characteristics at the toe of the spillway. It is a function of the head on the dam and the height of the dam above the water surface on the apron. For example:

$$\frac{d_1}{d_c} = \frac{\dfrac{Q}{V_1}}{\dfrac{Q^{\frac{2}{3}}}{g^{\frac{1}{3}}}} = \frac{Q^{\frac{1}{3}}\, g^{\frac{1}{3}}}{V_1} = \frac{(32.2)^{\frac{1}{3}}\, C^{\frac{1}{3}}\, (H^{\frac{3}{2}})^{\frac{1}{3}}}{(64.4)^{\frac{1}{2}}\, (H + h)^{\frac{1}{2}}} = 0.396\, C^{\frac{1}{3}} \left(\frac{H}{h + H}\right)^{\frac{1}{2}} \quad \ldots\ldots(33)$$

In Eq. 33, C is the spillway coefficient of discharge; H is the head on the dam, in feet; and h is the height of crest above water surface on the apron, in feet. Eq. 33 has the following significance: (a) When $\frac{d_1}{d_c} = 1.0$, the flow is at critical depth and the height of the jump is zero; (b) when $\frac{d_1}{d_c} > 1.0$, flow is tranquil and no jump can form; and (c) when $\frac{d_1}{d_c}$ approaches zero, the height of the jump approaches infinity. However, it can be shown that $\frac{d_1}{d_c}$ is a function of the Froude number, which was used in the SAF tests, and is equal to:

$$\frac{d_1}{d_c} = \frac{d_1}{\frac{Q^{\frac{2}{3}}}{g^{\frac{1}{3}}}} = \frac{g^{\frac{1}{3}}\, d_1}{V^{\frac{2}{3}}{}_1\, d^{\frac{2}{3}}{}_1} = \left(\frac{g\, d_1}{V^2{}_1}\right)^{\frac{1}{3}} = \frac{1}{\mathbf{F}^{\frac{1}{3}}} \dots\dots\dots\dots\dots (34)$$

The method of analyzing the test data was therefore similar, except that an attempt was made to find the effect on the tailwater depth of varying the sill height with the ratio $\frac{d_1}{d_c}$ (or **F**) maintained constant. Results shown in Fig. 33 indicate that increasing the sill height or increasing the ratio $\frac{d_1}{d_c}$ permits a reduction in the tailwater depth ratio. The minimum depth ratios with maximum height sills, and depth ratios for lower sills, were obtained during runs conducted as follows: With no sill in place, the inlet valve was opened and the inlet plate adjusted to produce a jet of the required initial depth, free from undulations and spray. The end of the jump was moved to the end of the apron by adjusting the tailwater control gate, and the "placing" was checked visually and by inspection of the depths recorded in the manometer tubes on and below the apron. The initial depth and discharge were measured and used to compute a theoretical depth d_2 which had to check the observed depth at the end of the jump. Without changing the discharge and initial depth, sills of increasing height were placed, successively, at the end of the apron. The most accurate values of d'_2 were obtained by starting with too low a tailwater depth, and raising it slowly. When the water was too shallow, a frothy fringe of water formed just below the sill. This fringe moved upstream as the tailwater was raised, and disappeared when all depths downstream from the sill were the same. When the tail gate was raised too far, the jump moved upstream on the apron.

Fig. 34 shows the comparison of the limiting curve of tailwater depth ratios to that proposed for the SAF basin. The variable ratio $\frac{c}{d_2}$ is the parameter linking the main variables $\frac{d'_2}{d_2}$ and $\frac{d_1}{d_c}$, whereas this parameter has been made equal to 0.07 in the SAF basin. The ratios $\frac{c}{d_2 - d_1}$, $\frac{c}{d'_2 - d_1}$, $\frac{c}{\frac{V^2{}_1}{2g}}$, and $\frac{c}{d'_2}$ were also tried to reduce the scatter resulting in the plot, but without profit. To obtain a comparison, the actual test points (without a factor of safety) from the SAF data in Fig. 11 were plotted as well as Eq. 6*b*. Results are in close agreement for the two basins, although the height of the rectangular sill for a given value of **F** would produce variable results.

Comparison of Height of End Sill.—Fig. 34 shows that, as the Froude number is increased, sills larger than 0.07 d_2 can be used safely without blocks in a rectangular basin. From Table 4 of the SAF tests, best results were obtained at $\mathbf{F} = 40$ with $\frac{c}{d_2} = 0.15$. The ratio of $\frac{d'_2}{d_2}$ for this sill height would be about 0.97 in the end sill basin, which is much higher than the value of 0.85 recommended for the SAF basin. However, the difference might be explained by the effectiveness of the blocks at the higher Froude numbers.

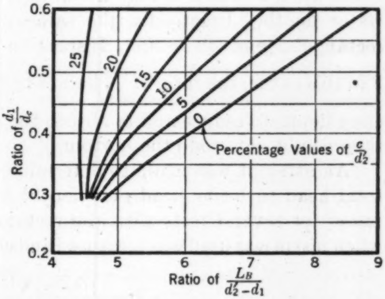


FIG. 35.—LENGTH OF JUMP TO HEIGHT OF JUMP RATIOS FOR RECTANGULAR END SILLS

The maximum height of end sill that could be used safely was obtained by increasing the height of sill, with the discharge and initial depth constant, until sufficient height was reached to make the sill begin to act as a control. The limit was reached when all depths downstream could not be made the same because of the drop down curve, which indicated water was directed toward the bottom.

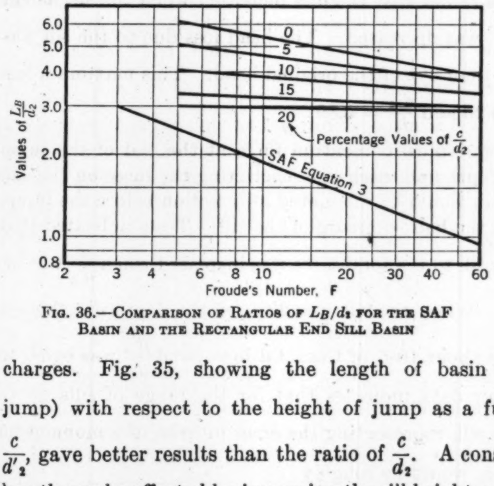


FIG. 36.—COMPARISON OF RATIOS OF L_B/d_2 FOR THE SAF BASIN AND THE RECTANGULAR END SILL BASIN

Comparison of Length of Basin.—The difficulty of expressing the basin length as a function of d_2 for all values of **F** was mentioned by the author when $L_B = 1.25$ d_2 gave results too short for small discharges and too long for higher discharges. Fig. 35, showing the length of basin (taken equal to length of jump) with respect to the height of jump as a function of $\frac{d_1}{d_c}$ (and **F**) and $\frac{c}{d'_2}$, gave better results than the ratio of $\frac{c}{d_2}$. A considerable reduction in basin length can be effected by increasing the sill height, using Figs. 35 and 36. This condition is especially true at the low Froude numbers where an almost 50%

reduction is possible because of the sill. Referring to Fig. 36, the SAF basin requires considerably less length, although with high rectangular end sills the lengths become more nearly the same. Floor and chute blocks therefore appear to be effective devices in reducing the length of jump.

Other Comparisons.—It was noted during the rectangular end sill tests that a boil formed over the sill for ratios of $\frac{d_1}{d_c} \leqq 0.32$ (equivalent to $\mathbf{F} \geqq 30$). This is interesting because in the SAF basin the roller was washed out of the stilling basin when $\mathbf{F}$ was less than about 30. The solution was to increase the tailwater depth. Practically the same procedure had to be followed for the rectangular end sill basin. Inspection of Fig. 34 shows that, with values of $\mathbf{F}$ in this range, the ratio of $\frac{d'_2}{d_2}$ had to be increased to almost unity to create sufficient depth to reduce and to absorb the downward velocity components so that they would not erode the bottom.

An attempt was made to determine the head loss created by the sill. The total head (velocity head plus depth) at the initial and final depths was computed for several tests with the rectangular sill. The head loss in the jump when no sill was used was measured and checked by the Stevens equation (36)(37),

$$i = \frac{6.25\,(\sqrt{1 + 16\,r} - 3)^3}{(1 + r)\,(\sqrt{1 + 16\,r} - 1)} \ldots\ldots\ldots\ldots\ldots\ldots (35)$$

in which i is the percentage of original head lost in the jump; and r is the ratio of velocity head to depth before the jump. The difference between the measured loss with a sill and that without a sill was expressed as a ratio of the original total head. It was found that the loss increased when the sill height was increased and when $\frac{d'_2}{d_2}$ was decreased. The head loss due to the sill was small, and was never more than 6% of the original head. This maximum loss was obtained when $\frac{c}{d'_2} = 0.25$ and $\frac{d'_2}{d_2} = 0.89$.

Calculation of the pressure-plus-momentum force at the end of the jump from the measured initial depth and discharge, neglecting the force on the sill, will give a smaller value than would be calculated at a section before the jump. The difference results from the back pressure of the sill. Tests indicated that the ratio of the force on the sill to the total force was increased when $\frac{c}{d'_2}$ was increased and $\frac{d'_2}{d_2}$ decreased. When the sill was 20% of the depth d'_2, the apparent force on the sill was about 10% of the total force, and $\frac{d'_2}{d_2}$ was equal to 0.90. Analysis of the other data indicates that for the range of sills up to $\frac{c}{d'_2} = 0.25$ the force on the sill, representing the error in using the momentum equation without correction, would be roughly

$$(P + M)_e = \left(1 - \frac{d'_2}{d_2}\right)(P + M)_1 \ldots\ldots\ldots\ldots\ldots (36)$$

in which $(P + M)_s$ is the force on the sill, in pounds; and $(P + M)_1$ is the pressure plus momentum force before the jump, in pounds.

The author is correct in stating that the SAF basin can be built shorter than the rectangular end sill type. For values of Froude numbers up to $\mathbf{F} = 30$, little difference seems to exist in the tailwater depth required for a jump. The indication is, however, that for the higher Froude numbers and the low tailwater depths, with a height of sill that would ordinarily be used, the rectangular basin would be inferior to one designed according to the SAF dimensions.

J. H. Douma,[8] Assoc. M. ASCE.—Although the detailed data and rules presented are for the design of stilling basins for small drop and spillway structures (used principally by the Soil Conservation Service of the United States Department of Agriculture in its soil and water conservation program), much of the author's discussion and some results of the Saint Anthony Falls (SAF) stilling basin tests are applicable to large stilling basins for flood-control and multiple-purpose projects.

In this discussion the author's data will be compared to model studies conducted in 1939 by the writer at the hydraulic laboratory of the Bureau of Reclamation, United States Department of the Interior. Hereinafter, these studies are called the USBR tests. The tests were conducted for the purpose of developing the rectangular stilling basin referred to in the paper and described by Jacob E. Warnock, M. ASCE (4). Comparisons also will be made with the current practice in stilling basin design for flood-control and multiple-purpose projects. The USBR tests are reported in an unpublished memorandum from the writer to the chief designing engineer of the Bureau of Reclamation on the subject, "Model Study of Wickiup Dam Outlet Works, Deschutes Project, Oregon," dated June 30, 1939.

The author has chosen to relate stilling basin dimensions to the hydraulic jump by means of the d_2-depth and a dimensionless term, $\frac{V^2_1}{g\,d_1}$, corresponding to the high-velocity flow at the stilling basin entrance. In the USBR studies, expressions were developed for stilling basin dimensions solely in terms of d_2. Others have expressed stilling basin dimensions in terms of either the height of jump,

$$d_j = d_2 - d_1 \qquad (37)$$

or the ratio,

$$J = d_2/d_1 \qquad (38)$$

As J. C. Stevens, Past-President, ASCE, has pointed out (15c), there are only two independent characteristics of the hydraulic jump—the initial depth, d_1, and the velocity, V_1—at the stilling basin entrance. From them are derived the initial energy head, the final depth, the final velocity, and the final energy head, as well as the lost energy head, the empirical characteristic length of jump, and the Froude number corresponding to the initial depth and velocity.

---

[8] Senior Hydr. Engr., Office, Chf. of Engrs., Washington, D. C.

All forms of the published formulas for stilling basin dimensions are functions of the two independent characteristics of the hydraulic jump; therefore, equally satisfactory criteria for the dimensions of a stilling basin could be obtained by use of any one of the forms, provided that appropriate constants were determined to reflect the various stilling basin conditions and the desired degrees of energy dissipation. For example, the variation of stilling basin lengths, as computed from published formulas, reflects the widely varying conditions. This results from the differences in the profiles of individual stilling basins, the relation of the provided tailwater depth to the d_2-depth required by momentum relations, and the ability of the foundation to withstand erosion downstream from the individual stilling basins. The nature of the stilling basin, whether for normal or emergency operation, and whether isolated or adjacent to a main structure, as well as the conception and judgment of the designer, also has some bearing on the final choice of stilling basin dimensions.

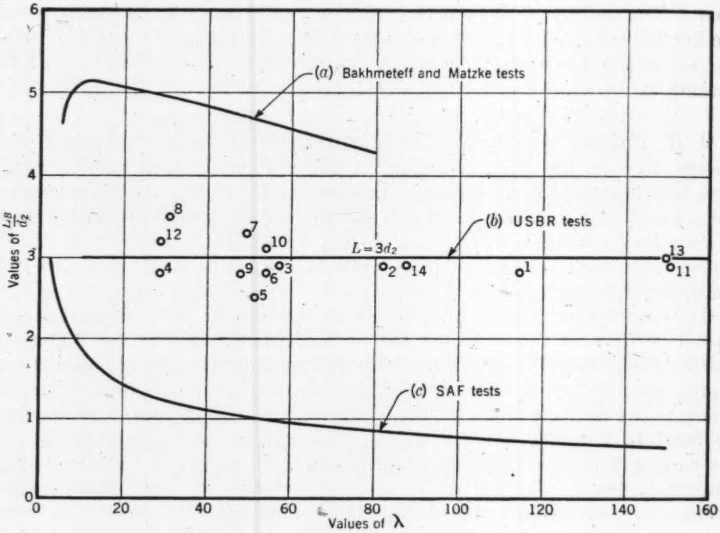


Fig. 37.—Comparison of Stilling Basin Test Results

In Fig. 37, the author's suggested curve for stilling basin length is compared to the design and model data available to the writer, as well as to the curve of B. A. Bakhmeteff, Hon. M. ASCE, and A. E. Matzke, Assoc. M. ASCE, for the length of hydraulic jump (15*d*). The criterion for stilling basin length,

$$L_B = 3\,d_2 \qquad (39)$$

which was developed by the USBR tests, is also shown in Fig. 37. Points 1 to 14, inclusive, are for stilling basins of the flood-control and multiple-purpose dams listed in Table 11. These stilling basins are of the horizontal-apron type with baffle blocks and end sills. All have been model tested for satisfactory operation under maximum design conditions. Those that have been constructed in prototype have operated in a fully satisfactory manner within the ranges of experienced discharge.

TABLE 11.—SUMMARY OF DATA FOR USBR STILLING BASINS

No. (1)	Dam (2)	Location (3)	L_B (ft) (4)	d_1 (ft) (5)	V_1 (ft/sec) (6)	d_2 (ft) (7)	λ (8)	L_B/d_2 (9)
1	Boca	Truckee, Calif.	59	1.5	73	21	115	2.8
2	Deer Creek	Provo, Utah	75	2.1	75	26	83	2.9
3	Vallecito	Durango, Colo.	92	3.1	76	33	58	2.9
4	Wash Overchute	Yuma, Ariz.	25	1.2	34	9	29	2.8
5	Wickiup	Bend, Ore.	55	2.3	62	22	52	2.5
6	Dorena	Dorena, Ore.	144	5.1	95	51	55	2.8
7	John Martin	Las Animas, Colo.	175	5.6	95	53	50	3.3
8	Santa Fe	Los Angeles, Calif.	86	3.3	58	25	32	3.5
9	Cherry Creek	Denver, Colo.	120	4.6	85	43	49	2.8
10	Enid	Enid, Miss.	100	3.3	76	33	55	3.1
11	Narrows	Murfreesboro, Ark.	120	2.5	110	42	151	2.9
12	Harlan County	Republican City, Nebr.	163	7.1	82	51	30	3.2
13	Conchas	Tucumcari, N. Mex.	134	2.6	112	44	150	3.0
14	Norfork	Buffalo, Ark.	185	4.9	118	63	88	2.9

Although the definition of the Froude number given by Eq. 1*a* is frequently used in the literature, the more convenient and usual form is

$$\mathbf{F} = \frac{V_1}{\sqrt{g\, d_1}} \qquad (40)$$

The term $\frac{V^2_1}{g\, d_1}$ is generally called the kinetic flow factor, λ. However, because λ is an appropriate dynamic characteristic of the Froude type, it may be used as a parameter in the dimensionless representation of stilling basin length.

The Bakhmeteff and Matzke curve for the length of hydraulic jump is based on the formation of the jump on a horizontal apron without baffle blocks or an end sill. This curve may be used for determining stilling basin lengths for those high dams in which baffle blocks are not used (because of the possible destruction of the blocks by cavitation), and in which unfavorable downstream channel conditions make it desirable to include the entire length of the jump in the stilling basin.

From Fig. 37, it may be concluded that for flood-control and multiple-purpose dams: (1) Baffle blocks and an end sill reduce considerably the required length of a stilling basin; (2) Eq. 39 is a satisfactory criterion for stilling basin length when baffle blocks and an end sill are used; and (3) the SAF stilling basin curve indicates stilling basin lengths dangerously short for the usual case of an erodible downstream channel.

The criterion, Eq. 39, is confirmed by Fig. 6 in which water-surface profiles show that the hydraulic jump is completely contained in a horizontal stilling basin whose length is approximately equal to 3 d_2. For shorter lengths, the crest of the jump occurs beyond the end of the stilling basin. Unless the stilling basin is located in a canyon composed of good rock, maintenance costs to provide protection against violent wave action (which occurs at the crest of the jump and for some distance downstream) may greatly exceed the additional cost of providing a stilling basin of adequate length. Since stilling basins for soil conservation structures are usually located in erodible material, it is questionable whether the short stilling basins given by the SAF curve would provide adequate protection for these structures.

The USBR tests included a test series similar to those made for the SAF stilling basin to determine the optimum size, spacing, and location of chute blocks, baffle blocks, and end sill. In general, the data obtained for the USBR tests are in agreement with the data for the SAF tests. One important difference, however, is the location of the baffle blocks; in the USBR studies it was concluded that the baffle blocks should be located $\frac{2}{3} L_B$ from the upstream end of the stilling basin to minimize cavitation possibilities. Tests on the longer USBR basin did not indicate that locating the baffle blocks $\frac{1}{3} L_B$ from the upstream end of the basin would improve the basin performance, as was the case for the shorter SAF basin.

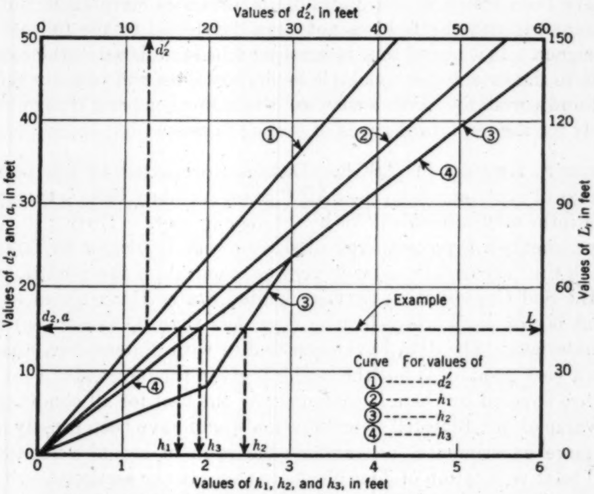
Design curves for stilling basin dimensions, based on the USBR tests, are presented in Fig. 38. In more recent plans chute blocks have not been used when V_1 is in the cavitation range, in which case two rows of baffle blocks are frequently used to obtain the desired energy dissipation.

The 15% reduction in d_2, to obtain d'_2, as given by Eq. 6*a*, was first suggested by the writer on the basis of the USBR tests. Subsequent model tests conducted by the United States Waterways Experiment Station at Vicksburg, Miss., and at the Bonneville (Ore.) and Los Angeles (Calif.) hydraulic laboratories of the Corps of Engineers, United States Army, have confirmed satisfactory stilling action for a 15% reduction in tailwater depth. In recent plans the practice of the Corps of Engineers has been to provide a tailwater depth equal to 0.9 d_2.

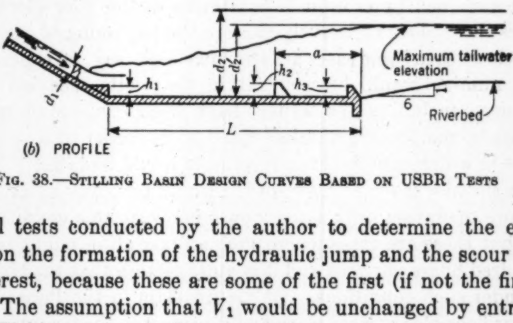
The USBR tests indicated, as did the SAF tests, that d'_2 could be reduced to about 0.75 d_2 before the jump would be swept out of the stilling basin. Thus, a design tailwater depth equal to 0.9 d_2 would provide a reasonable safety factor against excessive negative pressures on baffle blocks, against severe wave action downstream from the stilling basin, and against the jump sweeping out of the stilling basin because of unanticipated lowering of the tailwater.

Eq. 7, for stilling basin freeboard, is somewhat too conservative for flood-control and multiple-purpose dams. Usually, it is considered unnecessary to confine the maximum splash in stilling basins for those dams, because the infrequent occurrence and short duration of maximum discharge would not result in appreciable damage. Freeboard above the maximum tailwater level, equal to from 0.25 d_2 to 0.2 d_2, usually is considered ample protection for the stilling basin and the adjacent structures.

Model tests conducted at the United States Waterways Experiment Station at Vicksburg, Miss., for several large stilling basins showed that quadrant wing walls (located downstream from the end sill) most effectively reduce scour at the end of the stilling basin side walls. However, quadrant wing walls are more costly than perpendicular or 45° angle wing walls. Thus, quadrant wing walls should be used only when it is desirable to provide maximum protection against scour.



(a) DIMENSION CURVES



(b) PROFILE

FIG. 38.—STILLING BASIN DESIGN CURVES BASED ON USBR TESTS

The model tests conducted by the author to determine the effect of air entrainment on the formation of the hydraulic jump and the scour pattern are of special interest, because these are some of the first (if not the first) tests of this nature. The assumption that V_1 would be unchanged by entrained air is questioned. Field tests by the Bureau of Reclamation in 1938 on the Kittitas chute on the Yakima project in Washington indicated that the measured velocity of the water-air mixture was substantially less than the computed theoretical velocity based on Manning's formula. Field measurements on a wasteway at the Rutz Works in Austria, which were published by R. Ehrenberger in 1926 (38) clearly indicated the frictional effect of aeration by rapid increases in depth with sudden increases in chute slope. If air entrainment results in an added friction loss, then V_1 and d_1 at the entrance of large stilling basins would be somewhat smaller and larger, respectively, than theoretical values—thus requiring a smaller d_2-depth for the formation of the hydraulic jump. In that case significant savings might be affected in stilling basin dimensions.

Mr. Blaisdell presents a useful set of design rules for the SAF-type stilling basin. However, hydraulic design engineers should proceed with caution in applying these rules to other types of stilling basins. Many types of stilling basins have been tested in the hydraulic laboratories since 1930, but general design equations and charts have not been developed to the full satisfaction of the designer. It is hoped that interest may be stimulated in the engineering profession to the extent that available model test data will be more thoroughly analyzed and correlated, with a view to establishing general design equations and charts for the important types of stilling basins.

ROBERT F. EWALD,[9] M. ASCE.—Data are presented, in this paper, from a wide range of experiments on the SAF ("Saint Anthony Falls") type of stilling basin in a form very convenient for hydraulic engineers. During the course of these experiments a type of energy dissipator was developed for falling water that has all the "earmarks" of being very economical for the conditions usually met by the Soil Conservation Service, United States Department of Agriculture—that is, for moderate velocities (say, less than 50 ft per sec) on easily eroded materials. The data have been tied in with dimensionless parameters (which is a very good idea) but the reader is given the impression that the type of structure covered by them is economically the best for all conditions.

The value of Mr. Blaisdell's contribution would have been greatly improved by one more experimental parameter—the ratio of actual velocities at the preferred relative location of floor blocks to those in the section d_1. That ratio may set up limitations to the use of the SAF design in certain locations, the limits being governed by erosion or cavitation of the floor blocks. The SAF design places these blocks relatively close to the beginning of the jump in areas where the velocities in some cases are still very high (say, 50 ft per sec) although the Froude number would be well within the limits specified in the paper. Some designers, including the writer, have been confronted with situations where factors in the Froude number were on the order of $v_1 = 125$ ft per sec and $d_1 = 3$—in which case the Froude number would be 172±. The prototype velocities at the blocks for an SAF design would certainly exceed 60 ft per sec. In a structure operating only a few hours per year this condition should not be serious, but there are cases where such an operation might be expected for several hundred hours per year and it is quite certain that the blocks would not stand up under such conditions. At Gatun Dam in the Panama Canal Zone where similar floor blocks were placed with respect to the beginning of the jump, much as proposed in the SAF design, many of the blocks were destroyed within a few years. At Grand Coulee Dam in Washington the upcurved part of the bucket apparently is located at about the same relative position with respect to the jump; and, after less than 10 years of operation, this part of the structure required redesign when it was reconstructed because of the great erosive action.

The writer became interested in the subject of erosion at the toe of high overfall dams early in 1913 and in the fall of that year began a long series of tests on models of a proposed dam on the Little Tennessee River near the mouth of the Cheoah River in North Carolina. This dam was to be nearly 200 ft high and was to provide for possible maximum floods of 160,000 cu ft per sec with a net spillway crest length of about 600 ft. (The dam was actually completed 6 years later.)

---

[9] Pittsburgh, Pa.

The model tests very soon indicated the great significance of the hydraulic jump in the erosion problem and the fact that the design should fundamentally be based on the performance of the jump. As part of this investigation a series of tests was made using staggered baffle piers with a height equal to d_1 at the beginning of the jump, on the theory that these piers would materially reduce the pool depth required to bring the jump to the bucket (34*b*). Although the tests were rather crude, they indicated clearly that the amount of energy absorbed by impact and turbulence at the piers was so small, relative to the absorption inherent in the jump, as to be unimportant, and that the piers were undesirable in view of the difficulties at Gatun Dam which were then being discussed. The tests also indicated that the erosion accompanying the unmodified jump would never be structurally serious because of the highly resistant character of the material on which the dam was to be built and that nature would ultimately form an adequate cushion pool.

At the time the Cheoah Dam tests were made, the literature on the jump available in the United States was limited to a few lines by the late Mansfield Merriman, M. ASCE (39), based on an erroneous derivation of the formula. The derivation by William C. Unwin (40), of course, has been known for some years, but was not likely to be noticed by engineers. The literature on the subject now covers thousands of pages and the subject apparently has not yet been exhausted. Mr. Blaisdell's paper, for example, may lead to modifications of the jump formula to cover the conditions inherent in the SAF design.

FRED W. BLAISDELL,[10] ASSOC. M. ASCE.—The thoughtful and constructively critical comments of the discussers are evidence that the writer's aim of presenting the results of the SAF (Saint Anthony Falls) stilling basin investigation in such a way that each reader could independently evaluate and intelligently discuss the design rules developed therefrom was achieved. The discussions could not have been prepared without the expenditure of considerable time and effort, and the writer wishes to acknowledge this work and to express his sincere appreciation of the contributions of the discussers. They have materially increased the value of the paper to the engineering profession.

Of considerable interest are the energy losses and relationships in the hydraulic jump and stilling basin computed by Mr. Blotcky and summarized in Table 10. The proportion of the energy at Section 1, Fig. 18, that is lost in the hydraulic jump is least at small values of $\mathbf{F}_1$ (15*e*). This may be why the blocks are most effective at low values of $\mathbf{F}_1$, as Mr. Blotcky has noted. However, Col. 9, Table 10, and the results of the original tests as recalled by the writer seem to indicate that the energy loss ascribed to the blocks is far less than is warranted by their importance. For example, in Table 10 the median value of ϵ_b/ϵ_j is 0.11. For this value of ϵ_b/ϵ_j the energy loss due to the blocks alone is only 6% of the energy at Section 1, Fig. 18. Since the reduction in basin length achieved through the use of blocks and sills is in the order of 75%, it may be that the blocks have some function other than the dissipation of energy. However, it is probable that the location of Section 2′, as assumed by Mr. Blotcky, is incorrect. At the end of the SAF basin the depth of flow is increased by the boil and decreased by the end sill. Furthermore, the streamlines have considerable curvature at this point, and, as a result of the surface roller, there may be upstream components of the velocity at this section (see Fig. 1(*b*)). An additional reason why the location of Section 2′, Fig. 18(*b*), is incorrect is the fact that some energy dissipation occurs downstream from the end of the basin. Section 2′ should be selected at some point in the downstream channel. The writer has no data to indicate the depths and velocities at a point where normal flow conditions have been established in the downstream channel. In the absence of this information, Mr. Blotcky's computations summarized in Table 10 are accepted for their relative value.

---

[10] Project Supervisor, U. S. Dept. of Agriculture, SCS, St. Anthony Falls Hydr. Laboratory, Minneapolis, Minn.

These comments regarding Section 2′ also apply to the coefficient of form drag (C_d) computations summarized in Fig. 23. Although the magnitudes of C_d shown in Fig. 23 are probably inaccurate, Mr. Blotcky is to be commended for his most interesting analysis in which he introduces the effect of the floor blocks into the momentum equation for the hydraulic jump.

The use of blocks certainly modifies Eqs. 2, but there is some question as to whether or not their use invalidates it completely, as Mr. Blotcky maintains. In fact, in commenting on Fig. 21 he notes that certain ratios computed for the block jump "are comparable to those in a hydraulic jump * * *."

In view of Eq. 9 and of Mr. Blotcky's pertinent comments, it would seem that the use of d_c, in place of d_2 as used by the writer, is a matter of personal preference. The use of d_c was considered, but this parameter was not adopted because of the facts expressed in Eq. 9 and because its use would require the computation of an additional parameter when designing the SAF basin, without any apparent advantage being gained thereby.

In Fig. 19 Mr. Blotcky has used d_c as a parameter when replotting some of the data presented in Table 2 and Fig. 7. Through the data points he has drawn the curve represented by Eq. 11. The relationship between d_c and d_2 embodied in the hydraulic jump equation makes it difficult to compare mathematically Eq. 11 with Eq. 3. It is, however, possible to express Eq. 11 in terms of d_1. Substituting for q^2 in Eq. 10 its equivalent, $\mathbf{F}_1\, g\, d_1\, d^2{}_1$,

$$d_c = \sqrt[3]{\mathbf{F}_1\, g\, d_1\, d^2{}_1/g} = \sqrt[3]{d^3{}_1\, \mathbf{F}_1} = d_1\, (\mathbf{F}_1)^{0.33} \qquad (41)$$

Substituting Eq. 41 in Eq. 11,

$$\frac{L_B}{d_1\, (\mathbf{F}_1)^{0.33}} = \frac{6.2}{(\mathbf{F}_1)^{0.25}} \qquad (42)$$

or

$$\frac{L_B}{d_1} = 6.2\, (\mathbf{F}_1)^{0.08} \qquad (43)$$

B. A. Bakhmeteff, Hon. M. ASCE, and A. E. Matzke, Assoc. M. ASCE (15*f*), have shown that the relationship L/d_1 between the length of the hydraulic jump, L, and d_1 varies considerably with λ, or $\mathbf{F}_1$ as it is designated by the writer, whereas the ratio L/d_2 is approximately constant. Inasmuch as the action in the SAF stilling basin is considered to be a forced hydraulic jump and the basin length is related to the jump length, the writer prefers the ratio L/d_2.

Mr. Blotcky attempts to compare Eqs. 11 and 3 in Fig. 19. The writer strongly objects to this comparison. The comparison should be between the curve and its pertinent data, which are given for Eq. 3 in Fig. 7 and for Eq. 11 in Fig. 19. The writer has computed L_B by Eqs. 3 and 11 and has tabulated the results in Table 12, using tests selected from Table 6. The values of L_B listed in Table 6 were those used in the tests. These values have been recomputed by inserting in Eq. 3 the observed values of d_1, d_2, and $\mathbf{F}_1$, which differed slightly from values used in the design of the model. It will be noted in every case except two that L_B is shorter when computed by Eq. 11 than when Eq. 3 is used. In some cases the computed basin length is 20% shorter. This difference is great enough to make the use of Eq. 11 inadvisable. Although based on the same data, Eq. 3 has been thoroughly checked by model tests and its use is to be preferred.

On the basis of Fig. 20, the writer agrees with Mr. Blotcky that the form of Eq. 18 is more desirable than that of Eqs. 6. However, most of the data presented in Fig. 11 are not included in Fig. 20. The writer has recomputed the data shown in Fig. 11 and summarized them in Table 13. These data are plotted in Fig. 39, in which the curve shown has the equation:

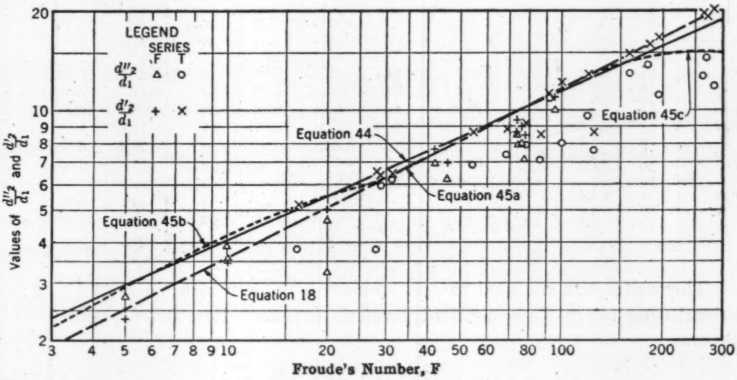


FIG. 39.—DIMENSIONLESS PLOT SHOWING TAILWATER DEPTH AS A FUNCTION OF THE FROUDE NUMBER

$$\frac{d'_2}{d_1} = 1.4\ (\mathbf{F}_1)^{0.45} \qquad (44)$$

Eq. 18 is also shown to permit comparison. The writer considers that Eq. 44, which is based on better data than is Eq. 18, is to be preferred, but that either Eqs. 6 or Eq. 44 will give good results. This contribution by Mr. Blotcky simplifies the design of the SAF stilling basin and is greatly appreciated by the writer. Mr. Blotcky's significant contribution in this instance shows the value of publishing papers in *Proceedings* where they may receive the consulting services of the entire membership of the Society.

TABLE 12.—COMPARISON OF BASIN LENGTHS COMPUTED BY EQS. 11 AND 3

Test No.	q	d_c	d_2	F_1	L_B		Col. 5/6
					Eq. 11	Eq. 3	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
T53.........	4.00	0.792	2.82	288	1.19	1.48	0.80
T63.........	9.42	1.40	4.875	252	2.18	2.68	0.81
T61.........	9.94	1.45	4.826	196	2.41	2.92	0.83
T52.........	4.00	0.792	2.410	121	1.48	1.75	0.85
T60.........	14.4	1.86	5.48	100.6	3.64	4.27	0.85
T59.........	14.8	1.90	4.98	54.6	4.32	4.91	0.88
T58.........	14.9	1.90	4.39	28.2	5.12	5.55	0.92
T54.........	15.0	1.91	3.96	16.5	5.88	6.14	0.96
T57.........	20.0	2.32	4.50	12.4	7.66	7.78	0.98
T56.........	21.0	2.39	4.09	7.0	9.12	8.79	1.04
T55.........	21.0	2.39	4.05	6.7	9.22	8.85	1.04

TABLE 13.—SUMMARY OF DATA ON TAILWATER DEPTH

Test No.	F_1 (1)	d'_2/d_1 (2)	d''_2/d_1 (3)	Test No.	F_1 (1)	d'_2/d_1 (2)	d''_2/d_1 (3)	Test No.	F_1 (1)	d'_2/d_1 (2)	d''_2/d_1 (3)
F9.....	96	10.80	10.00	T39....	183.1	15.77	13.57	T49....	68.7	8.63	7.29
F17....	74	9.22	8.50	T40....	29.2	6.29	5.88	T50....	121	12.83	9.59
F18....	74	8.58	7.88	T41....	31.8	6.39	6.14	T53....	288	19.98	11.83
F20....	76	8.85	7.98	T42....	93.7	11.06	10.87	T54....	16.5	5.12	3.73
F21....	42	7.58	6.92	T44....	161	14.72	12.95	T58....	28.2	6.50	3.73
F28....	78	8.43	7.07	T45....	87.0	8.47	7.02	T59....	54.6	8.59	6.77
F30....	46	6.83	6.20	T46....	274	19.16	14.40	T60....	100.6	12.02	7.90
F94....	20	4.97	4.66	T47....	126	8.51	7.57	T61....	196	16.60	11.08
F97....	10	3.40	3.53 <	T48....	78.5	9.10	7.80	T62....	267	19.57	12.52
F98....	5	2.29	2.70 <								
F99....	10	3.40	3.85 >								
F101...	20	4.96	3.22 >								

The writer has changed the parameters of Eqs. 6 into those of Eq. 44 by substituting for d_2 its equivalent given in Eq. 2*b*. Eqs. 6*a*, 6*b*, and 6*c* then become

$$\frac{d'_2}{d_1} = 0.425\left(-1 + \sqrt{8\,F_1 + 1}\right) \qquad (45a)$$

$$\frac{d'_2}{d_1} = \left(-1 + \sqrt{8\,F_1 + 1}\right)\left(0.55 - \frac{F_1}{240}\right) \qquad (45b)$$

and

$$\frac{d'_2}{d_1} = \left(-1 + \sqrt{8\,F_1 + 1}\right)\left(0.50 - \frac{F_1}{1{,}600}\right) \qquad (45c)$$

respectively. Eqs. 45 have been plotted in Fig. 39 to permit their comparison with Eq. 44.

The writer cannot agree with Mr. Blotcky that the undular type of jump occurs in the block jump simply because d'_2/d_2 is greater than one for small values of F_1. It is the recollection of the writer that the roller associated with the direct jump occurred at all values of $\mathbf{F}_1$ at which tests were made. Because a slope exists at the entrance to the SAF stilling basin, it is believed that increasing d'_2/d_2 is conducive to causing the entering stream to plunge under the tailwater, as it does for the direct jump, rather than to ride at the tailwater surface as it does for the undular jump. In any event, it is difficult to follow Mr. Blotcky's reasoning as to why Eq. 18 is invalid when $\mathbf{F}_1$ is less than 15. The writer feels that the form of Eq. 18 is excellent and represents the complete data at all values of $\mathbf{F}_1$, but that the constants used in Eq. 44 are to be preferred.

Mr. Culp followed closely the model studies leading to the development of the SAF stilling basin design and contributed freely from his experience in the design and construction of soil conservation structures. In matters regarding the feasibility of certain hydraulic features from the viewpoint of the structural designer and construction engineer, his advice was frequently sought, freely given, and invariably followed. It is a pleasure to be able to acknowledge his contribution publicly. Under the direction of Edwin Freyburger, chief of the Soil Conservation Service regional engineering division in Milwaukee, Wis., Mr. Culp designed the first SAF stilling basin before the final report was prepared. Since then he has designed and built several others. It was hoped that he would be able to report on some structures that had been tested by nature at about the design flow. However, since his discussion was written, the Stuart Perry spillway located in Crawford County, Iowa, has received a severe test. Fig. 40 shows a view of this structure taken prior to the storm described in a letter from Mr. Culp, who obtained his information from Floyd Nimmo, construction engineer on the job.

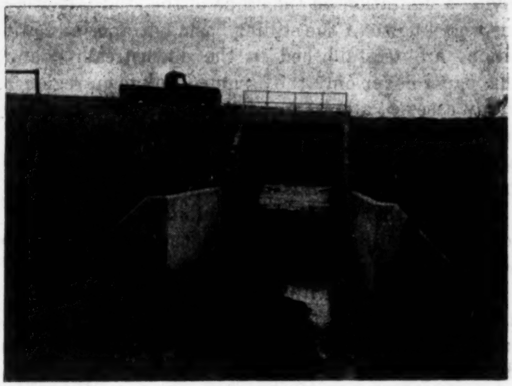


FIG. 40.—STUART PERRY SPILLWAY, CRAWFORD COUNTY, IOWA, PRIOR TO STORM OF JUNE 22, 1947

> "The storm occurred June 22, 1947, between 4:00 p.m. and 6:00 p.m. with most of the rainfall in the first hour. No standard or automatic gauges were in the vicinity, but rough measurements in buckets, etc. indicated a total rainfall of about seven inches. This storm produced a maximum head on the inlet of the spillway of almost four feet, or nearly a foot more than the design head. Hence, even though the inlet was somewhat re-

> stricted by poor placement of the earth fill * * * the flow through the spillway was probably equal to or greater than the design discharge [more than 50% in excess of the design discharge, assuming a 3 ft design head, a 4 ft actual head and unrestricted approach to the inlet]. * * * Mr. Nimmo tells me that the outlet functioned perfectly and that there was very little erosion in the immediate vicinity of the outlet."

Fig. 41 shows views of this structure taken on February 18, 1948. It is apparent that very little scour occurred as a result of the flood of June 22, 1947. The writer has seen three additional outlets which he has every reason to believe have passed heavy flows. These apparently functioned as anticipated, although the downstream channels had not eroded to a stable grade at two of the outlets when the writer saw them in 1945.

Mr. Baumann believes that it would have been helpful to include information regarding the energy gradient between the reservoir and the stilling basin. The SAF stilling basin will be used with drop inlet culverts as well as with overflow spillways. A different value of the coefficient ϕ would have to be determined, not only for each type of structure, but also for each size of structure.[9] To provide a limitation on the scope of the study, it was assumed that flow conditions were known at the stilling basin entrance and at a point where normal conditions are reestablished in the downstream channel; the tests reported in this paper cover only the assumed unknown region between these points. In many cases ϕ, a measure of the lumped-together losses up to the basin entrance, can be computed with sufficient precision from information presently available to the designer. High precision is unnecessary in designing a SAF stilling basin. This comment is no argument for sloppy work, but is rather a recognition of the fact that the magnitude of many of the factors involved can be determined only roughly. The foremost of these factors is probably the rate of runoff on which all subsequent design depends. It is certainly very difficult to determine the rate of runoff with any high degree of precision.

Mr. Baumann's attempt to determine d'_2 from a consideration of the amount of kinetic energy transformed into heat energy is most interesting, but it appears that additional work is required before this method can be used without resort to model studies. However, any attempt to develop an equation from theoretical considerations is always commendable and leads to a better understanding of the problem.

The writer shares most of the views expressed by Mr. Peterka. His comments are a welcome addition to the paper. Nevertheless, the writer cannot agree with Mr. Peterka's comment that the chute blocks and floor blocks of the SAF stilling basin can be dispensed with. It is true that much of the energy dissipation takes place in the channel downstream from the stilling basin. It is also true that the SAF stilling basin throws water upward and away from the structure as does the Fontana Dam (Tennessee Valley) energy dissipator, but the water does not reach a height sufficient to create spray except at the surface under some conditions. The action is such that energy dissipation beyond the basin exit takes place principally near the water surface, as at Fontana Dam, rather than at the bed. However, some energy is dissipated within the SAF basin, in contrast to the Fontana Dam distributor where all energy dissipation takes place beyond the deflector bucket. At high values of **F** a bucket could be used in place of the SAF stilling basin, but at low values of **F** there is insufficient energy available to throw the jet into the air. The writer feels that there is a need for both types of energy dissipator but that the "apron accessories" are required and are economical in the SAF design.

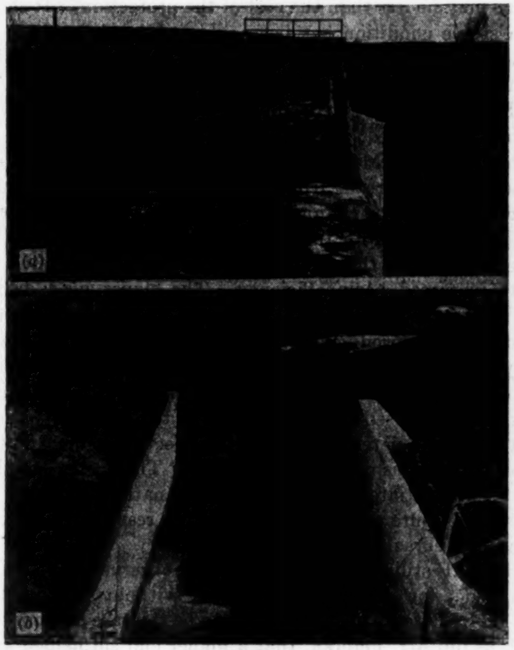


FIG. 41.—STUART PERRY SPILLWAY ON FEBRUARY 18, 1948
(*a*) Upstream View (Compare with Fig. 40)
(*b*) Downstream View; Bank Scour Occurred Before Spillway was Built

Mr. Peterka voices a common objection to blocks and piers, and the writer feels somewhat the same. It is significant, however, that piers are used almost invariably when the problem of energy dissipation is particularly difficult, or when the cost must be kept at a minimum. Such considerations were uppermost in the minds of those engaged in developing the SAF design.

Whether or not special protection from wave action is required in the downstream channel depends to some extent on the frequency with which the design flow is expected and on the duration of such flow. Little wave action was apparent in the models until the discharge approached the design value. At lesser discharges the surface disturbance might be described as "ripples." For soil conservation structures the commonly used design storm has an anticipated 50-yr frequency. For small watersheds the design flow occurs for only a few hours at a time. Under these conditions vegetation is undoubtedly sufficient to protect the banks. The writer shares Mr. Peterka's opinion that for structures subjected to the design flow for long periods, some bank protection from waves will probably be required. Whether it is more expensive to provide bank protection from wave wash for maximum flows or to use a larger basin, as suggested by Mr. Douma, is a problem that the designer of each structure must solve. In many cases the value of the land protected by riprap or other methods may be less than the cost of the protection. It should be noted that, after a stable condition has been achieved in an erodible channel, the normal channel is wider at the basin exit than it is farther downstream. This phenomenon should be taken into consideration if bank protection is used. Otherwise the bank protection would be subject to excessive velocities and might possibly be damaged.

The writer feels that Mr. Peterka need not be concerned about the high surface velocities downstream from the apron. Since the highest velocities are on the center line of the channel and are near the surface, it is not believed that protection is required for this reason alone.

Mr. Peterka suggests that steeper slopes at the basin entrance probably contribute to increased basin efficiency. The slopes used during the SAF tests varied from 1 on 2.6 to 1 on 1.5. In the SAF basin, however, the chute and floor blocks aid in producing a uniform velocity distribution at the basin exit and may offset the poorer results obtained by Mr. Peterka for the Horsetooth Dam outlet works in Colorado when the entering slope was decreased.

The range of slopes used at the entrances to the SAF stilling basin models is too narrow to permit a determination of the effect of changes in the transition profile on the proportions of the chute blocks. The writer can therefore offer no evidence to substantiate or to refute the explanation offered by Mr. Peterka, which he based on "pure conjecture" and on the results of tests on the Enders Dam spillway in Nebraska.

During the SAF stilling basin tests, it was observed that the use of chute blocks permitted a lowering of the tailwater without washing the jump out of the basin. Mr. Peterka also found this to be the case at Enders Dam. The writer found, as did Mr. Peterka, that a higher end sill helped keep the jump in the basin. However, the SAF end sill is subjected to higher velocities than is the Enders Dam sill; its height has a greater effect on the flow; and the end sill could not be raised without greatly increasing the boil height, which would necessitate raising the side walls. All these considerations were kept in mind when it was finally decided that the best solution for the SAF stilling basin was to leave the chute blocks in place and to determine the best end sill height for the assumed form of the stilling basin.

In Fig. 34 Mr. Laushey has attempted to compare d'_2/d_2 for the SAF stilling basin with certain results he obtained. The writer feels that there is only one point of comparison—at the intersection of Eq. 6*b* and Mr. Laushey's curve for $d'_2/d_2 = 0.07$. This intersection indicates only the Froude number (or d_1/d_c) at which the two basins have the same tailwater depth. The limiting curves labeled "SAF; d'_2/d_2 (no factor of safety)" and "limiting curve, rectangular sill" are not based on the same criteria and are therefore not comparable.

The writer does not believe that the height of the end sill in the two types of stilling basin can be compared as Mr. Laushey has done. Fairly high velocities existed at the end sill in the SAF basin, whereas in Mr. Laushey's basin the velocities were, without doubt, comparatively much lower. In the second place, the SAF basin is much shorter. These considerations and others, in addition to the effect of the blocks mentioned by Mr. Laushey, undoubtedly contribute to the differences in the recommended values of d'_2/d_2.

Mr. Laushey's curve for zero end sill height (Fig. 36) gives a basin length at variance with the length of jump determined by Messrs. Bakhmeteff and Matzke. Fig. 36 does, however, give an excellent picture of the basin length reduction that can be achieved through the use of sills, and also clearly shows the further reduction made possible by the block arrangement of the SAF stilling basin.

Mr. Laushey notes that in the SAF stilling basin the tailwater depth had to be increased at values of $\mathbf{F}_1$ less than 30, and states that a similar increase was required for the end sill basin. However, it can be seen in Fig. 34 that for a constant end sill height the relative tailwater depth is increased with increasing values of the Froude number for the end sill basin. Just the opposite is the case for the SAF basin, where Fig. 11 shows that the relative tailwater depth is increased with decreasing values of $\mathbf{F}_1$ when $\mathbf{F}_1$ is less than 30.

The results presented by Mr. Laushey are most interesting and valuable. Rather than deprecate his results, the writer has attempted to show that the proper magnitudes of the parameters are so dependent on the type of basin under consideration that it is not possible to make a valid comparison between many of the common elements making up these two types of basin. In this the writer agrees with Mr. Douma's statement, "* * * hydraulic design engineers should proceed with caution in applying these rules to other types of stilling basins." Mr. Peterka also has expressed somewhat the same thought in his discussion.

Mr. Douma objects to the writer's definition of $\mathbf{F}_1$ and prefers λ. There are three reasons why the writer defined $\mathbf{F}_1$ by Eq. 1*a*: First, he learned it that way; second, it has direct physical significance, because it represents the ratio of the force required to overcome inertia to the force of gravity; and, third, this definition is used (41*a*) in ASCE *Manual of Engineering Practice No. 25*. Many persons use the form preferred by Mr. Douma. However, when writing in an ASCE publication, the writer feels a certain obligation to use the form approved by the Society.

Mr. Douma concludes from Fig. 37 that Eq. 39 is a satisfactory criterion for the stilling basin length when blocks and an end sill are used. The writer agrees with this statement as it applies to the United States Bureau of Reclamation (USBR) tests. He cannot, however, agree that curve (*c*) in Fig. 37 in itself indicates that the SAF stilling basin is too short. In the first place, Mr. Douma presents neither data nor arguments to support his statement. Furthermore, the USBR data cannot alone be used to discredit another stilling basin design based on probably as thorough and exhausting study as was the USBR design. The writer went to great length in his paper to show that the SAF stilling basin gave entirely satisfactory results "for the usual case of an erodible downstream channel," and it can be stated that the SAF stilling basin, like the USBR basin, has given satisfactory service under field conditions.

The writer feels that generalized results of tests made at public expense should be made available to the public. This is one reason why he offered the SAF results for publication; and, for this reason also, it is well that Mr. Douma has presented the design criteria for the USBR stilling basin.

It should be noted that the boil in the SAF basin rises above normal tailwater level (in contrast to larger basins where the velocity at the floor blocks is lower). Therefore, the freeboard normally used by designers should be figured from the top of the boil rather than from the tailwater surface. With this fact in mind, the freeboard proposed by Mr. Douma is in reasonable agreement with that proposed by the writer. Spray will probably go over the top of the SAF basin walls at the design flow, but masses of water, which the writer associates with splash, should not.

Mr. Douma feels that the velocity in open channels with entrained air is less than that of a corresponding nonaerated flow computed by the accepted formulas. He bases his statement on the results obtained by R. Ehrenberger (38) and on field tests made by the Bureau of Reclamation. Offsetting these results are those reported by the late L. Standish Hall, M. ASCE (11), who finds that the velocity increases when air is entrained. The most recent, and undoubtedly the most accurate, series of tests to come to the attention of the writer are those reported by Warren W. DeLapp (42). From a large number of carefully performed experiments in a 1-ft-wide channel having three different roughnesses, Mr. DeLapp shows that, for all practical purposes, normal values of Manning's *n* can be used when computing flows with entrained air. He concludes that the velocities with and without air entrainment will be approximately the same. However, the depth of the flow will increase with air entrainment as a result of the greater volume rate of flow. In view of Mr. DeLapp's results and the conflicting results obtained by others, the writer's assumption of identical velocities without and with air entrainment does not seem unreasonable.

Mr. Ewald has called attention to one feature of the SAF stilling basin on which the experiments are incomplete: No observations were made of the velocities or pressures at the blocks. It is true that the velocities at the floor blocks are high and that cavitation may be expected if they are high enough. Until additional data become available, the designer should proceed with caution whenever he suspects that the velocities at the floor blocks will be such that cavitation can be expected.

After the turbine room tests had been completed, the basin for test T63 was left in place and operated intermittently for purposes of demonstration over a period of 2 years. When the basin was removed, scour was observed on the wood floor in the vicinity of the floor blocks. A view of this scour is shown in Fig. 42. The erosion around the blocks was the only scour noted, its maximum depth being 0.03 in.

Mr. Ewald feels that little energy is absorbed by the floor blocks and bases his opinion on tests performed in 1914 and 1915. He tried blocks only at the beginning and at the end of the hydraulic jump, and in his words (34*b*), "* * * the middle * * * was completely neglected." The best location for the floor blocks is probably somewhere between the ends of the jump. If Mr. Ewald had located the blocks in this region, it is probable that his present opinion would be somewhat revised. Certainly, the reduction in size of the SAF stilling basin could not have been achieved without the use of baffles and sills.

During the analysis of the SAF stilling basin data, it was noted that a significant reduction in basin size could be obtained for a given discharge if **F** could be increased. Because of this discovery, a method of increasing **F** was sought; and, as a result, a study of expanding transitions in high-velocity flows was initiated. This study, which is incomplete in January, 1948, was based on the initial assumption that the velocity would be unchanged along the transition, but that the depth of flow would be decreased. This decrease in depth causes an increase in **F**. The results to date (43) indicate that over-all savings up to 50% in concrete volume and 20% in excavation may be obtained through the use of a transition. This saving cannot ordinarily be accomplished at overflow dams, but it is possible when chutes or culverts are used. The use of a SAF stilling basin in conjunction with a transition may, under optimum conditions, result in a structure only one eighth as large as a comparable hydraulic jump stilling basin. This claim may seem to be fantastic and it may seem impossible to achieve a safe outlet structure of this size. The writer will not attempt to prove his contention here, but suggests that any person interested make the computations himself, and, if he doubts the adequacy of his design, subject it to verification by model tests.

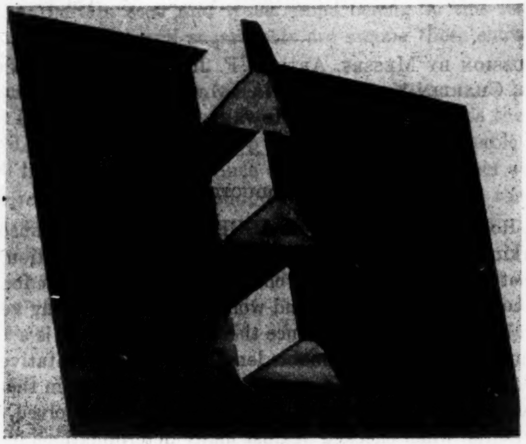


FIG. 42.—SCOUR AROUND FLOOR BLOCKS

The writer may have given the impression, as interpreted by Mr. Ewald, that the SAF stilling basin is economically the best for all conditions. If he has, he wishes emphatically to correct that impression. There are many locations where other types of outlet structure should be used. The writer is confident, however, that the SAF stilling basin can compete with alternate types of outlet which might be used in similar locations.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2343

FORECASTING PRODUCTIVITY OF IRRIGABLE LANDS

BY W. C. MULDROW,[1] ASSOC. M. ASCE

WITH DISCUSSION BY MESSRS. ARTHUR F. JOHNSON, H. W. VAN LOO, WILL H. NOBLE, CHARLES KIRBY FOX, M. R. LEWIS, HARRY F. BLANEY, R. L. PARSHALL, AND HOWARD N. MAGNESS.

INTRODUCTION

In 1941, Robert L. Lowry, Jr., M. ASCE, and Arthur F. Johnson, Assoc. M. ASCE, working with the United States Bureau of Reclamation, produced a heat-unit method of estimating the consumptive use of water for agriculture.[2] The writer considers this a sound and workable method, giving very consistent results from long-term records. Since the only requirement is a United States Weather Bureau record of reasonable length in, or representative of, the area under consideration, the method may be applied anywhere in the western part of the United States. The formula is easily and simply resolved, and the very consistent results, when checked against many determinations by other methods, give an accuracy satisfactory for use in preliminary estimates. As shown by J. L. Burkholder,[3] M. ASCE, in his discussion of the Lowry-Johnson paper, the line produced is the median of a band covering the range of divergence of individual years from the average.

The question of feasibility is always before the engineer on a proposed irrigation project, and feasibility is determined by the ratio between cost and returns. The three indispensable elements of a feasible project are soil, water, and climate. If an irrigable area has fertile soils, a water supply adequate to the production of full crops, and a growing season adapted to the cultivation of a pattern of profitable crops, it probably can afford a project with a relatively high per-acre price. The costs are determined by well-known engineering methods of survey and estimate. For the other side of the equation the irrigation engineer must solve the question: "What will be the returns?"

NOTE.—Published in February, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Engr., U. S. Engrs., Portland, Ore. (Mr. Muldrow died on January 14, 1947.)

[2] "Consumptive Use of Water for Agriculture," by Robert L. Lowry, Jr., and Arthur F. Johnson, *Transactions*, ASCE, Vol. 107, 1942, p. 1243.

[3] *Ibid.*, p. 1287

A wide variety of factors affect the crop patterns used in various irrigated communities. Accessibility to markets is an important one. Others are crop pests and diseases, the basic limitations of many crops as to their tolerance of heat and cold, etc. It is generally true that fruit and truck crops, which are most desirable in building up a diet of high nutritional value: (1) Require a long season with abundant heat units to produce high yields; (2) are high-value crops, resulting in the highest per-acre incomes where they can be grown; and (3) are also high-cost crops. Since the production of these crops uses many man-hours of labor, the farm operator is not overjoyed; but this feature is desirable in a national economy where the greatest problem ahead is to provide sufficient work for all. There are also many specialty crops which are sharply limited by their very narrow environmental adaptation. Apples, citrus fruits, peppermint, and certain seed and truck crops belong to this list. Alfalfa, potatoes, the small grains, and sugar beets are among those showing a very wide range of adaptability.

With all these and many other variations in crop patterns, it is submitted that the most important attribute differentiating farm lands is the volume of crops produced. Within the same crop pattern the most valuable and profitable lands will be those able to produce the greatest tonnage per acre. From a broad water-resource planning standpoint, it is also desirable that available water be used where it will do the most good—that is, where an acre-foot of water will produce the greatest amount of crop.

A simple extension of the heat-unit method from consumptive use of water to tonnage of crop produced will provide a method of estimating the productivity of lands on a proposed project. The writer submits Fig. 1 for this purpose.

GRAPH

The original Lowry-Johnson consumptive use curve[4] is shown in Fig. 1 with the consumptive use points of representative localities in the irrigated west superposed. These points range from the Valley of Henry's Fork, Upper Snake River, El. 6500, down by steps to Pasco, Wash., at the junction of the Snake and Columbia rivers, then to the Sacramento Valley and on down to the Imperial Valley in California, near sea level. This range, in available heat units (accumulated day degrees above 32° F), is from 4,300 to 20,000. Alfalfa is grown more universally than any other crop on these irrigated lands. In the Pacific Northwest it occupies from 35% to 60% of the acreage on nearly all projects, and is grown on substantial areas of lands in every locality shown in Fig. 1. The reason, of course, is its ability to gather nitrogen from the air in addition to its forage value. Another crop that is grown throughout this area in considerable amounts is Irish potatoes.

The acre yields used were taken from the 1940 Census,[5] where the statistics are presented by counties. The points shown in Fig. 1 were limited to localities where crops were grown in acreages large enough to give a dependable average.

---

[4] "Consumptive Use of Water for Agriculture," by Robert L. Lowry, Jr., and Arthur F. Johnson, *Transactions*, ASCE, Vol. 107, 1942, p. 1252, Fig. 2.

[5] 1940 Census, Section on Agriculture, Mountain and Pacific States, Vol. 1, Pt. 6.

Two or more counties were usually added together to give coverage for the locality, and the average was taken. Some checking was done through the limited amount of crop census data available to the writer from the records of projects of the United States Bureau of Reclamation. The annual crop census taken by the Bureau on all projects forms the most accurate and detailed body of crop data known to the writer. Some highly interesting and valuable conclusions can be drawn from this thorough analysis. Neither time nor comprehensive data were available to make a detailed and exhaustive study, yet it is felt that the material used is sufficient to place the yield lines shown with a fair degree of accuracy. About 2,000,000 acres of alfalfa are represented by these lines. A new irrigation project, wherever located, may be spotted on this chart by picking its total available heat units from the appropriate Weather Bureau record, and its relative productivity may be read from the yield lines. Also, the relative efficiency in the use of water may be noted from the top curve, "Consumptive Use per Ton of Alfalfa." The values are naturally those for average soils producing alfalfa in each region.

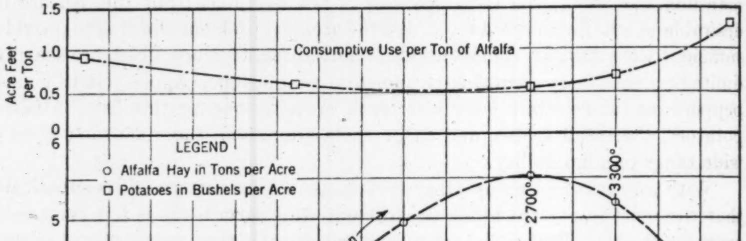


FIG. 1.—CROP YIELDS AS RELATED TO AVAILABLE HEAT UNITS AND CONSUMPTIVE USE OF WATER

It will be noted that, from Henry's Fork down to the Columbia River and Yakima Valley, the heat-unit yield relationship is a straight line, amounting to some 2,400 day degrees per ton. Practically this same value holds good in the Sacramento Valley in California; but, progressing south, into longer and hotter seasons, with more heat units, it appears that the weather becomes too hot for the crop to make effective use of all the heat available. These are questions of plant metabolism, which differs with each species. Thus the yield decreases and the consumption of water per ton increases. The writer was quite surprised by the record in the Imperial Valley in California, where alfalfa grows every month in the year and the farmers harvest eight cuttings; yet the census shows that Imperial County had 114,146 acres of alfalfa in 1939 and that the average yield was 2.99 tons per acre. Other crop yield lines show similar characteristics. The writer traced several, but omitted them from Fig. 1 for simplicity.

It is unfortunate that some common denominator cannot be found, like total digestive nutrients, into which all the crops can be converted for any typical valley pattern; but the problem is not that simple. It is impossible directly to compare the apple orchards on the rolling hillsides of Wenatchee, Wash., with the forage and general crops of a flat and spring-frosty near-by area that may have the same total of heat units. In Yakima Valley, the Parker Heights area might produce the same tonnage of hay, potatoes, and sugar beets as do the bottom lands across the river; but the Heights also produces peaches, cherries, and pears worth several times as much per acre, because they are frost free. There are similar cases in the southwest, where citrus fruits, olives, and avocados occupy limited areas, usually the rolling foothills, while the wide plains below them are devoted to dairy and general farm crops. Imperial Valley does not grow alfalfa to make money, but rather to restore nitrogen and humus to the soils. These elements are later taken out by crops of early-season lettuce, cantaloupes, etc., which sell for many dollars per acre.

Messrs. Lowry and Johnson emphasize that their consumptive-use-of-water curve is not a simple and automatic method of solving the problem but that a full knowledge of all modifying factors is needed to develop good judgment in its use, if the best results are to be obtained. This limitation is even more true with the crop prediction method presented in this paper, although it does offer a quick way to find the relative productivity, quite closely, in terms of ordinary general crop patterns. Also, the upper curve of consumptive use per ton undoubtedly gives a relative measure of efficiency in the use of water for any given crop pattern; but a wide and thorough knowledge of agronomy and of agricultural values in the region under consideration is needed to convert that information into terms of dollars per acre, possible or probable. Irrigation engineers must have practical data to determine the question of project feasibility.

It is recognized that the crop-yield lines are only the median lines of a band, as yields are influenced in some degree by soil fertility, pests, winds, farm methods, etc. Average soil fertility has probably the greatest influence. Klamath Falls, Ore., shows a higher potato yield than Buhl, Idaho, probably because potatoes are grown on newer and richer lands. The richness of the soil solution determines the number of pounds of water a plant must drink to obtain a pound of plant food. Since plant food comes from the breakdown of extremely fine soil particles, the amount of plant food available in any soil, above the clay line, is roughly proportional to the percentage of fines in that soil. Because the alluvial soils of the Columbia River system were formed by degradation, the coarser particles stopping first and the finer particles being carried farther downstream, it is not surprising to find this steady increase in fertility reflected in that part of this curve. In Washington the Walla Walla-Lower Yakima lands have a mean diversion duty of about 4 acre-ft; the Henry's Fork lands average 8 acre-ft; and the Lower Yakima lands consume little more than half as much water to produce a ton of alfalfa as do the Henry's Fork lands. The rise in use per ton in the hotter regions must be ascribed mainly to increased unavoidable ground evaporation which forms part of valley consumptive use. If a proposed project is restricted to high-grade lands, the yields will be greater than those read from Fig. 1, which are drawn from average lands.

SUMMARY

This paper is submitted as a first step toward establishing a set of relationships which clearly exist and which may furnish a worthwhile tool in planning the integrated groups of irrigation and multipurpose projects that will make the best use of land and water resources along the west coast region of the United States. It is hoped that workers in this field will test it by additional data, use their knowledge and experience to correct and refine it, and so make these basic relationships as useful as possible in solving practical problems.

It is not to be expected that this method will serve as an easy substitute for detailed economic studies, in which the agricultural history of near-by and comparable developed areas is used to predict production on the project in question. The method will serve as a check on the validity of the areas selected for comparison and as an indicator of reasonable adjustments where exact parallel conditions cannot be found.

DISCUSSION

Arthur F. Johnson,[6] Assoc. M. ASCE.—An ingenious method of estimating average crop yields from the relation of such yields to available heat in day degrees Fahrenheit using the method for computing available heat presented by Robert L. Lowry, Jr., M. ASCE, and the writer[2] is suggested by the author. The method will be useful for reconnaissance studies where data on crop yields under irrigation are not readily available. It is not sufficiently reliable to serve as a close criterion of project feasibility. As the author wisely states, the method is more limited in estimating productivity than in estimating consumptive use of water due to the greater number of factors contributing to the variability of the results.

The author has confined his treatment to individual crops but has related yields to the average available heat rather than to individual effective heat values for each crop as suggested by C. R. Hedke,[7] M. ASCE. Somewhat more reliable results might be expected by developing separate curves using appropriate temperature bases for each crop.

The yield curves for both alfalfa and potatoes (Fig. 1) are surprisingly well supported by the plotted points representing the areas studied. The marked drop in yield beyond optimum heat conditions is characteristic of most crops. Alfalfa grows continuously where temperatures do not drop below freezing but in the Imperial Valley (California) and in other areas of extremely high sustained summer temperatures its growth is inhibited during the hottest part of the season and it becomes almost dormant. The growing season for potatoes, on the other hand, is comparatively short and it is actually possible to produce two crops in a single year although this would not be an acceptable practice because of the drain on soil fertility and risk of fostering potato diseases. It is presumed that the potato curve in Fig. 1 applies to a single planting so that materially higher total yields might be expected if repeated crops in the same year were included where climatic conditions permit. Nevertheless, as the limit of tolerance to heat is approached, yields may be expected to fall off to a negligible amount.

In Fig. 1, the curve "Consumptive Use per Ton of Alfalfa" was evidently derived by dividing values on the curve "Consumptive Use of Water—Valley Average" by corresponding points on the alfalfa yield curve. This procedure is not proper because the curve of average valley consumptive use represents a composite of consumptive use requirements for crop distributions typical of each heat condition. The percentage of alfalfa is not the same throughout the range of the consumptive use curve but rises from zero in frosty mountain meadows to a maximum and then drops with increasing heat. The percentage of alfalfa is high for the Snake River Valley (Idaho) but is relatively low for the Mesilla Valley (New Mexico) where cotton is "king." Therefore, the derived curve is somewhat lower than would be the case with true values of consumptive use of alfalfa, and the error increases in some degree with increasing heat as alfalfa is a relatively higher consumer of water than the average crop distribution which the Lowry-Johnson curve represents.

---

[6] Acting Chf., Div. of Eng. and Surveys, Branch of Project Planning, Bureau of Reclamation, Washington, D. C.

[7] "Consumptive Use of Water by Crops," by C. R. Hedke, New Mexico Engr.'s Office, July, 1924 (unpublished).

H. W. Van Loo,[8] Jun. ASCE.—The heat-unit method of estimating the consumptive use of water for agriculture extended to provide a method of estimating the productivity of lands on a proposed project may provide a very satisfactory system of forecasting the returns to any proposed development. However, the relatively low yield of alfalfa in the Imperial Valley (California), about which the late Mr. Muldrow expressed surprise, may not be entirely justifiable.

TABLE 1.—Annual Yield of Alfalfa, in Tons (Total) and Tons per Acre (Unit), for Coachella Valley, California

Year	Acreage	Total yield	Unit yield
1939..........	1,276	5,104	4.0
1940..........	1,400	9,100	6.5
1941..........	1,600	12,000	7.5
1942..........	1,100	5,500	5.0
1943..........	1,310	9,170	7.0
1944..........	1,243	8,079	6.5
1945..........	1,541	7,705	5.0
1946..........	1,741	12,187	7.0
Average.....	...	...	(6.1)

Although the writer does not propose to question the total yield of alfalfa shown for Imperial Valley, there may have been some misinterpretation of conditions. In a region such as Imperial Valley where alfalfa is used on considerable acreage for rejuvenation of the soil, the total yield cannot be divided by the total acreage under cultivation, since the total acreage is not always producing alfalfa during an entire year. A second factor that may have affected the results is that a considerable part of the total acreage is used for pasturing various kinds of livestock. For instance, range cattle may be purchased in Arizona, shipped to Imperial Valley, and fattened on alfalfa. More than likely the acreage used for pasturing has been included in the total acreage since some cutting of alfalfa is usually made, but no correction has been made for the decreased cuttings of alfalfa.

As supporting data for the contention that some error has been made, Table 1 shows the yield of alfalfa for near-by Coachella Valley which is about sixty miles north of Imperial Valley and has nearly identical weather conditions.

The average yield of 6.1 tons of alfalfa per acre for Coachella Valley is about twice as high as the 2.99 tons shown for Imperial Valley. Possibly, the light sandy soils which predominate in Coachella Valley plus the generally well-drained farm lands would account for some of the difference. Whatever the reason for the discrepancy, it is suggested that further examination be made of the yield of crops grown in the warmer sections of the United States.

Will H. Noble,[9] Assoc. M. ASCE.—The law of a heat-productivity relation is set forth admirably in this paper. Perhaps the popular concept that water makes the crop should give way to a concept of heat as the productive agent, with commensurate water an adjunct. The author's demonstrated heat-productivity relation is one of a succession of relations applying to the actual problem of economic feasibility, namely: (1) Water requirement varies with heat, (2) crop production varies with heat, (3) crop value varies with production, hence with heat, and (4) economic feasibility is a function of crop value, hence a function of heat. If procedure toward item (4) is necessarily by way of item (2), the methods of finding the (2) relation should be explored for possible improvement and standardization.

---

[8] Civ. Engr., Coachella, Calif.

[9] State Center, Iowa.

For instance, the heat-productivity relations studied and graphed for the paper appear to be based on the 1940 Census; that is, on 1939 crops, and (presumably)1939 seasonal heat. Since economic feasibility rests on a balance of average annual costs versus average net returns, ratios of average yield to average seasonal heat are required. A compilation of average heat-yield relations for the commonly irrigated crops, with probable deviations from such averages, and a recording of the most favorable heat range for each crop, would be a valuable addition to engineering statistics. It could be based on the United States Weather Bureau value for normal mean monthly temperatures above 32° F, and the annual reports of crop yields and values on United States Bureau of Reclamation projects, as far as they have been published. These crop reports cover existing projects in a block of fifteen states extending eastward from the Pacific coast to include North Dakota, South Dakota, Nebraska, Colorado, New Mexico, and Texas on the east. Deductions from so great a range of data should be of general application in any part of the United States where irrigation may be practiced or proposed. This is true whether the proposal is purely for irrigation or is a multiple-purpose project with irrigation one of a number of interests.

The widely accepted Lowry-Johnson system of heat units, in day degrees of maximum temperatures above 32° F, has quite naturally been used by the author. This system was evolved for solving the heat versus consumptive-use-of-water relation, closely akin to evaporation. For this purpose, use of the maximum daily temperature was found to give a closer correlation than was found with average daily temperature. Unless a similarly superior correlation is found for the heat versus productivity relation by use of the maximum temperatures, it would appear a simpler and more practical procedure to refer directly to the normal mean monthly temperatures of the Weather Bureau's annual summaries, and to arrive directly at the normal seasonal heat in degree months. In this case, the productivity season would naturally terminate with the average date of first killing fall frost.

Since heat determines yield, and yield determines crop value per acre, should there not be a definite heat versus value-per-acre relation? Increased heat means larger crops, or a greater number of crops per season, or higher value crops, all of which mean more dollars per acre, for more heat per acre. Project income is a composite income from a diversity of crops whose seasonal heat ranges include the seasonal heat of the particular project. Must the engineer determine probable yield and price for each crop, then assume some pattern of crop diversity, and so estimate the composite crop value; or does nature itself determine the diversity largely through the available heat? Can a heat versus crop-value graph be drawn that will give the desired preliminary estimate of farm income? The writer's study of Bureau of Reclamation crop statistics for the decade from 1921 to 1930, inclusive (the most stable decade of reclamation history as to commodity prices and construction costs), indicates that such a relation may be definitely charted, with correlations that favorably compare with those for the heat-productivity relation. If so, such a chart or graph should be a logical sequence to the main argument of the paper. For its application, the general price structure would need to be projected forward to the years of construction and operation of the project under consideration; but this must be done whether prices are to be applied item by item, or as a composite per-acre figure.

This discussion is intended to imply acceptance and appreciation of the paper, and a desire to see the idea extended and put into most usable forms. Whatever the form in which it is expressed, the author's well-worded caution against undue reliance on the average ratios found will be applicable. The individual irrigator may not safely assume that his own farm will live up (or down) to the project average income per acre; nor can the irrigation engineer assume that his own project will be an average one. That point he must determine by more particular investigation.

CHARLES KIRBY FOX,[10] M. ASCE.—The question at the end of the second paragraph of this paper—"What will be the returns" [of an irrigation project]? —is timely and shows that engineers are becoming more "cost conscious."

Two aspects of the paper, however, are open to serious discussion, namely: (1) The use of 32° F as a base and (2) the adoption of a range of available heat units in terms of thousands of degree-days above 32° F. Most plants begin growing when the mean temperatures reach from 42° F to 48° F. Since the rate of growth at temperatures lower than 50° F is slow, a mean monthly temperature of 50° F is a logical base. It is an easy base for computation purposes and is readily correlated with data issued by the United States Weather Bureau. The use of thousands of degrees per day above 32° F would complicate computations considerably. Elsewhere the writer has called attention to the advisability of using the 50° F base.[11]

The late Mr. Muldrow is correct in his observation that some plants cease to grow when the weather becomes too warm.

Closely allied to the question of growing temperatures are the following six factors: (*a*) The soils; (*b*) the quantity of water used: (*c*) the character of the water; (*d*) the transportation facilities; (*e*) the value of fluctuating dollar; and (*f*) special conditions and study required by some, often highly profitable, crops.

(*a*) *The Soils.*—The University of California, at Berkeley, has issued a bulletin[12] in which a soil classification is analyzed from a production point of view. The Soil Survey Bureau, United States Department of Agriculture, had reported on fifty-nine California areas of which six areas (more than 2,500,000 acres) were not complete and nine areas (more than 4,750,000 acres) were not computed because of the alkali content of the soil. The University of California bulletin was concerned with the remaining forty-four areas (15,000,000 acres). Its classification, briefly, is as follows:

---

[10] Cons. Engr., Los Angeles, Calif.

[11] Discussion by Charles Kirby Fox of "'A National Reclamation Policy': Report of the Committee of the Irrigation Division," *Transactions*, ASCE, Vol. 95, 1931, p. 1357.

[12] "A Method of Rating California Soils," by Walter F. Weir and R. Earl Storie, *Bulletin No. 599*, College of Agriculture, Univ. of California, Berkeley, Calif.

Class	Percentage of maximum yield
1	> 80
2	60 to 79
3	40 to 59
4	20 to 39
5	10 to 19
6	< 10

The average total producibility of classes 1, 2, and 3 was 70% or, roughly, two thirds. Classes 4, 5, and 6 are not considered suitable agricultural soils for cultivation and irrigation. The East Mesa of Imperial Valley in California is rated on the same basis. The area of California is given as 158,693 sq miles; and, since the foregoing soil survey is supposed to embrace most of the arable land in California, the forty-four soil-class areas have been reduced to a percentage of California area, as shown in Table 2. All these data show that the areas having the best soils are very limited.

TABLE 2.—SUMMARY OF SOIL SURVEY RESULTS (PERCENTAGES)

Class of soil	PRODUCTION BASED ON:		
	Forty-four areas	California	East Mesa
Acres	(14,858,561)	(101,563,520)	(222,025)
1	13.6	1.95	0.2
2	10	1.43	3.6
3	19.3	2.75	26.2
(Total)	(42.9)	(6.13)	(30.0)
4	....	....	12.5
5	....	....	23.2
6	....	....	34.3

(*b*) *The Quantity of Water Used.*—By plotting more than 2,500 determinations of the duty of water (including many multiple tests), it was determined that, if the quantity of water required to produce the maximum yields were known, it could be assumed that two thirds of that water will produce yields of 90%; and one and one-half times that maximum quantity will only produce yields of 66%, or two thirds.

(*c*) *The Character of the Water.*—Alkali or saline waters, depending on relative concentrations, cause a decline in yields. As proof of this fact one has only to examine the thousands of acres of land in old irrigation districts, which were formerly productive but are now waterlogged and saturated with alkali. This is particularly noticeable in the great Central Valley (California) and the Imperial Valley, as well as in the Yuma Project in Arizona. The acreage yields (not dollar) of the Imperial Valley and the Yuma Project were plotted for the period from 1926 to 1940 and showed a decline of approximately one half, in relation to comparable crop yield data for Los Angeles County in California.

(*d*) *The Transportation Facilities.*—In some cases the transportation facilities will limit the yield of commercial crops.

M. R. LEWIS,[13] M. ASCE.—A very interesting method of forecasting the productivity of an area has been presented. However, it is based on climatic conditions; and, because climate is only one of a number of important factors in crop production, it is obvious that any forecast which does not include other factors, particularly soil characteristics, must be subject to wide variation when applied to a single project.

---

[13] With Region 1, U. S. Bureau of Reclamation, Boise, Idaho.

The author suggests that the crop census data secured by the United States Bureau of Reclamation would furnish a basis for a very interesting extension of his study. In the limited time available the writer has attempted to do this for the projects in Region 1 of the Bureau, which is essentially the Columbia River drainage area. Figs. 2(*a*), 2(*b*), and 2(*c*) show the relation between the yield of alfalfa, potatoes, and sugar beets, respectively, to the total heat units on those projects in the region where sufficient acreage of the given crops is grown to make the results significant. In each figure has been indicated, in addition, the comparable data for two Bureau projects outside Region 1. These two projects have been chosen to indicate the wide variation which is found when projects scattered over a wide area are used for this comparison. The yield data are the 5-year averages, 1941 to 1945.

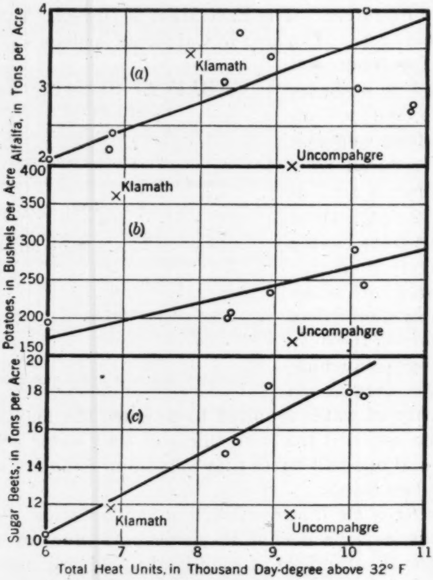


FIG. 2

It will be noted that the sugar beet and potato yields for Region 1 projects conform to a linear relationship. Alfalfa yields do not follow this relationship so closely. As a general proposition, it seems doubtful if any one crop can be used to indicate the productivity over the extremely wide range suggested by the author. It is well known that, in practice, alfalfa in such areas as the Imperial and Salt River valleys grows very little during the extremely hot months. Potatoes have such a short growing season that they can use only a part of the total heat in most sections of the west. It is possible to grow two crops of potatoes, one after the other, in a single season in areas where the total heat units amount to 10,000 day degrees or more. This is not a usual practice and it is therefore impossible to tell what the total production of potatoes might be if the whole of the long growing season were used in the hotter sections; nor do sugar beets require all the growing season in the more southern valleys. It, therefore, appears that no crop is available which could be used to show the productivity over a very wide range—although it seems probable that in the hotter climates sugar cane might be used for this purpose.

Harry F. Blaney,[14] **M. ASCE.**—For many years the writer has been interested in the relationships between temperature, consumptive use of water, and crop yield. The late Mr. Muldrow is to be complimented on his ingenious method of estimating the productivity of lands on a proposed irrigation project, and his paper should encourage further research on the subject. However, the writer does not agree with some of the statements and results presented.

The author states (under the heading, "Graph") that he was

> "* * * quite surprised by the record in the Imperial Valley in California, where alfalfa grows every month in the year and the farmers harvest eight cuttings; yet the census shows that Imperial County had 114,146 acres of alfalfa in 1939 and that the average yield was 2.99 tons per acre."

The writer does not question the census data, but they probably do not reflect the tremendous amounts of green alfalfa produced and fed to livestock shipped into Imperial Valley from other areas. The Imperial County Agricultural Commissioner reports that in 1939 approximately 75,000 head of cattle and 120,000 head of sheep were fattened. Table 3 is taken from records of the annual crop and livestock reports of the Imperial Irrigation District.

TABLE 3.—Head of Livestock Fed in Imperial Valley

Year	Cattle	Sheep	Dairy	Hogs
1942......	84,420	81,080	15,200	42,010
1943......	75,260	81,620	13,050	39.005
1944......	77,200	113,700	12,776	19,817
1945......	116,730	140,593	11,355	18,679

During the winter months livestock are brought into the valley, fed green alfalfa, and are then shipped out to market. The same fields which produce the baled hay also provide the winter pasturage. There is very little permanent pasture in Imperial Valley. In other words, the census data more nearly reveal the amounts of alfalfa harvested and sold, rather than produced; also, the actual yield of alfalfa in the valley is higher than 2.99 tons per acre.

A report on cost of production studies, conducted in 1943 on several representative farms by the Agricultural Extension Service of the University of California and the United States Department of Agriculture, in cooperation with alfalfa growers in Imperial County, states:

---

[14] Senior Irrig. Engr., Div. of Irrig., SCS, Research, U. S. Dept. of Agriculture, Los Angeles, Calif.

"The average yield for all records including that utilized as pasture was 4.64 tons per acre, ranging from 3.5 to 6.21. Average cost per acre was $72.75; income $104.17 and the net profit $31.42."[15]

Mr. Muldrow's statement that "Imperial Valley does not grow alfalfa to make money * * *" may have been true some years ago. However, in recent years the value of Imperial Valley's alfalfa crop was seldom exceeded by that of any other single crop produced in the valley. For example, the Imperial Irrigation District reports that, in 1943, 114,085 acres produced 16,585 carloads of alfalfa valued at $5,970,600; and, in 1945, when the price of alfalfa was good and hay was scarce, 158,713 acres produced 34,343 carloads valued at $11,848,335. Owners are leasing land suitable for growing alfalfa for more than $40.00 per acre per yr.

TABLE 4.—CONSUMPTIVE USE OF WATER BY ALFALFA IN IRRIGATED AREAS

Location (1)	Mean annual temperature (degrees F) (2)	Season (3)	Consumptive use[a] (ft) (4)
Spokane, Wash.	48	Apr. 12–Oct. 13	2.5
Walla Walla, Wash.	53	Mar. 31–Nov. 5	3.1
American Falls, Idaho	46	May 26–Sep. 16	1.7
Boise, Idaho	50	May 1–Oct. 5	2.5
Cheyenne, Wyo.	45	May 15–Oct. 2	1.9
Worland, Wyo.	44	May 10–Sep. 27	2.1
Missoula, Mont.	45	May 18–Sep. 23	1.8
Logan, Utah	48	May 7–Oct. 11	2.3
Saint George, Utah	....	Apr. 10–Oct. 23	3.1
Sacramento, Calif.	60	Apr. 1–Oct. 31	3.1
Bakersfield, Calif.	65	Apr. 1–Oct. 31	3.3
Antelope Valley, Calif.	62	Apr. 1–Oct. 31	3.2
Imperial Valley, Calif.	73	Jan. 1–Nov. 30[b]	3.7
Phoenix, Ariz.	72	Mar. 1–Nov. 30[b]	3.7
Mesilla, New Mex.	60	Apr. 1–Oct. 31	3.2
Carlsbad, New Mex.	63	Apr. 1–Oct. 31	3.2
Lubbock, Tex.	60	Apr. 1–Oct. 31	3.2
Fort Stockton, Tex.	65	Apr. 1–Oct. 31	3.3
Altus, Okla.	62	Apr. 1–Oct. 31	3.3

[a] Computed from mean monthly temperatures (t), per cent of daytime hours (p), and consumptive-use coefficient (k); consumptive use $= t\,p\,k$. [b] Month of July excluded for rest period.

As indicated by the author, the "consumptive-use-of-water" curve in Fig. 1 represents average valley crops during the growing season. Studies made by the writer in the Upper Rio Grande Basin in New Mexico[16] and in other areas throughout the west, indicate that consumptive use of water shown at the upper end of this curve is probably more representative of the annual consumptive use by alfalfa. For instance, the annual consumptive use of water by alfalfa in Mesilla Valley in New Mexico[16] has been estimated at 4.0 ft, whereas the average annual valley consumptive use is only 2.8 ft. Alfalfa uses more water than other crops. The consumptive use of water by potatoes should not exceed 2 ft. Table 4 has been prepared to show seasonal consumptive use by alfalfa in various areas of the west. Briefly, the procedure[17] is to correlate existing consumptive use data with monthly temperature, per cent of daytime hours, and growing season. The coefficients so developed for different crops are used to transfer consumptive use data from one section to other areas where only climatological data are available.

---

[15] "1943 Alfalfa Production Cost and Efficiency Analysis, Imperial County, Calif.," Agri. Extension Service, Univ. of Calif., Berkeley, and U. S. D. A., cooperating with alfalfa growers, Imperial County, Calif. (mimeographed).

[16] "Water Utilization, Upper Rio Grande Basin, Part III," by Harry F. Blaney, P. A. Ewing, O. W. Israelsen, C. Rohwer, and F. C. Scobey, National Resources Committee, Washington, D. C., February, 1938.

[17] "A Method of Estimating Water Requirements in Irrigated Areas from Climatological Data," by Harry F. Blaney and Wayne D. Criddle, SCS, U. S. D. A., June, 1945 (mimeographed).

The curve "Tons per acre, alfalfa hay = 2,400 day degrees per ton" (Fig. 1) is probably incorrect inasmuch as the author did not have all the facts. For instance, the alfalfa yield data are county averages, whereas temperatures may not represent the area where most of the crop is grown. Most of the alfalfa in Los Angeles County is grown in Antelope Valley and other interior valleys, where the temperatures are similar to those for Bakersfield and Riverside, Calif., rather than for those of Los Angeles. The available heat units in Antelope Valley are about the same as in Bakersfield, and are somewhat higher than those in Los Angeles. If the yield of 5.5 tons of alfalfa per acre, shown for Los Angeles, is moved to the left in Fig. 1 to about 16,500 available degree days, and the yield for Imperial Valley is increased from 3.0 to 4.5 tons per alfalfa, the writer believes the resulting curve would more nearly represent field conditions.

The writer agrees with the thoughts expressed in the "Summary." It is a challenge to both engineers and research workers to improve procedure for forecasting the productivity of irrigable lands and the feasibility of proposed irrigation projects.

R. L. PARSHALL,[18] Assoc. M. ASCE.—The production of crops may be said to depend largely on the nature of the soil. However, three other elements are essential, such as moisture, food, and temperature. From the standpoint of forecasting productivity or yield from the land, it is necessary to have in mind the crop to be grown. Some soils, covering rather large areas, are of such a nature as to possess the essential food elements. They are capable of utilizing irrigation water efficiently and economically, and are located geographically so as to give a sufficiently long growing season to permit the various crops to mature. This is, more or less, an ideal condition, and many such areas exist throughout the irrigated west. There are limited areas where food, moisture, and temperature are favorable but where the soil is not suitable for general crops, regardless of how economically adaptable it is to the growing of special crops (such as fruit, trees, or shrubs).

The title of this paper, "Forecasting Productivity of Irrigable Lands" seems to infer that the problem involved is to determine the economic return through agricultural crops to be grown on lands not yet developed. It is quite obvious that, to arrive at an approximation, many factors are to be considered. Since none of these factors are definitely fixed, when they are brought together in the final analysis the conclusion can only be generally stated. The ultimate aim of the idea of productivity is that of approaching maximum yield of crop; and from the standpoint of irrigation it might be inferred that more water means more crop.

Under favorable soil conditions, food and temperature tend to bear out this contention for certain crops, but generally this is not true. Since productivity implies plant growth and development, it is known that the ordinary field crops have no conservation characteristics as far as soil moisture is concerned. As long as moisture is present the physiological development of the plant proceeds and, to promote a profitable yield, water as irrigation must be applied to prevent the growth from suffering a dormant stage or a setback in development. High temperatures augument this condition. The water supply available for application at timely or critical use is vastly more important than would be the factor of temperatures in the evaluation of forecasting productivity of irrigable lands.

---

[18] Senior Irrig. Engr., Bureau of Agri. Eng., U. S. Dept. of Agriculture, Colorado Agri. Experiment Station, Fort Collins, Colo.

HOWARD N. MAGNESS,[19] ESQ.—It is gratifying that the paper by Mr. Muldrow has resulted in such appropriate and beneficial discussion. The writer feels his inability to comment on the discussions and to prepare a closing discussion in the way Mr. Muldrow would have done. Nonetheless, in view of his professional association and fellowship with Mr. Muldrow, the writer welcomes the opportunity to conclude the discussion.

After many years of experience on irrigation planning, management and accomplishments and observations on changes in land use with specialization of crops and increased production, Mr. Muldrow frequently stated that he believed preliminary forecasts of production and basic economic estimates in support of projects were too conservative. The comments of the several discussers indicate a growing desire for more complete economic consideration of returns from proposed improvements. This tendency is rapidly assuming greater importance, especially in the more costly projects involving major engineering developments that affect large segments of land and population, and can result only in a firmer foundation for future planning.

It is realized that the material used by Mr. Muldrow was not as complete as might be desirable. The comments of Messrs. Lewis and Johnson concerning the use of additional crops are pertinent, as are those of Messrs. Blaney and Van Loo, concerning the discrepancy of yields in the Imperial Valley in California. The point raised by Mr. Fox relative to most desirable temperature base is considered reasonable, and comparison of the effect of the two temperature bases mentioned would be desirable.

The suggestion of Mr. Noble that further investigation be made toward evolving a final cash return per acre is especially helpful. It is true that, in general, crops produced in areas of high temperatures have as high a value—or perhaps an even higher value—per pound or other unit of production, as those grown in cooler areas. For this reason, where the temperatures exceed the optimum for the growth of potatoes or alfalfa, they should be replaced by other crops adapted to these temperatures, which possibly have an even higher yield value per acre.

It is realized that the basic paper and the discussions concerning it are too brief to present, fully, even this one feature of irrigation planning—that is, considerations of temperature in relation to crop production. Although vital, temperature is still only one of the factors affecting a project. Mr. Muldrow realized that his paper was only introductory to a complex and very interesting point of investigation. Further study and development of the subject would be most desirable.

---

[19] Engr. (Soils), Portland Dist., Corps of Engrs., Dept. of the Army, Portland, Ore.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2344

CONTINUOUS FRAME ANALYSIS BY ELASTIC SUPPORT ACTION

By J. Charles Rathbun,[1] M. ASCE, and C. W. Cunningham,[2] Assoc. M. ASCE

With Discussion by Messrs. L. J. Mensch, Frederick S. Merritt, I. Oesterblom, Thomas C. Kavanagh, A. Floris, Tao King, Leroy A. Beaufoy, A. A. Eremin, Robert B. B. Moorman, Eduardo Agramonte, Stephen J. Fraenkel and Robert L. Janes, Phil M. Ferguson, Ralph W. Stewart, Maurice Barron, Edwin H. Gaylord, William A. Conwell, Harry A. Garfinkel, and J. Charles Rathbun and C. W. Cunningham.

Synopsis

The purpose of this paper is to present a "classical" or "exact" method of analyzing continuous frames which requires the direct solution of very few, if any, simultaneous equations. The greatest objection to methods of slope deflection, least work, and virtual work is the labor involved in the solution of these equations. There is some parallelism in form with the moment distribution method but the fundamental ideas are entirely different. The repeated cycles are entirely avoided. Reference is made to this parallelism to clarify the derivations and to help in visualizing the several steps in the solution of the problems.

The procedure recommended for the solution of problems requires the calculation of certain quantities which are analogous to the stiffness factors, carry-over factors, and fixed-end moments of moment distribution. Equations are developed for these quantities using a beam with elastic supports. This support condition is a result of the elasticity of the members of the frame adjacent to the supports. The equations have been previously developed using the flexibility of the connections as the elasticity.[3] The method is particularly well adapted to the analysis of frames with flexible connections.

The necessary equations are derived, the general method of procedure is outlined, and then the solution of a progressive series of problems is given in detail or outlined so that it can be followed readily.

Note.—Published in April, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Prof., Civ. Eng., College of the City of New York, New York, N. Y.

[2] Asst. Prof., Civ. Eng., College of the City of New York, New York, N. Y.

[3] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 101, 1936, pp. 548–552 and 558.

INTRODUCTION

The method is presented as a series of problems beginning with a simple case and then progressing to more complicated cases as the technique of solution becomes more apparent. A tabular arrangement of required values may be used since the solution follows a fixed pattern.

PART A. DERIVATION OF EQUATIONS

The letter symbols in this paper are defined where they first appear, in the text or by illustration.

The properties of the conjugate beam as presented in standard texts on strength of materials[4,5] are used in the derivations. If the conjugate beam is loaded with the $\frac{M}{E\,I}$-diagram of the given beam, the resulting shear and bending moment at any point are equal, respectively, to the slope and deflection at the corresponding point on the elastic curve of the given beam.

The conjugate beam has the same span length as the given beam. The reactions must be consistent with the $\frac{M}{E\,I}$-loading, and they must also produce shears and moments consistent with the conditions imposed on the given beam. The sign convention will be correct if the following notation is used:

(1) The shear (slope) at any section is positive if the resultant of all forces $\left(\frac{M}{E\,I}\right)$ to the left of the section acts in a positive or upward direction;

(2) The bending moment (deflection) at any section is positive if it causes compression in the upper fibers of the beam;

(3) The $\frac{M}{E\,I}$-diagram has the same sign as the M-diagram and is taken as an upward load if positive; and

(4) The left reaction of the conjugate beam is equal to the slope of the elastic curve at the left support. Similarly, the right reaction is equal to the slope at the right support with its sign changed.

In any continuous frame one or both of the supporting points of a member are elastic by virtue of the flexibility of the other members framing into these points. This elasticity (which is assumed to be the result of flexural deformation only) may be measured by the angular change of a section at the point resulting from a unit moment, assuming that the member under consideration is not acting.

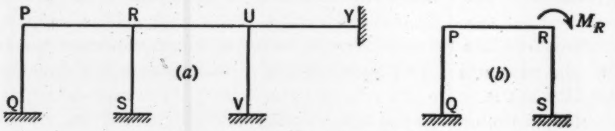


FIG. 1.—TYPICAL CONTINUOUS FRAME

---

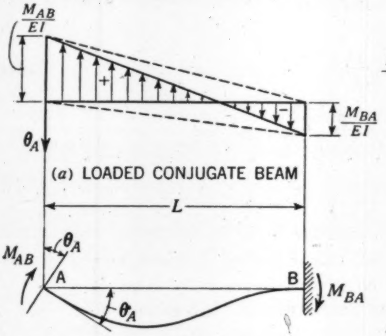
[4] "Resistance of Materials," by Fred B. Seely, John Wiley & Sons, Inc., New York, N. Y., 2d Ed., 1935, pp. 298 and 312.

[5] "Elements of Strength of Materials," by S. Timoshenko and G. H. McCullough, D. Van Nostrand Co., Inc., New York, N. Y., 2d Ed., 1940, pp. 167 and 183.

Member RU, Fig. 1(*a*), represents a beam on elastic supports. The elasticity of support U depends on the flexibility of members UV and UY and the elasticity of support R depends on the flexibility of members RS, RP, and PQ. At each of these supports these flexibilities can be combined into one quantity, called the "elastic factor," Z. The method of computing and using this quantity and others required for the analysis will be shown.

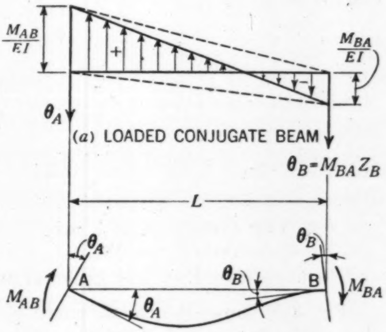
1. The Carry-Over Factor and the Elastic Factor for a Prismatic Beam Having a Fixed Support at One End

Fig. 2(*b*) shows the elastic curve of a given beam AB having a span of length L in which point B is a support assumed to be fixed. Beam UY in Fig. 1(*a*) is such a member. The beam is rotated at point A by an applied moment M_{AB} which produces an induced moment M_{BA} at support B. The loaded conjugate beam is shown in Fig. 2(*a*). Since the slope of the given beam AB at point B is zero, the reaction of the conjugate beam B is also zero. For convenience the $\frac{M}{EI}$-diagram of Fig. 2(*a*) has been divided into two triangles (one positive and one negative) by the dotted lines.


$\frac{M_{AB}}{EI}$
+
−
$\frac{M_{BA}}{EI}$
θ_A
(*a*) LOADED CONJUGATE BEAM
L
M_{AB}
θ_A
A
B
M_{BA}
θ_A

(*b*) LOAD AND DEFLECTION DIAGRAM

Fig. 2.—Fixed Support at One End


$\frac{M_{AB}}{EI}$
+
$\frac{M_{BA}}{EI}$
−
θ_A
(*a*) LOADED CONJUGATE BEAM
$\theta_B = M_{BA} Z_B$
L
θ_A
θ_B
θ_B
A
B
M_{BA}
M_{AB}
θ_A

(*b*) LOAD AND DEFLECTION DIAGRAM

Fig. 3.—Elastic Support at Both Ends

Taking moments about point A on the conjugate beam: $\frac{M_{AB}}{EI} \times \frac{L}{2} \times \frac{L}{3} - \frac{M_{BA}}{EI} \times \frac{L}{2} \times \frac{2L}{3} = 0$; or

$$M_{BA} = \tfrac{1}{2} M_{AB} \qquad (1)$$

In Eq. 1, the fraction $\frac{1}{2}$ is the well-known "carry-over factor" for a fixed support at point B. The "elastic factor" is found by computing the reaction θ_A of the conjugate beam. Since θ_B is zero, θ_A is equal to the area of the area of the $\frac{M}{EI}$-diagram; or

$$\theta_A = \frac{M_{AB}}{EI} \times \frac{L}{2} - \frac{M_{BA}}{EI} \times \frac{L}{2} \qquad (2a)$$

from which, by Eq. 1,

$$\theta_A = \frac{M_{AB}\,L}{4\,E\,I} \quad \text{(2b)}$$

In Eq. 2b $\frac{L}{4\,E\,I}$ is the "elastic factor" or angular change at point A due to the application of a unit moment. Representing the elastic factor at point A by Z_A, with the far end, B, fixed:

$$Z_A = \frac{L}{4\,E\,I} \quad \text{(3)}$$

If point B is a hinged support, the $\frac{M}{E\,I}$-diagram is a single triangle since M_{BA} is zero, and

$$\theta_A = \frac{M_{AB}\,L}{3\,E\,I} \quad \text{(4)}$$

Since the elastic factor Z_A is equal to $\frac{\theta_A}{M_{AB}}$, it follows:

$$Z_A = \frac{L}{3\,E\,I} \quad \text{(5)}$$

2. THE CARRY-OVER FACTOR AND THE ELASTIC FACTOR FOR A PRISMATIC BEAM HAVING AN ELASTIC SUPPORT AT END B, THE END FARTHEST FROM THE APPLIED MOMENT

Fig. 3(b) shows the elastic curve of a beam AB similar in every respect to that of Fig. 2 except that the slope θ_B at end B is proportional to M_{BA}. From the definition of Z:

$$\theta_B = M_{BA}\,Z_B \quad \text{(6)}$$

Taking moments about end A on the conjugate beam: $-\theta_B\,L - \frac{M_{BA}}{E\,I} \times \frac{L}{2} \times \frac{2\,L}{3} + \frac{M_{AB}}{E\,I} \times \frac{L}{2} \times \frac{L}{3} = 0$. Substituting $M_{BA}\,Z_B$ for θ_B and multiplying by $-\frac{3\,E\,I}{L}$: $3\,E\,I\,M_{BA}\,Z_B + M_{BA}\,L - \frac{M_{AB}\,L}{2} = 0$; $(L + 3\,E\,I\,Z_B)\,M_{BA} = \frac{M_{AB}\,L}{2}$; and

$$M_{BA} = \frac{M_{AB}}{2} \times \frac{L}{L + 3\,E\,I\,Z_B} = \frac{M_{AB}}{2} \times \frac{L}{L_{AB}} \quad \text{(7)}$$

in which $\frac{L}{2\,L_{AB}}$ is the carry-over factor when support B is elastic. The quantity $(L + 3\,E\,I\,Z_B)$ will be called the "elastic length" of the beam with respect to the support B and will be denoted by L_{AB} to simplify the form of the equations involving this expression; thus:

$$L_{AB} = L + 3\,E\,I\,Z_B \quad \text{(8)}$$

The "elastic factor" is obtained by computing the reactions, θ, of the conjugate beam in Fig. 3(a):

$$-\theta_A L + \frac{M_{AB}}{E I} \times \frac{L}{2} \times \frac{2 L}{3} - \frac{M_{BA}}{E I} \times \frac{L}{2} \times \frac{L}{3} = 0 \ldots\ldots\ldots (9a)$$

and

$$\theta_A = \frac{M_{AB} L}{3 E I} - \frac{M_{BA} L}{6 E I} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9b)$$

From Eq. 7,

$$\theta_A = \frac{M_{AB} L}{3 E I} - \frac{M_{AB} L^2}{12 E I L_{AB}} \ldots\ldots\ldots\ldots\ldots\ldots (10a)$$

That is,

$$\theta_A = M_{AB} \left(\frac{4 L L_{AB} - L^2}{12 E I L_{AB}} \right) \ldots\ldots\ldots\ldots\ldots\ldots (10b)$$

and

$$Z_A = \frac{\theta_A}{M_{AB}} = \frac{4 L L_{AB} - L^2}{12 E I L_{AB}} = \frac{L}{3 E I} \left[\frac{L_{AB} - \frac{L}{4}}{L_{AB}} \right] \ldots\ldots\ldots (11)$$

Eq. 11 reduces to Eq. 3 when point B is fixed since Z_B is zero and L_{AB} becomes L.

The general use of Eq. 11 will be shown by outlining the procedure for computing the elastic factor for the frame RPQ of Fig. 1(a). The elastic factor Z_{RPQ} depends on the flexibility of the column of PQ which may be obtained from Eq. 3. Therefore, Z_{RPQ} is found by substituting Z_{PQ} from Eq. 3 in Eq. 11. Inasmuch as Z_A in Eq. 11 is a function of Z_B, the latter must be computed first. Thus, one must start with a point such as a fixed or hinged joint whose rigidity is known.

The procedure for computing the elastic factor for part of a frame such as QPRS of Fig. 1(a) will now be given. Fig. 1(b) shows the part QPRS isolated as a free body and subjected to an external positive moment M_R. The thrust and shear have been omitted from the sketch for simplicity. Since the sum of the moments about a joint is zero, the sum of the moments on joint R from RPQ and RS is equal but opposite in sign to the moment M_R. The elastic factor representing the combined flexibility of RPQ and RS is found by noting that each of the aforementioned moments is proportional to the stiffness factor of the member which is equal to the reciprocal of the elastic factor. The moment relation at joint R is

$$M_R = -(M_{RS} + M_{RP}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Since $\theta = M Z$,

$$\frac{\theta_R}{Z_R} = -\left(-\frac{\theta_R}{Z_{RS}} - \frac{\theta_R}{Z_{RPQ}} \right) \ldots\ldots\ldots\ldots\ldots\ldots (13a)$$

and

$$\frac{1}{Z_R} = \frac{1}{Z_{RS}} + \frac{1}{Z_{RPQ}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13b)$$

in which Z_R is the angular change at joint R when a unit moment is applied to the part QPRS at that joint.

3. End Moments Due to Transverse Loading for a Prismatic Beam on Elastic Supports

Fig. 4(*b*) represents the elastic curve of a beam AB of span length *L*, loaded transversely and having two flexible supports A and B. The moment diagram for the given transverse loads acting on a simple beam of length *L* is shown in Fig. 4(*c*). The moment diagrams due to M_{AB} and M_{BA} are shown in Figs. 4(*d*) and 4(*e*), respectively. The principle of superposition of the simple-beam moment diagrams is used to obtain the moment diagram for the loads and moments acting simultaneously. The resulting curve, divided by EI, is shown in Fig. 4(*a*).

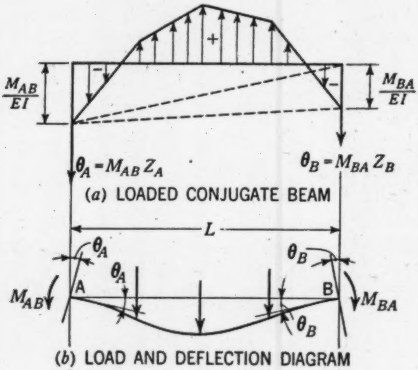

+
−
−
$\frac{M_{AB}}{EI}$
$\frac{M_{BA}}{EI}$
$\theta_A = M_{AB}Z_A$
$\theta_B = M_{BA}Z_B$
(a) LOADED CONJUGATE BEAM
L
θ_A
θ_A
θ_B
θ_B
A
B
M_{AB}
M_{BA}
(b) LOAD AND DEFLECTION DIAGRAM
$\bar{x}_A$
$\bar{x}_B$
+
Centroid
Area = A
(c) M− DIAGRAM DUE TO GIVEN LOADS
$\frac{L}{3}$
$\frac{2L}{3}$
−
Centroid
M_{AB}
(d) M_{AB}-DIAGRAM
$\frac{2L}{3}$
$\frac{L}{3}$
−
Centroid
M_{BA}
(e) M_{BA}-DIAGRAM

Fig. 4.—Beam on Elastic Supports, Loaded Transversely

Using the $\frac{M}{EI}$-diagram in Fig. 4(*a*) as a load, the reactions, θ_A and θ_B, of the conjugate beam are found by taking moments about supports B and A, respectively. The rotation θ in each equation is then replaced by its corresponding value, MZ, and the equations are solved simultaneously to obtain formulas for the end moments. Taking moments about end B: $-\theta_A L - \frac{M_{AB}}{EI} \times \frac{L}{2} \times \frac{2L}{3} - \frac{M_{BA}}{EI} \times \frac{L}{2} \times \frac{L}{3} + \frac{A}{EI} \times \bar{x}_B = 0$; $3EI\theta_A + M_{AB}L + \frac{M_{BA}L}{2} - \frac{3A\bar{x}_B}{L} = 0$; $3EIM_{AB}Z_A + M_{AB}L + \frac{M_{BA}L}{2} - \frac{3A\bar{x}_B}{L} = 0$; and

$$M_{AB}(L + 3EIZ_A) + \frac{M_{BA}L}{2} = \frac{3A\bar{x}_B}{L} \qquad \text{(14a)}$$

For brevity, let:

$$L + 3EIZ_A = L_{BA} \qquad \text{(14b)}$$

—from which:

$$M_{AB}L_{BA} + \frac{M_{BA}L}{2} = \frac{3A\bar{x}_B}{L} \qquad \text{(15a)}$$

Similarly, by taking moments about point A:

$$M_{BA}\,L_{AB} + \frac{M_{AB}\,L}{2} = \frac{3\,A\,\bar{x}_A}{L} \qquad (15b)$$

The solution of the simultaneous equations (Eqs. 15*a* and 15*b*) is given by Eqs. 16, as follows:

$$M_{AB} = \frac{6\,A}{L}\left(\frac{2\,L_{AB}\,\bar{x}_B - L\,\bar{x}_A}{4\,L_{AB}\,L_{BA} - L^2}\right) \qquad (16a)$$

and

$$M_{BA} = \frac{6\,A}{L}\left(\frac{2\,L_{BA}\,\bar{x}_A - L\,\bar{x}_B}{4\,L_{AB}\,L_{BA} - L^2}\right) \qquad (16b)$$

If one end, such as B, is a hinged support, $M_{BA} = 0$. The numerator and denominator of Eq. 16*a* are then divided by L_{AB} producing

$$M_{AB} = \frac{6\,A}{L}\left[\frac{2\,\bar{x}_B - \dfrac{L\,\bar{x}_A}{L_{AB}}}{4\,L_{BA} - \dfrac{L^2}{L_{AB}}}\right] \qquad (16c)$$

For a hinge, $Z = \infty$; consequently, $L_{AB} = \infty$ by Eq. 8. The last term in both the numerator and the denominator of Eq. 16*c* becomes zero. This equation then reduces to

$$M_{AB} = \frac{3\,A\,\bar{x}_B}{L_{BA}\,L} \qquad (16d)$$

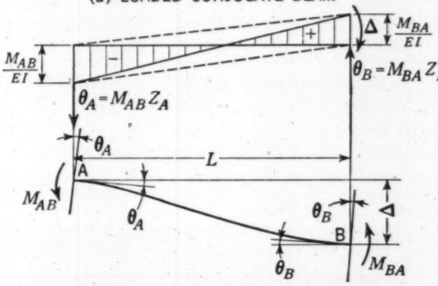
If ends A and B are fixed supports, L_{AB} and L_{BA} of Eqs. 16*a* and 16*b* are replaced by L since Z_A and Z_B are both zero. The resulting equations are the well-known formulas for fixed-end moments in the moment distribution method:

$$M_{AB} = \frac{2\,A}{L^2}\,(2\,\bar{x}_B - \bar{x}_A) \qquad (17a)$$

and

$$M_{BA} = \frac{2\,A}{L^2}\,(2\,\bar{x}_A - \bar{x}_B) \qquad (17b)$$

(*a*) LOADED CONJUGATE BEAM



(*b*) LOAD AND DEFLECTION DIAGRAM

FIG. 5.—UNEQUAL SETTLEMENT OF SUPPORTS

4. END MOMENTS DUE TO THE RELATIVE DISPLACEMENT OF THE ENDS OF A PRISMATIC BEAM ON ELASTIC SUPPORTS

These displacements are perpendicular to the longitudinal axis of the beam. Fig. 5(*b*) shows the elastic curve of a beam AB of span length L when the end B is displaced a distance Δ relative to end A. The induced moments are M_{AB} and M_{BA}. The loaded conjugate beam is shown in Fig. 5(*a*). The displacement of end B, shown as a negative quantity in Fig. 5(*b*), is represented as a negative moment Δ acting on the conjugate beam at support B. Taking moments about end B on the conjugate beam: $-\theta_A L - \frac{M_{AB}}{E I} \times \frac{L}{2} \times \frac{2 L}{3} + \frac{M_{BA}}{E I} \times \frac{L}{2} \times \frac{L}{3} + \Delta = 0$; and

$$M_{AB} L + 3 E I \theta_A - \frac{M_{BA} L}{2} = \frac{3 E I \Delta}{L} \quad \text{(18)}$$

Substituting the value of θ_A and also L_{BA} for $L + 3 E I Z_A$:

$$M_{AB} L_{BA} - \frac{M_{BA} L}{2} = \frac{3 E I \Delta}{L} \quad \text{(19a)}$$

Similarly, taking moments about end A and substituting the corresponding values for θ_B and $L + 3 E I Z_B$ (Eq. 8):

$$M_{BA} L_{AB} - \frac{M_{AB} L}{2} = \frac{3 E I \Delta}{L} \quad \text{(19b)}$$

The solution of the simultaneous equations (Eqs. 19) is given by Eqs. 20, as follows:

$$M_{AB} = \frac{6 E I \Delta}{L} \left(\frac{2 L_{AB} + L}{4 L_{AB} L_{BA} - L^2} \right) = \frac{3 E I \Delta}{2 L} \left[\frac{2 L_{AB} + L}{L_{AB} L_{BA} - \frac{L^2}{4}} \right] \quad \text{(20a)}$$

and

$$M_{BA} = \frac{6 E I \Delta}{L} \left(\frac{2 L_{BA} + L}{4 L_{AB} L_{BA} - L^2} \right) = \frac{3 E I \Delta}{2 L} \left[\frac{2 L_{BA} + L}{L_{AB} L_{BA} - \frac{L^2}{4}} \right] \quad \text{(20b)}$$

If points A and B are fixed supports, L_{AB} and L_{BA} of Eqs. 20 are replaced by L since Z_A and Z_B are both zero. Then these equations reduced to the joint translation equations of moment distribution:

$$M_{AB} = M_{BA} = \frac{6 E I \Delta}{L^2} \quad \text{(21)}$$

5. Flexible Connections

In general, the flexibility of the connections can be taken into consideration by increasing the elastic length of the beams. The "elastic length" of the member used in the computations will then be

$$L' = L + 3 E I Z_J + 3 E I Z_C \quad \text{(22)}$$

in which Z_J refers to the joint and Z_C to the connection. The value of Z_C for the connection must be obtained by experiment or by other methods, as explained elsewhere.[3]

However, Eq. 11 cannot be used for flexible connections because θ_A in this formula is the result of joint rotation only. Therefore, it will be necessary to derive a new expression for the "elastic factor."

Since the rotation at the joint is equal to the rotation of the end of the beam plus the angular change in the connection:

$$\theta_{AJ} = \frac{M_{AB}\,L}{3\,E\,I}\left[\frac{L_{AB} - \dfrac{L}{4}}{L_{AB}}\right] + \theta_{AC} \ldots\ldots\ldots\ldots\ldots (23a)$$

and

$$Z_{AJ} = \frac{\theta_{AJ}}{M_{AB}} = \frac{L}{3EI}\left[\frac{L_{AB} - \dfrac{L}{4}}{L_{AB}}\right] + Z_{AC} = \frac{L}{3EI}\left[\frac{L_{AB} + 3\,E\,I\,\dfrac{L_{AB}}{L}\,Z_{AC} - \dfrac{L}{4}}{L_{AB}}\right]$$

$$= \frac{L\,L_{AB} + 3\,E\,I\,Z_{AC}\,L_{AB} - \dfrac{L^2}{4}}{3\,E\,I\,L_{AB}} = \frac{(L + 3\,E\,I\,Z_{AC})\,L_{AB} - \dfrac{L^2}{4}}{3\,E\,I\,L_{AB}} \ldots (23b)$$

For brevity, let:

$$L + 3\,E\,I\,Z_{AC} = L'_{AC} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

from which

$$Z_{AJ} = \frac{L'_{AC}\,L_{AB} - \dfrac{L^2}{4}}{3\,E\,I\,L_{AB}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

If the connection of AB at end B is flexible—

$$L_{AB} = L + 3\,E\,I\,(Z_B + Z_{BC}) \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

—as stated in Eq. 22.

6. Method of Procedure

If the frame is one in which there is no joint translation (sidesway), the solution of the problem may be divided into six distinct steps. In those cases where the solution involves joint translation, a solution is first obtained assuming that this motion is prevented. Later this assumption is corrected by additional steps.

In the derivation of formulas it was emphasized that the calculation of the elastic factors must begin with a member adjacent to a primary support (a joint whose rigidity is known). However, in some structures (the Vierendeel truss being an example), this is not possible. The computations for the elastic factors cannot be begun where a path can be followed to its starting point. This condition can be avoided temporarily by securing certain joints against rotation. These "fixed joints" can later be released and the effect of their adjustment superimposed on the solution already obtained. In many frames the "fixed joints" are not required.

Step (*a*).—Select a minimum number of "fixed joints," dividing the frame into sections such that in any section one can move on the frame from any point in the section to any other by one, and only one, route.

Step (*b*).—Compute the elastic factors Z_A and Z_B for each member of the frame (see Eqs. 3, 5, or 11). The elastic lengths and the carry-over factors are also computed.

Step (*c*).—Compute the moments at the ends of the members connected to each "fixed joint" when the joint is rotated a unit amount. Distribute the resulting moments throughout the frame.

Step (*d*).—Compute and distribute the end moments of the loaded members by Eqs. 16.

Step (*e*).—The temporary "fixed joints" are eliminated by forming simultaneous equations from the resulting steps (*c*) and (*d*). Solve these equations for the rotation correction factors.

Step (*f*).—Eliminate the fixed condition of the joints by substituting by proportion the results of step (*e*) in step (*c*), and combine the results with those of step (*d*). If there is no joint translation (sidesway), the result of step (*f*) is the final solution of the problem.

7. JOINT TRANSLATION

Each problem contains a definite number of shear requirements that must be satisfied. These result in deformations across the section where they occur. Literal values are assigned to these deformations and an equation of condition is set up for each section. The literal terms of each equation represent the shears across the sections due to the several (unknown) deformations. The solution of the simultaneous equations will give the values of the unknowns. The final moments are found by the principle of superposition, adding the change in the moments due to these deformations to those found by step (*f*). The shear corrections require three additional steps.

Step (*g*).—Translate the joints, successively, along each shear section an equal, arbitrary amount allowing no rotation except that due to the flexibility of the supporting members; and then compute the resulting end moments from Eqs. 20. Distribute these moments in the same manner as those due to a load.

Step (*h*).—Compute the shear on each shear section due to the joint translations of step (*g*). Form equations of condition for the several shear sections noting that these shears are the coefficients of the literal deformations. The solution of these equations gives the amount of deflection necessary to satisfy the shear requirements of the problem.

Step (*i*).—Multiply the moments of step (*g*) by the shear correction factors obtained from the solution of the equations of step (*h*) and add the results to the moments found by step (*f*). The final result is the complete solution of the problem.

The steps in the illustrative problems are lettered to correspond to the foregoing step letters. If a step is not required for a problem, it will be omitted.

A study of Eqs. 3 to 20 will show that E is eliminated from the calculations when the elastic factor from Eq. 3 or Eq. 5 is substituted in the succeeding equations. For convenience in computations involving individual equations, E will be assumed equal to unity. The moment of inertia, I, will be given a proportional value since it is seldom the same for all members of the structure.

In evaluating the formulas, the customary sign convention for moments used in beam analysis has been followed—that is, a moment is positive if it produces compression on the upper fibers of the beam. This notation presents difficulties in frame analysis so, as in moment distribution, the convention used in the problems will be: A positive end moment is one that will tend to turn the end of a member going into a joint in a clockwise direction.

PART B. ILLUSTRATIVE EXAMPLES

EXAMPLE 1 (*a*)

A four-span continuous beam ABCDE (Fig. 6) carries a 4-kip concentrated load at the center of span BC. The left end, A, is fixed and the right end, E, is hinged. The moment of inertia, I, is taken as 15 and is included in the computations for illustrative purposes although the solution is independent of its value. The moments at points A, B, C, and D are the required quantities.

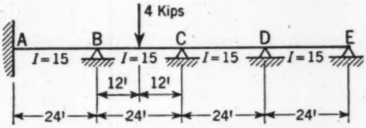


FIG. 6.—FOUR-SPAN CONTINUOUS BEAM WITH ONE CONCENTRATED LOAD

Step (*a*).—Since there are no closed circuits and since the computations for the elastic factors begin with the single members AB and DE, no joints fixed against rotation are needed.

Step (*b*).—To provide a starting point for the computations, assume that points B and C are isolated elastic supports, the elasticity of point B resulting from the flexibility of member BA and that of point C from the flexibility of member CDE. The required elastic factors will be calculated first: For member BA, Eq. 3 gives $Z_{BA} = \frac{24}{4 \times 15} = \frac{1}{2.500}$; and, for member DE, Eq. 5 gives $Z_{DE} = \frac{24}{3 \times 15} = \frac{1}{1.875}$. The elastic factor Z_{CDE} for member CDE is calculated from Eq. 11 which involves the elastic length L_{CD} of member CD; thus (see Eq. 8): $L_{CD} = L + 3\,E\,I\,Z_{DE} = 24 + 3 \times 15 \times \frac{1}{1.875} = 24 + 24 = 48.00$; and (see Eq. 11) $Z_{CDE} = \frac{24}{3 \times 15}\left[\frac{48 - \frac{24}{4}}{48}\right] = \frac{1}{2.143}$.

The elastic lengths, L_{BC} and L_{CB}, needed in step (*d*) are (see Eq. 8): $L_{BC} = L + 3\,E\,I\,Z_{CDE} = 24 + 3 \times 15 \times \frac{1}{2.143} = 45.00$; and $L_{CB} = L + 3\,E\,I\,Z_{BA} = 24 + 3 \times 15 \times \frac{1}{2.500} = 42.00$. The required carry-over factors are: Points B to A (Eq. 1), 0.50; points C to D (Eq. 7) $\frac{24}{2 \times 48} = 0.25$; and points D to E $= 0$ (as is known) because (see Eq. 7) $L_{DE} = \infty$.

The elastic length of member DE is infinite since $Z_E = \infty$ due to the hinge.

Step (*d*).—The maximum ordinate of the moment diagram for the 4-kip concentrated load is $\frac{P\,L}{4} = \frac{4 \times 24}{4} = 24$ ft-kips. Therefore, $A = \frac{1}{2} \times 24 \times 24 = 288$; and $\bar{x}_B = \bar{x}_C = 12$. Then, by Eq. 16*a*,

$$M_{BC} = \frac{6 \times 288}{24}\left(\frac{2 \times 45 \times 12 - 24 \times 12}{4 \times 45 \times 42 - 24 \times 24}\right) = 8.165;$$

and, by Eq. 16*b*,

$$M_{CB} = \frac{6 \times 288}{24}\left(\frac{2 \times 42 \times 12 - 24 \times 12}{4 \times 45 \times 42 - 24 \times 24}\right) = 7.423.$$

Moments M_{BC} and M_{CB} for this problem are tabulated and distributed in Table 1, with the sign notation as mentioned in "Part A: 6. Method of Procedure."

TABLE 1.—MOMENTS AT THE SUPPORTS OF THE CONTINUOUS BEAM ABCDE, FIG. 6

AB	BA	BC	CB	CD	DC	DE	ED
−4.082	−8.165	+8.165	−7.423	+7.423	+1.856	−1.856	0

EXAMPLE 1(*b*)

The same four-span continuous beam as in Example 1(*a*) (Fig. 6) carries a 4-kip load at the center of span BC and a load of 8 kips uniformly distributed over span CD.

Step (*b*).—Compute the following values in addition to those already found in Example 1(*a*): For member CBA, $Z_{CBA} = \frac{24}{3 \times 15}\left[\frac{42 - \frac{24}{4}}{42}\right] = \frac{1}{2.188}$ (see Eq. 11); and L_{DC} (see Eq. 8) $= 24 + 3 \times 15 \times \frac{1}{2.188} = 24 + 20.57 = 44.57$.

The carry-over factor from points C to B is $\frac{24}{2 \times 42} = 0.286$.

Step (*d*).—Find the end moments due to the 8-kip uniform load. The maximum ordinate of the moment-curve diagram (a second degree parabola) is $\frac{W\,L}{8} = \frac{8 \times 24}{8} = 24$ ft-kips. The area under a second degree parabola is equal to two thirds of the area of the circumscribed rectangle. Therefore, $A = \frac{2}{3} \times 24 \times 24 = 384$ and $\bar{x}_C = \bar{x}_D = 12$.

Then, by Eq. 16*a*,

$$M_{CD} = \frac{6 \times 384}{24}\left(\frac{2 \times 48 \times 12 - 24 \times 12}{4 \times 48 \times 44.57 - 24 \times 24}\right) = 10.392;$$

and, by Eq. 16*b*,

$$M_{DC} = \frac{6 \times 384}{24}\left(\frac{2 \times 44.57 \times 12 - 24 \times 12}{4 \times 48 \times 44.57 - 24 \times 24}\right) = 9.402.$$

Moments M_{BC} and M_{CB} from Example 1(*a*), and moments M_{CD} and M_{DC} in this example, are tabulated and distributed in Table 2, the last line of which gives the moments at the several joints, created by the two loads.

TABLE 2.—TABULATION AND DISTRIBUTION OF MOMENTS, EXAMPLE 1(*b*)

Line	Description	AB	BA	BC	CB	CD	DC	DE	ED
1	Moments created by elasticity of supports	....	....	+8.165	− 7.423	+10.392	−9.402	....	....
2	Joint distribution.....	....	−8.165	....	−10.392	+ 7.423	....	+9.402	....
3	Carry-over*..........	−2.598	+2.969	−2.969	....	....	+1.856	−1.856	0
4	Final moments.......	−2.598	−5.196	+5.196	−17.815	+17.815	−7.546	+7.546	0

* For member AB, 5.196 × 0.50 = 2.598; for member BA, 10.392 × 0.286 = 2.969; and, for member DC, 7.423 × 0.25 = 1.856.

EXAMPLE 2

A rigid frame (Fig. 7) carries a 4-kip concentrated load at the center of the beam AC. The moments at points A, B, C, and D are required.

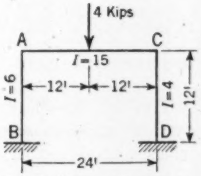


FIG. 7.—RIGID SYMMETRICAL FRAME

Step (*b*).—It will be assumed that points A and C are isolated elastic supports. The elasticity of joint A is a function of the flexibility of member AB and the elasticity of joint C is a function of the flexibility of member CD. Eq. 3 yields for member AB, $Z_{AB} = \frac{12}{4 \times 6} = \frac{1}{2}$; and, for member CD, $Z_{CD} = \frac{12}{4 \times 4} = \frac{3}{4}$. The elastic lengths (Eq. 8) are: $L_{AC} = 24 + 3 \times 15 \times \frac{3}{4} = 24 + 33.75 = 57.75$; and $L_{CA} = 24 + 3 \times 15 \times \frac{1}{2} = 24 + 22.50 = 46.50$.

By Eq. 11: For member ACD—$Z_{ACD} = \frac{24}{3 \times 15}\left[\frac{57.75 - \frac{24}{4}}{57.75}\right] = \frac{1}{2.092}$, and, for member CAB, $Z_{CAB} = \frac{24}{3 \times 15}\left[\frac{46.50 - \frac{24}{4}}{46.50}\right] = \frac{1}{2.153}$.

Step (*g*).—The elastic lengths required in step (*g*) are:

$$L_{AB} = L + 3\,E\,I\,Z_B \quad = 12 + 3 \times 6 \times 0 \quad = 12.00$$

$$L_{BA} = L + 3\,E\,I\,Z_{ACD} = 12 + 3 \times 6 \times \frac{1}{2.092} = 20.60$$

$$L_{CD} = L + 3\,E\,I\,Z_D \quad = 12 + 3 \times 4 \times 0 \quad = 12.00$$

$$L_{DC} = L + 3\,E\,I\,Z_{CAB} = 12 + 3 \times 4 \times \frac{1}{2.153} = 17.56$$

The required carry-over factors are: By Eq. 1—points A to B = 0.50 and points C to D = 0.50; and, by Eq. 7—points A to C = $\frac{24}{2 \times 57.75} = 0.208$ and points C to A = $\frac{24}{2 \times 46.50} = 0.258$.

Step (*d*).—From Example 1(*a*), $A = 288$ and $\bar{x}_A = \bar{x}_C = 12$. Then, by Eqs. 16: $M_{AC} = \frac{6 \times 288}{24}\left(\frac{2 \times 57.75 \times 12 - 24 \times 12}{4 \times 57.75 \times 46.50 - 24 \times 24}\right) = 7.777$; and $M_{CA} = \frac{6 \times 288}{24}\left(\frac{2 \times 46.50 \times 12 - 24 \times 12}{4 \times 57.75 \times 46.50 - 24 \times 24}\right) = 5.865$. These moments are tabulated and distributed in the first line of Table 3(*b*).

Step (*g*).—According to statics the shear equation

$$\frac{M_{AB} + M_{BA} + M_{CD} + M_{DC}}{12}$$

should equal zero; but the actual shear across members AB and CD is

$$\frac{-7.777 - 3.888 + 5.865 + 2.932}{12} = \frac{-2.868}{12};$$

and, therefore, as should be expected, joints A and C must move laterally until the induced moments produce a shear that will counteract the effect of the shear $\frac{-2.868}{12}$. Joint A will be translated a distance Δ, which for convenience is taken as $\frac{12}{E}$, laterally to the right of joint B. However, the direction of the relative movement is entirely arbitrary. In a like manner joint C will be translated the same distance to the right of point D. The induced moments (see Eq. 17) are:

TABLE 3.—TABULATION AND DISTRIBUTION OF MOMENTS, EXAMPLE 2

Line	Description	BA	AB	AC	CA	CD	DC
	(*a*) STEP (*g*)						
1	Moments created by elasticity of supports.	+2.267	+1.534	....	....	+1.236	+1.618
2	Joint distribution....................	....	....	−1.534	−1.236	....	....
3	Carry-over[a]........................	+0.160	+0.319	−0.319	−0.319	+0.319	+0.160
4	Final moments........................	+2.427	+1.853	−1.853	−1.555	+1.555	+1.778
	(*b*) STEPS (*d*) AND (*i*)						
5	Moments created by the load, assuming that sideway is prevented (step (*d*))....	−3.888	−7.777	+7.777	−5.865	+5.865	+2.932
6	Correction for sidesway (line 4 × 0.376)...	+0.914	+0.698	−0.698	−0.586	+0.586	+0.670
7	Final moments (step (*i*))...............	−2.974	−7.079	+7.079	−6.451	+6.451	+3.602

[a] For members BA and DC, 0.319 × 0.50 = 0.160; for member AC, 1.236 × 0.258 = 0.319; and, for member CA, 1.534 × 0.208 = 0.319.

$$M_{AB} = \frac{6 \times 6 \times 12}{12}\left(\frac{2 \times 12 + 12}{4 \times 12 \times 20.62 - 12 \times 12}\right) = 1.534$$

$$M_{BA} = \frac{6 \times 6 \times 12}{12}\left(\frac{2 \times 20.62 + 12}{4 \times 12 \times 20.62 - 12 \times 12}\right) = 2.267$$

$$M_{CD} = \frac{6 \times 4 \times 12}{12}\left(\frac{2 \times 12 + 12}{4 \times 12 \times 17.60 - 12 \times 12}\right) = 1.236$$

$$M_{DC} = \frac{6 \times 4 \times 12}{12}\left(\frac{2 \times 17.60 + 12}{4 \times 12 \times 17.60 - 12 \times 12}\right) = 1.618$$

These moments are tabulated and distributed in Table 3(*a*). The shear across members AB and CD due to these moments is:

$$\frac{+1.853 + 2.427 + 1.555 + 1.778}{12} = \frac{+7.613}{12}.$$

Step (*h*).—The shear correction factor is: $\frac{2.868}{12} \div \frac{7.613}{12} = \frac{2.868}{7.613} = 0.376.$ The sign of the shear from step (*g*) has been changed to produce reverse balancing moments.

Step (*i*).—Line 7, Table 3(*b*) gives the final moments for the frame including the effect of joint translation.

EXAMPLE 3

A four-span rigid frame is loaded as shown in Fig. 8. The end moments of all members are required.

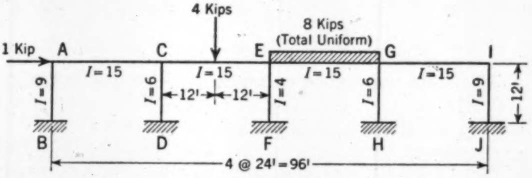


FIG. 8.—FOUR-SPAN RIGID FRAME

Step (*b*).—The necessary elastic lengths and elastic factors are found, first, by progressing from joint I to joint A and then from joint A to joint I. In the computations in Table 4 it will be found convenient to express the elastic factor in its reciprocal form, which is the "stiffness factor." The remaining values are found by symmetry; thus:

TABLE 4.—COMPUTATIONS FOR STEP (*b*), EXAMPLE 3

Member	ELASTIC LENGTH			STIFFNESS FACTOR			
	Symbol	Computation	Feet	Symbol	Computation	Feet	Eq.
IJ......	L_{IJ}	....	12.00	$1/Z_{IJ}$	$4 \times 9/12$	3.000	3
GI......	L_{GI}	$24 + \frac{3 \times 15}{3.000}$	39.00	$1/Z_{GI}$	$\frac{3 \times 15 \times 39}{24(39 - 6)}$	2.216	11
GH.....	L_{GH}	....	12.00	$1/Z_{GH}$	$4 \times 6/12$	2.000	3
EG.....	L_{EG}	$24 + \frac{3 \times 15}{2.216 + 2.000}$	34.67	$1/Z_{EG}$	$\frac{3 \times 15 \times 34.67}{24(34.67 - 6)}$	2.267	11
EF.....	L_{EF}	....	12.00	$1/Z_{EF}$	$4 \times 4/12$	1.333	3
CE.....	L_{CE}	$24 + \frac{3 \times 15}{2.267 + 1.333}$	36.50	$1/Z_{CE}$	$\frac{3 \times 15 \times 36.50}{24(36.50 - 6)}$	2.244	11
CD.....	L_{CD}	....	12.00	$1/Z_{CD}$	$4 \times 6/12$	2.000	3
AC.....	L_{AC}	$24 + \frac{3 \times 15}{2.244 + 2.000}$	34.60	$1/Z_{AC}$	$\frac{3 \times 15 \times 34.60}{24(34.60 - 6)}$	2.268	11
BA.....	L_{BA}	$12 + \frac{3 \times 9}{2.268}$	23.90	....	....	....	....

Member	L	$1/Z$
AB..............	12.00	3.000
CA..............	39.00	2.216
EC..............	34.67	2.267
GE..............	36.50	2.244
IG	34.60	2.268
JI	23.90	....

The following elastic lengths are required for the sidesway calculations and are based on stiffness factors in Table 4: For member DC,

$$L_{DC} = 12 + \frac{3 \times 6}{2.216 + 2.244} = 16.04;$$

for member FE, $L_{FE} = 12 + \dfrac{3 \times 4}{2 \times 2.267} = 14.65$; and, for member HG, $L_{HG} = 16.04$.

The carry-over factors are found by Eq. 7; thus: From point A to point C, $\dfrac{24}{2 \times 34.60} = 0.347$; from point C to point E, $\dfrac{24}{2 \times 36.45} = 0.329$; from point E to point G, $\dfrac{24}{2 \times 34.70} = 0.346$; and, from point G to point I, $\dfrac{24}{2 \times 39.00} = 0.308$. Carry-over factors, from I to G, G to E, E to C, and C to A, are found by symmetry; and the carry-over factors from the column tops to the bases are all 0.500.

Step (*d*).—The values of A, $\bar{x}_A$, and $\bar{x}_B$, for the 4-kip concentrated load and the 8-kip uniform load, are taken from Examples 1(*a*) and 1(*b*). The end moments from each of these loads are computed by Eqs. 16*a* and 16*b*:

$$M_{CE} = \frac{6 \times 288}{24}\left(\frac{2 \times 36.50 \times 12 - 24 \times 12}{4 \times 36.50 \times 34.67 - 24 \times 24}\right) = 9.436$$

$$M_{EC} = \frac{6 \times 288}{24}\left(\frac{2 \times 34.67 \times 12 - 24 \times 12}{4 \times 36.50 \times 34.67 - 24 \times 24}\right) = 8.734$$

$$M_{EG} = \frac{6 \times 384}{24}\left(\frac{2 \times 34.67 \times 12 - 24 \times 12}{4 \times 34.67 \times 36.50 - 24 \times 24}\right) = 11.645$$

$$M_{GE} = \frac{6 \times 384}{24}\left(\frac{2 \times 36.50 \times 12 - 24 \times 12}{4 \times 34.67 \times 36.50 - 24 \times 24}\right) = 12.582$$

To facilitate the distribution of moments, the stiffness factors of the members framing into joints C, E, and G are listed as follows:

Member	Stiffness factor
Joint C—	
CA	2.216
CD	2.000
CE	2.244
Joint E—	
EC	2.267
EF	1.333
EG	2.267
Joint G—	
GE	2.244
GH	2.000
GI	2.216

The end moment from any member is distributed to the other members at the joint in direct proportion to its stiffness. The moments caused by the loads, tabulated in line 6, Table 5(*b*), are distributed first to the right (line 7) and then to the left (line 8) by the proper stiffness and carry-over factors. Since the moments in line 6 are the result of elastic support action, there is no carry-over for $M_{EC} = -8.734$ to joint C or for $M_{EG} = -11.645$ to joint G, the moment $M_{EC} = -8.734$ being distributed to members EF and EG as $+3.234$ and $+5.500$, respectively. The part of $+5.500$ carried over to joint G is $+1.903$. This is combined with -12.582 to produce a net moment $M_{GE} = -10.679$ which is distributed to members GH and GI. Line 9, Table 5(*b*), gives the final moments for this step.

Step (*g*).—If the system is in equilibrium, the sum of the end moments for the columns should equal 1 kip × 12 ft or 12.000 ft-kips. However, the actual sum of these moments listed in line 9, Table 5(*b*), is $+0.157$, showing that an additional $+11.843$ must be furnished. To effect this adjustment the joints A, C, E, G, and I, Fig. 8, will be translated a convenient distance $\Delta = \frac{12}{E}$ laterally to the right. Using Eqs. 20*a* and 20*b*, the induced moments are:

$$M_{AB} = \frac{6 \times 9 \times 12}{12} \times \left(\frac{2 \times 12 + 12}{4 \times 12 \times 23.90 - 12 \times 12} \right) = 1.938$$

$$M_{BA} = \frac{6 \times 9 \times 12}{12} \times \left(\frac{2 \times 23.90 + 12}{4 \times 12 \times 23.90 - 12 \times 12} \right) = 3.219$$

$$M_{CD} = \frac{6 \times 6 \times 12}{12} \times \left(\frac{2 \times 12 + 12}{4 \times 12 \times 16.04 - 12 \times 12} \right) = 2.071$$

$$M_{DC} = \frac{6 \times 6 \times 12}{12} \times \left(\frac{2 \times 16.04 + 12}{4 \times 12 \times 16.04 - 12 \times 12} \right) = 2.536$$

$$M_{EF} = \frac{6 \times 4 \times 12}{12} \times \left(\frac{2 \times 12 + 12}{4 \times 12 \times 14.65 - 12 \times 12} \right) = 1.546$$

$$M_{FE} = \frac{6 \times 4 \times 12}{12} \times \left(\frac{2 \times 14.65 + 12}{4 \times 12 \times 14.65 - 12 \times 12} \right) = 1.773$$

$$M_{GH} = \quad \cdots\cdot \quad = 2.071$$

$$M_{HG} = \quad \cdots\cdot \quad = 2.536$$

$$M_{IJ} = \quad \cdots\cdot \quad = 1.938$$

$$M_{JI} = \quad \cdots\cdot \quad = 3.219$$

These moments are tabulated and distributed in Table 5(*a*). The sum of the end moments for the columns is 25.204.

Step (*h*).—The shear correction factor is: $\frac{11.843}{25.204} = 0.470$.

Step (*i*).—Multiply the moments in line 5, Table 5(*a*), by 0.470 and tabulate the results in line 10, Table 5(*b*). The addition of lines 9 and 10 gives the final moments for the frame including the effect of joint translation.

EXAMPLE 4

A single-panel, open-web (Vierendeel) truss is loaded as shown in Fig. 9. The end moments of all members are required.

TABLE 5.—TABULATION AND DIS-

Line	Description	COLUMNS					JOINT A	
		BA	DC	FE	HG	JI	AB	AC
	(a) STEP (g)							
1	Moments resulting from elasticity of supports	+3.219	+2.536	+1.773	+2.536	+3.219	+1.938	....
2	Joint distribution	....	....	....	....	....	....	−1.938
3	Carry-over to the right	....	+0.158	+0.042	+0.052	+0.141	....	....
4	Carry-over to the left	+0.141	+0.052	+0.042	+0.158	....	+0.281	−0.281
5	Final moments	+3.360	+2.746	+1.857	+2.746	+3.360	+2.219	−2.219
	(b) STEP (f) AND STEP (i)							
6	Moments resulting from elasticity of supports, for 4-kip and 8-kip loads	....	....	....	....	....	....	....
7	Joint distribution and carry-over (right)	....	....	+1.617	+2.533	−0.863	....	....
8	Joint distribution and carry-over (left)	+0.558	−1.636	−2.156	....	....	+1.116	−1.116
9	Final moments with no sidesway	+0.558	−1.636	−0.539	+2.533	−0.863	+1.116	−1.116
10	Correction for sidesway (line 5 × 0.470)	+1.579	+1.290	+0.873	+1.290	+1.579	+1.043	−1.043
11	Final moments including sidesway	+2.137	−0.316	+0.334	+3.823	+0.716	+2.159	−2.159

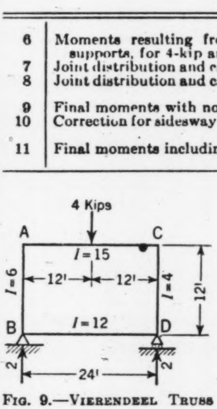


FIG. 9.—VIERENDEEL TRUSS

Step (*a*).—To compute the elastic factors and the elastic lengths for the several members it is necessary to start with a single member having a fixed or hinged joint at one end. The idea of the "fixed joint" is now introduced for the purpose of establishing, temporarily, a joint of known rigidity. Joint B is assumed to be temporarily secured against rotation.

Step (*b*).—Computations for step (*b*) are as shown in Table 6. The elastic and carry-over factors are as follows:

Joint	Computation	Carry-over factor	Joint	Computation	Carry-over factor
A to B	....	0.500	D to B	....	0.500
C to A	$\frac{24}{2 \times 46.50}$	0.258	C to D	$\frac{12}{2 \times 18.00}$	0.333
D to C	$\frac{12}{2 \times 17.57}$	0.341	A to C	$\frac{24}{2 \times 61.50}$	0.195
B to D	$\frac{24}{2 \times 53.85}$	0.223	B to A	$\frac{12}{2 \times 20.66}$	0.290

Step (*c*).—Eq. 11 is used to obtain a general expression for the moments induced in the ends of the members framing into a joint when the joint is rotated a unit amount. If θ_A is equal to unity, Eq. 11 becomes:

$$M_{AB} = \frac{1}{Z_{AB}} \qquad \text{(27)}$$

TRIBUTION OF MOMENTS, EXAMPLE 3

Joint C			Joint E			Joint G			Joint I		Line
CA	CD	CE	EC	EF	EG	GE	GH	GI	IG	IJ	
					(a) Step (g)						
....	+2.071	....	....	+1.546	....	....	+2.071	....	....	+1.938	1
-1.029	....	−1.042	−0.773	....	−0.773	−1.042	....	−1.029	−1.938	....	2
-0.672	+0.317	+0.355	−0.226	+0.084	+0.142	−0.218	+0.104	+0.115	−0.281	+0.281	3
+0.115	+0.103	−0.218	+0.142	+0.084	−0.226	+0.355	+0.317	−0.672	....	....	4
-1.586	+2.491	−0.905	−0.857	+1.714	−0.857	−0.905	+2.492	−1.586	−2.219	+2.219	5
					(b) Step (d) and Step (i)						
....	....	+9.436	−8.734	....	+11.645	−12.582	....	....	....	....	6
....	....	....	....	+3.234	+5.500	+1.903	+5.066	+5.613	+1.727	−1.727	7
-3.626	−3.273	−2.538	−7.333	−4.312	....	....	....	....	....	....	8
-3.626	−3.273	+6.898	−16.067	−1.078	+17.145	−10.679	+5.066	+5.613	+1.727	−1.727	9
-0.745	+1.171	−0.425	−0.403	+0.806	−0.403	−0.425	+1.171	−0.746	−1.043	+1.043	10
-4.371	−2.102	+6.473	−16.470	−0.272	+16.742	−11.104	+6.237	+4.867	+0.684	−0.684	11

The "fixed joint" B is rotated a unit amount producing the following moments which are assumed positive for convenience: $M_{BA} = 1.755$ and $M_{BD} = 1.688$. The moment M_{BA} (= 1.755) is tabulated and distributed in line 1, Table 7(*a*), assuming that member BD is secured against rotation during the operation; $M_{BD} = 1.688$ is tabulated and distributed in line 2; and line 3 contains the final moments for this step.

The external moment required to rotate joint B a unit amount is found as follows: Let M_B be the external moment. Then $M_B + M_{BA} + M_{BD} = 0$; and $M_B = -(M_{BA} + M_{BD}) = -(1.738 + 1.671) = -3.409$. The quantity M_B is required for the calculation of the rotation correction factor of steps (*e*) and (*g*).

Step (*d*).—From Example 1(*a*), $A = 288$ and $\bar{x}_A = \bar{x}_C = 12$. Then

$$M_{AC} = \frac{6 \times 288}{24}\left(\frac{2 \times 61.50 \times 12 - 24 \times 12}{4 \times 61.50 \times 46.50 - 24 \times 24}\right) = 7.874;$$

and

$$M_{CA} = \frac{6 \times 288}{24}\left(\frac{2 \times 46.50 \times 12 - 24 \times 12}{4 \times 61.50 \times 46.50 - 24 \times 24}\right) = 5.488.$$

Moments M_{AC} and M_{CA} are tabulated and distributed in line 10, Table 7(*c*).

Step (*e*).—The "fixed joint" B is eliminated in the following manner: The external moment required to hold joint B fixed against rotation, as assumed in the computations for step (*d*), is equal but opposite in sign to the sum of the internal moments, M_{BA} and M_{BD} in line 10, Table 7(*c*). This external moment is $-(-3.937 - 0.914) = +4.851$; and, to release the joint, an additional external moment of -4.851 is also applied at joint B. The rotation correction factor is $\dfrac{-4.851}{-3.409} = +1.423$.

Step (*f*).—To find the moments in the members due to the external moment of -4.852 applied at joint B, multiply the moments in line 3, Table 7(*a*), by $+1.423$ and enter the product in line 11, Table 7(*c*). Line 12, Table 7(*c*), gives the final moments if sidesway is prevented.

TABLE 6.—COMPUTATIONS FOR STEP (*b*), EXAMPLE 4

Member	ELASTIC LENGTH			ELASTIC FACTOR		
	Symbol	Computation	Feet	Symbol	Computation	Feet
AB.......	L_{AB}	....	12.00	Z_{AB}	$\frac{12}{4 \times 6}$	$\frac{1}{2.000}$
CA.......	L_{CA}	$24 + \frac{3 \times 15}{2.000}$	46.50	Z_{CA}	$\frac{24}{3 \times 15}\left(\frac{46.50 - 6}{46.50}\right)$	$\frac{1}{2.153}$
DC.......	L_{DC}	$12 + \frac{3 \times 4}{2.153}$	17.57	Z_{DC}	$\frac{12}{3 \times 4}\left(\frac{17\ 57 - 3}{17.57}\right)$	$\frac{1}{1.206}$
BD.......	L_{BD}	$24 + \frac{3 \times 12}{1.206}$	53.85	Z_{BD}	$\frac{24}{3 \times 12}\left(\frac{53.85 - 6}{53.85}\right)$	$\frac{1}{1.688}$
DB.......	L_{BD}	....	24.00	Z_{DB}	$\frac{24}{4 \times 12}$	$\frac{1}{2.000}$
CD.......	L_{CD}	$12 + \frac{3 \times 4}{2.000}$	18.00	Z_{CD}	$\frac{12}{3 \times 4}\left(\frac{18.00 - 3}{18.00}\right)$	$\frac{1}{2.000}$
AC.......	L_{AC}	$24 + \frac{3 \times 15}{1.200}$	61.50	Z_{AC}	$\frac{24}{3 \times 15}\left(\frac{61.50 - 6}{61.50}\right)$	$\frac{1}{2.078}$
BA.......	L_{BA}	$12 + \frac{3 \times 6}{2.078}$	20.66	Z_{BA}	$\frac{12}{3 \times 6}\left(\frac{20\ 66 - 3}{20.67}\right)$	$\frac{1}{1.755}$

Step (*g*).—For equilibrium the sum of the end moments for members AB and CD should equal 12×0 or 0 since there are no lateral loads. The actual sum of these moments from line 12, Table 7(*c*), is -1.872, showing that an additional $+1.872$ must be furnished by the lateral movement of joints A and C relative to joints B and D. For the purpose of finding the lateral deformation required for equilibrium, joints A and C are translated a convenient distance $\Delta = 12/E$ laterally to the right. The induced moments are:

$$M_{AB} = \frac{6 \times 6 \times 12}{12}\left(\frac{2 \times 12 + 12}{4 \times 12 \times 20.66 - 12 \times 12}\right) = 1.528$$

$$M_{BA} = \frac{6 \times 6 \times 12}{12}\left(\frac{2 \times 20.66 + 12}{4 \times 12 \times 20.66 - 12 \times 12}\right) = 2.264$$

$$M_{CD} = \frac{6 \times 4 \times 12}{12}\left(\frac{2 \times 18 + 12}{4 \times 18 \times 17.57 - 12 \times 12}\right) = 1.027$$

$$M_{DC} = \frac{6 \times 4 \times 12}{12}\left(\frac{2 \times 17.57 + 12}{4 \times 18 \times 17.57 - 12 \times 12}\right) = 1.009$$

These moments are entered in line 4 and distributed in lines 5 and 6, Table 7(*b*), assuming joint B is fixed against rotation. Line 7 gives the moments before releasing joint B. The releasing moment is + 1.843 or 2.397 − 0.554. The rotation correction factor is $\frac{+\,1.843}{-\,3.409}$ or − 0.541.

TABLE 7.—Tabulation and Distribution of Moments, Example 4

Line	Description	BA	AB	AC	CA	CD	DC	DB	BD
	(*a*) Step (*c*) (Distribution After Rotating Joint B)								
1	Rotation of member BA only..	+1.755	+0.509	−0.509	−0.099	+0.099	+0.033	−0.033	−0.017
2	Rotation of member BD only..	−0.017	−0.033	+0.033	+0.128	−0.128	−0.376	+0.376	+1.688
3	Final moments, step (*c*).......	+1.738	+0.476	−0.476	+0.029	−0.029	−0.343	+0.343	+1.671
	(*b*) Steps (*g*) and (*h*)								
4	Moments caused by the elasticity of the supports, assuming joint B fixed against rotation.....................	+2.264	+1.528	....	....	+1.027	+1.009	....	....
5	Distribution to the right......	....	....	−1.528	−0.298	+0.298	+0.099	−1.108	−0.554
6	Distribution to the left........	+0.133	+0.265	−0.265	−1.027	....	....	....	....
7	Final moments resulting from translation, assuming joint B fixed against rotation	+2.397	+1.793	−1.793	−1.325	+1.325	+1.108	−1.108	−0.554
8	Moments after release of joint B (line 3 × − 0.541).......	−0.939	−0.257	+0.257	−0.016	+0.016	+0.185	−0.185	−0.904
9	Final moments resulting from translation................	+1.458	+1.536	−1.536	−1.341	+1.341	+1.293	−1.293	−1.458
	(*c*) Steps (*d*), (*f*), (*g*), and (*i*)								
10	Joint B fixed against rotation and no sidesway............	−3.937	−7.874	+7.874	−5.488	+5.488	+1.829	−1.829	−0.914
11	Joint B released (line 3 × + 1.423)................	+2.473	+0.678	−0.678	+0.041	−0.041	−0.488	+0.488	+2.378
12	Final moments without sidesway......................	−1.464	−7.196	+7.196	−5.447	+5.447	+1.341	−1.341	+1.464
13	Correction for sidesway (line 9 × + 0.333)..............	+0.485	+0.511	−0.511	−0.446	+0.446	+0.430	−0.430	−0.485
14	Final moments...............	−0.979	−6.685	+6.685	−5.893	+5.893	+1.771	−1.771	+0.979

Multiply the moments in line 3, Table 7(*a*), by − 0.541 and enter the product in line 8, Table 7(*b*). Line 9 gives the moments due to the lateral displacement of $12/E$ including the effect of releasing joint B.

Step (*h*).—The sum of the end moments for members AB and CD, from line 9, Table 7(*b*), is + 5.628. The shear correction factor is $\frac{+\,1.872}{+\,5.628} = +\,0.333$.

Step (*i*).—Multiply the moment in line 9, Table 7(*b*), by + 0.333 and tabulate the result in line 13, Table 7(*c*). The addition of lines 12 and 13 gives the final moments for the frame including the effect of joint translation laterally (see line 14, Table 7(*c*)).

EXAMPLE 5

A double-panel, open-web truss is loaded as shown in Fig. 10. The end moments of all members are required.

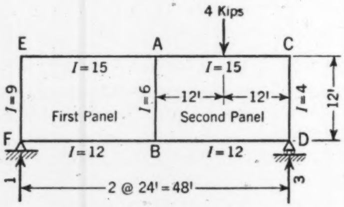


FIG. 10.—DOUBLE-PANEL OPEN-WEB TRUSS

Step (*a*).—Joint B is assumed to be temporarily secured against rotation.

TABLE 8.—COMPUTATIONS FOR STEP (*b*), EXAMPLE 5

Member	ELASTIC LENGTH			STIFFNESS FACTOR		
	Symbol	Computation	Feet	Symbol	Computation	Feet
DB.......	L_{DB}	....	24.00	$1/Z_{DB}$	....	2.000
CD.......	L_{CD}	....	18.00	$1/Z_{CD}$	....	1.200
AC.......	L_{AC}	....	61.50	$1/Z_{AC}$	....	2.078
AB.......	L_{AB}	....	12.00	$1/Z_{AB}$	....	2.000
EA.......	L_{EA}	$24 + \frac{3 \times 15}{2.000 + 2.078}$	35.04	$1/Z_{EA}$	$\frac{3 \times 15}{24}\left(\frac{35.04}{35.04 - 6}\right)$	2.262
FE.......	L_{FE}	$12 + \frac{3 \times 9}{2.262}$	23.93	$1/Z_{FE}$	$\frac{3 \times 9}{12}\left(\frac{23.93}{23.93 - 3}\right)$	2.572
BF.......	L_{BF}	$24 + \frac{3 \times 12}{2.572}$	37.99	$1/Z_{BF}$	$\frac{3 \times 12}{24}\left(\frac{37.99}{37.99 - 6}\right)$	1.781
FB.......	L_{FB}	....	24.00	$1/Z_{FB}$	.....	2.000
EF.......	L_{EF}	$12 + \frac{3 \times 9}{2.000}$	25.50	$1/Z_{EF}$	$\frac{3 \times 9}{12}\left(\frac{25.50}{25.50 - 3}\right)$	2.550
AE.......	L_{AE}	$24 + \frac{3 \times 15}{2.550}$	41.65	$1/Z_{AE}$	$\frac{3 \times 15}{24}\left(\frac{41.65}{41.65 - 6}\right)$	2.191
BA.......	L_{BA}	$24 + \frac{3 \times 6}{2.191 + 2.078}$	16.22	$1/Z_{BA}$	$\frac{3 \times 6}{12}\left(\frac{16.22}{16.22 - 3}\right)$	1.840
CA.......	L_{CA}	$24 + \frac{3 \times 15}{2.000 + 2.191}$	34.74	$1/Z_{CA}$	$\frac{3 \times 15}{24}\left(\frac{34.74}{34.74 - 6}\right)$	2.266
DC.......	L_{DC}	$12 + \frac{3 \times 4}{2.266}$	17.29	$1/Z_{DC}$	$\frac{3 \times 4}{12}\left(\frac{17.29}{17.29 - 3}\right)$	1.210
BD.......	L_{BD}	$24 + \frac{3 \times 12}{1.210}$	53.76	$1/Z_{BD}$	$\frac{3 \times 12}{24}\left(\frac{53.76}{53.76 - 6}\right)$	1.088

Step (*b*).—In Table 8, the detailed computations are omitted for quantities that have been computed in Example 4. The elastic and carry-over factors are:

Joint	Computation	Carry-over factor	Joint	Computation	Carry-over factor
D to B.....	....	0.500	F to B.....	....	0.500
C to D.....	....	0.333	E to F.....	$\frac{12}{2 \times 25.50}$	0.235
A to C.....	....	0.195	A to E.....	$\frac{24}{2 \times 41.65}$	0.288
A to B.....	....	0.500	B to A.....	$\frac{12}{2 \times 16.22}$	0.370
E to A.....	$\frac{24}{2 \times 35.04}$	0.343	C to A.....	$\frac{24}{2 \times 34.74}$	0.346
F to E.....	$\frac{12}{2 \times 23.93}$	0.251	D to C.....	$\frac{12}{2 \times 17.29}$	0.347
B to F.....	$\frac{24}{2 \times 37.99}$	0.316	B to D.....	$\frac{24}{2 \times 53.76}$	0.223

Step (*c*).—The "fixed joint" B is rotated a unit amount producing the following moments: $M_{BF} = 1.781$, $M_{BA} = 1.840$, and $M_{BD} = 1.688$. These moments are tabulated and distributed in Table 9(*a*). In line 1, the moment due to the rotation of member BA is distributed as far as members AE and AC. In line 2 the moment due to the rotation of member BF is distributed to member AC where it is combined with that due to the rotation of BA. The resultant moment is then distributed on the remainder of the line. A similar condition exists in line 3 where the distribution is begun with the rotation of member BD. The final moments for this step are given in line 4, Table 9(*a*). The external moment M_B is equal to $-(1.767 + 1.816 + 1.676) = -5.259$.

Step (*d*).—From Example 1(*a*) $A = 288$ and $\bar{x}_A = \bar{x}_C = 12$. Then,

$$M_{AC} = \frac{6 \times 288}{24}\left(\frac{2 \times 61.50 \times 12 - 24 \times 12}{4 \times 61.50 \times 34.74 - 24 \times 24}\right) = 10.733;$$

and

$$M_{CA} = \frac{6 \times 288}{24}\left(\frac{2 \times 34.74 \times 12 - 24 \times 12}{4 \times 61.50 \times 34.74 - 24 \times 24}\right) = 4.980.$$

These moments are tabulated and distributed in line 17, Table 9(*d*).

Step (*e*).—The external moment necessary to release joint B is the sum of the internal moments at point B which are listed in line 17, Table 9(*d*). This releasing moment is -3.573. The rotation correction factor is $\frac{-3.573}{-5.259} = +0.679$.

Step (*f*).—Multiply the moments in line 4, Table 9(*a*), by $+0.679$ to obtain the moments due to -3.573 applied externally at point B and enter the product in line 18, Table 9(*d*). Line 19, Table 9(*d*), gives the final moments if vertical shear deformation is prevented.

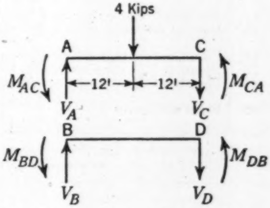


FIG. 11.—FREE-BODY DIAGRAM, SECOND PANEL OF FIG. 10

Step (*g*).—In the first panel of Fig. 10 the sum of the end moments for members EA and FB should equal 1 kip × 24 ft for equilibrium, but the actual sum from line 19, Table 9(*d*), is − 6.428. In the second panel the sum of the end moments for members AC and BD should equal − 1 kip × 24 ft, but the actual sum is + 4.548.

The proof of the statement that the sum of the moments should be − 24 is given as follows: Fig. 11 shows a free-body sketch of members AC and BD. Since they do not enter into the computations, the direct stresses are omitted. In Fig. 11, $V_A = \dfrac{M_{AC} + M_{CA}}{24} + \dfrac{4 \times 12}{24}$; and $V_B = \dfrac{M_{BD} + M_{DB}}{24}$.

TABLE 9.—TABULATION AND DIS-

Line	Description	JOINT B	JOINT F		JOINT E	
		BF	FB	FE	EF	EA
	(*a*) STEP					
	Distribution After the Rotation of:					
1	Member BA	....	....	....	....	....
2	Members BF and BA	+1.781	+0.563	−0.563	−0.141	+0.141
3	Members BD and BA	−0.014	−0.025	+0.025	+0.107	−0.107
4	Final moments	+1.767	+0.538	−0.538	−0.034	+0.034
	(*b*) STEP (*g*) (TRANSLATION					
5	Moments (elasticity of supports)^{a}	+2.344	+1.688	....	....	+1.609
6	Distribution of moments, to the right	....	....	−1.688	−0.423	+0.423
7	Distribution of moments, to the left	+0.189	+0.379	−0.379	−1.609	....
8	Translation; joint B fixed against rotation^{b}	+2.533	+2.067	−2.067	−2.032	+2.032
9	Joint B released^{c} (line 4 × −0.383)	−0.677	−0.206	+0.206	+0.013	−0.013
10	Final moments, translation of member EF	+1.856	+1.861	−1.861	−2.019	+2.019
	(*c*) STEP (*g*) (TRANSLATION					
11	Moments (elasticity of supports)^{a}	....	....	....	....	....
12	Distribution of moments, to the right	....	....	....	....	....
13	Distribution of moments, to the left	+0.032	+0.064	−0.064	−0.271	+0.271
14	Translation; joint B fixed against rotation^{b}	+0.032	+0.064	−0.064	−0.271	+0.271
15	Joint B released^{c} (line 4 × +0.339)	+0.598	+0.182	−0.182	−0.011	+0.011
16	Final moments, translation of member CD	+0.630	+0.246	−0.246	−0.282	+0.282
	(*d*) STEPS (*d*),					
17	Moments created by load^{d} (joint B fixed)	−0.190	−0.380	+0.380	+1.617	−1.617
18	Joint B released^{c} (line 4 × +0.679)	+1.200	+0.365	−0.365	−0.023	+0.023
19	Final moments, vertical movement prevented	+1.010	−0.015	+0.015	+1.594	−1.594
20	Member EF^{c} (line 10 × +2.980)	+5.532	+5.545	−5.515	−6.018	+6.018
21	Member CD^{c} (line 16 × +3.555)	+2.240	+0.873	−0.873	−1.003	+1.003
22	Final moments	+8.782	+6.403	−6.403	−5.427	+5.427

[a] Moments resulting from the elasticity of supports. [b] Final moments resulting from translation, assuming joint B fixed against rotation and vertical movement. [c] Correction for the translation of the member indicated.

The sum of the shears $V_A + V_B$ is equal to 1 from the first panel and therefore: $\dfrac{M_{AC} + M_{CA} + M_{BD} + M_{DB}}{24} + 2 = 1$; and $M_{AC} + M_{CA} + M_{BD} + M_{DB} = -1 \times 24$.

TRIBUTION OF MOMENTS, EXAMPLE 5

JOINT A			JOINT C		JOINT D		JOINT B		Line
AE	AB	AC	CA	CD	DC	DB	BD	BA	
c)									
−0.350	+0.681	−0.331	....	....	....	....	....	+1.840	1
+0.048	−0.024	−0.021	−0.070	+0.070	+0.023	−0.023	−0.012	−0.012	2
−0.021	−0.024	+0.045	+0.131	−0.131	−0.377	+0.377	+1.688	−0.012	3
−0.323	+0.633	−0.310	+0.061	−0.061	−0.354	+0.354	+1.676	+1.816	4
OF MEMBER EF)									
+1.836	....	....	....	....	....	....	....	....	5
+0.145	−0.972	−1.009	−0.197	+0.197	+0.065	−0.065	−0.033	....	6
....	....	....	....	....	....	....	....	−0.486	7
+1.981	−0.972	−1.009	−0.197	+0.197	+0.065	−0.065	−0.033	−0.486	8
+0.121	−0.243	+0.119	−0.024	+0.024	+0.136	−0.136	−0.641	−0.696	9
+2.105	−1.215	−0.890	−0.221	+0.221	+0.201	−0.201	−0.674	−1.182	10
OF MEMBER CD)									
....	....	−1.660	−1.056	....	....	−1.131	−2.066	....	11
....	....	....	....	+1.056	+0.352	−0.352	−0.176	....	12
+0.938	+0.857	−0.136	−0.392	+0.392	+1.131	..	..	+0.428	13
+0.938	+0.857	−1.796	−1.448	+1.448	+1.483	−1.483	−2.242	+0.428	14
−0.109	+0.214	−0.105	+0.021	−0.021	−0.120	+0.120	+0.569	+0.615	15
+0.829	+1.071	−1.901	−1.427	+1.427	+1.363	−1.363	−1.673	+1.043	16
(*f*), AND (*i*)									
−5.610	−5.123	+10.733	−4.930	+4.930	+1.643	−1.643	−0.822	−2.561	17
−0.219	+0.430	−0.211	+0.042	−0.042	−0.240	+0.240	+1.139	+1.234	18
−5.829	−4.693	+10.522	−4.888	+4.888	+1.403	−1.403	+0.317	−1.327	19
+6.271	−3.618	−2.653	−0.657	+0.657	+0.599	−0.599	−2.011	−3.521	20
+2.948	+3.809	−6.757	−5.074	+5.074	+4.845	−4.845	−5.950	+3.710	21
+3.390	−4.502	+1.112	−10.619	+10.619	+6.847	−6.847	−7.644	−1.138	22

that joint B is fixed against rotation. [c] Moments after releasing joint B. [d] Moments created by load, assuming

Since the frame is not in equilibrium vertically, deformation sufficient to satisfy the shear requirements must take place across each panel. To compute these deformations, first translate member EF a convenient distance $\Delta = 24/E$ vertically upward relative to member AB. The induced moments are:

$$M_{EA} = \frac{6 \times 15 \times 24}{24}\left(\frac{2 \times 35.04 + 24}{4 \times 35.04 \times 41.65 - 24 \times 24}\right) = 1.609$$

$$M_{AE} = \frac{6 \times 15 \times 24}{24}\left(\frac{2 \times 41.65 + 24}{4 \times 35.04 \times 41.65 - 24 \times 24}\right) = 1.836$$

$$M_{FB} = \frac{6 \times 12 \times 24}{24}\left(\frac{2 \times 24.00 + 24}{4 \times 24.00 \times 37.99 - 24 \times 24}\right) = 1.688$$

$$M_{BF} = \frac{6 \times 12 \times 24}{24}\left(\frac{2 \times 37.99 + 24}{4 \times 24.00 \times 37.99 - 24 \times 24}\right) = 2.344$$

These moments are entered in line 5 and distributed in lines 6 and 7, Table 9(*b*), assuming joint B fixed against rotation. Line 8 gives the moments before releasing joint B. The releasing amount is + 2.014.

The rotation correction factor is $\frac{+\,2.014}{-\,5.259} = -\,0.383$. Multiply the moments in line 4, Table 9(*a*), by − 0.383 and tabulate the product in line 9, Table 9(*b*). Line 10 yields the moments due to the translation of member EF including the effect of releasing the joint.

Next, to compute the deformations, translate member CD a convenient distance $\Delta = 24/E$ vertically upward relative to member AB. The induced moments are:

$$M_{CA} = \frac{6 \times 15 \times 24}{24}\left(\frac{2 \times 34.74 + 24}{4 \times 34.74 \times 61.50 - 24 \times 24}\right) = 1.056$$

$$M_{AC} = \frac{6 \times 15 \times 24}{24}\left(\frac{2 \times 61.50 + 24}{4 \times 34.74 \times 61.50 - 24 \times 24}\right) = 1.660$$

$$M_{DB} = \frac{6 \times 12 \times 24}{24}\left(\frac{2 \times 24.00 + 24}{4 \times 24.00 \times 53.76 - 24 \times 24}\right) = 1.131$$

$$M_{BD} = \frac{6 \times 12 \times 24}{24}\left(\frac{2 \times 53.76 + 24}{4 \times 24.00 \times 53.76 - 24 \times 24}\right) = 2.066$$

These moments are entered in line 11 and distributed in lines 12 and 13, Table 9(*c*). Line 14 gives the moments before releasing joint B. The releasing moment is − 1.782. The rotation correction factor is − 1.782/− 5.259 = + 0.339. Multiply the moments in line 4, Table 9(*a*), by + 0.339 and list the product, line 15, Table 9(*c*). Line 16 gives the moments due to the translation of member CD including the effect of releasing the joint.

Step (*h*).—Let *X* be the shear correction factor (required deformation across the panel) for the first panel; and let *Y* be the shear correction factor (required deformation across the panel) for the second panel. The sum of the end moments from the tables for the following members is:

Reference	Members EA and FB	Members AC and BD
Line 10, Table 9(*b*)	+ 7.841	− 1.987
Line 16, Table 9(*c*)	+ 1.987	− 6.364
Line 19, Table 9(*d*)	− 6.428	+ 4.547

From the foregoing, the shear equations can be written as follows: For the first panel—

$$7.841\,X + 1.987\,Y - 6.428 = 24.00 \quad \text{(28a)}$$

and, for the second panel—

$$-1.987\,X - 6.364\,Y + 4.547 = -24.00 \quad \text{(28b)}$$

Solving Eqs. 28 simultaneously, $X = 2.980$ and $Y = 3.555$.

Step (*i*).—Multiply the moments in line 10, Table 9(*b*), by + 2.980 and those in line 16, Table 9(*c*), by + 3.555. Tabulate the results in lines 20 and 21, Table 9(*d*). The addition of lines 19, 20, and 21 gives the final moments including the effect of joint translation (see line 22).

Example 6

A four-panel, open-web truss is loaded as shown in Fig. 12. The end moments of all members are required.

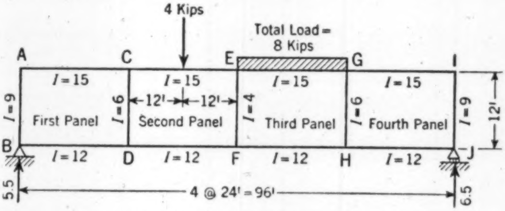


Fig. 12.—Four-Panel Open-Web Truss

Step (*a*).—Joints D and H are assumed to be temporarily secured against rotation.

Step (*b*).—Computations for step (*b*) are arranged in Table 10, the elastic lengths and stiffness factors for JH through DB, HG, and HF being determined by symmetry. The carry-over factors are:

Joint	Computation	Carry-over factor	Joint	Computation	Carry-over factor
B to D....	....	0.500	G to E....	$\frac{24}{2 \times 36.797}$	0.326
A to B....	$\frac{12}{2 \times 25.500}$	0.235	G to H....	....	0.500
C to A....	$\frac{24}{2 \times 41.647}$	0.288	I to G....	$\frac{24}{2 \times 34.612}$	0.347
C to D....	....	0.500	J to I....	$\frac{12}{2 \times 23.904}$	0.251
E to C....	$\frac{24}{2 \times 34.738}$	0.345	H to J....	$\frac{24}{2 \times 37.992}$	0.316
F to D....	....	0.500	D to C....	$\frac{12}{2 \times 16.062}$	0.374
F to H....	....	0.500	F to E....	$\frac{12}{2 \times 14.647}$	0.410
E to F....	$\frac{12}{2 \times 15.000}$	0.400	D to F....	$\frac{24}{2 \times 35.051}$	0.342

TABLE 10.—COMPUTATIONS FOR STEP (*b*), EXAMPLE 6

Member	ELASTIC LENGTH			STIFFNESS FACTOR		
	Symbol	Computation	Feet	Symbol	Computation	Feet
BD.......	L_{BD}	....	24.000	$1/Z_{BD}$	$\frac{4 \times 12}{24}$	2.0000
AB.......	L_{AB}	$12 + \frac{3 \times 9}{2.0000}$	25.500	$1/Z_{AB}$	$\frac{3 \times 9}{12}\left(\frac{25.500}{25.500 - 3}\right)$	2.5500
CA.......	L_{CA}	$24 + \frac{3 \times 15}{2.5500}$	41.647	$1/Z_{CA}$	$\frac{3 \times 15}{24}\left(\frac{41.647}{41.647 - 6}\right)$	2.1906
CD.......	L_{CD}	....	12.000	$1/Z_{CD}$	$\frac{4 \times 6}{12}$	2.0000
EC.......	L_{EC}	$24 + \frac{3 \times 15}{2.1906 + 2.0000}$	34.738	$1/Z_{EC}$	$\frac{3 \times 15}{24}\left(\frac{34.738}{34.738 - 6}\right)$	2.2665
FD.......	L_{FD}	....	24.000	$1/Z_{FD}$	$\frac{4 \times 12}{24}$	2.0000
FH.......	L_{FH}	....	24.000	$1/Z_{FH}$	$\frac{4 \times 12}{24}$	2.0000
EF.......	L_{EF}	$12 + \frac{3 \times 4}{2.0000 + 2.0000}$	15.000	$1/Z_{EF}$	$\frac{3 \times 4}{12}\left(\frac{15.000}{15.000 - 3}\right)$	1.2500
GE.......	L_{GE}	$24 + \frac{3 \times 15}{1.2500 + 2.2665}$	36.797	$1/Z_{GE}$	$\frac{3 \times 15}{24}\left(\frac{36.797}{36.797 - 6}\right)$	2.2403
GH.......	L_{GH}	....	12.000	$1/Z_{GH}$	....	2.0000
IG.......	L_{IG}	$24 + \frac{3 \times 15}{2.2403 + 2.0000}$	34.612	$1/Z_{IG}$	$\frac{3 \times 15}{24}\left(\frac{34.612}{34.612 - 6}\right)$	2.2682
JI.......	L_{JI}	$12 + \frac{3 \times 9}{2.2682}$	23.904	$1/Z_{JI}$	$\frac{3 \times 9}{12}\left(\frac{23.904}{23.904 - 3}\right)$	2.5729
HJ.......	L_{HJ}	$24 + \frac{3 \times 12}{2.5729}$	37.992	$1/Z_{HJ}$	$\frac{3 \times 12}{24}\left(\frac{37.992}{37.992 - 6}\right)$	1.7813
JH.......	L_{JH}	....	24.000	$1/Z_{JH}$	....	2.0000
IJ.......	L_{IJ}	....	25.500	$1/Z_{IJ}$	....	2.5500
GI.......	L_{GI}	....	41.647	$1/Z_{GI}$	....	2.1906
EG.......	L_{EG}	....	34.738	$1/Z_{EG}$	....	2.2665
CE.......	L_{CE}	....	36.797	$1/Z_{CE}$	....	2.2403
AC.......	L_{AC}	....	34.612	$1/Z_{AC}$	....	2.2682
BA.......	L_{BA}	....	23.904	$1/Z_{BA}$	....	2.5729
DB.......	L_{DB}	....	37.992	$1/Z_{DB}$	....	1.7813
DC.......	L_{DC}	$12 + \frac{3 \times 6}{2.1906 + 2.2403}$	16.062	$1/Z_{DC}$	$\frac{3 \times 6}{12}\left(\frac{16.062}{16.062 - 3}\right)$	1.8445
HG.......	L_{HG}	....	16.062	$1/Z_{HG}$	....	1.8445
FE.......	L_{FE}	$12 + \frac{3 \times 4}{2 \times 2.2665}$	14.647	$1/Z_{FE}$	$\frac{3 \times 4}{12}\left(\frac{14.647}{14.647 - 3}\right)$	1.2576
DF.......	L_{DF}	$24 + \frac{3 \times 12}{1.2576 + 2.0000}$	35.051	$1/Z_{DF}$	$\frac{3 \times 12}{24}\left(\frac{35.051}{35.051 - 6}\right)$	1.8098
HF.......	L_{HF}	....	35.051	$1/Z_{HF}$	....	1.8098

By symmetry:

Joint	Factor	Joint	Factor	Joint	Factor
J to H....	0.500	H to G....	0.374	A to C....	0.347
I to J	0.235	E to G....	0.345	B to A....	0.251
G to I	0.288	C to E....	0.326	D to B....	0.316
				H to F....	0.342

Step (*c*).—The "fixed joint" D is rotated a unit amount producing the following moments: $M_{DB} = 1.7813$, $M_{DC} = 1.8445$, and $M_{DF} = 1.8098$. These moments are tabulated and distributed as shown in Table 11 under "(*a*) Unit

TABLE 11.—MOMENTS DUE TO UNIT JOINT ROTATION—STEPS (*c*), (*f*), AND (*g*), EXAMPLE 7

Line	Distribution after rotation of:	(*a*) UNIT ROTATION OF JOINT D							
		BA	BD	DB	DC	DF	FD	FE	FH
		(*b*) UNIT ROTATION OF JOINT H							
		JI	JH	HJ	HG	HF	FH	FE	FD
1	Member DC...........	+0.0231	−0.0231	−0.0115	+1.8415	−0.0040	−0.0081	+0.0162	−0.0081
2	Member DF...........	+0.0006	−0.0006	−0.0003	−0.0040	+1.8098	+0.6196	−0.2392	−0.3804
3	Member DB...........	−0.5626	+0.5626	+1.7813	−0.0115	−0.0003	−0.0006	+0.0012	−0.0006
4	Final moments.........	−0.5389	+0.5389	+1.7695	+1.8290	+1.8055	+0.6109	−0.2218	−0.3891

TABLE 11.—(*Continued*)

Line	(*a*) UNIT ROTATION OF JOINT D								
	HF	HG	HJ	JH	JI	AB	AC	CA	CD
	(*b*) UNIT ROTATION OF JOINT H								
	DF	DC	DB	BD	BA	IJ	IG	GI	GH
1	−0.0010	−0.0060	−0.0005	−0.0009	+0.0009	+0.0982	−0.0982	−0.3406	+0.6890
2	−0.1902	−0.0040	−0.0003	−0.0006	+0.0006	+0.0026	−0.0026	−0.0089	−0.0080
3	−0.0003	−0.0005	0	−0.0001	+0.0001	−0.1412	+0.1412	+0.0190	−0.0231
4	−0.1945	−0.0105	−0.0008	−0.0016	+0.0016	−0.0404	+0.0404	−0.3005	+0.6579

TABLE 11.—(*Continued*)

Line	(*a*) UNIT ROTATION OF JOINT D								
	CE	EC	EF	EG	GE	GH	GI	IG	IJ
	(*b*) UNIT ROTATION OF JOINT H								
	GE	EG	EF	EC	CE	CD	CA	AC	AB
1	−0.3484	−0.1136	+0.0401	+0.0732	+0.0253	−0.0121	−0.0132	−0.0038	+0.0038
2	+0.0169	+0.0190	−0.0980	+0.0490	+0.0169	−0.0080	−0.0089	−0.0026	+0.0026
3	−0.0259	−0.0084	+0.0030	+0.0054	+0.0019	−0.0009	−0.0010	−0.0003	+0.0003
4	−0.3574	−0.0730	−0.0546	+0.1276	+0.0441	−0.0210	−0.0231	−0.0067	+0.0067

Rotation of Joint D." Joint H is assumed fixed against rotation during this operation.

Next the "fixed joint" H is rotated a unit amount producing the following moments: $M_{HJ} = 1.7813$, $M_{HG} = 1.8445$, and $M_{HF} = 1.8098$. From symmetry it is evident that the values given in Table 11 under "(*b*) Unit Rotation of Joint H" are the result of distributing these moments throughout the frame. The final moments for this step are given in line 4, Table 11. To compute the external moments M_D and M_H resulting from a unit rotation of joints D and H proceed as in Table 12.

TABLE 12.—COMPUTATION OF EXTERNAL MOMENTS, STEP (*c*), EXAMPLE 6

Symbol	Joint rotated	Computation	External moment
M_D	D	$-(1.7695 + 1.8290 + 1.8055)$	-5.4040
M_H	D	$-(0.1945 - 0.0105 - 0.0008)$	$+0.2058$
M_D	H	. . . .	$+0.2058$
M_H	H	. . . .	-5.4040

Step (*d*).—The values of A and x for the 4-kip concentrated load and the 8-kip uniform load are taken from Examples 1(*a*) and 1(*b*). Then:

$$M_{CE} = \frac{6 \times 288}{24}\left(\frac{2 \times 36.797 \times 12 - 24 \times 12}{4 \times 36.797 \times 34.738 - 24 \times 24}\right) = 9.444$$

$$M_{EC} = \frac{6 \times 288}{24}\left(\frac{2 \times 34.738 \times 12 - 24 \times 12}{4 \times 36.797 \times 34.738 - 24 \times 24}\right) = 8.660$$

$$M_{EG} = \frac{6 \times 384}{24}\left(\frac{2 \times 34.738 \times 12 - 24 \times 12}{4 \times 34.738 \times 36.797 - 24 \times 24}\right) = 11.547$$

$$M_{GE} = \frac{6 \times 384}{24}\left(\frac{2 \times 36.797 \times 12 - 24 \times 12}{4 \times 34.738 \times 36.797 - 24 \times 24}\right) = 12.592$$

These moments are entered in line 15 and distributed in line 16, Table 13(*c*), assuming that joints D and H are fixed against rotation and that vertical movement of all joints is prevented. Note that -4.105 listed in line 15, Table 13(*c*), under member EF is the proportional part of 11.547 which is distributed to member EF. Line 17 gives the moments before releasing joints D and H.

Step (*e*).—The external moment necessary to release joints D and H is the sum of the internal moments at each of these joints, as listed in line 17, Table 13(*c*); thus: The releasing moment for $D = -0.122 - 1.640 + 0.103 = -1.659$; and the releasing moment for $H = +0.103 + 2.545 + 0.189 = +2.836$. The external moments at each of these joints due to a unit rotation of each joint have been tabulated at the end of step (*c*). Similar to step (*h*), Example 5, let X_L be the rotation correction factor for joint D, or the amount that joint D must rotate to release both joints, assumed fixed during the load computations for step (*d*); and let Y_L be a similar rotation correction factor for joint H.

TABLE 13.—TABULATION AND DISTRIBUTION OF MOMENTS, EXAMPLE 6

Line	Description	Joint B		Joint D			Joint F		
		BA	BD	DB	DC	DF	FD	FE	FH
	(a) STEPS (g) AND (i)								
1	Moments (elasticity of supports)[a]........	....	+1.6879	+2.3440	....	....	....	....	....
2	Distribution to the right......	−1.6879	....	....	−0.4733	−0.0123	−0.0246	+0.0492	−0.0246
3	Distribution to the left........	−0.3801	+0.3801	+0.1900	....	....	....	....	....
4	Translation; joints D and H fixed[b].........	−2.0680	+2.0680	+2.5340	−0.4733	−0.0123	−0.0246	+0.0492	−0.0246
	Moments After Releasing:								
5	Joint D.....	+0.0245	−0.0245	−0.0713	−0.6939	−0.6850	−0.2318	+0.0842	+0.1467
6	Joint H.....	0	0	0	+0.0001	+0.0017	+0.0033	+0.0019	−0.0052
7	Final moments, translation of AB...........	−1.8635	+1.8635	+1.8627	−1.1671	−0.6956	−0.2531	+0.1353	+0.1178
	(b) STEPS (g) AND (i)								
8	Moments (elasticity of supports)[a].......	....	....	....	....	+2.4294	+1.8588	....	....
9	Distribution of EC to EG and EF; and EF to joints D and H	....	....	....	....	+0.0659	+0.1318	−0.2636	+0.1318
10	Distribution of FD, CE, and EG...........	+0.0704	−0.0704	−0.0352	−0.4741	....	....	−0.7176	−1.1412
11	Translation; joints D and H fixed[b].........	+0.0704	−0.0704	−0.0352	−0.4741	+2.4953	+1.9906	−0.9812	−1.0094
	Moments After Releasing:								
12	Joint D.....	+0.1968	−0.1968	−0.6461	−0.6678	−0.6593	−0.2231	+0.0810	+0.1421
13	Joint H.....	+0.0010	−0.0010	0	−0.0007	−0.0121	−0.0242	−0.0138	+0.0380
14	Final moments, translation of CD...........	+0.2682	−0.2682	−0.6813	−1.1426	+1.8239	+1.7433	−0.9140	−0.8293
	(c) STEPS (d), (e), (f), (h), AND (i)								
15	Load moments[a]..	....	....	....	....	....	....	....	....
16	Distribution.....	+0.244	−0.244	−0.122	−1.640	+0.103	+0.205	−0.410	+0.205
17	Final moments; joints D and H fixed[c].........	+0.244	−0.244	−0.122	−1.640	+0.103	+0.205	−0.410	+0.205
18	After releasing joint D........	−0.155	+0.155	+0.509	+0.526	+0.519	+0.176	−0.064	−0.112
19	After releasing joint H.......	−0.001	+0.001	0	+0.005	+0.100	+0.200	+0.114	−0.314
20	Final moments; no vertical movement.....	+0.088	−0.088	+0.387	−1.109	+0.722	+0.581	−0.360	−0.221
	Shear Correction for the:								
21	First panel..	−41.118	+41.118	+41.099	−25.751	−15.348	−5.583	+2.984	+2.509
22	Second panel	+4.188	−4.188	−10.640	−17.843	+28.183	+27.224	−14.274	−12.950
23	Third panel.	+0.527	−0.527	−1.313	−2.292	+3.605	+9.308	+10.260	−19.568
24	Fourth panel	−0.164	+0.164	+0.394	+0.732	−1.126	−2.877	−3.303	+6.180
25	Final moments..	−36.479	+36.479	+29.927	−46.263	+16.336	+28.653	−4.693	−23.960

[a] Moments resulting from the elasticity of supports. [b] Final moments resulting from translation, assuming joints D and H to be fixed against rotation. [c] Final moments resulting from applied load, assuming joints D and H fixed against rotation and all joints fixed against vertical movement.

TABLE 13.—(*Continued*)

Line	JOINT H			JOINT J		JOINT A		JOINT C	
	HF	HG	HJ	JH	JI	AB	AC	CA	CD
					(*a*) STEPS (*g*) AND (*i*) (*Continued*)				
1	....	....	....	....	....	....	+1.6156	+1.8602	....
2	−0.0123	−0.0184	−0.0013	−0.0027	+0.0027	−0.4237	+0.4237	+0.1469	−0.9467
3	....	....	....	....	....	−1.6156	....	....	....
4	−0.0123	−0.0184	−0.0013	−0.0027	+0.0027	−2.0393	+2.0393	+2.0071	−0.9467
5	+0.0738	+0.0040	+0.0003	+0.0006	−0.0006	+0.0154	−0.0154	+0.1140	−0.2496
6	−0.0154	−0.0156	−0.0151	−0.0046	+0.0046	−0.0001	+0.0001	+0.0002	+0.0002
7	+0.0461	−0.0300	−0.0161	−0.0067	+0.0067	−2.0240	+2.0240	+2.1213	−1.1961
					(*b*) STEPS (*g*) AND (*i*) (*Continued*)				
8	....	....	....	....	....	....	....	....	....
9	+0.0659	....	....	....	....	....	....	....	....
10	−0.5706	+0.0864	+0.0064	+0.0128	−0.0128	+0.2992	−0.2992	−1.0385	−0.9482
11	−0.5047	+0.0864	+0.0064	+0.0128	−0.0128	+0.2992	−0.2992	−1.0385	−0.9482
12	+0.0710	+0.0038	+0.0003	+0.0006	−0.0006	+0.0148	−0.0148	+1.1097	−0.2402
13	+0.1125	+0.1140	+0.1103	+0.0336	−0.0336	+0.0004	−0.0004	−0.0014	−0.0013
14	−0.3212	+0.2042	+0.1170	+0.0470	−0.0470	+0.3144	−0.3144	−0.9302	−1.1897
					(*c*) STEPS (*d*), (*e*), (*f*), (*h*), AND (*i*) (*Continued*)				
15	....	....	....	....	....	....	....	....	....
16	+0.103	+2.545	+0.189	+0.378	−0.378	+1.035	−1.035	−3.593	−3.280
17	+0.103	+2.545	+0.189	+0.378	−0.378	+1.035	−1.035	−3.593	−3.280
18	−0.056	−0.003	0	0	0	−0.012	+0.012	−0.086	+0.189
19	−0.928	−0.910	−0.910	−0.277	+0.277	−0.003	+0.003	+0.012	+0.011
20	−0.881	+1.602	−0.721	+0.101	−0.101	+1.020	−1.020	−3.667	−3.080
21	+1.017	−0.661	−0.356	−0.148	+0.148	−44.654	+44.654	+46.805	−26.391
22	−5.015	+3.189	+1.826	+0.734	−0.734	+4.911	−4.911	−11.527	−18.579
23	−20.473	+12.825	+7.648	+3.010	−3.010	+0.668	−0.668	−2.009	−2.486
24	+16.989	+28.505	−45.494	−45.515	+45.515	−0.230	+0.230	+0.707	+0.839
25	−8.363	+45.460	−37.097	−41.818	+41.818	−38.285	+38.285	+27.309	−49.697

Equations can be written: For joint D—

$$-\ 5.404\ X_L + 0.206\ Y_L = -\ 1.659 \qquad (29a)$$

and, for joint H—

$$+\ 0.206\ X_L - 5.404\ Y_L = +\ 2.836 \qquad (29b)$$

Solving simultaneously: $X_L = 0.288$ and $Y_L = -\ 0.514$.

Step (f).—Multiply the moments in line 4, Table 11, by + 0.288 and enter the product in line 18, Table 13(*c*), under the same moment subscripts as those given under "(*a*) Unit Rotation of Joint D" in Table 11. Next multiply the moments in line 4, Table 11, by − 0.514 and enter the result in line 19, Table 13(*c*), under the same moment subscripts as those given under "(*b*) Unit Rotation of Joint H" in Table 11. Line 20, Table 13(*c*), gives the final moments if deformation due to vertical shear is prevented.

Step (g).—To compute the required shear deformations translate member AB a distance $\Delta = 24/E$ vertically upward with respect to CD. The induced

TABLE 13.—(*Continued*)

Line	JOINT C	JOINT E			JOINT G			JOINT I	
	CE	EC	EF	EG	GE	GH	GI	IG	IJ
	(*a*) STEPS (*g*) AND (*i*) (*Continued*)								
1	....	....	....	....	....	....	....	....	....
2	−1.0604	−0.3458	+0.1229	+0.2229	+0.0770	−0.0368	−0.0402	−0.0116	+0.0116
3	....	....	....	....	....	....	....	....	....
4	−1.0604	−0.3458	+0.1229	+0.2229	+0.0770	−0.0368	−0.0102	−0.0116	+0.0116
5	+0.1356	+0.0277	+0.0207	−0.0484	−0.0167	+0.0080	+0.0087	+0.0025	−0.0025
6	−0.0004	−0.0011	+0.0005	+0.0006	+0.0030	−0.0056	+0.0026	−0.0003	+0.0003
7	−0.9252	−0.3192	+0.1441	+0.1751	+0.0633	−0.0344	−0.0289	−0.0094	+0.0094
	(*b*) STEPS (*g*) AND (*i*) (*Continued*)								
8	+1.9359	+1.8543	....	....	....	....	....	....	....
9	....	....	−0.6592	−1.1951	....	....	....	....	....
10	+0.0508	+0.1470	−0.2940	+0.1470	−0.3621	+0.1728	+0.1893	+0.0545	−0.0545
11	+1.9867	+2.0013	−0.9532	−1.0481	−0.3621	+0.1728	+0.1893	+0.0545	−0.0545
12	+0.1305	+0.0266	+0.0200	−0.0466	−0.0161	+0.0077	+0.0084	+0.0024	−0.0024
13	+0.0027	+0.0080	−0.0034	−0.0046	−0.0223	+0.0110	−0.0187	+0.0025	−0.0025
14	+2.1199	+2.0359	−0.9366	−1.0993	−0.4005	+0.2215	+0.1790	+0.0594	−0.0594
	(*c*) STEPS (*d*), (*e*), (*f*), (*h*), AND (*i*) (*Continued*)								
15	+9.444	−8.660	−4.105	+11.547	−12.502	....	....	....	....
16	−2.571	−7.442	+3.078	+5.582	+1.928	+5.090	+5.574	+1.606	−1.606
17	+6.873	−16.102	−1.027	+17.129	−10.664	+5.090	+5.574	+1.606	−1.606
18	−0.103	−0.021	−0.016	+0.037	+0.013	−0.006	−0.007	−0.002	+0.002
19	−0.023	−0.066	+0.028	+0.038	+0.184	−0.338	+0.154	−0.021	+0.021
20	+6.747	−16.189	−1.015	+17.204	−10.467	+4.746	+5.721	+1.583	−1.583
21	−20.414	−7.043	+3.181	+3.862	+1.397	−0.759	−0.638	−0.208	+0.208
22	+33.106	+31.792	−14.625	−17.167	−6.254	+3.458	+2.796	+0.929	−0.929
23	+4.495	+12.339	+10.512	−22.851	−23.795	+13.354	+19.441	+3.530	−3.530
24	−1.546	−4.276	−3.521	+7.707	+22.597	+29.213	−51.810	−49.429	+49.429
25	+22.388	+16.623	−5.468	−11.155	−16.522	+50.012	−33.490	−43.595	+43.505

moments are:

$$M_{AC} = \frac{6 \times 15 \times 24}{24}\left(\frac{2 \times 34.612 + 24}{4 \times 34.612 \times 41.647 - 24 \times 24}\right) = 1.6156$$

$$M_{CA} = \frac{6 \times 15 \times 24}{24}\left(\frac{2 \times 41.647 + 24}{4 \times 34.612 \times 41.647 - 24 \times 24}\right) = 1.8602$$

$$M_{BD} = \frac{6 \times 12 \times 24}{24}\left(\frac{2 \times 24.000 + 24}{4 \times 24.000 \times 37.992 - 24 \times 24}\right) = 1.6879$$

$$M_{DB} = \frac{6 \times 12 \times 24}{24}\left(\frac{2 \times 37.992 + 24}{4 \times 24.000 \times 37.992 - 24 \times 24}\right) = 2.3440$$

These moments are entered in line 1 and distributed in lines 2 and 3, Table 13(*a*), assuming joints D and H fixed against rotation. Line 4, Table 13(*a*), gives the moments before releasing joints D and H. The releasing line moment for *D* from line 4, Table 13(*a*), is + 2.5341 − 0.4733 − 0.0123 = + 2.0485;

and, similarly, for joint H the releasing moment is $- 0.0123 - 0.0184 - 0.0014 = - 0.0321$.

If X_{T1} and Y_{T1} are the correction factors for joints D and H, respectively, equations can be written: Let X_{T1} equal the rotation correction factor for joint D; and Y_{T1} equal the rotation correction factor for joint H. For joint D—

$$- 5.4040\,X_{T1} + 0.2058\,Y_{T1} = + 2.0485 \qquad (30a)$$

and, for joint H—

$$+ 0.2058\,X_{T1} - 5.4040\,Y_{T1} = - 0.0321 \qquad (30b)$$

Solving Eqs. 30, $X_{T1} = - 0.3794$ and $Y_{T1} = - 0.0085$.

Multiply the moments in line 4, Table 11, by $- 0.3794$ and enter the result in line 5, Table 13(*a*), under the same moment subscripts as those given under "(*a*) Unit Rotation of Joint D," in Table 11. Next multiply the moments in line 4, Table 11, by $- 0.0085$ and tabulate in line 6, Table 13(*a*), under the same moment subscripts as those given under "(*b*) Unit Rotation of Joint H," in Table 11. Line 7, Table 13(*a*), gives the moments due to the translation of AB, including the effect of releasing joints D and H.

Next, CD is translated a distance $\Delta = 24/E$ vertically upward relative to EF. The induced moments are:

$$M_{CE} = \frac{6 \times 15 \times 24}{24}\left(\frac{2 \times 36.797 + 24}{4 \times 36.797 \times 34.738 - 24 \times 24}\right) = 1.9359$$

$$M_{EC} = \frac{6 \times 15 \times 24}{24}\left(\frac{2 \times 34.738 + 24}{4 \times 36.797 \times 34.738 - 24 \times 24}\right) = 1.8543$$

$$M_{DF} = \frac{6 \times 12 \times 24}{24}\left(\frac{2 \times 35.051 + 24}{4 \times 35.051 \times 24.000 - 24 \times 24}\right) = 2.4294$$

$$M_{FD} = \frac{6 \times 12 \times 24}{24}\left(\frac{2 \times 24.000 + 24}{4 \times 35.051 \times 24.000 - 24 \times 24}\right) = 1.8588$$

These moments are written in line 8, and distributed in lines 9 and 10, Table 13(*b*), assuming joints D and H fixed against rotation. Line 11 gives the moments before releasing the joints D and H.

The releasing moment for joint D from line 11, Table 13(*b*), is: $- 0.0352 - 0.4741 + 2.4953 = + 1.9860$; and, similarly, for joint H: $- 0.5047 + 0.0864 + 0.0064 = - 0.4119$. Let X_{T2} and Y_{T2} be the rotation correction factors for joints D and H, respectively. Then equations can be written: For joint D—

$$- 5.4040\,X_{T2} + 0.2058\,Y_{T2} = + 1.9860 \qquad (31a)$$

and, for joint H—

$$+ 0.2058\,X_{T2} - 5.4040\,Y_{T2} = - 0.4119 \qquad (31b)$$

Solving Eqs. 31 simultaneously: $X_{T2} = - 0.3651$ and $Y_{T2} = + 0.0623$.

Multiply the moments in line 4, Table 11, by $- 0.3651$ and tabulate in line 12, Table 13(*b*), under the same moment subscripts as those given under "(*a*) Unit Rotation of Joint D," in Table 11. Next multiply the moments in line 4, Table 11, by + 0.0623 and tabulate in line 13, Table 13(*b*), under the same moment subscripts as those given under "(*b*) Unit Rotation of Joint H," in Table 11. Line 14, Table 13(*b*), gives the moments due to the translation of CD including the effect of releasing joints D and H.

The final moments due to the translation of GH a distance of $\Delta = 24/E$ vertically downward relative to EF are given in line 1, Table 14. These values are obtained from line 14, Table 13(*b*), as a result of the symmetry of the frame.

The final moments due to the translation of IJ a distance $\Delta = 24/E$ relative to GH are given in line 2, Table 14. These values are obtained from Table 13(*a*) as a result of symmetry.

TABLE 14.—FINAL MOMENTS AFTER THE TRANSLATION OF JOINTS GH (LINE 1) AND IJ (LINE 2)

Line	Member	Joint J		Joint H			Joint F			Joint D
		JI	JH	HJ	HG	HF	FH	FE	FD	DF
1	GH......	+0.2682	−0.2682	−0.6813	−1.1426	+1.8239	+1.7433	−0.9140	−0.8293	−0.3212
2	IJ.......	−1.8635	+1.8635	+1.8627	−1.1671	−0.6956	−0.2531	+0.1353	+0.1178	+0.0461

TABLE 14.—(*Continued*)

Line	Member	Joint D (*Continued*)		Joint B		Joint I		Joint G	
		DC	DB	BD	BA	IJ	IG	GI	GH
1	GH..........	+0.2042	+0.1170	+0.0470	−0.0470	+0.3144	−0.3144	−0.9302	−1.1897
2	IJ...........	−0.0300	−0.0161	−0.0067	+0.0067	−2.0240	+2.0240	+2.1213	−1.1961

TABLE 14.—(*Continued*)

Line	Member	Joint G	Joint E			Joint C			Joint A	
		GE	EG	EF	EC	CE	CD	CA	AC	AB
1	GH......	+2.1199	+2.0359	−0.9366	−1.0993	−0.4005	+0.2215	+0.1790	+0.0594	−0.0594
2	IJ.......	−0.9252	−0.3192	+0.1441	+0.1751	+0.0633	−0.0344	−0.0289	−0.0094	+0.0094

TABLE 15.—SUM OF END MOMENTS, STEP (*h*), EXAMPLE 6

Members	Table 13(*a*)	Table 13(*b*)	Table 14	Table 14	Table 13(*c*)	End moments should be:
	Line 7	Line 14	Line 1	Line 2	Line 20	
AC and BD.......	+7.8715	−2.1041	+0.4024	−0.0611	−4.388	+132.000
CE and DF.......	−2.1931	+7.7230	−2.6503	+0.4023	−8.139	+84.000
EG and FH.......	+0.4023	−2.6503	+7.7230	−2.1931	+5.635	−60.000
GI and HJ.......	−0.0611	+0.4024	−2.1941	+7.8715	+6.684	−156.000

Step (*h*).—Let W, X, Y, and Z represent the shear correction factors for the first, second, third, and fourth panels, respectively. Coefficients for the solution are assembled as shown in Table 15 and are used for the construction of shear equations as follows: For the first panel—

$$7.8715\,W - 2.1941\,X + 0.4024\,Y - 0.0611\,Z - 4.388 = +132.000 \quad (32a)$$

for the second panel—

$$-2.1931\,W + 7.7230\,X - 2.6503\,Y + 0.4023\,Z - 8.139 = +84.000 \quad (32b)$$

for the third panel—

$$0.4023\,W - 2.6503\,X + 7.7230\,Y - 2.1931\,Z + 5.635 = -60.000 \quad (32c)$$

and, for the fourth panel—

$$-0.0611\,W + 0.4024\,X - 2.1941\,Y + 7.8715\,Z + 6.684 = -156.000 \quad (32d)$$

Solving Eqs. 28, simultaneously: $W = 22.064$; $X = 15.617$; $Y = -11.224$; and $Z = -24.423$.

Step (*i*).—Multiply the moments in line 7, Table 13(*a*), by $W = 22.064$ and tabulate in line 21, Table 13(*c*). Multiply the moments in line 14, Table 13(*b*), by $X = 15.617$ and tabulate in line 22, Table 13(*c*). Multiply the moments in line 1, Table 14, by $Y = -11.224$ and tabulate in line 23, Table 13(*c*). Multiply the moments in line 2, Table 14, by $Z = -24.423$ and tabulate in line 24, Table 13(*c*). The addition of lines 20 to 24, inclusive, gives the final moments including the effect of joint translation.

EXAMPLE 7

A multiple story transverse bent is loaded as shown in Fig. 13. The end moments of all members are required.

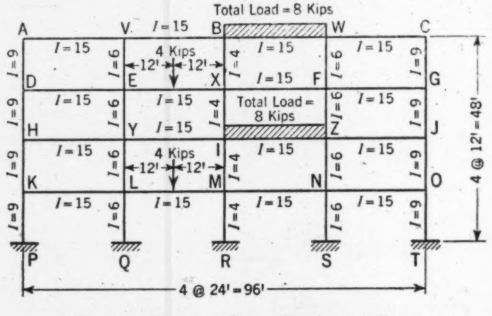


FIG. 13.—MULTIPLE STORY TRANSVERSE BENT

Step (*a*).—Joints V, W, X, Y, and Z are assumed to be secured against rotation temporarily.

Step (*b*).—Compute the stiffness factors, elastic lengths, and carry-over factors.

Step (*c*).—Rotate each of the "fixed joints" a unit amount. Then tabulate and distribute the induced moments for each "fixed joint" in a separate table.

Because of symmetry, only three tables are necessary. Compute the unbalanced moments at each of the "fixed joints" in each table.

Step (*d*).—Compute the end moments for the loaded members. Tabulate and distribute these moments in a fourth table.

Step (*e*).—Compute the releasing moments for each of the "fixed joints" from the table of step (*d*). With unbalanced moments from step (*c*), and these releasing moments, form and solve the five simultaneous equations required for the computation of the rotation correction factors for load.

Step (*f*).—Tabulate the rotation correction moments in the table of step (*d*).

Step (*g*).—Translate each story laterally an amount $\Delta = 12/E$. Tabulate and distribute the induced moments in a separate table for each story. Five simultaneous equations are required for each of the four story tables to correct for the rotation of the five "fixed joints." However, it should be noted that the coefficients of the corresponding unknowns for each set of equations are identical, the only difference being in the constants. Therefore, the four sets of equations may be solved with only slightly more labor than that involved for one set. Tabulate the rotation correction moments in the proper story tables.

Step (*h*).—From the final moments resulting from translation in step (*g*) shear correction equations are formed for the four stories. The solution of these four simultaneous equations gives the required shear correction factors.

Step (*i*).—Tabulate the shear correction moments in the table of step (*d*). The final moments are found by summing up the effect of load, rotation, and translation. Thus, the exact solution of the seventy-two unknown moments is obtained with two sets of simultaneous equations. One set consists of four equations to compensate for sidesway and the other of five equations to correct for the "fixed joints." The latter set will have a column of constants for each load considered.

SUMMARY

A method has been developed for analyzing continuous frames with prismatic members. The extension to members having a variable cross section may be made in a manner similar to that used in previous publications.[6] The method is particularly adapted to the solution of frames such as the Vierendeel truss. The effect of flexible connections which is considered in aircraft analysis is included in the paper.

---

[6] "Lateral Loads and Members of Variable Section," *Bulletin No. 66*, Eng. Experiment Station, Ohio State University, Columbus, Ohio, 1931.

DISCUSSION

L. J. Mensch,[7] M. ASCE.—The method of balancing fixed-end moments—connected with the name of Hardy Cross,[8] M. ASCE,—has had an immense success with teachers and students in the analysis of framed structures. Such members were formerly designed by the conventional moment formulas $\left(\frac{w\,L^2}{10}, \frac{w\,L^2}{12}, \text{and } \frac{PL}{6}\right)$ as first adopted by the Hennebique firm of Paris, France, before the turn of the century, and as since adopted by many building codes, except for the cases of very long spans, for which engineers of mathematical ability used Clapeyron's theorem, or a more or less complicated variation of it.

Professor Cross's method did not require deep studies or high mathematical skill, which explains its immediate success, but, at best, it is only a trial-and-error method requiring considerable time. Shorter methods have been known ever since Saint-Venant demonstrated that distant members have no appreciable effect on the member under investigation. For example, referring to Fig. 8, in the analysis of girder CE, the restraint of members AC, CD, EF, and EG only have to be taken into account.

The authors have shown Clapeyron's so-called exact method with new nomenclature simplified by previous calculation of certain elastic constants. However, even this simplified method requires too much effort for everyday use. Elsewhere[9,10,11,12] the writer has shown that a great deal more time and effort may be saved by the same theorem if Saint-Venant's principle is used. For girder CE in Fig. 8, the primary moments[10] are given by:

$$M_{CE} = \frac{3\,P\,L}{8} \frac{1 + 2\,N_E}{4\,(1 + N_C)\,(1 + N_E) - 1} \quad \text{.............} (33a)$$

$$M_{EC} = \frac{3\,P\,L}{8} \frac{1 + 2\,N_C}{4\,(1 + N_C)\,(1 + N_E) - 1} \quad \text{.............} (33b)$$

when

$$N_E = \frac{3\,K}{\Sigma m\,K_E} \quad \text{......................} (34a)$$

and

$$N_C = \frac{3\,K}{\Sigma m\,K_C} \quad \text{......................} (34b)$$

In these equations m is an elastic factor depending on the restraint of each adjacent member at the far end. When the end is hinged, $m = 3$; when the end is fixed, $m = 4$; when the restraint lies between these two conditions, $m = 3.6$ is a good guess.

---

[7] Gen. Contractor, Evanston, Ill.

[8] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, ASCE, Vol. 96, 1932, p. 1.

[9] *Ibid.*, Vol. 83, 1919–1920, p. 1682.

[10] *Ibid.*, Vol. 102, 1937, p. 1050.

[11] *Ibid.*, p. 897 and p. 438.

[12] *Journal*, ACI, February, 1932, p. 387.

For girders, $K = I/L$; for columns, $K = I/h$. For girder CE, $K = 15/24$; $K'_C = 15/24$; $K''_C = 6/12$; $K'_E = 15/24$; $K''_E = 4/12$;

$$N_C = \frac{3 \times 15/24}{3.6 \times 15/24 + 4 \times 6/12} = 0.442;$$

and

$$N_E = \frac{3 \times 15/24}{3.6 \times 15/24 + 4 \times 4/12} = 0.522.$$

These values, when introduced in Eqs. 33*a* and 33*b* with those for $P = 4{,}000$ and $L = 24$, give $M_{CE} = 9{,}615$ ft-lb and $M_{EC} = 8{,}660$ ft-lb, which agree within 1% with the values given in Table 5. These moments will be modified by the action of the 8,000-lb uniform load on span EG.

The moments in EG are found from the formulas:

$$M_{EG} = \frac{W\,L}{4}\,\frac{1 + 2\,N_G}{4\,(1 + N_E)\,(1 + N_G) - 1} \qquad \ldots\ldots\ldots\ldots (35a)$$

$$M_{GE} = \frac{W\,L}{4}\,\frac{1 + 2\,N_E}{4\,(1 + N_E)\,(1 + N_G) - 1} \qquad \ldots\ldots\ldots\ldots (35b)$$

It is easily found that $N_E = 0.522$ and $N_G = 0.422$, so that

$$M_{EG} = \frac{8{,}000 \times 24}{4}\,\frac{1 + 2 \times 0.422}{4 \times 1.422 \times 1{,}522 - 1} = 11{,}550 \text{ ft-lb}$$

and

$$M_{GE} = \frac{8{,}000 \times 24}{4}\,\frac{1 + 2 \times 0.522}{4 \times 1.422 \times 1.522 - 1} = 12{,}820 \text{ ft-lb},$$

which again closely agree with the tabulated results. Moment M_{EG} must be distributed to EC and EF in relation to stiffness, thus, $M'_{EC}:M'_{EF} = 3.6 \times 15/24 : 4 \times 4/12$, from which $M'_{EC} = 7{,}250$ ft-lb and $M'_{EF} = 4{,}300$ ft-lb. The maximum moment is $M_{EC} = 8{,}660 + 7{,}250 = 15{,}910$ ft-lb, as compared with 16,067 ft-lb given in the table.

The maximum value of M_{EG} is found by a similar distribution of the moment of M_{EC}, and is $M_{EG} = 11{,}550 + 8{,}660 \times 54/86 = 16{,}990$ ft-lb, as compared with 17,145 ft-lb in Table 5.

If the value of $m = 4$ for the columns be replaced by $m = 3.6$, there is a slight modification of the value of the moments, usable for Example 6, Fig. 12.

To show a short cut for analyzing sidesway, Example 2, Fig. 7, is chosen. Then,

$$M_{AC} = \frac{3\,P\,L}{8}\,\frac{1 + 2\,N_C}{4\,(1 + N_A)\,(1 + N_C) - 1} \qquad \ldots\ldots\ldots\ldots (36a)$$

$$M_{CA} = \frac{3\,P\,L}{8}\,\frac{1 + 2\,N_A}{4\,(1 + N_A)\,(1 + N_C) - 1} \qquad \ldots\ldots\ldots\ldots (36b)$$

In this problem $N_C = \dfrac{3 \times 15/24}{4 \times 4/12} = 1.408$ and $N_A = \dfrac{3 \times 15/24}{4 \times 6/12} = 0.938$, so that $M_{AC} = 7{,}720$ ft-lb and $M_{CA} = 5{,}830$ ft-lb when no sidesway is possible. Then the shears acting in the columns will be: In AB $(7{,}720 + 7{,}720/2)/12 = 965$ lb; in CD $(5{,}830 + 5{,}830/2)/12 = 729$ lb. These shears should be equal. They can be made so if an imaginary force of $965 - 729 = 236$ lb be assumed as acting in a horizontal direction (AC) at the top of the column. The problem is to find the proportionate parts to be assigned to each column, and, furthermore, how these shears modify the moments at the top and bottom of the columns.

By figuring a number of multicolumn bents the writer found that the proportionate part taken by column AB closely equals $\frac{K_{AB} + K_{AC}}{K_{AB} + 2\,K_{AC} + K_{CD}}$ $= \frac{6/12 + 15/24}{6/12 + 2 \times 15/24 + 4/12} = 0.54$, and that the shear in AB equals 236 × 0.54 = 127 lb, and, in CD = 238 − 127 = 111 lb. The zero point of moments produced in the fixed-ended columns by these shears is located between 0.4 h and 0.5 h from the column top. In many cases a good average for this distance is 0.45 h. Hence, the moment at the top of column AB is diminished by 127 × 0.45 × 12 = 686 ft-lb, and the moment at the top of column CD is increased by 111 × 0.45 × 12 = 600 ft-lb, which values closely agree with those in Table 3.

As another example for finding the shear, it is proposed to find the proportionate part of the shear for columns CD and AB of Fig. 8. For column CD, it is

$$\frac{K_{CA} + K_{CE} + K_{CD}}{\Sigma K \text{ of columns} + 2\,\Sigma K \text{ of all girders}}$$

$$= \frac{15/24 + 15/24 + 6/12}{2 \times 9/12 + 2 \times 6/12 + 4/12 + 2 \times 4 \times 15/24} = 0.223,$$

and, for column AB, it is $\frac{K_{AC} + K_{AB}}{\Sigma K \text{ of columns} + 2\,\Sigma K \text{ of girders}} = 0.176.$

The authors deserve great credit for having shown clearly the influence of sidesway in their examples. They have, however, side-stepped the issue of the influence of the width of the columns and the depth of the girders on the calculated moments.

Tests to near destruction of properly designed frames have consistently shown reserve strengths from 30% to more than 100% greater than those predicted by the classical "exact" theories. This was partly because the clear spans were smaller than the center-to-center distances, and partly because the plasticity of materials of construction under high stresses and long time loading made the negative moments considerably smaller. This discrepancy has long been known, but the use of the center-to-center distances and the use of a constant modulus of elasticity is still preached; it seems to have become an article of faith.

The writer cannot refrain from emphasizing that the deluge of literature on the analysis of framed structures since World War I has drowned the works of the few true thinking pioneers and the investigators of genius. In addition, this mass of literature has so dazzled students that they have lost their heads in the belief that everything old was wrong and everything new was right, and that a study of these new methods will lead them to an advanced standing in their profession, and to prosperity. The fact is that they are being robbed not only of their time, and their chance of acquiring useful knowledge, but also of their money.

Frederick S. Merritt,[13] Assoc. M. ASCE.—Several papers such as this one have been published in recent years demonstrating the feasibility of one-cycle moment distribution. They are proof that open-minded designers recognize that multicycle distribution has many shortcomings in both theoretical and practical applications. Although it can be proved that, in general, one-cycle distribution is a more powerful tool than multicycle distribution, most of these papers, including this one, do not conclusively show superiority over the multicycle method in solving practical problems. The reasons are quite evident:

---

[13] Asst. Editor, *Engineering News-Record,* New York, N. Y.

Either method can be used to analyze statically indeterminate structures, and the papers on one-cycle distribution published in the past in *Proceedings* have been unable to prove that the method is simpler, quicker, or easier to visualize—because too many extraneous concepts are introduced, requiring much preliminary computation. For example, in this paper "elastic length" is an unnecessary concept, as far as practical problems in general are concerned; results in extra computations when evaluating elastic properties of every span; and increases the number of required formulas, as well as complicating them. Thus, the superiority of one-cycle distribution will not be conclusive until the preliminary computations are reduced to a minimum and then by comparison it is shown that less work is involved than in, say, two cycles of fixed-end distribution.

The first step in streamlining one-cycle moment distribution is to base the method on the fewest necessary and sufficient fundamental terms for evaluating the elastic properties of each span, independent of the rest of the structure. Experience shows that the least number of terms required is three. Although these may be selected in many ways, the best ones are those that yield the simplest and fewest formulas.

A group that satisfies this criterion is $C^F{}_L$, $C^F{}_R$, and K. The first two are the "moment ratios" to left and right, respectively, for fixed ends—the ratio of the moment applied at one end of a beam to the moment carried over or induced at the other end when rotation is completely prevented. The third term is the moment to be applied at one end of the beam, considering it simply supported, to produce a unit rotation at the other end. These terms apply to beams of variable moment of inertia as well as to prismatic sections.

For beams of constant moment of inertia, $C^F{}_L = C^F{}_R = 2$ and $K = 6\,E\,I/L$. Since relative values of K are used in most cases, the coefficient 6 may often be disregarded.

Secondary terms needed to solve specific problems may be evaluated using the fundamental terms. For example, the moment to be applied at the left end of a beam, considering it simply supported, to produce unit rotation there is $K/C^E{}_L$.

The second streamlining step is to evaluate the end restraints due to the rest of the structure in the least number of terms. Instead of using "flexibility" for distribution of moments at a joint, requiring manipulation of reciprocal relationships, as the authors did, it is preferable to use stiffness. The stiffness S of the end of an unloaded span is defined as the moment producing a rotation of unity at the end where it is applied when the other end of the beam is restrained against rotation by other members of the structure.

Simpler formulas result if they are expressed in terms of "moment ratios."[14] The moment ratio C is defined for an unloaded member of a continuous structure as the ratio of the moment introduced at one end of the span to the moment carried over to the other end. Thus, moment ratio is the reciprocal of the authors' carry-over factor.

Equivalent to "combined flexibility," restraint R at the end of an unloaded beam at a joint is the moment applied at that end to produce a unit rotation in all the members at the joint. By definition of continuity, R is equal to the sum of the stiffnesses of the adjacent ends of the other members at the joint. Furthermore, the moment induced in any of these other members bears the same ratio to the applied moment as the stiffness of the member does to the restraint. Consequently, end moments are distributed at a joint in proportion to the stiffnesses of the members.

Stiffness can be expressed in terms of C as follows:

$$S_L = \frac{K}{C^F{}_L - 1/C_R} \qquad \text{(37a)}$$

$$S_R = \frac{K}{C^F{}_R - 1/C_L} \qquad \text{(37b)}$$

For beams of constant moment of inertia,

$$S_L = \frac{K}{2 - 1/C_R}\,; \qquad S_R = \frac{K}{2 - 1/C_L} \qquad \text{(37c)}$$

In turn, moment ratios can be computed in terms of the stiffnesses of adjacent spans:

$$C_L = C^F{}_L + \frac{K}{R_L} \qquad \text{(38a)}$$

$$C_R = C^F{}_R + \frac{K}{R_R} \qquad \text{(38b)}$$

Toward hinged supports, for which $R = 0$, C is infinitely large; toward fixed ends, for which R is infinitely large, $C = C^F$. For beams of constant moment of inertia,

$$C_L = 2 + \frac{K}{R_L}\,; \qquad C_R = 2 + \frac{K}{R_R} \qquad \text{(38c)}$$

A physical interpretation of moment ratios, in addition to the carry-over concept, can be obtained in terms of fixity, defined as the ratio C^F/C, which varies from zero for simple supports to 100% for fixed ends. If θ_P is the rotation of a beam at a hinged support and θ is the rotation when restraint is present, it can be shown that fixity is equal to $1 - \frac{\theta}{\theta_P}$ and, consequently, is a straight-line function of the angular rotation. For example, if a column footing rotates 0.4 as much as it would if it were unrestrained by the soil, the fixity is 1 − 0.4, or 0.6. Also, assuming $C^F = 2$, the moment ratio for the column in the direction of the footing may be taken as 2/0.6, or 3.33. In addition, this concept can be used in the case of flexible connections.

---

[14] "Moment Ratios Simplify Continuity Analysis," by Frederick S. Merritt, *Engineering News-Record*, November 1, 1945, p. 112, and November 15, 1945, p. 102.

Corresponding to Eqs. 16 for prismatic beams, the following formulas apply for end moments in loaded spans of variable moment of inertia (clockwise end moments and rotations being assumed positive):

$$M_L = -\frac{K(C_R\,\phi_L + \phi_R)}{C_R\,C_L - 1} \qquad \text{(39a)}$$

$$M_R = -\frac{K(C_L\,\phi_R + \phi_L)}{C_R\,C_L - 1} \qquad \text{(39b)}$$

in which ϕ represents the simple beam end rotation due to the load on the span. However, it is generally easier to begin distribution with fixed-end moments.

To indicate how this streamlined version works, Example 1(*a*) is recalculated. Referring to Fig. 6 and noting that the relative value of K is unity for all spans:

Step (*a*).—Same as the authors.

Step (*b*).—$S_{BA} = \frac{1}{2 - 1/2} = 0.667$; $C_{CB} = 2 + \frac{1}{0.667} = 3.5$. Similarly, $S_{DB} = \frac{1}{2 - 1/\infty} = 0.5$; $C_{CD} = 2 + \frac{1}{0.5} = 4$; $S_{CD} = \frac{1}{2 - 1/4} = 0.571$; and $C_{BC} = 2 + \frac{1}{0.571} = 3.75$.

Step (*d*).—For a concentrated load at the center of a simply supported span, $\phi = \pm \frac{3\,P\,L}{8\,K}$. Therefore, $M_{BC} = -\frac{3.75 \times 0.375 - 0.375}{3.75 \times 3.5 - 1} \times 4 \times 24 = -8.16$; and $M_{CB} = \frac{3.5 \times 0.375 - 0.375}{3.75 \times 3.5 - 1} \times 4 \times 24 = 7.42$. Then, $M_{AB} = \frac{8.16}{2} = 4.08$ and $M_{DC} = \frac{-7.42}{4} = -1.86$.

Example 4 (Fig. 9) is also solved to illustrate a more general technique.

Step (*a*).—For prismatic beams, the moment ratio can be estimated from

$$C = 2 + \frac{\gamma\,K_1}{\Sigma K} \qquad \text{(40)}$$

in which K_1 is the K-value of the member whose C-value is to be assumed; ΣK is the sum of the K-values of the other members of the joint toward which the ratio is directed; and γ is a constant varying from 1.5 for full restraint to 2.0 for no restraint at the far ends of these other members. Since γ varies between such narrow limits, it has only a very small effect on C.

From Fig. 9, the relative values of K are: $K_{AB} = 6$; $K_{CD} = 4$; $K_{AC} = 7.5$; and $K_{BD} = 6$. Assume $C_{AB} = 2 + \frac{1.75 \times 6}{6} = 3.75$ and $C_{CD} = 2 + \frac{1.75 \times 4}{6} = 3.17$.

Step (*b*).—Through the use of these assumed moment ratios, the distribution constants for the other members may be found. $S_{AB} = \frac{6}{2 - \frac{1}{3.75}} = 3.46$;

$C_{CA} = 2 + \frac{7.5}{3.46} = 4.17; S_{CA} = \frac{7.5}{2 - \frac{1}{4.17}} = 4.26; C_{DC} = 2 + \frac{4}{4.26} = 2.94; S_{DC} = \frac{4}{2 - \frac{1}{2.94}} = 2.41$; $C_{BD} = 2 + \frac{6}{2.41} = 4.49$; and $S_{BD} = \frac{6}{2 - \frac{1}{4.49}} = 3.38$. The corrected $C_{AB} = 2 + \frac{6}{3.38} = 3.77$ and S_{AB} is practically unchanged.

Similarly, $S_{CD} = \frac{4}{2 - \frac{1}{3.17}} = 2.37; C_{AC} = 2 + \frac{7.5}{2.37} = 5.16; S_{AC} = \frac{7.5}{2 - \frac{1}{5.16}} = 4.15$; $C_{BA} = 2 + \frac{6}{4.15} = 3.45$; $S_{BA} = \frac{6}{2 - \frac{1}{3.45}} = 3.51$; $C_{DB} = 2 + \frac{6}{3.51} = 3.71$; and $S_{DB} = \frac{6}{2 - \frac{1}{3.71}} = 3.47$. The corrected $C_{CD} = 2 + \frac{4}{3.47} = 3.15$ and S_{CD} is practically unchanged.

Next, a moment of + 1,000 is applied at each joint separately, and distributed. For example, for a moment of 1,000 at joint A, $M_{AB} = \frac{3.46}{3.46 + 4.15} \times 1{,}000 = 455$; $M_{BA} = \frac{455}{3.77} = 121$; $M_{BD} = -121$; $M_{DB} = -\frac{121}{4.49} = -27$; $M_{DC} = 27; M_{CD} = \frac{27}{2.94} = 9; M_{CA} = -9; M_{AC} = -\frac{9}{4.17} = -2$. Moments thus determined are given in Table 16.

Step (*d*).—From Example 1(*a*), ϕ for the 4-kip load is $\pm \frac{3\,P\,L}{8\,K}$, and the fixed-end moments are numerically equal to $\frac{2 \times 0.375 - 0.375}{4 - 1} \times 4 \times 24 = 12$. These moments are then distributed by multiplying the values in Col. 2, Table 16, by 0.012, and those in Col. 4 by − 0.012. Moments for the 4-kip load with sidesway prevented are shown in Table 17(*a*).

TABLE 16.—INFLUENCE FACTORS FOR MOMENT OF 1,000 APPLIED AT EACH JOINT, EXAMPLE 4

Member (1)	Point of Moment Application Joint A (2)	 Joint B (3)	 Joint C (4)	 Joint D (5)
AC....	546	−139	145	13
AB....	454	139	−145	−13
BA....	112	510	−11	−150
BD....	−112	490	11	150
DB....	6	100	−104	591
DC....	−6	−100	104	409
CD....	−97	−8	357	130
CA....	+97	8	643	−130

Step (*g*).—Under sidesway, joints A and C deflect equal amounts horizontally. Hence, the fixed-end moments in AB and CD caused by a horizontal movement, being proportional to the *K*-values, are in the ratio of 6 to 4. To eliminate the forces that were assumed to prevent sidesway to permit distribution of moments for the 4-kip load, apply fixed-end moments of −1,000 *M* at the top and bottom of CD and of −1,500 *M* at the top and bottom of AB.

Then, using Table 16, distribute the moments and assemble the results as in Table 17(*b*).

Step (*h*).—The shear in AB and CD due to these moments plus the moments previously obtained for the 4-kip load must be zero.

$$+7.19 - 769\,M + 1.48 - 728\,M - 5.44 - 670\,M - 1.32 - 646\,M = 0$$

$$2{,}813\,M = 1.91$$

$$M = 0.00068$$

TABLE 17.—TABULATION AND DISTRIBUTION OF MOMENTS, EXAMPLE 4

Line	Description	MEMBER							
		AC	AB	BA	BD	DB	DC	CD	CA
(*a*) STEP (*d*) (4-KIP LOAD, NO SIDESWAY)									
1	Fixed-end moment........	−12.0	....	....	....	....	....	....	+12.0
2	Distribution of A moment[a].	+ 6.55	+5.45	+1.35	−1.35	+0.07	−0.07	−1.16	+ 1.16
3	Distribution of C moment[a].	− 1.74	+1.74	+0.13	−0.13	+1.25	−1.25	−4.28	− 7.72
4	Total....................	− 7.19	+7.19	+1.48	−1.48	+1.32	−1.32	−5.44	+ 5.44
(*b*) STEP (*g*) (SIDESWAY DISTRIBUTION)									
5	Fixed-end moment........	....	−1,500	−1,500	....	....	−1,000	−1,000	....
6	Distribution of *A* moment[a].	+ 819	+ 681	+ 168	− 168	+ 9	− 9	− 145	+ 145
7	Distribution of *B* moment[a].	− 208	+ 208	+ 765	+ 735	+ 150	− 150	− 12	+ 12
8	Distribution of *C* moment[a].	+ 145	− 145	− 11	+ 11	− 104	+ 104	+ 357	+ 643
9	Distribution of *D* moment[a].	+ 13	− 13	− 150	+ 150	+ 591	+ 409	+ 130	− 130
10	Total....................	+ 769	− 769	− 728	+ 728	+ 646	− 646	− 670	+ 670
(*c*) STEP (*i*) (SIDESWAY CORRECTIONS AND FINAL MOMENTS)									
11	Sidesway corrections......	+ 0.52	−0.52	−0.49	+0.49	+0.44	−0.44	−0.46	+ 0.46
12	Final moments...........	− 6.67	+6.67	+0.99	−0.99	+1.76	−1.76	−5.90	+ 5.90

[a] Distribution of moment applied at the joint designated.

Step (*i*).—Multiplication of the values in line 10, Table 17(*b*), by this value of M gives the sidesway correction for the 4-kip load (line 11, Table 17(*c*)), and the final moments (line 12) are obtained by addition of these corrections to the moments for sidesway prevented (line 4).

I. OESTERBLOM,[15] M. ASCE.—For some years structural frames have been analyzed by the relaxation method so as to avoid the innumerable equations involved in the energy methods, with their attendant confusion and labor. In other words, the classical methods were dead when it came to the matter of analyzing frames. In the face of this trend, the authors have resurrected the classical treatment of frames—with a modification.

The first few pages set up a foundation by borrowing, chiefly from the classics, but also from the modern "relaxationists." A new element, the "elasticity factor," L, is introduced. There follow two pages of real substance and twenty-seven pages of illustrative examples. The first pages can be passed over quickly after they have been recognized as mostly old acquaintances. So also can the last pages, when it is found how simple the problems are; for any method will solve a primitive problem fairly and with little labor.

---

[15] Engr., Carbide and Carbon Chemicals Corp., South Charleston, W. Va.

As to the real substance of the paper, it is claimed that the new method provides very well for "flexible connections," meaning such as have a small but limited rotation; and that it is applicable to Vierendeel type trusses and frames (see heading, "Summary"). In a sense this is true; but the same may be claimed for the relaxation method when used intelligently; seemingly the latter can be adapted more simply and with less labor. The elaborate calculations shown for the easy problems in this paper are thus not at all inviting.

Among the many who have tried to improve upon the relaxation method, or to overcome the discouraging algorithms of the classics in the solution of frames, it has become the custom to illustrate with primitive examples. This is not said in a derogatory sense because such exploration is essential to progress. A few simple problems are in order. In a classroom, no doubt, all the examples had better be primitive, for young minds should not be burdened too heavily; but the larger audience of practicing engineers, will ask of any new method: "Does the method 'work' for an extensive and irregular framework, for short and long spans or columns mixed, for an offset layout, or for an irregular footing plan?" "Will it 'work' with a mixture of all kinds of loads—point loads, spread loads, sway loads, with loads sometimes off and sometimes on?"

With one exception the illustrations are not convincing. Even in the exception the authors offer no solution; they merely present an outline of how the method should be applied. In a practical sense, the exception, Example 7, is not a realistic problem. It does not have the usual complications, but consists of four equal spans in a four-story frame, in which all stories are the same. Loading is confined to two point loads and two spread loads only; no wind is involved. The loads are on the inner spans and are alternating; the point loads are at midspan. The layout is a simple one never found in real life.

Even so, the authors have not presented an answer. Possibly, the elasticity factors L (which should be formulated from all the restraining elements of the frame and not approximated roughly from only those most adjacent) are so complicated that they are entirely beyond human ken, and thus in the same class as the multiple equations in the original classical methods. Perhaps no practical solution is possible; or the answer would be a loose approximation at best. This is thus a challenge to the authors: Can they complete the analysis they have outlined and obtain true answers? If they can they should do so and thus prove that the method has practical value. If they win, their service to the profession is real and they are to be highly honored.

In expressing himself thus forcefully on this subject the writer has had in mind not only the present paper but several others—all erring in the sense that they outline the technique without exploring the underlying philosophy. In a philosophic sense the modern methods are far removed from the classic methods of analyzing elastic frames. Any investigator, pioneering in new ideas as to frames, must therefore anchor his promises to either of the two—

unless he is able to offer an entirely new philosophic approach—for there can be no such thing as a mixture of, or an average between, "energy" and "relaxation" methods; to state the question in more practical terms: Do the current explorers in this field realize the stupendous task they set themselves? The vastly superior relaxation method has its roots in the work of master mathematicians and philosophers, who discovered and developed it under the stress of urgent necessity during World War I; and the development has continued ever since.

When an author therefore makes a claim for a new and better method there must be (and this is a check on his own work) at least one critical and outstanding departure from known and accepted forms. He should analyze and describe this departure and submit a rigorous mathematical proof—either by explicit detail in the text or by tying his thesis by reference to authoritative and readily available sources. If this cannot be done the method is not ready for publication.

A technical paper is a record of constructive thoughts designed for continuing and recurrent study in future time; and it should be sufficiently inclusive to be definitely conclusive both when written and for all time to come. It should be as nearly perfect as possible.

These parenthetical observations, while suggested by this topic, are not aimed exclusively or even primarily at the present paper. They apply to all treatments of statically indeterminate frames, for which claims of complete originality or novelty are advanced.

Thomas C. Kavanagh,[16] Assoc. M. ASCE.—This method of frame analysis is of considerable interest in connection with the calculation of buckling restraints of axially loaded bars in a rigid jointed framework. Methods employing restraint coefficients similar to the authors' "elastic factors" have been employed for this purpose in various forms since the beginning of the twentieth century. All such methods involve solution of the problem of an indeterminate framework wherein the various elastic properties of the members are modified by the presence of axial loads. For the special cases wherein the axial loads are small, their effect may be disregarded as is the custom in ordinary methods of indeterminate analysis.

One of the earliest and most comprehensive uses of restraint coefficients was by H. Zimmermann, who developed the properties of these coefficients (*Spannziffer*) in a long series of publications dating from 1906, summarized in his book[17] on the buckling of frameworks. Eq. 11 is but a slightly different form of the general equation for an axially unloaded bar with rotational end restraints; for on substitution of Eq. 8 it may be rewritten:

$$1 + \frac{L}{3\,E\,I}\left(\frac{1}{Z_B} - \frac{1}{Z_A}\right) - \frac{3}{4}\left(\frac{L}{3\,E\,I}\right)^2 \frac{1}{Z_A}\,\frac{1}{Z_B} = 0 \ldots\ldots\ldots (41)$$

or, by substitution of Mr. Zimmermann's definitions, modified in accordance with the authors' sign convention,

---

[16] Instr. in Civ. Eng., New York Univ., New York, N. Y.

[17] "Knickfestigkeit der Stabverbindungen," by H. Zimmermann, Wilhelm Ernst & Son, Berlin, 1925.

$$m' = -\frac{1}{Z_A}; \qquad m'' = \frac{1}{Z_B} \quad \text{(42)}$$

Eq. 41 takes the basic form,

$$1 + A\,(m' + m'') + \tfrac{3}{4}\,A^2\,m'\,m'' = 0 \quad \text{(43)}$$

in which

$$A = \frac{L}{3\,E\,I} \quad \text{(44)}$$

In the case of the axially loaded member, the coefficients are modified, but the form remains the same, and the equation will be found repeated in most literature pertaining to elastic stability of bars. It is significant to note that Eq. 43 can be rewritten in the form,

$$(A\,m' + 4/3)\,(A\,m'' + 4/3) = (\tfrac{2}{3})^2 \quad \text{(45)}$$

which is an equilateral hyperbola, whose properties and physical interpretation were investigated at length by Mr. Zimmermann. Special cases like Eqs. 3 and 5, or others such as equal end restraints, are readily obtainable from these equations.

In 1935–1936, N. J. Hoff[18,19] developed along different lines the concept of the "relative stiffness factor," defined by:

$$\Omega = \frac{E\,I}{L}\,\frac{\theta}{M} \quad \text{(46)}$$

and used it for both axially loaded or axially unloaded members. The symbols on the right are as defined in the authors' paper.

In 1938, the moment distribution approach was utilized by Eugene E. Lundquist,[20] Assoc. M. ASCE, using the concepts of stiffness (elastic factor) and carry-over factor. That the approach is expressible in terms of Eq. 39 was demonstrated by W. R. Osgood, M. ASCE, in a discussion[21] of the same paper. It might be noted here that since the elastic factor is simply the summation of the stiffnesses of the restraining members, it would seem preferable to retain the widely understood term "stiffness" in place of the new term "elastic factor."

In all the above problems, as related to elastic stability analysis, one is faced with the problem of selecting joints with known or assumed restraints (stiffnesses), after which the remaining joints are resolved in a step-by-step process.

The authors are to be commended upon their clear presentation and the systematic procedure outlined, particularly as related to the problems of flexible connections and joint translation. Although the approach considering joint translation has been solved[22] for axially loaded members, it has never been systematically exploited.

---

[18] "Transversely Loaded Framework Members," by N. J. Hoff, *Journal*, Royal Aeronautical Soc., August, 1935, p. 718.

[19] "Elastically Encastred Struts," by N. J. Hoff, *ibid.*, September, 1936, p. 663.

[20] "Principles of Moment Distribution Applied to Stability of Structural Members," by E. E. Lundquist, *Proceedings*, 5th International Cong. for Applied Mechanics, 1938, p. 145.

[21] *Ibid.*, p. 149.

[22] "Elastic Stability of Plane Frameworks," by W. Prager, *Journal of the Aeronautical Sciences*, September, 1936, p. 388.

A. Floris,[23] Esq.—Introducing a method of analysis of hyperstatic structures, the authors claim that the theory presented in this paper is classical, or exact, requiring the solution of very few, if any, simultaneous elastic equations. This statement needs a close examination.

Classical methods are those which do not require successive or final corrections. The redundant quantities (that is, forces, moments, displacements, and rotations) of the hyperstatic structure are determined by the aid of an algorism in a systematic and unequivocal manner leading always to simple but troublesome linear simultaneous equations. It follows that the classical methods, although tedious and time consuming, are direct and the results obtained are final.

The authors' analysis, on the other hand, is preeminently a relaxation method. In certain cases, relaxation methods require also the solution of simultaneous equations. It is true, that by introducing the elastic moment area (conjugate beam), the authors side-step the iteration process of the moment-distribution method. Nevertheless, their analysis is an indirect one.

In accordance with the usage of American technical literature, the authors call Mohr's elastic moment-area method, the conjugate beam theory. Mohr in his writings[24, 25] never mentioned it, and it is not known in German technical literature by this name. It belongs, instead, to the elastic weights methods. Furthermore, this terminology is mathematically incorrect; for to be called conjugate there must be two paired quantities equal but differing in sign. The moment and the elastic moment areas are unequal and both of the same sign, positive or negative as the case may be. It is perhaps better to call this method the associate beam theory.

In the moment-distribution method the base system is a beam fixed at both ends. By releasing joints one at a time, the carry-over stiffness factors, fixed-end moments, and final moments are determined. By analogy, the base system of the authors' method is a beam elastically supported at both ends. By releasing the fixed points, after the carry-over and elastic factors have been determined, the moments of the elastic moment area, and, finally, the actual moments are found. As in the relaxation methods, the sidesway produced by unbalanced moments and shears is corrected to satisfy the equilibrium of the structure.

In the moment-distribution method the fixed-end moments can be taken from reference books, and are independent of the elasticity of the structure. The bending moments of the authors' base system (Eqs. 16) depend on the elasticity of the whole structure and cannot be determined in advance.

An attentive reader will search in vain for a proof of the analysis developed in this paper; he must be satisfied with heuristic reasoning. Only comparing them with the solutions reached by other well-known methods can the validity of the results obtained by the authors' method be established.

In Fig. 2 and Fig. 3 the beams and moments are not drawn properly, and the interaction of forces and moments is not clear. Support B is fixed in Fig. 2

---

[23] Asst. to Structural Engr., Pacific Elec. Ry. Co., Los Angeles, Calif.

[24] "Beitrag zur Theorie der Holz-und Eisenkonstruktionen," by Otto Mohr, *Zeitschrift des Architekten-und-Ingenieur-Vereins zu Hannover*, 1868, p. 19.

[25] "Abhandlungen aus dem Gebiete der Technischen Mechanik," by Otto Mohr, Wilhelm Ernst & Sohn, Berlin, 1914, p. 342.

and elastic in Fig. 3. In both cases, point A is hinged but vertically restrained. This is not mentioned explicitly. In end B of the fixed and elastically supported beams, the moment produced by M_{AB} is a rectangle; the moment area of the vertical reaction at A is a triangle with opposite sign. By superimposing the two moment areas, Figs. 2(*a*) and 3(*a*) are obtained.

In addition, the formulas are presented in a rather unsatisfactory and impractical manner. The general expression (Eq. 10*b*), for instance, can be written in the more convenient form—

$$\theta_A = M_{AB}\left(\frac{L}{3\,E\,I} - \frac{L^2}{12\,E\,I\,L_{AB}}\right) \ldots\ldots\ldots\ldots\ldots (47)$$

—in which the influence of support B is clearly indicated. The first term within the parentheses is the elastic factor at end A when end B can rotate freely, which is the basic case. The second term is the elastic factor that shows the influence of the degree of fixity at end B on the rotation of the beam at support A. If B is fixed, then $Z_B = 0$ and, from Eq. 8, $L_{AB} = L$. Thus the expression within the parentheses becomes equal to Eq. 3. On the other hand, if B is hinged, then $Z_B = \infty$, hence $L_{AB} = \infty$ and the second term within the parentheses vanishes, as it should, to agree with Eq. 4.

In conclusion, the proposed method offers no real advantages over the moment-distribution method, although it can be of some value in cases involving only sidesway. It is more tedious, more complicated, and certainly more difficult to remember and apply than the classical methods. The claim that iteration (cycles of distribution) is avoided and that the method is thus rendered briefer is rather illusory.

TAO KING,[26] M. ASCE.—The "classical" method of analyzing continuous frames, presented in this paper, is at least of theoretical interest. Compared with the Hardy Cross method of moment distribution,[27] the authors claim the advantage of avoiding repeated cycles (but the solution of a "few" simultaneous equations is still required), and express the belief that the method is particularly adapted to the solution of frames such as the Vierendeel truss.

All examples given in the paper can be solved readily by the Cross method of distributing fixed-end moments with the application of proper "correction moments" where required. For practical purposes, only three or four cycles are necessary. If convergency is rather slow, five or six cycles may be required. Another advantage of the Cross method lies in the fact that the angles of joint rotations, $E\,\theta$, and those of bar rotations, $E\,R = \frac{E\,\Delta}{L}$, may easily be calculated from the table of moment distribution. These values, when known, will help one in visualizing the deformed structure under the external loads.

---

[26] Head Prof., Civ. Eng., Peiping Polytechnic College, Peiyang Univ., Peiping, China.

[27] *Transactions*, ASCE, Vol. 96, 1932, p. 1.

The Vierendeel trusses in Examples 4 to 7 have rectangular panels and can be solved best by the Cross method. Example 6 is perhaps the most complicated one given by the authors. The writer has solved it by fixed-end moment distribution, as shown in Table 18.

In Table 18 (*a*), line 1 gives the distributing factors, and line 2 shows the fixed-end moments resulting from the loads applied on the top chords CE and EG, Fig. 12. In lines 3, 6, 9, etc., are given the correction moments to be applied so as to balance the unbalanced moments in the panels, resulting from the external loads or reactions, or from the newly introduced hypothetical moments after each cycle of distribution and carry-over. The balancing moments in each panel are divided between the top and bottom chords in proportion to their stiffness factors. Lines 4, 7, 10, etc., show the distributed moments, and lines 5, 8, 11, etc., give the carried-over moments. Final moments are indicated in line 20, which sums up all the moments in each column. For practical results, one might have stopped at the end of the fourth cycle.

No attempt has been made to balance the shears in the verticals; it can be proved easily that, when the moments around all the joints are balanced and when the proper "correction moments" are applied after each cycle, the final shears in the verticals must necessarily be balanced.

If one wishes to obtain still more "exact" values for the final end moments, Table 18(*b*) may be prepared. The values given in the last horizontal line of Table 18(*b*) agree almost exactly with those in line 25, Table 13. The signs of the moments as used by the writer are exactly opposite to those used by the authors.

The products $E\,\theta$ for each joint and $E\,R$ for each member can be readily computed from Table 18. In Example 6, it is evident that: (1) The $(E\,R)$-value for the top chord in any panel should be the same as that for the bottom chord in the same panel; (2) the $(E\,R)$-value for all the verticals should be the same; and (3) the sum of the $(E\,R)$-values for all the bottom chords should be zero, since the supports B and J are unyielding.

Let $E\,R_1$, $E\,R_2$, $E\,R_3$, and $E\,R_4$ be the angles of bar rotation for top and bottom chords in the first, second, third, and fourth panels (see Fig. 12), and let $E\,R_5$ be the angle of bar rotation for each and every vertical. The tentative values of $E\,R_1$, $E\,R_2$, $E\,R_3$, and $E\,R_4$ are computed as follows:

$$E\,R_{BD} = \frac{\text{sum of correction moments}}{-\,6 \times K_{BD}}$$

$$= \frac{-\,29.333 - 23.371 - 7.987 - 3.229 - 1.315 - 0.541 - 0.379}{-\,6 \times \frac{1}{2}} = +\,22.052.$$

(In the foregoing, the value $-\,0.379$ is the last correction moment, 0.541, multiplied by the ratio $\frac{r}{1-r}$ for member BD.) Similarly, $E\,R_{DF} = +\,15.592$; $E\,R_{FH} = -\,11.257$; and $E\,R_{HJ} = -\,24.422$.

The sum of these ($E\,R$)-values, being equal to + 1.965, is not zero. This discrepancy of + 1.965 must be divided equally among the four panels, since the latter are of equal lengths. In other words, all the verticals must rotate through an angle $E\,R_5$ of − 0.491, which is equal to one fourth of this discrepancy with the sign changed. All the tentative ($E\,R$)-values must be combined with this value of $E\,R_5$ in order to obtain the true values of $E\,R_1$, $E\,R_2$, $E\,R_3$, and $E\,R_4$, thus: + 21.561, + 15.101, − 11.748, and − 24.913, respectively. The ($E\,\theta$)-value for any joint may now be computed from:

TABLE 18.—SOLUTION OF EXAMPLE 6 BY THE

Line[a]	JOINT B		JOINT A		JOINT C			JOINT E
	BD	BA	AB	AC	CA	CD	CE	EC
(a) CUSTOMARY CYCLES OF								
1	2/5	3/5	6/11	5/11	5/14	4/14	5/14	15/38
2	....	....	....	....	....	....	−12.000	+12.000
3	−29.333	....	....	−36.667	−36.667	....	−23.333	−23.333
4	+11.733	+17.600	+20.000	+16.667	+25.714	+20.572	+25.714	+ 4.210
5	+ 8.000	+10.000	+ 8.800	+12.857	+ 8.333	+ 8.000	+ 2.105	+12.857
6	−23.371	....	....	−29.214	−29.214	....	−19.968	−19.968
7	+ 2.148	+ 3.223	+ 4.122	+ 3.435	+10.980	+ 8.784	+10.980	+ 1.826
8	+ 3.699	+ 2.061	+ 1.611	+ 5.490	+ 1.717	+ 3.699	+ 0.913	+ 5.490
9	− 7.987	....	....	− 9.983	− 9.984	....	− 8.815	− 8.815
10	+ 0.891	+ 1.336	+ 1.572	+ 1.310	+ 4.454	+ 3.563	+ 4.453	+ 0.860
11	+ 1.516	+ 0.786	+ 0.668	+ 2.227	+ 0.655	+ 1.516	+ 0.430	+ 2.227
12	− 3.229	....	....	− 4.037	− 4.036	....	− 3.665	− 3.666
13	+ 0.371	+ 0.556	+ 0.623	+ 0.519	+ 1.821	+ 1.457	+ 1.822	+ 0.424
14	+ 0.618	+ 0.311	+ 0.278	+ 0.910	+ 0.259	+ 0.618	+ 0.212	+ 0.911
15	− 1.316	....	....	− 1.644	− 1.644	....	− 1.549	− 1.548
16	+ 0.155	+ 0.232	+ 0.249	+ 0.207	+ 0.752	+ 0.601	+ 0.751	+ 0.220
17	+ 0.254	+ 0.124	+ 0.116	+ 0.376	+ 0.103	+ 0.254	+ 0.110	+ 0.375
18	− 0.540	....	....	− 0.675	− 0.676	....	− 0.668	− 0.669
19	+ 0.065	+ 0.097	+ 0.100	+ 0.083	+ 0.313	+ 0.251	+ 0.313	+ 0.118
20	−36.326	+36.326	+38.139	−38.139	−27.120	+49.315	−22.195	−16.481
(b) ADJUSTMENTS FOR								
m_1	−17.600	+17.600	+20.000	−20.000	−10.953	+20.572	− 9.619	− 7.123
m_2	−13.223	+13.223	+12.922	−12.922	− 9.901	+16.784	− 6.883	− 5.285
m_3	− 3.397	+ 3.397	+ 3.183	− 3.183	− 3.813	+ 7.262	− 3.449	− 2.465
m_4	− 1.342	+ 1.342	+ 1.291	− 1.291	− 1.560	+ 2.973	− 1.413	− 1.015
m_5	− 0.543	+ 0.543	+ 0.527	− 0.527	− 0.633	+ 1.219	− 0.586	− 0.417
m_6	− 0.221	+ 0.221	+ 0.216	− 0.216	− 0.260	+ 0.505	− 0.245	− 0.176
m_7	− 0.152	+ 0.152	+ 0.149	− 0.149	− 0.184	+ 0.357	− 0.173	− 0.128
Σ	−36.478	+36.478	+38.288	−38.288	−27.304	+49.672	−22.368	−16.609

[a] The lines m_1, m_2, etc., are the algebraic sums of carried-over moments, correction moments, and distributed

$$E\,\theta = \frac{\text{sum of distributed moments at the joint}}{\text{four times the sum of stiffness factors around the joint}} + E\,R_5 \quad (48)$$

Thus, $E\,\theta_B$ is found as follows:

$$
\begin{aligned}
E\,\theta_B = &+ 11.733 + 17.600 + 2.148 + 3.223 + 0.891 + 1.336 + 0.371 \\
&+ 0.556 + 0.155 + 0.232 + 0.065 + 0.097 \\
&+ 0.113/4\,(1/2 + 3/4) - 0.491 = + 7.213
\end{aligned}
$$

HARDY CROSS METHOD OF CONVERGING APPROXIMATIONS

JOINT E		JOINT G			JOINT I		JOINT J	Line*
EF	EG	GE	GH	GI	IG	IJ	JI	
MOMENT DISTRIBUTION								
8/38	15/38	5/14	4/14	5/14	5/11	6/11	3/5	1
....	−16.000	+16.000	....	....	....	....	....	2
....	+16.667	+16.667	....	+43.333	+43.333	....	....	3
+2.246	+ 4.210	−27.143	−21.714	−27.143	−19.697	−23.636	−20.800	4
+0.667	−13.571	+ 2.105	− 8.000	− 9.848	−13.571	−10.400	−11.818	5
....	+15.388	+15.389	....	+31.961	+31.961	....	....	6
+0.974	+ 1.827	−11.288	− 9.031	−11.288	− 3.632	− 4.358	− 3.450	7
+0.317	− 5.644	+ 0.913	− 3.515	− 1.816	− 5.644	− 1.725	− 2.179	8
....	+ 6.475	+ 6.474	....	+10.104	+10.104	....	....	9
+0.458	+ 0.859	− 4.343	− 3.474	− 4.343	− 1.243	− 1.492	− 1.434	10
+0.150	− 2.171	+ 0.429	− 1.346	− 0.621	− 2.171	− 0.717	− 0.746	11
....	+ 2.385	+ 2.385	....	+ 3.847	+ 3.846	....	....	12
+0.226	+ 0.425	− 1.677	− 1.341	− 1.676	− 0.435	− 0.523	− 0.592	13
+0.078	− 0.838	+ 0.212	− 0.499	− 0.217	− 0.838	− 0.296	− 0.261	14
....	+ 0.840	+ 0.841	....	+ 1.460	+ 1.460	....	....	15
+0.117	+ 0.220	− 0.642	− 0.513	− 0.642	− 0.148	− 0.178	− 0.245	16
+0.042	− 0.321	+ 0.110	− 0.181	− 0.074	− 0.321	− 0.122	− 0.089	17
....	+ 0.275	+ 0.275	....	+ 0.548	+ 0.548	....	....	18
+0.063	+ 0.117	− 0.242	− 0.194	− 0.242	− 0.048	− 0.057	− 0.101	19
+5.338	+11.143	+16.465	−49.808	+33.343	+43.504	−43.504	−41.715	20
MORE EXACT SOLUTION								
+2.246	+ 4.877	+ 5.524	−21.714	+16.190	+23.636	−23.636	−20.800	m_1
+1.611	+ 3.644	+ 6.206	−17.031	+10.825	+14.758	−14.758	−15.268	m_2
+0.775	+ 1.690	+ 3.044	− 6.989	+ 3.945	+ 3.217	− 3.217	− 3.613	m_3
+0.376	+ 0.639	+ 1.137	− 2.687	+ 1.550	+ 1.240	− 1.240	− 1.338	m_4
+0.195	+ 0.222	+ 0.411	− 1.012	+ 0.601	+ 0.474	− 0.474	− 0.506	m_5
+0.105	+ 0.071	+ 0.143	− 0.375	+ 0.232	+ 0.179	− 0.179	− 0.190	m_6
+0.076	+ 0.052	+ 0.084	− 0.220	+ 0.136	+ 0.109	− 0.109	− 0.114	m_7
+5.414	+11.195	+16.549	−50.028	+33.479	+43.613	−43.613	−41.829	Σ

moments in cycles 1, 2, etc.; m_7 is obtained by multiplying m_6 by $\frac{r}{1-r}$, in which $r = \frac{m_6}{m_5}$.

(In the foregoing the value $+ 0.113$ is $(+ 0.065 + 0.097) \times \frac{r}{1-r}$ for the members around that joint.) Similarly: $E\,\theta_D = + 13.733$; $E\,\theta_F = + 1.451$; $E\,\theta_H = - 14.135$; $E\,\theta_J = - 9.385$; $E\,\theta_A = + 8.420$; $E\,\theta_C = + 17.211$; $E\,\theta_E = + 2.606$; $E\,\theta_G = - 18.682$; and $E\,\theta_I = - 10.584$.

TABLE 18.—(*Continued*)

Line[a]	JOINT J	JOINT H			JOINT F			JOINT D		
	JH	HJ	HG	HF	FH	FE	FD	DF	DC	DB
					(a)—(Continued)					
1	2/5	1/3	1/3	1/3	3/8	2/8	3/8	1/3	1/3	1/3
2	....	....	....	....	....	....	....	....	....	....
3	+34.667	+34.667	....	+13.333	+13.333	....	−18.667	−18.667	....	−29.333
4	−13.867	−16.000	−16.000	−16.000	+ 2.000	+1.334	+ 2.000	+16.000	+16.000	+16.000
5	− 8.000	− 6.933	−10.857	+ 1.000	− 8.000	+1.123	+ 8.000	+ 1.000	+10.286	+ 5.866
6	+25.568	+25.569	....	+12.311	+12.311	....	−15.975	−15.975	....	−23.371
7	− 2.300	− 7.030	− 7.030	− 7.030	+ 0.953	+0.635	+ 0.953	+ 7.398	+ 7.398	+ 7.398
8	− 3.515	− 1.150	− 4.515	+ 0.476	− 3.515	+0.487	+ 3.699	+ 0.476	+ 4.392	+ 1.074
9	+ 8.084	+ 8.083	....	+ 5.180	+ 5.179	....	− 7.052	− 7.053	....	− 7.987
10	− 0.956	− 2.691	− 2.692	− 2.691	+ 0.451	+0.300	+ 0.451	+ 3.033	+ 3.032	+ 3.033
11	− 1.345	− 0.478	− 1.737	+ 0.225	− 1.345	+0.229	+ 1.516	+ 0.225	+ 1.781	+ 0.445
12	+ 3.077	+ 3.078	....	+ 1.908	+ 1.908	....	− 2.932	− 2.932	....	− 3.229
13	− 0.394	− 0.999	− 0.998	− 0.999	+ 0.234	+0.156	+ 0.234	+ 1.237	+ 1.237	+ 1.236
14	− 0.499	− 0.197	− 0.670	+ 0.117	− 0.499	+0.113	+ 0.618	+ 0.117	+ 0.728	+ 0.185
15	+ 1.168	+ 1.167	....	+ 0.672	+ 0.672	....	− 1.239	− 1.239	....	− 1.315
16	− 0.163	− 0.363	− 0.363	− 0.363	+ 0.125	+0.084	+ 0.126	+ 0.508	+ 0.508	+ 0.508
17	− 0.181	− 0.081	− 0.256	+ 0.062	− 0.181	+0.058	+ 0.254	+ 0.063	+ 0.300	+ 0.077
18	+ 0.438	+ 0.439	....	+ 0.220	+ 0.220	....	− 0.535	− 0.535	....	− 0.541
19	− 0.067	− 0.128	− 0.128	− 0.128	+ 0.069	+0.046	+ 0.069	+ 0.212	+ 0.212	+ 0.212
20	+41.715	+36.953	−45.246	+ 8.293	+23.915	+4.565	−28.480	−16.132	+45.874	−29.742
					(b)—(Continued)					
m_1	+20 800	+18.667	−16.000	− 2.667	+15.333	+1.334	−16.667	− 2.667	+16.000	−13.333
m_2	+15.268	+11.606	−17.887	+ 6.281	+ 5.264	+1.758	− 7.022	− 7.577	+17.684	−10.107
m_3	+ 3.613	+ 4.242	− 7.207	+ 2.965	+ 2.115	+0.787	− 2.902	− 3.544	+ 7.424	− 3.880
m_4	+ 1.338	+ 1.601	− 2.735	+ 1.134	+ 0.797	+0.385	− 1.182	− 1.470	+ 3.018	− 1.548
m_5	+ 0.506	+ 0.607	− 1.033	+ 0.426	+ 0.298	+0.197	− 0.495	− 0.614	+ 1.236	− 0.622
m_6	+ 0.190	+ 0.230	− 0.384	+ 0.154	+ 0.108	+0.104	− 0.212	− 0.260	+ 0.512	− 0.252
m_7	+ 0.114	+ 0.136	− 0.227	+ 0.091	+ 0.081	+0.078	− 0.159	− 0.184	+ 0.362	− 0.178
Σ	+41.829	+37.089	−45.473	+ 8.384	+23.996	+4.643	−28.639	−16.316	+46.236	−29.920

[a] The lines m_1, m_2, etc., are the algebraic sums of carried-over moments, correction moments, and distributed moments in cycles 1, 2, etc.; m_7 is obtained by multiplying m_6 by $\frac{r}{1 - r}$, in which $r = \frac{m_6}{m_5}$.

LEROY A. BEAUFOY,[28] ESQ.—The method offered by Professors Rathbun and Cunningham is presented in three divisions in terms of its application to different structural types:

(*a*) Structures in which fixed joints need not be introduced, such as continuous beams on unyielding supports, or single story frames of a type in which only a single sidesway may occur;

(*b*) Vierendeel trusses; and

(*c*) Building frames.

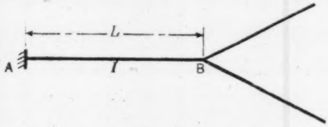


FIG. 14

The writer will direct his discussion at the same three divisions.

In respect of the first, attention is drawn to certain similarities and differences as compared with an earlier paper by Otto Gottschalk.[29] It is easy to show that some of the authors' expressions are, in fact, the same as expressions derived (from a fundamentally different point of view) by Mr. Gottschalk. Thus, consider a beam AB, fixed at end A and connected at joint B to one or more members (Fig. 14). In adapting the Gottschalk symbols to the authors' notation let S_A be the relative stiffness value for end A; S_B be the sum of the relative stiffness values of all members at end A except member BA; $-m$ be the rotation at joint B induced by unit rotation at end A; k be the stiffness ($E\,I/L$) of a member; and c be the carry-over factor from point A to point B. Then, among the expressions derived by Mr. Gottschalk are:

---

[28] Univ. Reader in Civ. Eng., King's College, Univ. of London, London, England.

$$m = \frac{0.5\,k}{S_B + k} \qquad \text{(49a)}$$

$$S_A = (1 - 0.5\,m)\,k \qquad \text{(49b)}$$

and

$$c = \frac{1 - 2\,m}{2 - m} \qquad \text{(49c)}$$

In Eq. 49*a*, writing for *k* its equivalent $\frac{E\,I}{L}$:

$$m = \frac{1}{2\,\frac{S_B\,L}{E\,I} + 2} \qquad \text{(50a)}$$

and substituting this in Eq. 49*c*:

$$c = \frac{1}{2}\left[\frac{L}{L + \frac{3\,E\,I}{4\,S_B}}\right] \qquad \text{(50b)}$$

Transposing to the authors' symbols through their definition for Z_B in Eq. 6, that is,

$$Z_B = \frac{1}{4\,S_B} \qquad \text{(51a)}$$

—and substituting in Eq. 50*b*—

$$c = \frac{1}{2} \times \frac{L}{L + 3\,E\,I\,Z_B} = \frac{1}{2} \times \frac{L}{L_{AB}} \qquad \text{(51b)}$$

which is the same as the corresponding quantity in Eq. 7.

Again, by substituting for *m* from Eq. 50*a*, and writing $E\,I/L$ for *k*, Eq. 49*b* can be rewritten as

$$S_A = \left[\frac{4\,\frac{S_B\,L}{E\,I} + 3}{4\,\frac{S_B\,L}{E\,I} + 4}\right]\frac{E\,I}{L} \qquad \text{(52a)}$$

Substituting $Z_A = \frac{1}{4\,S_A}$ and $Z_B = \frac{1}{4\,S_B}$:

$$Z_A = \frac{L}{3\,E\,I}\left[\frac{(3\,E\,I\,Z_B + L) - \frac{L}{4}}{L + 3\,E\,I\,Z_B}\right] = \frac{L}{3\,E\,I}\left(\frac{L_{AB} - \frac{L}{4}}{L_{AB}}\right) \qquad \text{(52b)}$$

which is the same as Eq. 11.

---

[29] "Structural Analysis Based Upon Principles Pertaining to Unloaded Models," by Otto Gottschalk, *Transactions*, ASCE, Vol. 103, 1938, p. 1019.

Line	Description	A	(0.63)		C	(0.63)		E	(0.63)		G	(0.63)		I
(1)	Fixed-end moment		–	–		-12.0	12.0		-16.0	16.0		–	–	
(2)	Shear requirements		-36.67	-36.67		-23.33	-23.33		16.67	16.67		43.33	43.33	
(3)	Arbitrary joint displacement	36.0	5.0	8.75	42.0	8.75	1.32	4.2	1.32	-10.0	-48.0	-10.0	-5.83	-42.0
			-8.75	-5.0		-1.32	-8.75		10.0	-1.32		5.83	10.0	
(4)	Balancing	2.41	2.02	6.72	5.38	6.72	1.02	0.54	1.02	-4.47	-3.57	-4.47	-2.5	-3.0
(5)	Carry-over	-1.1	3.36	1.01	-0.49	0.51	3.36	-0.24	-2.24	0.51	2.06	-1.25	-2.24	1.2
(2)			-2.62	-2.62		-2.51	-2.51		0.02	0.02		0.51	0.52	
(3)		1.2	0.17	0.62	3.0	0.62	0.24	0.75	0.24	-0.25	-1.2	-0.25	-0.06	-0.4
			-0.63	-0.17		-0.24	-0.62		0.25	-0.24		0.06	0.25	
(4)		-0.2	-0.17	0.1	0.08	0.09	0.29	0.16	0.3	0.01		0.02	0.33	0.4
	TOTALS	38.31	-38.31	-27.26	49.97	22.71	-16.98	5.41	11.58	16.93	-50.71	33.78	43.8	-43.8
		(0.75)			(0.5)			(0.33)			0.5			0.75
	TOTALS	36.53	-36.53	-30.16	46.32	-16.16	-28.43	4.56	23.87	8.55	-45.97	37.41	41.8	-41.8
(4)		0.33	0.21	-0.39	-0.39	-0.39	-0.27	-0.17	-0.27	0.88	0.89	0.88	-0.44	-0.3
			-0.50	-0.13		-0.19	-0.5		0.2	-0.19		0.05	0.2	
(3)		1.2	0.13	0.5	3.0	0.5	0.19	0.75	0.19	-0.2	-1.2	-0.2	-0.05	-0.4
(2)			-2.09	-2.09		-2.0	-2.0		0.02	0.02		0.41	0.42	
(5)		1.21	-0.49	-0.74	2.69	-0.37	-0.49	0.27	2.07	-0.37	-1.78	0.8	2.07	-1.5
(4)		-2.2	-1.47	-0.98	-0.98	-0.99	-0.74	-0.49	-0.73	4.13	4.12	4.13	1.6	2.4
			-7.0	-4.0		-1.05	-7.0		8.0	-1.05		4.67	8.0	
(3)		36.0	4.0	7.0	42.0	7.0	1.05	4.2	1.05	-8.0	-48.0	-8.0	-4.67	-42.0
(2)			-29.33	-29.33		-18.67	-18.67		13.33	13.33		34.67	34.67	
(1)			–	–		–	–		–	–		–	–	
		B	(0.5)		D	(0.5)		F	(0.5)		H	(0.5)		J

FIG. 15.—COMPLETE SOLUTION OF EXAMPLE 6 BY MOMENT DISTRIBUTION

However, although the expressions derived by Professors Rathbun and Cunningham are basically the same as those derived by Mr. Gottschalk, the procedure recommended by the authors is quite different. They aim at obtaining a direct solution for a given case of loading, whereas Mr. Gottschalk, by considering the unloaded structure as an elastic body and determining its shape when deformed in a particular manner, obtains influence lines from which stress values for any loading condition can be obtained. Mr. Gottschalk's method, however, can be used for direct stress analysis. The labor involved in the authors' method is no less than that employed in Mr. Gottschalk's method, and the results are tied to a particular loading instead of being readily applied to any loading. In comparing the two methods, therefore, advantage would appear to be with the earlier one.

In fact, taking Example 3 as an instance and comparing the authors' solution with that by the Gottschalk method, there is added labor in the former because of the necessity for calculating the end moments when the ends are elastically supported (what one might call "elastic end moments"); whereas Mr. Gottschalk uses the more easily determined fixed-end moments. Subsequent processes in both methods are comparable and involve approximately the same amount of labor.

The second application, to Vierendeel trusses, is regarded as quite unsatisfactory. As presented, the method involves so many operations that it will not appeal to many engineers who, by using other methods, can much more easily obtain solutions to structures of the types dealt with. Compare the authors' solution to Example 6, which occupied nearly ten pages of the paper, with Fig. 15, which gives, on a single page, the complete solution by moment distribution, using the device of arbitrary joint displacements.[30] It is recognized that the paper sets out specifically to give an "exact" method, but it is submitted that such a method should offer some advantage over existing methods. Whatever the facts may be as regards the claim that the number of simultaneous equations is largely reduced as compared with the classical methods, it remains true to say that many simultaneous equations are left. For instance, Example 6 involves three sets of two simultaneous equations plus one set of four, and it should be noted that for the solution of these equations an ordinary slide rule is inadequate. As the method does not appear to be adapted to the determination of influence lines, separate solutions involving much the same labor would presumably be needed for each case of loading, whenever more than one loading condition has to be investigated. It must also be emphasized that in the case of a Vierendeel truss with non-parallel chords any method utilizing the shearing forces in the panels will encounter special difficulties because the inclined chord mem-

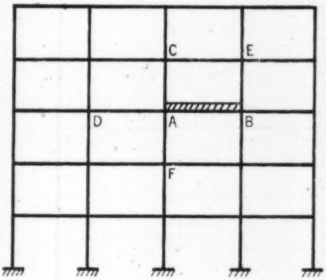


Fig. 16

---

[30] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York and London, 1932, p. 228.

bers participate in the resistance to shearing force. It is not considered, therefore, that the authors' claim (that their method is particularly adapted to the solution of Vierendeel trusses) is borne out by the contents of their paper.

The application of the authors' method to building frames may be justified only from the point of view of obtaining an exact solution, for which purpose it has certain merits. Otherwise, it will be found accurate enough in dealing, for example, with the problem in Fig. 16, to start by determining the stiffnesses of the ends AC, AD, and AF, as if all members at points C, D, and F (other than CA, DA, and FA) have their far ends either hinged or fixed. The actual stiffnesses and carry-over factors are determined accordingly, and a solution is readily obtained by the Gottschalk method. It would be easier to assume the far ends to be hinged rather than fixed, since there would then be no unbalanced moments there. The differences between the actual moment and the moments obtained by either of these assumptions will be slight in any case. It may be added that an even quicker solution can be obtained using moment distribution and speeding up the convergence by suitable arbitrary joint displacements.

A final comment relates to a detail in the paper. In Section 7, step (*g*) would be clarified, it is suggested, by adding "the fixed joints must later be released and the moments corrected." In fact the authors do this in Example 6, step (*g*), and mention it again in Example 7, step (*g*). Incidentally, this process alone involves the solution of *a*-groups of *b*-equations each, in which *a* denotes the number of fixed joints and *b* the number of shear displacements.

A. A. EREMIN,[31] Assoc. M. ASCE.—A direct algebraic distribution of moments in continuous frames is shown in this paper by solving the illustrative examples. However, distribution of moments may also be clarified by the graphical construction of moments.

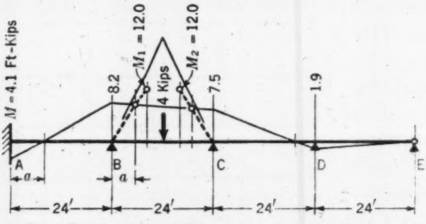


FIG. 17.—GRAPHICAL DETERMINATION OF MOMENTS FOR EXAMPLE 1(*a*)

The moments in Example 1(*a*), determined by the graphics, are shown in Fig. 17. The distance to the inflection point, a, is

$$a = \frac{e\,L}{e + e_0\,Z_A\,\dfrac{I}{L}} \quad \ldots\ldots (53)$$

and the elastic factor is

$$Z_B = \left(e - e_0\,\frac{a}{L - a}\right)\frac{L}{I} \quad \ldots (54)$$

The constants e and e_0 in Eqs. 53 and 54 vary with the shape of the member and may be read from the diagrams prepared by the writer.[32] In a member with a constant section, $e = 0.333$ and $e_0 = 0.167$.

---

[31] Associate Bridge Engr., Bridge Dept., Div. of Highways, State Dept. of Public Works, Sacramento, Calif.

[32] "Analysis of Continuous Frames by Graphical Distribution of Moments," by A. A. Eremin, Sacramento, Calif., 1943.

The moment closing line in span BC is constructed through the intersection points on the inflection point verticals cut by lines drawn through the central moments M_1 and M_2, as shown in Fig. 17. Similarity of moment distribution shown in Table 1 and that shown by the graphics in Fig. 17 is obvious.

Final moments produced by a combined loading have been computed by the authors by superposition of moments in Tables 2 and 5. The summation operation may be eliminated by application of the "cross lines." The moments determined by the "cross lines" for Example 3 with no sidesway are shown in Fig. 18. Final moments determined by the graphics in Figs. 17 and 18 differ by less than 2% from those computed by the authors.

By applying Eqs. 53 and 54 to the authors' method it may be extended to continuous frames composed of members with varying sections. Application of the "cross lines" and graphical construction of sidesway effect are shown elsewhere.[33,34]

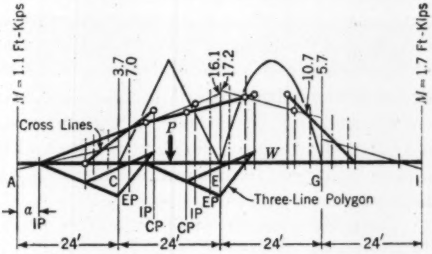


FIG. 18.—GRAPHICAL DETERMINATION OF MOMENTS FOR EXAMPLE 3 WITH NO SIDESWAY

It is still generally believed that the exact computation of moments follows only from algebraic methods. Therefore, it is gratifying that the authors have shown an algebraic method which may be visualized and checked by graphical constructions, not only in the final result but also in successive steps of the computation.

ROBERT B. B. MOORMAN,[35] M. ASCE.—One of the disadvantages in the use of the method of analysis proposed in this paper is the evaluation of $A\bar{x}_A$ and $A\bar{x}_B$ in Eqs. 16 and 17. This may be overcome to a certain degree, for prismatic sections, for uniform loads extending over the span and for concentrated loads by use of the expressions:

$$A\bar{x}_A = \frac{w\,L^4}{24} + \tfrac{1}{6}\,\Sigma P(L - x)(L + x)x$$

$$= \frac{w\,L^4}{24} + \tfrac{1}{6}\,\Sigma P(L^2 - x^2)x \quad (55a)$$

and

$$A\,\bar{x}_B = \frac{w\,L^4}{24} + \tfrac{1}{6}\,\Sigma P(L - x)(2\,L - x)x$$

$$= \frac{w\,L^4}{24} + \tfrac{1}{6}\,\Sigma P(2\,L^2 - 3\,L\,x + x^2)x \quad (55b)$$

The notation is defined in Fig. 19.

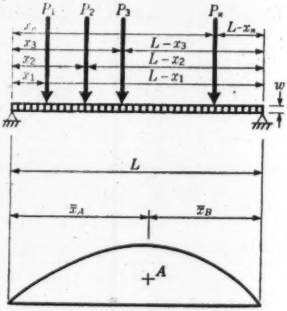


FIG. 19

---

[33] *Transactions*, ASCE, Vol. 110, 1945, p. 1542.

[34] *Ibid.*, Vol. 111, 1946, p. 887.

[35] Prof., Civ. Eng., College of Eng., Univ. of Missouri, Columbia, Mo.

It may be of interest to note that methods of analysis along somewhat the same line have been proposed by several others.[36,37,38,39,40]

Eduardo Agramonte,[41] Esq.—The authors have achieved their purpose in presenting a method of direct analysis requiring the solution of few, if any, simultaneous equations; it requires computing stiffness and carry-over factors in order to distribute and to carry over moments. This is more than casual parallelism with other moment-distribution methods and, in fact, warrants the inclusion of the authors' method in the same group.

A single set of basic principles governs slope deflection, iterative and noniterative moment distribution, fixed points, balancing angles, traversing elastic curves, and similar methods. Procedures may vary in many ways, hence the diversity of methods. Some principles and their related operative factors play a prominent part in one group of methods whereas other principles and factors have a greater role in others.

Continuous frame analysis was in need of an expeditious method when Hardy Cross, M. ASCE, presented moment distribution.[8] Universal welcome and adoption are the best proof of its merits; since it is neither unorthodox nor inexact, there is no reason for considering it less classical, precise, or exact than others. Methods of the types presented by T. Y. Lin,[39] Assoc. M. ASCE, J. A. Wise,[42] M. ASCE, and the authors may adequately be grouped as direct, noniterative, or single-cycle moment distribution. Development in this field is in progress; closed frames were a stumbling block, as cyclic computation of stiffnesses, followed by cyclic distribution of moments, appeared inevitable. However, since rapid convergence of values to close degrees of precision was no comfort, the authors' "fixed joint" expedient is a valuable step forward.

Usual moment distribution is simple because prismatic members require only one K-value each, and carry-over factors are known to be 0.5 throughout. For symmetrical members with variable moment of inertia, one stiffness factor and one carry-over factor suffice. Unsymmetrical members require two of each; as a contrast, direct distribution starts with such factors and proceeds to determine two additional stiffnesses, and two more carry-over factors for every member. This amount is quite enough. Auxiliary factors, such as rotation ratios, moment ratios, the authors' "elastic length," and possibly the writer's J-factor (explained hereafter) defeat their purpose.

Direct distribution is especially convenient when dealing with variable moment of inertia. Iterative moment distribution becomes less convergent and its carry-over factors are no longer 0.5. In such cases, direct methods compare to best advantage. For these reasons, the writer feels justified in extending the authors' method to cover nonprismatic members.

---

[36] "Structural Frameworks," by Thomas F. Hickerson, Univ. of North Carolina Press, Chapel Hill, N. C., 1934.

[37] "Neue Statik," by Max Mayer Bauwelt-Verlag, Berlin, 1937 and 1942.

[38] "Method of Moment Distribution by Means of Increased Moments," by Tsu Wu Chao, *Bulletin*, College of Eng., Kung Shang Univ., Tientsin, China, Vol. XI, No. 1, p. 13.

[39] "A Direct Method of Moment Distribution," by T. Y. Lin, *Transactions*, ASCE, Vol. 102, 1937, p. 561.

[40] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, p. 118.

[41] Civ. Engr., Ministry of Public Works, Buenos Aires, Argentina.

[42] "A Precise Moment Distribution Method," by Joseph A. Wise, *Journal*, ACI, November, 1938, p. 89.

Analysis of Individual Members.—If c_{AB} is assumed to be the ordinary carry-over factor from A to B (B fixed), Eq. 1 becomes, for the conditions shown in Fig. 2,

$$M_{BA} = c_{AB}\, M_{AB} \tag{56}$$

Also, if S_{AB} denotes the stiffness of end A when end B is fixed, the flexibility of end A (Eq. 3) will be:

$$Z_{AB(f)} = \frac{1}{S_{AB}} \tag{57}$$

Likewise, if U_{AB} denotes the stiffness of end A when end B is hinged, its value is:

$$U_{AB} = (1 - c_{AB}\, c_{BA})\, S_{AB} \tag{58}$$

Inasmuch as

$$c_{AB}\, S_{AB} = c_{BA}\, S_{BA} \tag{59a}$$

$$c_{AB}\, U_{AB} = c_{BA}\, U_{BA} \tag{59b}$$

Eq. 5 becomes:

$$Z_{AB(h)} = \frac{1}{U_{AB}} \tag{60}$$

Fixed-end moments and c-factors and S-factors for various types of nonprismatic members are available in current literature. As hinged end stiffness factors, U, are decidedly superior to S-factors in direct distribution methods, their appearance in future diagrams or charts for nonprismatic members would be convenient.

The column analogy furnishes the same data for any type of loading and shape. Flexible connections may be treated as incidental to the elastic properties of the member to which they belong, by determining final fixed-end moments and c-factors, U-factors, or S-factors which take the connections into account.

Letting $Z^R{}_A$ and $Z^R{}_B$ be the flexibility of each connection of member AB and $c^{\circ}{}_A$, $c^{\circ}{}_B$, $U^{\circ}{}_A$, $U^{\circ}{}_B$, $\mathrm{FEM}^{\circ}{}_A$, and $\mathrm{FEM}^{\circ}{}_B$, be the factors and fixed-end moments for the member considered deprived of connections, final factors and fixed-end moments become:

$$U_A = \frac{U^{\circ}{}_A}{1 + U^{\circ}{}_A\, Z^R{}_A} \tag{61a}$$

$$c_A = \frac{c^{\circ}{}_A}{1 + U^{\circ}{}_B\, Z^R{}_B} \tag{61b}$$

and

$$\mathrm{FEM}_A = \mathrm{FEM}^{\circ}{}_A - \frac{U_A}{1 - c_A\, c_B}\left(Z^R{}_A\, \mathrm{FEM}^{\circ}{}_A + c_A\, Z^R{}_B\, \mathrm{FEM}^{\circ}{}_B\right) \tag{61c}$$

which are valid for end B by transposition of subindexes.

When one or both ends of a member should be considered infinitely stiff ($I = \infty$) and the central section is prismatic ($I = I_o$) the factors may be computed from Fig. 20, obtained by application of the column analogy.

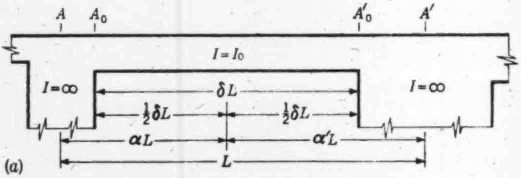
Frame Analysis.—To prevent confusion, the following sign convention is used throughout this discussion:

(*a*) Clockwise rotation is positive.

(*b*) End moments (couples) take the sign of rotation they produce or tend to produce; their sign is opposite to that of the rotations they resist.

(*c*) Bending moments along the span are positive where they produce tension at the under surface of beams or the right side of columns; therefore, when making moment diagrams the sign of all left-end beam moments or bottom-end column moments should be inverted.

(*d*) Shear is positive, where, should transverse shear rupture occur, particles imprisoned between the sliding fracture planes would be made to roll clockwise.



(*b*) STIFFNESS AND CARRY-OVER FACTORS

$$3\,\alpha^2 + \left(\frac{\delta}{2}\right)^2 = m \qquad\qquad m' = 3\,(\alpha')^2 + \left(\frac{\delta}{2}\right)^2$$

$$\frac{m}{\delta^3}\,\frac{4\,E\,I_o}{L} = S \qquad\qquad S' = \frac{m'}{\delta^3}\,\frac{4\,E\,I_o}{L}$$

$$\frac{1}{\delta\,m'}\,\frac{3\,E\,I_o}{L} = U \qquad\qquad U' = \frac{1}{\delta\,m}\,\frac{3\,E\,I_o}{L}$$

$$\frac{3\,\alpha}{m} - 1 = c \qquad\qquad c' = \frac{3\,\alpha'}{m'} - 1$$

$$\text{Check:}\quad \frac{S}{S'} = \frac{U}{U'} = \frac{c'}{c} = \frac{m}{m'}$$

(*c*) FIXED-END MOMENTS FOR UNIFORM LOAD, *w*, ON FULL SPAN

$$(2\,m - \delta^2)\,\frac{w\,L^2}{12} = M_{fAA'} \qquad\qquad M_{fA'A} = (2\,m' - \delta^2)\,\frac{w\,L^2}{12}$$

FIG. 20.—CONSTANTS FOR MEMBERS HAVING INFINITELY STIFF END SECTIONS

In Fig. 21, a positive external moment M_{A-B} acts at end A of member AB causing its positive rotation, θ_A, which is resisted by end B with intensity $M_{AB} = -\,M_{A-B}$. Flexure, thus produced, causes end BA to exert a negative moment M_{BA} at joint B, which in turn induces negative rotation, θ_B. All other members (BC, BD, and BE) react in unison, opposing an aggregate negative moment $M_{B-A} = -\,M_{BA}$. The writer propounds subindexes such as "B–A" as illustrative for moments or factors at joint B, pertaining to all members meeting at this joint except member BA.

If moment M_{A-B}, in Fig. 21, would originate rotation $\theta_A = +1$ radian, it would equal T_{AB}, the stiffness of end A of member AB. The reciprocal of T_{AB} is Z_{AB}; the rotation originated by unit moment $M_{A-B} = +1$—the flexibility of end A. If members AB, BD, and BE are suppressed and an external moment is applied at B to end BC, stiffness T_{BC}, or flexibility Z_{BC} of member BC, would be similarly illustrated.

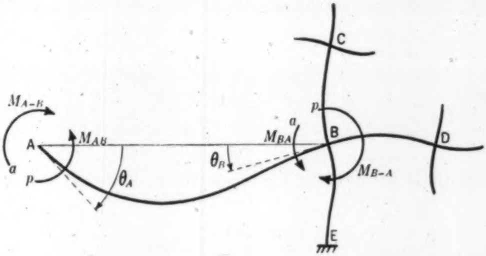


FIG. 21.—EXTERNAL MOMENT APPLIED AT END A OF MEMBER AB

Resistance to rotation at B, provoked by such a moment as M_{BA}, coming from member AB, depends on the aggregate stiffness of ends BC, BD, and BE, which is

$$T_{B-A} = T_{BC} + T_{BD} + T_{BE} = \frac{1}{Z_{B-A}} \qquad (62)$$

which in turn is equivalent to Eq. 13*b*. The total stiffness at B would be:

$$\begin{aligned} T_{B+} &= T_{BA} + T_{BC} + T_{BD} + T_{BE} = T_{B-A} + T_{BA} \\ &= T_{B-C} + T_{BC} = \cdots = T_{B-N} + T_{BN} \qquad (63) \end{aligned}$$

The authors provide no symbol for the carry-over or transmission factor.[43] Let such a factor be g; then $M_{BA} = g_{AB}\,M_{AB}$ in Fig. 21. Elastic lengths, flexibility, and carry-over factors for flexibly connected members are given in the following expressions—respectively, equivalent to Eqs. 24, 22, and 25:

$$L_{2AB} = (1 + Z^R{}_{AB}\,U^o{}_{AB})\,L \qquad (64a)$$

$$L_{3AB} = [1 + (Z^R{}_{BA} + Z_{B-A})\,U^o{}_{BA}]\,L \qquad (64b)$$

and

$$Z_{AB} = \frac{L_{2AB}\,L_{3AB} - c^o{}_{AB}\,c^o{}_{BA}\,L^2}{U^o{}_{AB}\,L_{3AB}\,L} \qquad (64c)$$

Also,

$$g_{AB} = c^o{}_{AB}\,\frac{L}{L_{3AB}} \qquad (65)$$

However, if values of c and U are previously computed to take into account the flexibility of connections, as in Eqs. 61, no further attention is necessary in that respect and Eqs. 66 to Eq. 86 apply directly.

Analysis of the frame may be accomplished by:

$$L_{AB} = (1 + U_{BA}\,Z_{B-A})\,L \qquad (66a)$$

and

$$Z_{AB} = \frac{L_{AB} - c_{AB}\,c_{BA}\,L}{U_{AB}\,L_{AB}} \qquad (66b)$$

---

[43] "Moving Loads on Restrained Beams and Frames," by R. C. Brumfield, *Transactions*, ASCE, Vol. 111, 1946, p. 807.

by following the operative sequence, Eqs. 13*b*, 66*a*, and 66*b*, obtaining *g*-factors independently from:

$$g_{AB} = c_{AB} \frac{L}{L_{AB}} \quad \text{(66c)}$$

Or, by the operative sequence, Eqs. 62, 67*a*, and 67*b*

$$g_{AB} = \frac{c_{AB}\, T_{B-A}}{U_{BA} + T_{B-A}} \quad \text{(67a)}$$

and

$$T_{AB} = \frac{U_{AB}}{1 - c_{BA}\, g_{AB}} \quad \text{(67b)}$$

Or, since products *g T* are as useful as single *g*-factors:

$$g_{AB}\, T_{AB} = \frac{c_{BA}\, U_{BA}\, T_{B-A}}{U_{BA} + (1 - c_{AB}\, c_{BA})\, T_{B-A}} \quad \text{(68a)}$$

and

$$T_{AB} = U_{AB} + c_{BA}\,(g_{AB}\, T_{AB}) \quad \text{(68b)}$$

may be found by the operative sequence, Eqs. 62, 68*a*, and 68*b*.

It may be found less tiresome, however, to avoid the repeated alternative use of expressions so dissimilar as types *a* and *b* of Eqs. 66, 67, or 68 by dealing, in the first place, with stiffnesses,

$$T_{AB} = \frac{(U_{BA} + T_{B-A})\, U_{AB}}{(U_{BA} + T_{B-A}) - c_{AB}\, c_{BA}\, T_{B-A}} \quad \text{(69a)}$$

by using Eqs. 62 and 69*a*, and to compute *g*-factors or (*g T*)-factors afterwards:

$$g_{AB} = \frac{T_{AB} - U_{AB}}{T_{AB}\, c_{BA}} \quad \text{(69b)}$$

or

$$g_{AB}\, T_{AB} = \frac{T_{AB} - U_{AB}}{c_{BA}} \quad \text{(69c)}$$

As a check on *T*-factors, *g*-factors, or (*g T*)-factors computed, obtaining for each member at the same time a new factor, *J*, that is useful for loading and sidesway analyses, these expressions hold true:

$$\frac{g_{AB}\, T_{AB}\, T_{A-B}}{T_{A+}} = J = \frac{g_{BA}\, T_{BA}\, T_{B-A}}{T_{B+}} \quad \text{(70)}$$

If B is fixed, $T_{B-A} = T_{B+} = \infty$, and

$$\frac{g_{AB}\, T_{AB}\, T_{A-B}}{T_{A+}} = J = g_{BA}\, T_{BA} \quad \text{(71)}$$

If B is hinged, $T_{B-A} = g_{AB} = J = 0$, but a check is offered by

$$\frac{c_{AB}\, T_{AB}\, T_{A-B}}{T_{A+}} = g_{BA}\, U_{BA} \quad \text{(72)}$$

Factor *J* is also equivalent to

$$J = \frac{U_{AB}}{c_{BA}} \frac{g_{AB}\, g_{BA}}{1 - g_{AB}\, g_{BA}} = \frac{U_{AB}}{c_{BA}} \frac{L^2}{\dfrac{L_{AB}\, L_{BA}}{c_{AB}\, c_{BA}} - L^2} \quad \text{(73)}$$

Transverse Loading.—Referring to Fig. 4, the moment diagram Fig. 4(*c*) considering the member as simply supported, must first be reduced, dividing its ordinates M by the local values of $E\,I$ along the span. Area A' and the location of the centroid (x_A, x_B) of the resulting diagram must be determined. Then, member AB, as actually connected to the frame, will exert moments M_{AB} at A and M_{BA} at B (opposite to those shown in Fig. 4(*b*) which amount to:

$$M_{AB} = A' \frac{U_{AB}}{c_{BA}} \frac{x_A\,L - \dfrac{x_B}{c_{AB}}\,L_{AB}}{\dfrac{L_{AB}\,L_{BA}}{c_{AB}\,c_{BA}} - L^2} \qquad (74)$$

which is equivalent to Eq. 16*a*. By inverting the subindexes M_{BA} is given. If the J-factor for the span is known,

$$M_{AB} = J \quad A' \left(\frac{x_A}{L} - \frac{x_B}{g_{AB}\,L} \right) \qquad (75a)$$

and

$$M_{BA} = J \quad A' \left(\frac{x_A}{g_{BA}\,L} - \frac{x_B}{L} \right) \qquad (75b)$$

When fixed-end moments are obtainable, no moment diagrams are necessary, since

$$M_{AB} = \frac{\text{FEM}_{AB}}{T_{A+}}\,T_{A-B} - \frac{\text{FEM}_{BA}}{T_{B+}}\,g_{BA}\,T_{BA} \qquad (76)$$

which also is valid for M_{BA} by transposition of the subindexes.

Sidesway.—Eqs. 20 take any of these alternative forms:

$$M_{AB} = \Delta U_{AB} \frac{L_{AB} + c_{AB}\,L}{L_{AB}\,L_{BA} - c_{AB}\,c_{BA}\,L^2} \qquad (77)$$

$$M_{AB} = \frac{\Delta}{L} \frac{1 + g_{AB}}{g_{AB}}\,J \qquad (78)$$

$$M_{AB} = \frac{\Delta}{L} \frac{(T_{AB} + g_{AB}\,T_{AB})\,T_{A-B}}{T_{A+}} \qquad (79)$$

and

$$M_{AB} = \frac{\Delta}{L} \left(\frac{T_{AB}}{g_{AB}\,T_{AB}} + 1 \right) J \qquad (80)$$

Transmission of Moments.—Stiffness T is, by definition, moment capable of provoking unit rotation; moments M_{A-B} and M_{BA} (Fig. 21) provoke rotations θ_A and θ_B, and the moment M_{CB} transmitted to C will provoke rotation θ_C. Hence,

$$\theta_A = \frac{M_{A-B}}{T_{AB}} \qquad (81a)$$

$$\theta_B = \frac{M_{BA}}{T_{B-A}} \qquad (81b)$$

and

$$\theta_C = \frac{M_{CB}}{T_{C-B}} \qquad (81c)$$

and such rotations awake the passive reactions:

$$M_{AB} = -\theta_A T_{AB} \dots\dots (82a)$$

$$M_{B-A} = -\theta_B T_{B-A} \dots\dots (82b)$$

$$M_{BC} = -\theta_B T_{BC} \dots\dots (82c)$$

and

$$M_{BD} = -\theta_B T_{BD} \dots\dots (82d)$$

and so on, for the other moments.

Distribution results from combining Eqs. 81 and Eqs. 82, which give

$$M_{B-A} = -M_{BA} \dots\dots (83a)$$

$$M_{BC} = -\frac{M_{BA}}{T_{B-A}} T_{BC} \dots\dots (83b)$$

and

$$M_{BD} = -\frac{M_{BA}}{T_{B-A}} T_{BD} \dots\dots (83c)$$

Carry-over (active) moments result either from distributed (passive) moments at the far end ($M_{CB} = g_{BC} M_{BC}$) or directly, in a single slide-rule movement, when distributing from active moments outside the far end, thus:

$$M_{CB} = -\frac{M_{BA}}{T_{B-A}} g_{BC} T_{BC} \dots\dots (84a)$$

and

$$M_{DB} = -\frac{M_{BA}}{T_{B-A}} g_{BD} T_{BD} \dots\dots (84b)$$

in which case products $g\,T$ apply throughout instead of single g-factors (see Eqs. 69, 70, 71, 76, 79, and 80). This also allows abbreviated distribution—by computing only active moments for the whole frame first, and passive moments when and where desired.

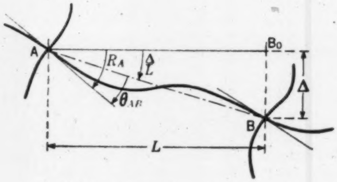


FIG. 22.—ROTATION AT JOINTS

Check on Final Moments.—The tangent at end A of a member AB (Fig. 22) forms an angle θ_{AB} with the straight line joining A and B such that:

$$\theta_{AB} = -\frac{(M_{AB} - \mathrm{FEM}_{AB}) - c_{BA}(M_{BA} - \mathrm{FEM}_{BA})}{U_{AB}} \dots\dots (85)$$

in which the fixed-end moments (FEM) are those due to all transverse loads on the member, and M_{AB} and M_{BA} are the actual moments at each end. This expression holds true with or without relative displacement of the joints. Rotation for any joint, computed separately by applying Eq. 85 to the end of each member framing into the joint, affords the desired check. In the case of displacements or sidesway, the quantity $\frac{\Delta}{L}$ must be noted as the check is in total rotation:

$$\left(\frac{\Delta}{L}\right)_{AB} + \theta_{AB} = R_A = \left(\frac{\Delta}{L}\right)_{AC} + \theta_{AC} \cdots = \left(\frac{\Delta}{L}\right)_{AN} + \theta_{AN} \dots (86)$$

If actual rotations are desired in radians, proper values must be used for E and I; if not, arbitrary values do just as well.

In the practical solution of a problem, ordinary (iterative) distribution requires only one step less than that outlined herein; that is, frame analysis does not apply.

STEPHEN J. FRAENKEL,[44] ESQ., and ROBERT L. JANES,[45] JUN. ASCE.—An interesting new approach to the problem of calculating moments in rigid frames is presented in this paper. At the same time, the presentation of another method for solving stiff-jointed structures raises the basic question as to whether the proposed modifications of the commonly used analytical procedures offer such a significant improvement as to supplant present practices.

The advantage of this method as compared with the classical method of analysis is said to be the elimination of solving simultaneous equations in order to reach a solution. It is possible, however, that the difficulty of solving simultaneous equations has been somewhat exaggerated. In many cases the number of unknowns does not exceed five or six (see Example 3) and, in fact, two or three simultaneous equations will often suffice in a "least work" solution. Although the use of a slope deflection method in the case of a tall building becomes an absurdity, its application in a case of five or six unknowns does not seem unreasonable. It can be assumed, furthermore, that a computing machine is available in engineering offices where structural analyses are executed frequently, and a solution of six simultaneous equations on a machine is a small task. Even where longhand computations are required, the use of a method of successive approximation,[46] makes the solution of such a system of equations a relatively simple and quick matter.

A comparison between various methods of analysis, used in the Examples given by the authors seems an obvious method of judging relative applicability of each. To a small extent, such a comparison may be unfair to the authors, in that the examples chosen were for the purpose of illustrating the method, rather than for illustrating the type of problem most readily solved by it. Even a cursory check of Example 1, however, indicates that it could have been solved with much less effort by the three moment equation, moment distribution, or slope deflection. Example 2 could likewise have been done by the method of least work (requiring two simultaneous equations) or even more simply by a single slope deflection equation.

In view of present methods of analysis it is perhaps more pertinent to compare this method with those based on moment distribution, or with the method of moment distribution itself. It has not been claimed by the authors that this method affords a saving in time over the method of moment distribution, or, for that matter, over any other method, yet unless some advantage can be claimed there seems little practical benefit to be obtained. It is admittedly difficult to judge the relative merit, from a time standpoint, of various methods for accomplishing the same result, principally because the length of time required is intimately connected with the engineer's relative familiarity with the various methods. In spite of this inherent difficulty there appears to be no other satisfactory basis of comparison than a direct application to a particular problem. A comparison has therefore been made using the authors' Example 4 as a basis. The methods used included the authors' method, that of moment distribution,[8] the modification or extension of this method as suggested by J. Wise,[47] and the column analogy.[48] The time required for a solution was considerably greater when using the method presented by the authors than by the use of any other methods. Granting a certain amount of lost motion due to unfamiliarity with the method, it seems doubtful, to say the least, that this time could have been equaled with the authors' method, even if one were thoroughly familiar with it. The discrepancy in time would probably be less in Examples 5, 6, or 7. In other words, it is believed that the procedure, as devised, sacrifices simplicity in simpler problems to applicability, and possibly superiority, when applied to complex frames. It must be remembered that the great majority of engineering calculations for indeterminate frames is relatively elementary and the cases in which the authors' procedure might excel constitute but a small minority of the total amount of work passing through a design office.

---

[44] Res. Engr., Structural Research, Armour Research Foundation, Chicago, Ill.

[45] Supv. of Structural Research, Armour Research Foundation, Chicago, Ill.

[46] "Analysis of Rigid Frames," by A. Amirikian, U. S. Govt. Printing Office, Washington, D. C., 1942.

The suggested procedure would appear, therefore, to be a useful tool only to those who are constantly engaged in the analysis of complex stiff-jointed structures; the average structural engineer who deals with a wide variety of problems would in all likelihood not remember this procedure from one proper application to the next. This discussion is not meant to be a brief for mediocrity but a mere recognition of facts learned in structural design offices.

PHIL M. FERGUSON,[49] M. ASCE.—Although a different vocabulary and a somewhat different language are used in this paper, nevertheless, it seems to resolve itself into a restatement of the method introduced by T. Y. Lin,[39] Assoc. M. ASCE. The fundamental difference apparent between the two papers is the authors' use of an "elastic factor," Z. For the same purpose, Professor Lin uses a "modified stiffness," K_m. (He actually uses relative stiffness, the equivalent of substituting I/L for $4\ E\ I/L$.) If the unmodified K is assumed to represent $4\ E\ I/L$, the authors' Z is simply the reciprocal of K_m, the Lin modified stiffness factor. As a matter of fact, in the actual examples of this paper one finds Z always expressed as 1/2.500, 1/1.875, and so on. In other words, the authors find it convenient to tabulate the "stiffness factor" $1/Z$ instead of Z itself; and $1/Z$ is simply the modified stiffness, K_m, introduced by Professor Lin.

In like fashion, the authors' carry-over factors are identically those "modified carry-over factors" used in the Lin paper. It is true that this paper seems to use different formulas; but the formulas are different only because the authors begin with flexibility instead of its reciprocal, stiffness; also, they define and use a new quantity called "elastic length." Only a few simple algebraic steps are needed to show that the "different" formulas are, in reality, identities. The sequence of calculations for the frame constants also seems to be identical in the two methods.

---

[47] "A Precise Moment Distribution Method," by J. A. Wise, Univ. of Minnesota, Minneapolis (mimeographed).

[48] "The Column Analogy," by Hardy Cross, *Bulletin No. 215*, Eng. Experiment Station, Univ. of Illinois, Urbana, 1930.

[49] Chairman, Civ. Eng., Univ. of Texas, Austin, Tex.

Likewise, in the handling of load terms, there is but one difference, which is a rather minor one. Professor Lin begins with fixed-end moments and must follow these with a single distribution and carry-over from each unbalanced joint before he obtains the authors' starting moments. The procedures seem to be identical from this point on.

A statement by the authors would be of interest, regarding the relative merits of these two approaches to this same general method—that is, the relative ease with which calculations or solutions can be formulated for more complicated problems. The writer is of the opinion that the basic moment-distribution process (essentially the same as that developed by Hardy Cross,[8] M. ASCE) will henceforth be the chief tool of designers in this field, and that all other methods will be secondary. In this situation he notes some advantages in the Lin approach to this problem. Modified stiffness and carry-over factors are closely related to the basic method in meaning and usage; and Professor Lin develops the factors very simply from it. This makes the development simpler for the student, and easier for the engineer who uses it only occasionally. Furthermore, the Lin formulas are adapted to members of variable section, inasmuch as all published tabulations of stiffness, carry-over factor, and fixed-end moment are directly usable. For variable sections, it would appear that the authors' method requires a return, each time, to the derivations based on the conjugate beam concept.

In the experience of the writer, very little use has been found for the term "flexibility," and it is about to be dropped from his teaching vocabulary. Since flexibility in algebraic terms is the inverse of stiffness, a new symbol and definition does not seem preferable to writing simply $1/K$ or $1/(4\,E\,K)$.

Part B of the authors' paper, with several examples of closed frames, is worthwhile in showing applications of this theory to frames more complicated than those previously covered by such methods. The device of a limited number of joints artifically restrained from rotation in the early stages is ingenious. A statement would be desirable as to whether the direct determination of the frame constants with all joints free to rotate was found to be a longer process, or whether it was ruled out as a method of successive approximation, and therefore not suitable for an "exact" solution. The tabulation forms are interesting and helpful; in fact, the authors are to be congratulated on a paper that is unusually clear and particularly well presented.

As a matter of general procedure, it appears that the multiple-panel Vierendeel truss of Example 6 could be more easily analyzed by the method of successive corrections. If one uses the criterion ratio procedure in such a solution (as was developed for wind stress calculations by L. E. Grinter,[50] M. ASCE, and if one is satisfied with errors as great as 1%, the time required should not be more than one third of that required for Example 6. Since this would not be an exact method, the comparison may not be fair.

Ralph W. Stewart,[51] M. ASCE.—A method is presented in this paper which is efficient in computing the bending moments in complex continuous frames such as Vierendeel trusses, and which is as accurate as the common theory of flexure. This gives the paper considerable value as a reference.

---

[50] "Theory of Modern Steel Structures," by Linton E. Grinter, The Macmillan Co., New York, N. Y., 1936–1937, pp. 168–178.

[51] Bridge Engr., City of Los Angeles, Los Angeles, Calif.

In developing the method by applying certain old theorems of structural mechanics, it appears to the writer that the preliminary work is cumbersome. For example, in order to obtain Eq. 14*a* the authors show five drawings (Figs. 4(*a*) to 4(*e*), inclusive). By simply adding two lines to Fig. 4(*b*) and transferring to it the longitudinal dimensions as shown in Figs. 4(*c*), 4(*d*), and 4(*e*), it is converted into Fig. 23, which shows all the elements of the flexure and makes the other four drawings unnecessary. The authors' procedure for deriving Eq. 14*a* can now be duplicated term by term by referring to Fig. 23 and using its geometrical properties. This reduces the labor involved in the derivation and also eliminates the use of the conjugate beam theorem. Similarly, by adding two lines to each of Figs. 2(*b*), 3(*b*), and 5(*b*), reference to the conjugate beam theory and also the moment-area drawings can be omitted.

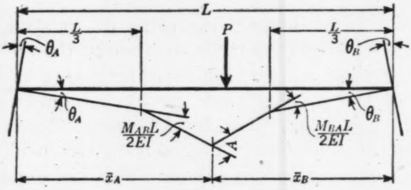


FIG. 23

Beginning with Clapeyron's discovery of the three-moment theorem, various special formulas or constants have been proposed for facilitating the analysis of stresses in continuous structures. These include, among others, the slope deflection equations, the conjugate beam theorem, the carry-over and special stiffness factors used in the end-moment distribution method, and the transmission coefficients. The authors now offer the "elastic length" and the "elastic factor" to add to the list. These special formulas or constants are derivatives obtained by combining part or all the six basic constants of flexure in one form or another.

TABLE 19.—TABULATION

Line	Description	Factor	Columns				
			BA	DC	FE	HG	JI
1	Moments due to FEM in span GE..........	16/4,462	−0.03	+0.10	−0.34	+2.51	−0.85
2	Moments due to unbalance at point E.......	4/830.8	−0.04	+0.13	−0.45	+0.13	−0.04
3	Moments due to FEM in span CE..........	12/4,462	+0.63	−1.89	+0.26	−0.07	+0.02
4	Balanced fixed-end moments...............	....	....	....	....	....	....
5	Final moments (without sidesway)..........	....	+0.56	−1.66	−0.53	−2.57	−0.87

Because the propriety of using the adjective "basic" in identifying the constants about to be described has been questioned,[52] it is necessary to explain why this is done. A suitable explanation can be presented by chemical analogy. The most important discovery in chemistry prior to atomic fission was that all molecules could be broken down into components (atoms) which could not at that time be further subdivided. Combinations of atoms became known as "derivatives"; for example, CH_4, C_2H_2, and C_8H_{18} are called hydrocarbon derivatives, and hydrogen and carbon may be considered as their basic components. A similar condition exists in machines, for which the basic components may be listed as shafts, gears, keys, pulleys, belts, and the like.

---

[52] *Transactions*, ASCE, Vol. 108, 1943, p. 1230.

All conditions of flexure may be reduced to an assembly of the following constants, which are designated as basic:[53]

(1) The moment necessary to produce a unit change in the direction of the elastic curve of a beam when applied at the left end, the right end being hinged;

(2) The moment necessary to produce a unit change in the direction of the elastic curve of a beam when applied at the right end, the left end being hinged;

(3) The abscissa of the intersection of the end tangents to the elastic curve of the beam (or the center of gravity of the $\frac{M}{E\,I}$-diagram) for condition (1); and

(4) The corresponding abscissa for condition (2).

These four constants are known as the beam constants. They are interrelated so that, if three are given, the fourth can be computed by Maxwell's theorem of reciprocal deflections (angular). Constants (1) and (2) are the same as the term preceding the parentheses in any slope deflection equation.

The two additional basic constants, known as the load constants, are:

(5) The total curvature in the beam (equal to the area of its $\frac{M}{E\,I}$-diagram), when it is considered as simply supported and subjected to the external loading, which acts on it between the supports; and

(6) The abscissa of the intersection of the end tangents to the elastic curve for condition (5).

OF MOMENTS, EXAMPLE 6

JOINT A		JOINT C			JOINT E			JOINT G			JOINT I		Line
AB	AC	CA	CD	CE	EC	EF	EG	GE	GH	GI	IG	IJ	
−0.06	+0.06	+0.21	+0.19	−0.40	−1.15	−0.68	+1.83	+5.55	+4.95	−10.50	+1.69	−1.69	1
−0.09	+0.09	+0.28	+0.25	−0.53	+2.45	−0.91	−1.54	−0.53	+0.25	+0.28	+0.09	−0.09	2
+1.27	−1.27	−4.12	−3.71	+7.93	−1.37	+0.51	+0.86	+0.30	−0.14	−0.16	−0.05	+0.05	3
....	....	....	....	....	−12.00	....	+12.00	....	....	....	....	....	4
+1.12	−1.12	−3.63	−3.27	+7.00	−12.07	−1.08	+13.15	+5.32	+5.06	−10.38	+1.73	−1.73	5

Although constants (1) and (2), which are called the basic stiffnesses, can be further subdivided into force times distance, neither force (which might be axial), nor distance acting alone can produce flexure. The simplest condition which can produce flexure is a combination of force and distance; therefore, constants (1) and (2) have been reduced to the lowest possible terms as far as direct relationship to flexure is concerned.

---

[53] "The Mechanics of Creep for Structural Analysis," by Joseph Marin, *Transactions*, ASCE, Vol. 108, 1943, p. 477.

If a quantic, such as the stiffness value of a member as defined for use in end-moment distribution, contains more than one of these basic constants, it is a derived function and not a basic constant. With these basic constants (Fig. 23) well in mind, a computer does not need to burden his tool kit with extra items like the conjugate beam theorem.

The authors' Example 3 (Fig. 8) is easily solved by the application of only four of these constants, as illustrated by Fig. 24 and Table 19. In Fig. 24 all

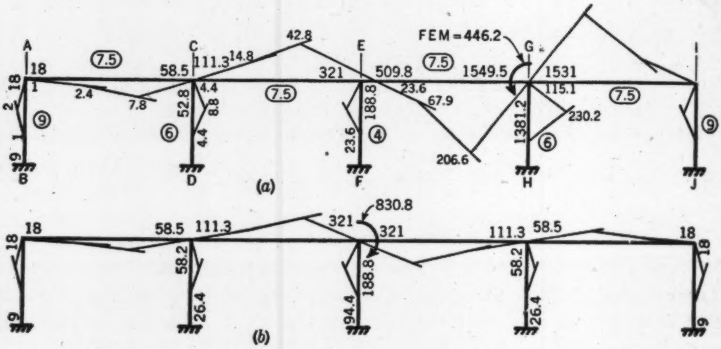


FIG. 24.—MOMENTS FOR EXAMPLE 3

angles of flexure are at the third points of the members. This solution involves a minimum number of slide rule settings and requires no preliminary computation of special constants. It may be assumed that the authors presented problems like Example 3 for the purpose of illustrating the use of the "elastic length" and the "elastic factor," rather than to shorten the solutions by using these special functions.

The authors' solution of Example 3 (Fig. 8), shows that $\frac{2\,I}{L}$ was used for the relative stiffness values of the members (constants (1) and (2) in the list of basic constants of flexure with E omitted). In Fig. 24, a set of values equal to $\frac{12\,I}{L}$ was used in order to avoid fractional values as much as possible. In Fig. 24, the series of angle values will remain identical for any proportional set of stiffness values but the moment values will vary directly as the stiffness values. The authors' "stiffness factors" can therefore be taken from Fig. 24. For example, the authors' stiffness factor at end E of member CE is $\frac{3.21}{6 \times 23.6} = 2.267$, which checks the value in Table 4. If Fig. 24 had been constructed by using $\frac{2\,I}{L}$ values for the relative member stiffnesses, the reduction factor 6 would disappear from this computation and the authors' elastic factors and stiffness factors could be obtained by simply dividing the end slope angles at the activated ends of the members by the moments at these activated ends, and vice versa.

It becomes troublesome to describe flexural analysis procedures with the words "stiffness" or "stiffness factor" meaning any one of the three following separate properties:

(1) Basic stiffness: Used in the slope deflection and traverse methods.
(2) End slope stiffness of a member with the opposite end fixed: Used in end-monent distribution.
(3) End slope stiffness of a member or an assembly of members for whatever remote end conditions exist: Used by the authors of this paper. It is the reciprocal of the "elastic factor."

Figs. 6, 7, and 8 are so easily solved by the use of basic constants that the use of the authors' special devices for such problems seems inadvisable. However, the solution of the Vierendeel truss shown in Fig. 12 is not easy by any method, and for structures of this class the authors' method appears to be useful.

Maurice Barron,[54] M. ASCE.—The authors are to be commended for bringing to the attention of the profession the fact that a method of direct moment distribution is possible, and that it is no longer necessary to use the methods of successive approximation. Unfortunately, the proposed method is elementary when compared to more advanced methods of direct moment distribution.

One such method[55] is that developed by Harris Epstein. Another method,[56] somewhat similar to the Epstein method, has been fully developed by R. C. Brumfield, M. ASCE. It appears that many of the refinements of the Brumfield method could be incorporated into the authors' method. For instance, their concept and use of the artificial fixed joint is evidence of a "stumbling block" that Professor Brumfield has ingeniously eliminated. This problem is encountered in every re-entrant frame. The difficulty arises from the fact that there is no starting point of known restraint. The authors, therefore, fix as many joints as are required and later release these joints, with a further corrective moment distribution. Professor Brumfield has developed a method whereby he can "jump into" any such re-entrant frame and very accurately estimate the transmission coefficient necessary to complete the analysis and, at the same time, check the estimated values. This means that a starting point is available and no corrective distribution is required because of arbitrary fixing of joints. Furthermore, this becomes a very useful tool in analyzing an individual member of a complicated frame without making an elaborate analysis of the surrounding members. The problem of sidesway, or the adjustment for unbalanced shears, is solved by very much the same procedure in both methods.

---

[54] Chf., Structural Design Div., Clarke, Rapuano and Holleran, Cons. Engrs. and Landscape Architects, New York, N. Y.

[55] "Standards of Design for Concrete," *No. 3Yb*, Bureau of Yards and Docks, U. S. Navy Dept., November 15, 1929, pp. 148-177.

[56] "Moving Loads on Restrained Beams and Frames," by R. C. Brumfield, *Transactions*, ASCE, Vol. 111, 1946, p. 805.

In order to illustrate the relatively weak points of the authors' paper the writer has used the Brumfield method to solve Examples 4 and 5. Additional symbols necessary for adaptation of this method to the authors' examples include:

The flexibility factor, Q, similar to the reciprocal of the ordinary stiffness factor, for uniform prismatic members measured by L/I; the transmission coefficient, C, similar to the ordinary carry-over factor and denoting the ratio of the moment arriving at one end of a member to the moment applied at the opposite end of the same member; the stiffness index, i_s, which is a measure of the resistance a member offers to the rotation of the joint to which it is attached (each member has two stiffness indexes, one for each end); and the restraint index, R, of a member radiating from a joint, a measure of the resistance to rotation offered to that member when it is used to actuate the joint by the other members radiating from the same joint. The relations between these factors are expressed by:

$$i_s = Q\,(2 - C) \qquad \text{(87)}$$

in which the "going away" transmission coefficient is

$$C = \frac{1}{2 + \dfrac{R}{Q}} \qquad \text{(88)}$$

and

$$\frac{1}{R} = \frac{1}{i_{s1}} + \frac{1}{i_{s2}} + \frac{1}{i_{s3}} + \cdots + \frac{1}{i_{sn}} \qquad \text{(89)}$$

In Example 4 (Fig. 9), the authors have arbitrarily chosen joint B as a fixed joint, thereby making the carry-over factors toward joint B of members AB and DB equal to 0.500. The transmission coefficient corresponding to the first of these two carry-over factors can be estimated as follows: $C_{AB} = \dfrac{1}{2 + \dfrac{Q_{BD}\,(2 - 0.25)}{Q_{AB}}} = \dfrac{1}{2 + \dfrac{2\,(2 - 0.25)}{2}} = 0.266.$ The approximate coefficient in the denominator is the average "going away" coefficient of all the restraining members at the joint toward which the desired estimated coefficient is transmitting.

In the computation for the estimated value of C_{AB}, only a rough estimate is required for the value of C_{BD} ($= 0.25$) which appears in the denominator. The value of 0.25 was chosen because the member BD is transmitting toward member DC, which is more flexible than member BD. If these two members were of equal flexibility, the value of 0.3 might be used, whereas if joint D had several restraining members, or one of great rigidity, the value of 0.35 might be used to estimate C_{AB}.

Similarly, C_{BA} is estimated to be: $C_{BA} = \dfrac{1}{2+\dfrac{Q_{AC}\,(2-0.2)}{Q_{BA}}} = \dfrac{1}{2+\dfrac{1.6\,(2-0.2)}{2}}$ $= 0.291$. The value of 0.2, as an approximation of C_{AC}, is used because member AC is transmitting toward the more flexible member CD. If the value 0.25 had been used, C_{BA} would equal 0.294; thus, a 25% change in the estimated coefficient results in only 1% change in the final coefficient, indicating that a crude guess is sufficient for very accurate results. The two estimated coefficients are entered in the tabulated computation of Table 20, and all the transmission coefficients are computed, including the exact values of C_{AB} and C_{BA}. It is interesting to note the agreement between estimated and final coefficients. Also, a comparison of the Brumfield transmission coefficients and the authors' carry-over factors demonstrates that the two are identical at joints twice removed from the fixed joint, showing a rapid convergence to the true value.

TABLE 20.—COMPUTATION OF TRANSMISSION COEFFICIENTS AND TORQUE SPLITS, EXAMPLE 4

Joint	Member	Stiffness Index, i_s		Restraint index, R	Transmission Coefficient		Authors' carry-over factor	Torque split[a]
		Computation	Value		Symbol	Value		
A	AB	2.0 (2 − 0.266)[b]	3.47	2.89	C_{BA}	0.290	0.290	0.454
	AC	1.6 (2 − 0.194)	2.89	3.47	C_{CA}	0.240	0.258	0.546
C	CD	3.0 (2 − 0.317)	5.05	2.82	C_{DC}	0.340	0.341	0.358
	CA	1.6 (2 − 0.240)	2.82	5.05	C_{AC}	0.194	0.195	0.642
D	DB	2.0 (2 − 0.270)	3.46	4.98	C_{BD}	0.223	0.223	0.410
	DC	3.0 (2 − 0.340)	4.98	3.46	C_{CD}	0.317	0.333	0.590
B	BA	2.0 (2 − 0.291)[b]	3.42	3.55	C_{AB}	0.265	0.500	0.510
	BD	2.0 (2 − 0.223)	3.55	3.42	C_{DB}	0.270	0.500	0.490

[a] $R/(R + i_s)$. [b] Estimated coefficients.

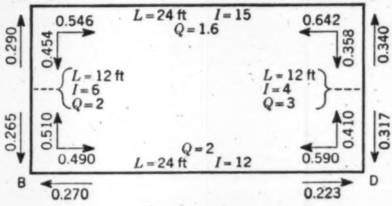


FIG. 25.—FRAME CHARACTERISTICS FOR EXAMPLE 4 (FIG. 9)

The frame characteristics for Example 4 are found to be as shown in Fig. 25, from which the end moments in member AC, with joint translation pre-

Line	Description								
				A				C	
		←AB→		←AC→		←CA→		←CD→	
1		0.14	7.30	7.30	0.14	0.15	5.57	5.57	0.15
2			0.04	0.04			0.03	0.03	
3	Uncorrected moment		7.20	7.20			5.45	5.45	
4	Sidesway correction	0.51			0.51		0.45	0.45	
5	Final moment		6.69	6.69			5.90	5.90	
					(*a*)				
		←BA→		←BD→		←DB→		←DC→	
6		0.48	1.93	1.93	0.48	0.43	1.77	1.77	0.43
7			0.01	0.01					
8	Uncorrected moment		1.46	1.46			1.34	1.34	
9	Sidesway correction	0.48			0.48		0.43	0.43	
10	Final moment (kip-ft)		0.98	0.98			1.77	1.77	
				B			D		
				A				C	
11			1,000 *M*						667 *M*
		←AB→		←AC→		←CA→		←CD→	
12		454 AB[a]	3 AC	546 AC	2 AB	106 AC	9 AB	9 AB	106 AC
13		2 AB	9 BD	3 AC	148 BA	1 AC	29 BA	29 BA	1 AC
14		148 BA	103 CA	9 BD	1 BA	37 BD	1 CD	1 CD	37 BD
15		1 BA	31 DB	103 CA	6 CD	428 CA	93 DC	239 CD	2 CA
16		6 CD		31 DB	22 DC	2 CA		93 DC	6 DB
17		22 DC				6 DB			
18			513 *M*	513 *M*		448 *M*			448 *M*
19			1,000 *M*		(*b*)				667 *M*
		←BA→		←BD→		←DB→		←DC→	
20		120 AB	9 AC	9 AC	120 AB	34 AC	27 AB	27 AB	34 AC
21		1 AB	2 BD	2 BD	1 AB	109 BD	9 BA	9 BA	109 BD
22		2 BA	27 CA	27 CA	2 BA	1 BD	76 CD	76 CD	1 BD
23		20 CD	106 DB	106 DB	6 DC	6 CA	1 DC	1 DC	6 CA
24		6 DC	1 DB	1 DB	20 CD	2 DB			2 DB
25		510 BA		490 BD		393 DB		274 DC	
26			486 *M*	486 *M*		432 *M*			432 *M*
				B			D		

[a] Letters following moment values indicate the joint of origin.

FIG. 26.—MOMENT COMPUTATIONS FOR EXAMPLE 4
(*a*) Final Moments
(*b*) Sidesway Distribution

vented, are determined as follows:

$$M_{AC} = -\frac{[(2\,k - 3\,k^2 + k^3) - C_{AC}\,(k - k^3)]\,P\,L\,C_{CA}}{1 - C_{AC}\,C_{CA}}$$

$$= -\frac{[(2 \times \tfrac{1}{2} - 3 \times \tfrac{1}{2}^2 + \tfrac{1}{2}^3) - 0.194\,(\tfrac{1}{2} - \tfrac{1}{2}^3)]\,4 \times 24 \times 0.240}{1 - 0.194 \times 0.240}$$

$= -$ 7.30 kip-ft; and

$$M_{CA} = -\frac{[(k - k^3) - C_{CA}\,(2\,k - 3\,k^2 + k^3)]\,P\,L\,C_{AC}}{1 - C_{CA}\,C_{AC}}$$

$$= -\frac{[(\tfrac{1}{2} - \tfrac{1}{2}^3) - 0.240\,(2 \times \tfrac{1}{2} - 3 \times \tfrac{1}{2}^2 + \tfrac{1}{2}^3)]\,4 \times 24 \times 0.194}{1 - 0.240 \times 0.194}$$

$= -$ 5.57 kip-ft.

TABLE 21.—COMPUTATION OF TRANSMISSION COEFFICIENTS, MOMENT SPLITS, AND TORQUE SPLITS, IN EXAMPLE 5

Joint	Member	STIFFNESS INDEX, i_s		Restraint index, R	TRANSMISSION COEFFICIENT		Authors' carry-over factor	Moment split[a]			Torque split[b]
		Computation	Value		Symbol	Value					
F	FE	1.33 (2 −0.252)[c]	2.33	3.30	C_{EF}	0.224	0.235	....	....	....	0.586
	FB	2.0 (2 −0.351)	3.30	2.33	C_{BF}	0.316	0.316	....	....	....	0.414
E	EF	1.33 (2 −0.224)[c]	2.37	2.66	C_{FE}	0.250	0.251	....	....	....	0.530
	EA	1.6 (2 −0.337)	2.66	2.37	C_{AE}	0.287	0.288	....	....	....	0.470
A	AB	2.0 (2 −0.353)	3.29	1.41	C_{BA}	0.370	0.370	AB	0.467	0.454	0.299
	AE	1.6 (2 −0.287)	2.74	1.54	C_{EA}	0.337	0.343	0.513	AE	0.546	0.360
	AC	1.6 (2 −0.195)	2.89	1.49	C_{CA}	0.341	0.346	0.487	0.533	AC	0.341
C	CD	3.0 (2 −0.323)[c]	5.03	2.65	C_{DC}	0.347	0.347	....	....	....	0.345
	CA	1.6 (2 −0.341)	2.65	5.03	C_{AC}	0.195	0.195	....	....	....	0.655
D	DB	2.0 (2 −0.354)	3.29	4.96	C_{BD}	0.224	0.223	....	....	....	0.601
	DC	3.0 (2 −0.347)[c]	4.96	3.29	C_{CD}	0.323	0.333	....	....	....	0.399
B	BF	2.0 (2 −0.316)	3.37	1.70	C_{FB}	0.351	0.500	BF	0.514	0.492	0.335
	BA	2.0 (2 −0.370)	3.26	1.73	C_{AB}	0.319	0.500	0.521	BA	0.508	0.347
	BD	2.0 (2 −0.224)	3.55	1.66	C_{DB}	0.354	0.500	0.479	0.486	BD	0.318

[a] R/i_s. [b] $R/(R + i_s)$. [c] Estimated coefficients.

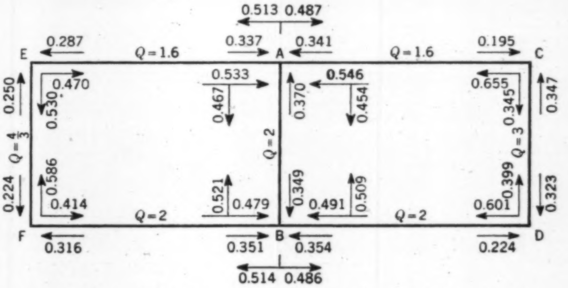

0.513 0.487
E
0.287
Q=1.6
0.337
A
0.341
Q=1.6
0.195
C
0.470
0.530
0.250
Q=4/3
0.586
0.224
0.414
0.533
0.467
Q=2
0.521
0.479
0.370
0.546
0.454
0.349
0.491
0.509
0.655
0.345
Q=3
0.399
0.601
0.347
0.323
Q=2
Q=2
F
0.316
0.351
B
0.354
D
0.224
0.514 0.486

FIG. 27.—FRAME CHARACTERISTICS FOR EXAMPLE 5 (FIG. 10)

Distribution of moments in the frame is shown in Fig. 26, in which arrows to the right indicate positive moments (authors' convention), and those to the left indicate negative moments. This sign convention for moments has been used throughout the writer's solution of Examples 4 and 5.

(a)

E A

EF		EA		AE		AB
	1,000 M_1					
530 EF		470 EA	1 EF		4 EF	8 EF
1 EF	1 EA	1 EA		158 EA		
	69 AE	69 AE		240 AE		200 AB
3 AB			3 AB			
16 CA	1 CA	1 CA	16 CA		54 CA	3 CD
	1 CD	1 CD			2 CD	
146 FE	4 FB	4 FB	146 FE	14 FE	49 FE	23 FE
				1 FB		1 FB
12 BA	18 BF	18 BF	12 BA	6 BF	42 BA	83 BA
1 BD			1 BD		3 BD	
4 DB	3 DB	3 DB	4 DB	9 DB		
	3 DC	3 DC		10 DC		11 DC
	387 M_1	387 M_1		284 M_1		

F B

FE		FB		BF		BA
119 EF	1,000 M_1		119 EF	1 EA	42 EF	22 EF
4 DB	4 EA	4 EA	4 DB	13 EA		1 EA
	15 AE	15 AE		5 AE		
1 AC			1 AC		3 AC	
11 AB			11 AB		36 AB	70 AB
3 CA	3 CA	3 CA	3 CA	8 CA		9 CD
	3 CD	3 CD		9 CD		
586 FE		414 FB	1 FE	145 FB	4 FE	8 FE
1 FE	1 FB	1 FB				
3 BA	73 BF	73 BF	3 BA	232 BF		223 BA
15 DB	1 DB	1 DB	15 DB		47 DB	
1 DC	1 DC	1 DC	1 DC		2 DC	4 DC
	357 M_1	357 M_1		279 M_1		

(b)

E A

EF		EA		AE		AB
		1,000 M_2		1,000 M_2		
	530 EF	1 EF	470 EA	4 EF	158 EA	74 EA
1 EA	1 EF		1 EA			
	117 FE	117 FE		40 FE	1 FB	
3 FB			3 FB		12 FB	22 FB
1 CD	2 CD	2 CD	1 CD		4 CD	7 CD
35 CA			35 CA		122 CA	
	1 CA	1 CA				
7 DB			7 DB		2 DB	1 DB
	5 DB	5 DB		16 DB		
	6 DC	6 DC		21 DC		17 DC
	615 M_2	615 M_2		782 M_2		

F B

FE		FB		BF		BA
		800 M_2		800 M_2		
	119 EF	119 EF		42 EF		26 EA
4 EA			4 EA		13 EA	
					1 EA	
	469 FE	1 FE		3 FE		
1 FB	1 FE		331 FB		116 FB	61 FB
			1 FB			
8 CA	6 CD	6 CD	8 CA	19 CD	3 CA	20 CD
	6 CA	6 CA		18 CA		2 CA
26 DB	1 DB	1 DB	26 DB		83 DB	
1 DC	1 DC	1 DC	1 DC		3 DC	6 DC
	562 M_2	562 M_2		663 M_2		

Fig. 28.—Moment Compu-
(a) Sideway Dis-
(b) Vertical Sway

C

→	← AC →		← CA →		← CD →	
667 M_1						444 M_1
		4 EF	2 EF	1 EF	1 EF	2 EF
74 EA		84 EA	1 EA	16 EA	16 EA	1 EA
	227 AC		44 AC			44 AC
				3 AB	3 AB	
45 CA	99 CA		291 CA		153 CD	1 CA
		1 CD	1 CA			
28 FB	26 FE		5 FE			5 FE
	14 FB	2 FB	3 FB	5 FB	5 FB	3 FB
3 BF		3 BF		1 BF	1 BF	
3 BD	6 BD	41 BA	16 BD	8 BA	8 BA	16 BD
18 DB	9 DB	21 DC	2 DB	61 DC	61 DC	2 DB
509 M_1	225 M_1		270 M_1			270 M_1

D

→	← BD →		← DB →		← DC →	
667 M_1	20 EF		4 EF			444 M_1
26 EA	13 EA	2 EA	3 EA	5 EA	5 EA	4 EF
						3 EA
3 AE	5 AC	2 AE	14 AC			14 AC
2 AC						
		34 AB		8 AB	8 AB	
16 CA	8 CA	18 CD	2 CA	49 CD	49 CD	2 CA
75 FB		4 FE	2 FE	1 FE	1 FE	2 FE
		70 FB	1 FB	16 FB	16 FB	1 FB
	212 BD		48 BD	3 BA	3 BA	48 BD
48 DB	95 DB		267 DB		177 DC	
		2 DC		1 DC	1 DC	
500 M_1	221 M_1		258 M_1			258 M_1

C

→	← AC →		← CA →		← CD →	
		1,000 M_2		1,000 M_2		
8 EF	4 EF		1 EF	2 EF	2 EF	1 EF
	84 EA		16 EA	1 EA	1 EA	16 EA
19 FE		21 FE		4 FE	4 FE	
1 FB	2 FB	10 FB	4 FB	2 FB	2 FB	4 FB
	223 CD	3 CD	655 CA	1 CD	345 CD	
101 CA			1 CA		1 CD	1 CA
	1 DB					
32 DB	16 DB	38 DC	3 DB	111 DC	111 DC	3 DB
40 M_2		742 M_2		441 M_2	441 M_2	

D

→	← BD →		← DB →		← DC →	
		800 M_2		800 M_2		
22 EF		20 EF		4 EF	4 EF	
1 EA	2 EA	13 EA	5 EA	3 EA	3 EA	5 EA
6 FE	3 FE		1 FE	1 FE	1 FE	1 FE
	55 FB		12 FB	1 FB	1 FB	12 FB
	1 CA	39 CD		111 CD	111 CD	
35 CA	17 CA		4 CA			4 CA
87 DB	170 DB		481 DB		319 DC	1 DB
		3 DC	1 DB	1 DC	1 DC	
36 M_2		627 M_2		417 M_2	417 M_2	

TATIONS FOR EXAMPLE 5
tribution
Distribution

E A C

←EF	EF→	←EA	EA→	←AE	AE→	←AB	AB→	←AC	AC→	←CA	CA→	←CD	CD→
1.66 AE	0.07 AB	0.07 AB	1.66 AE	0.02 AB	5.78 AE	0.02 AE	4.80 AB	10.58 AC	0.01 AE	0.06 AB	4.95 CA	4.95 CD	0.06 AB
0.02 CD	0.02 CD	0.02 CD	0.02 CD		0.01 AE	0.11 CD	0.01 AB	0.02 AB	0.01 AB				
					0.01 AB		0.01 AB		0.05 CD		0.01 CD	0.01 CD	
					0.06 CD								
1.59			1.59		5.84		4.69	10.53			4.90	4.90	
0.49^a			0.49^a		0.36^a	0.65^a			0.29^a		0.35^a	0.35^a	
	7.53^b	7.53^b		9.56^b			0.49^b		9.07^b		5.40^b	5.40^b	
Σ =	5.45	5.45		3.36			4.53	1.17			10.65	10.65	

F B D

←FE	FE→	←FB	FB→	←BF	BF→	←BA	BA→	←BD	BD→	←DB	DB→	←DC	DC→
0.37 AE	0.27 AB	0.27 AB	0.37 AE	0.86 AB	0.13 AE	0.07 AE	1.68 AB	0.06 AE	0.57 CD	0.01 AE	1.60 CD	1.60 CD	0.01 AE
	0.09 CD	0.09 CD		0.28 CD		0.29 CD		0.82 AB		0.18 AB			0.18 AB
0.01			0.01	1.01			1.32	0.31			1.41	1.41	
0.46^a			0.46^a		0.35^a	0.64^a			0.28^a		0.33^a	0.33^a	
	6.88^b	6.88^b		8.12^b			0.44^b		7.68^b		5.12^b	5.12^b	
Σ =	6.41	6.41		8.78			1.12		7.65		6.86	6.86	

EF, AB, and CD		EA and FB		EF, AB, and CD			EA and FB	
←	→	←	→	←	→	→	←	←
1.59	4.69		1.59	441 M_2	615 M_2	387 M_1	387 M_1	615 M_2
0.01	1.32		5.84	417	562	357	357	782
4.90		1.01	0.01		40	509	284	562
1.41					36	500	279	663
7.91	6.01	1.01	7.44	858	1,253	270	1,307 M_1	2,622 M_2
6.01			1.01		858	258		
Σ = 1.90		Σ =	6.43		395 M_2	2,281 M_1		

[a] Sideway correction. [b] Vertical sway correction.

FIG. 29.—FINAL MOMENT COMPUTATIONS FOR EXAMPLE 5

The sidesway, or unbalanced shear problem, has also been worked out by the Brumfield method. From Fig. 26(*a*), the total unbalanced moments equal -1.87, which is twelve times the unbalanced shear. The moment corresponding to this unbalanced shear, from Fig. 26(*b*), is $+1{,}879\ M$. Equating these two values, $1{,}879\ M = -1.87$, from which $1{,}000\ M = -0.996$. Application of this factor as a sidesway correction (lines 4 and 9, Fig. 26(*a*)) to the moments for no sidesway (lines 3 and 8) gives the final moments (lines 5 and 10).

In Example 5 (Fig. 10), the authors again use the expedient of fixing joints —in this case joint B. After the preliminary distribution, joint B is released and a corrective distribution is computed. The correction for unbalanced shears in this problem is more complicated than in the previous one, because there is not only a sidesway, but also a vertical sway. The authors' procedure in step (*g*) is to deform the structure a distance, $\Delta = 24/E$, and then to find the induced moments resulting from this distortion. Eight moments are thus required for distribution.

In the Brumfield procedure (Table 21 and Figs. 27, 28, and 29) member AB is translated vertically downward with no rotation of joints. Since the translation for point A of member AE and AC is the same, and the Q-values and L-values are identical, the induced fixed-end moments, M_{EA}, M_{AE}, M_{AC}, and M_{CA}, will be the same. Similarly, the induced fixed-end moments for BD and DB will be the same. Furthermore, the induced fixed-end moments of the top chord members will bear a constant relationship to the fixed-end moments induced in the bottom chord members. The estimated transmission coefficients, shown in Table 20 for members EF and CD, were computed by the same method as used for Example 4. Using the transmission coefficients of Table 20, it is found that: $M_{CA} = -\dfrac{4 \times 24 \times 0.195\ (0.659 \times 3/8)}{1 - 0.0665}$ $= -4.95$ kip-ft; and $M_{AC} = -\dfrac{4 \times 24 \times 0.341\ (0.805 \times 3/8)}{1 - 0.0665} = -10.58$ kip-ft.

Since all horizontal members in this example are the same length (Fig. 10), the induced fixed-end moments will be proportional to the moments of inertia of the respective members, or inversely proportional to the Q-values. Therefore, if 1,000 units of moment ($1{,}000\ M_2$) are distributed at each joint E, A, and C, $800\ M_2$ must be distributed at each joint F, B, and D. These six moments—three at $1{,}000\ M_2$ and three at $800\ M_2$—may be distributed concurrently, and the totals found at all joints. Since the moments at the center are equal and opposite they balance and need not be distributed. In like fashion, joints E, A, and C may be translated horizontally without joint rotation. Again, the induced fixed-end moments without joint rotation will be proportional to the moment of inertia of each vertical member. If 1,000 units of moment ($1{,}000\ M_1$) are distributed at joints E and F, $667\ M_1$ must be distributed at joints A and B, and $444\ M_1$ must be distributed at joints C and D. These six moments—two at $1{,}000\ M_1$, two at $667\ M_1$, and two at $444\ M_1$—may be distributed concurrently, and the moments summed up for each joint.

The final horizontal shears in the vertical members must equal zero, and the vertical shears in the horizontal members must be consistent with the loading. Therefore, it is necessary to write the two simultaneous equations, $+ 2{,}281\ M_1 + 395\ M_2 - 1.90 = 0$; $+ 1{,}307\ M_1 + 2{,}622\ M_2 - 30.43 = 0$. When solved simultaneously, these two equations will yield values of M_1 and M_2 that will balance the shears both vertically and horizontally. By this method $1{,}000\ M_2 = 12.24$ and $1{,}000\ M_1 = -1.28$. Application of the corrections are shown in Fig. 29.

At this point, it should be noted that sidesway results from either a dissymmetry of structure, a dissymmetry of loading, or a combination of both. When vertical sway and horizontal sway can occur, simultaneously, they must be treated simultaneously. The authors have not done so in Examples 5, 6, and 7. The fact that their answers, for Example 5, are so close to the correct ones is due to the unfortunate choice of frame and loading, where dissymmetry of structure is almost balanced by the dissymmetry of loading. The fact that the sum of the authors' moments at the top and bottom of the columns does not quite add up to zero is not due to computational inaccuracies, but to the unbalancing effect of separately correcting for horizontal sway and vertical sway. When a structure has several vertical members to which sidesway may be distributed, the correction will be of little significance, but a fallacy may result, such as in this example, if such structures are used to demonstrate a principle. The question is then raised as to whether the authors' computation of sway is theoretically exact or a practical approximation.

In step (*d*) of Examples 4 and 5, the authors find the end moments of the loaded member by using the moment areas and conjugate beam, which were used in the derivation. The Brumfield method finds these moments directly from the transmission coefficients. Using the authors' carry-over factors and the Brumfield equations, the end moments for Example 4 are found to be: $M_{AC} = -\dfrac{4 \times 24 \times 0.258}{1 - 0.258 \times 0.195}\left(\tfrac{3}{8} - 0.196 \times \tfrac{3}{8}\right) = -7.80$ kip-ft (authors' value 7.874 kip-ft) and $M_{CA} = -\dfrac{4 \times 24 \times 0.195}{0.9495}\left(\tfrac{3}{8} - 0.258 \times \tfrac{3}{8}\right) = -5.48$ kip-ft) (authors' value 5.488 kip-ft). Thus, Professor Brumfield's equation is a more direct solution. In addition, his equations have been extended for uniform loads, triangular loads, and complex loads.

The authors' concept of "elastic length" is interesting but its practical usefulness is questionable. It is evident that the method could have been developed without introducing this new concept.

Many problems confronting the practicing structural engineer are readily solved by the Brumfield method. Some of these problems cannot be solved by any other method, and it is improbable that the authors' method can be extended to be as all-inclusive. If it can be extended, however, it might be useful if the authors would indicate how the extension can be accomplished.

Fig. 30 shows typical examples of some of these problems. In each of the examples shown the moment of inertia may be a variable with little additional work.

Figs. 30(*a*) to 30(*f*) show various types of rectangular and nonrectangular frames. Fig. 30(*g*) shows three curved members with a common point A, and Fig. 30(*h*) shows a frame with both curved and straight members. In Fig. 30(*j*), the intrusion of the column into the ordinary building beam is unimportant, but, in Fig. 30(*k*), the column intrusion is a large part of the beam span and must be considered in the analysis."

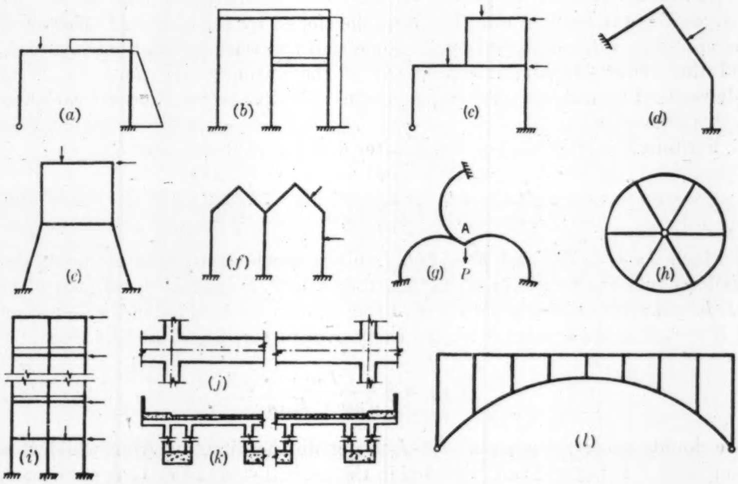


FIG. 30

Fig. 30 (*l*) shows an arch rib with spandrel columns and deck. The analysis of this structure is complicated by the fact that the deck, the spandrel columns, and the ribs will participate in supporting the loads and in resisting temperature stresses. The problem is so complex that, ordinarily, bridge designers will avoid the analysis by judicious use of expansion joints in the deck structure. This procedure obviates the beneficial effects that the rib derives from deck participation.

The Brumfield method uniquely solves the problems presented. It is universal in its application in that it develops transmission coefficients for curved members, as well as for straight members and for members with variable moment of inertia. These transmission coefficients take into account the actual restraints at the ends of the member. If the connection of a member yields and the amount of yielding is known, it may be incorporated into the analysis with little difficulty.

The outstanding feature of all methods of direct moment distribution is that the moments "flow away" from the point of origin in one direction—never returning to that point except in re-entrant frames. The structural "keenness" developed by use of direct methods far exceeds that developed by methods employing successive approximation. This in itself is a strong argument for wide adoption of direct methods such as those proposed by the authors and discussed herein.

Edwin H. Gaylord,[57] Assoc. M. ASCE.—An ingenious method for the analysis of continuous frames has been developed by the authors. However, the procedure is handicapped by the somewhat awkward calculations involved, and the writer suggests modifications of the author's equations which he believes lead to fairly simple computations. Flexible connections are excluded in this discussion.

Forming the reciprocal of Eq. 8, after dividing each term by L:

$$\frac{L}{L_{AB}} = \frac{1}{1 + 3\,E\,I\,Z_B/L} = \frac{J_B}{J_B + K} \qquad (90)$$

in which $J_B = 1/Z_B$, and $K = I/L$. (Since continuous frame moments depend on relative moments of inertia, the ratio I/L may be used instead of $3\,I/L$. If different materials are involved, E must be included in the definition of K.) Using the symbol F for the carry-over factor, Eqs. 7 and 90 give

$$F_{AB} = \tfrac{1}{2}\,\frac{J_{BC}}{J_{BC} + K_{AB}} \qquad (91)$$

The double subscript is used with J to denote the direction from which it is computed. If the division indicated in the parentheses of Eq. 11 is performed, the following equation for J is obtained:

$$J_{AB} = \frac{K_{AB}}{1 - F_{AB}/2} \qquad (92)$$

Once the values of F and J are computed for a given frame, end moments due to transverse loads can be determined rather simply. Consider the moment at point C in span CE, Fig. 8. If the values of x_A and x_B as given by Eqs. 17 are substituted in Eqs. 16a, the following equation is obtained:

$$M_{CE} = \frac{M_{FCE}\,(4\,L_{CE}\,L - L^2) - 2\,M_{FEC}\,(L_{CE}\,L - L^2)}{4\,L_{CE}\,L_{EC} - L^2} \qquad (93a)$$

In Eq. 93a, the subscript F denotes fixed-end moments. The second term in the numerator is negative to conform to the convention which denotes moments as positive when they are clockwise on the joint (or member). The second term is positive for those who prefer the moment sign convention used in strength of materials. Eq. 93*a* can be written:

---

[57] Prof. and Chairman, Dept. of Civ. Eng., Ohio Univ., Athens, Ohio.

$$M_{CE} = 4\,M_{FCE}\,\frac{L - \frac{1}{4}\,\dfrac{L^2}{L_{CE}}}{4\,L_{EC} - \dfrac{L^2}{L_{CE}}} - 2\,M_{FEC}\,\frac{\dfrac{L}{L_{EC}}\,(L_{CE} - L)}{4\,L_{CE} - \dfrac{L^2}{L_{EC}}} \dots\dots (93b)$$

From Eq. 11,

$$L - \frac{1}{4}\,\frac{L^2}{L_{CE}} = \frac{3\,E\,I}{J_{CE}} \dots\dots\dots\dots\dots\dots\dots (93c)$$

From Eq. 93*c*,

$$\frac{L^2}{L_{CE}} = 4\,L - \frac{12\,E\,I}{J_{CE}} \dots\dots\dots\dots\dots\dots\dots (93d)$$

and, therefore,

$$4\,L_{EC} - \frac{L^2}{L_{CE}} = 4\,(L_{EC} - L) + \frac{12\,E\,I}{J_{CE}} \dots\dots\dots\dots (93e)$$

Now, from Eqs. 8 and 13*b*,

$$L_{EC} - L = \frac{3\,E\,I}{J_{CA} + J_{CD}} \dots\dots\dots\dots\dots\dots\dots (93f)$$

Substituting from Eq. 93*f* into Eq. 93*e*

$$4\,L_{EC} - \frac{L^2}{L_{CE}} = 12\,E\,I\left(\frac{1}{J_{CA} + J_{CD}} + \frac{1}{J_{CE}}\right) \dots\dots\dots (93g)$$

The right member of Eq. 93*b* can now be expressed in terms of stiffness factors, by substitution from Eqs. 93*c* and 93*g* into the first term, and, with proper change in notation, by substitution from Eqs. 93*f* and 93*g* into the second term. Finally, observing that $L/L_{EC} = 2\,F_{EC}$, Eq. 93*b* reduces to

$$M_{CE} = M_{FCE}\,\frac{J_{CA} + J_{CD}}{J_{CA} + J_{CD} + J_{CE}} - M_{FEC}\,\frac{F_{EC}\,J_{EC}}{J_{EC} + J_{EF} + J_{EG}} \dots (93h)$$

Denote by the symbol R the ratio of the stiffness factor of any member to the sum of the stiffnesses of all members framing into the joint. Then, for M_{AB}, Eq. 93*h* may be written:

$$M_{AB} = (1 - R_{AB})\,M_{FAB} - F_{BA}\,R_{BA}\,M_{FBA} \dots\dots\dots (94a)$$

or

$$M_{AB} = M_{FAB} - R_{AB}\,M_{FAB} - F_{BA}\,R_{BA}\,M_{FBA} \dots\dots\dots (94b)$$

Eq. 94*a* has the advantage of requiring the addition of only two terms. However, slide rule work is simplified in Eq. 94*b*, since the CI-scale may be used to obtain the second term of the equation for M_{AB}, and the third term of the equation for M_{BA} may then usually be found with either the C-scale or the CF-scale without changing the position of the sliding scale.

TABLE 22.—MODIFIED SOLUTION OF EXAMPLE 1(*a*)

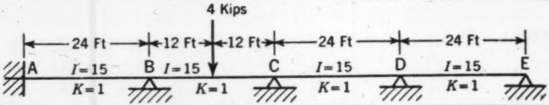
Line	Quantity	 4 Kips 24 Ft 12 Ft 12 Ft 24 Ft 24 Ft A B C D E $I=15$ $I=15$ $I=15$ $I=15$ $K=1$ $K=1$ $K=1$ $K=1$							
1	F		0.50	0.267	0.286	0.250		0	
2	J		1.333	1.156	1.167	1.143		1.000	
3	R			0.464	0.505				
4	M_F			+12	−12				
5	$(1 - R)\ M_F$			+6.43	−5.94				
6	$-F\ R\ M_F$			+1.74	−1.49				
7	M	−4.08	−8.17	+8.17	−7.43	+7.43	+1.86	−1.86	0

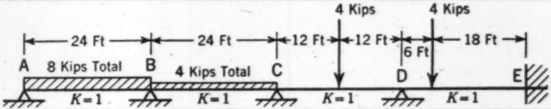
Table 22 shows the solution of Example 1(*a*) by Eqs. 91, 92, and 94*a*. Since the I/L-ratios are the same for all members, the value $K = 1$ was chosen for simplicity. The computations start by entering in line 1 the known value $F_{BA} = 0.5$. (In a strict sense, the computations start with the known joint stiffness $J_A = \infty$. Eq. 91 gives $F = 0.5$ when $J \rightarrow \infty$.) With F_{BA} known, the value of J_{BA} is calculated from Eq. 92 and entered in line 2, Table 22. With J_{BA} known, F_{CB} is determined from Eq. 91 and recorded in its proper place. Calculations to the right stop with J_{CB} since the DC-factors are not needed for this example. The known value $F_{DE} = 0$ is recorded next, and computations to the left proceed as outlined, stopping with the factors for span BC. It will be observed that the additions in Eqs. 91 and 92 can be performed mentally, and the equations are easily remembered after using them a few times.

Only two values of R are needed for this example, and they are computed and recorded in line 3, Table 22. The remainder of the calculations is self-explanatory. However, it should be noted that the second term of Eq. 94*a* is a "carry-over" and must be so entered in the table.

The example in Table 23 is given to show the procedure when more than one span is loaded. In this example, Eq. 94*b* is used. It is unnecessary to calculate values of J at the end supports, since $R = 0$ for a fixed-end support and $R = 1$ for a hinged-end support. After the values of R and M_F are determined, $M_{FAB} = 16$ is multiplied by the R-value immediately above, and the negative of the result recorded immediately below in line 5. This product is then multiplied by F_{AB} and the result, -4.29, recorded in line 6 at point B. After lines 5 and 6 have been completed, line 7 is found by adding lines 4, 5, and 6.

The first step in calculating carry-overs to the right is to record at point B of span BC in line 8 the balancing moment for $M_{BA} = -12.86$. The carry-over +3.46 is then recorded at point C. This value added to the moment immediately above gives the balancing monent (+1.74) to be recorded at point C of span CD. The carry-over to point D is then computed, and the process continued until point E is reached. Line 9 is computed similarly, and the final moments in line 10 are determined by adding rows 7, 8, and 9.

TABLE 23.—Illustrative Example; More Than One Span Loaded

Line	Quantity	 4 Kips 4 Kips 24 Ft 24 Ft 12 Ft 12 Ft 6 Ft 18 Ft A 8 Kips Total B 4 Kips Total C D E K=1 K=1 K=1 K=1							
1	F	0.268	0	0.269	0.250	0.286	0.267	0.500	0.268
2	J		1.000	1.156	1.143	1.167	1.156	1.333	
3	R	1.000	0.464	0.536	0.495	0.505	0.464	0.536	0
4	M_F	+16	−16	+8	−8	+12	−12	+13.5	−4.5
5	$-R\,M_F$	−16	+7.43	−4.29	+3.96	−6.06	+5.57	−7.24	0
6	$-F\,R\,M_F$	0	−4.29	+0.99	−1.16	+1.49	−1.73	0	−3.62
7	M	0	−12.86	+4.70	−5.20	+7.43	−8.16	+6.26	−8.12
8	Right[a]			+12.86	+3.46	+1.74	+0.50	+7.66	+3.83
9	Left[a]		−3.26	−1.44	−5.76	−1.67	−6.26		
10	M	0	−16.12	+16.12	−7.50	+7.50	−13.92	+13.92	−4.29

[a] Carry-over to the right or left.

The tabular solution can be extended to frames of the type shown in Fig. 8. For sidesway calculations, Eqs. 20 can be expressed in terms of the factors *F*, *J*, and *R*. Consider the leg CD of the frame of Fig. 8. From Eq. 20*a*,

$$M_{CD} = \frac{3\,E\,I\,\Delta}{2\,L}\left[\frac{2}{L_{DC} - \dfrac{L^2}{4\,L_{CD}}} + \frac{L/L_{DC}}{L_{CD} - \dfrac{L^2}{4\,L_{DC}}}\right] \quad \ldots\ldots\ldots (95a)$$

Substituting from Eq. 93*g*, with proper changes in notation, Eq. 95*a* becomes, for M_{AB},

$$M_{AB} = \frac{\Delta}{L}\left[J_{AB}\,(1 - R_{AB}) + F_{BA}\,J_{BA}\,(1 - R_{BA})\right] \quad \ldots\ldots\ldots (95b)$$

The solution of the frame of Fig. 8, with sidesway restrained, is given in Table 24. As in the previous examples in this discussion, relative values of *K* are used. The computation starts with the known values F_{AB}, F_{CD}, F_{EF}, etc., from which the values of J_{AB}, J_{CD}, J_{EF}, etc., are computed by Eq. 92. The values of F_{CA} and J_{CA} are determined next, using the known value of F_{AB}. In calculating F_{EC}, it must be remembered that the stiffness factor *J* in Eq. 91 is given by Eq. 13*b* as the sum of J_{CA} and J_{CD}—that is,

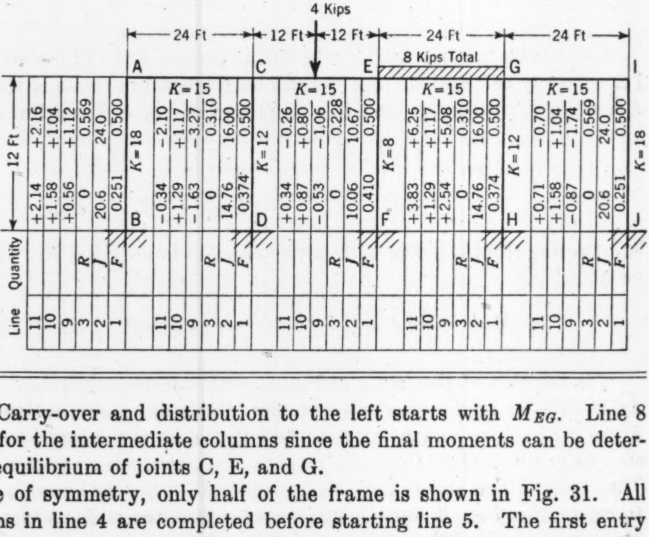
$$F_{EC} = \tfrac{1}{2}\,\frac{J_{CA} + J_{CD}}{J_{CA} + J_{CD} + K_{EC}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (96)$$

Calculations proceed in this manner until the JI-values have been determined. The process must then be repeated, starting with F_{IJ} and working to the left until joint B is reached. Because of the symmetry of the frame in this example, values from point J to the left are the same as those from point B to the right. Finally, values of F and J for joints D, F, and H are determined.

The first step in determining line 8 is the distribution of M_{EC} to members EF and EG in proportion to their stiffness factors (J_E). The carry-over from point E to point G is calculated next and the final moment $M_{GE} = -10.71$ is distributed. Carry-over and distribution are continued until joint J is reached. Carry-over and distribution to the left starts with M_{EG}. Line 8 is omitted for the intermediate columns since the final moments can be determined by equilibrium of joints C, E, and G.

TABLE 24.—MODIFIED SOLUTION OF EXAMPLE 3

Line	Quantity	A	C		E		G		I
11	Final moment	−2.16	−4.38	+6.49	−16.48	+16.74	−11.13	+4.89	+0.70
10	Correction for sidesway	−1.04	−0.74	−0.42	−0.40	−0.40	−0.42	−0.74	−1.04
9	Final moment; no sidesway	−1.12	−3.64	+6.91	−16.08	+17.14	−10.71	+5.63	+1.74
8	Distribution and carry-over			−2.54	−7.34	+.550	+1.90		
7	M			+9.45	−8.74	+11.64	−12.61		
6	$-F\,R\,M_F$			+1.60	−1.37	+1.82	−2.14		
5	$-R\,M_F$			−4.15	+4.63	−6.18	+5.53		
4	M_F			+12	−12	+16	−16		
3	R	0.431	0.344	0.346	0.386	0.386	0.346	0.344	0.431
2	J	18.15	17.73	17.95	18.14	18.14	17.95	17.73	18.15
1	F	0.347	0.308	0.329	0.346	0.346	0.329	0.308	0.347


4 Kips
24 Ft
12 Ft
12 Ft
24 Ft
24 Ft
8 Kips Total
A
C
E
G
I
B
D
F
H
J
12 Ft
K=18
K=15
K=12
K=15
K=8
K=15
K=12
K=15
K=18
+2.16
+1.04
+1.12
0.569
24.0
0.500
+2.14
+1.58
+0.56
0
20.6
0.251
−2.10
+1.17
−3.27
0.310
16.00
0.500
−0.34
+1.29
−1.63
0
14.76
0.374
−0.26
+0.80
−1.06
0.228
10.67
0.500
+0.34
+0.87
−0.53
0
10.06
0.410
+6.25
+1.17
+5.08
0.310
16.00
0.500
+3.83
+1.29
+2.54
0
14.76
0.374
−0.70
+1.04
−1.74
0.569
24.0
0.500
+0.71
+1.58
−0.87
0
20.6
0.251
Quantity
R
J
F
Line
11
10
9
3
2
1

Because of symmetry, only half of the frame is shown in Fig. 31. All distributions in line 4 are completed before starting line 5. The first entry in line 5 is the carry-over to point C from point A. This carry-over is distributed, giving $M_{CE} = +2.85$. The carry-over to point E is based on the sum of the moments +2.85 and −8.33 at point C. The carry-over from point E to point F is recorded in line 6 at point C in span CE. The process continues until joint B is reached. As in Table 24, no distributions are recorded for the intermediate columns. The moment M_{CD} in line 9 is calculated from equilibrium of joint C, and the total distribution, +3.37, is the difference between lines 3 and 9. The carry-over to point D can then be found, and the final moment M_{DC} determined. No explanation of the final steps, lines 10 and 11, Table 24, is necessary.

The writer has found the procedure described especially simple and easily remembered for continuous beams. It can be used very effectively in the construction of influence lines, and the advantage of basing the procedure on fixed-end moments becomes evident in this application. For example, the M_F-values of line 4, and the final moments of line 7, Table 22, can be quickly determined for a sufficient number of positions of load between point B and point C to plot the part BC of the influence lines for moment at point B and point C. Parts BC of the influence lines for M_A and M_D can then be found by multiplying ordinates of the M_B and M_C influence lines by F_{BA} and F_{CD}, respectively. As an alternative procedure, the equations of the influence lines for M_B and M_C may be found. For example:

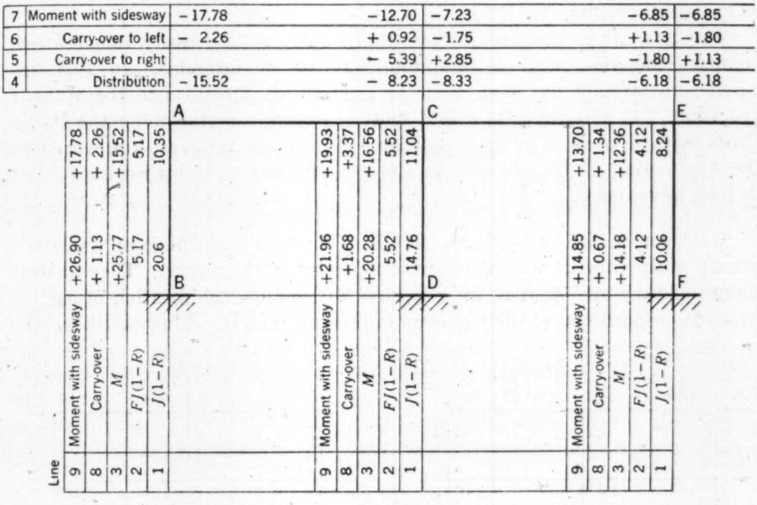


FIG. 31

		A	C	C	E	
7	Moment with sidesway	− 17.78	− 12.70	− 7.23	− 6.85	− 6.85
6	Carry-over to left	− 2.26	+ 0.92	− 1.75	+ 1.13	− 1.80
5	Carry-over to right		− 5.39	+ 2.85	− 1.80	+ 1.13
4	Distribution	− 15.52	− 8.23	− 8.33	− 6.18	− 6.18

Line		A	B	C	D	E	F
9	Moment with sidesway	+17.78	+26.90	+19.93	+21.96	+13.70	+14.85
8	Carry-over	+ 2.26	+ 1.13	+3.37	+1.68	+ 1.34	+ 0.67
3	M	+15.52	+25.77	+16.56	+20.28	+12.36	+14.18
2	$FJ(1-R)$	5.17	5.17	5.52	5.52	4.12	4.12
1	$J(1-R)$	10.35	20.6	11.04	14.76	8.24	10.06

$$M_{FBC} = 24\,k\,(1 - k)^2 \quad \text{.................... (97a)}$$

and

$$M_{FCB} = 24\,k^2\,(1 - k) \quad \text{.................... (97b)}$$

in which $k = x/L$, and x is the abscissa of the unit load position measured from point B. Following the calculations of lines 5, 6, and 7, Table 22, the final moments are

$$M_{BC} = 9.46\,k\,(1 - k)\,(1.36 - k) \quad \text{.............. (98a)}$$

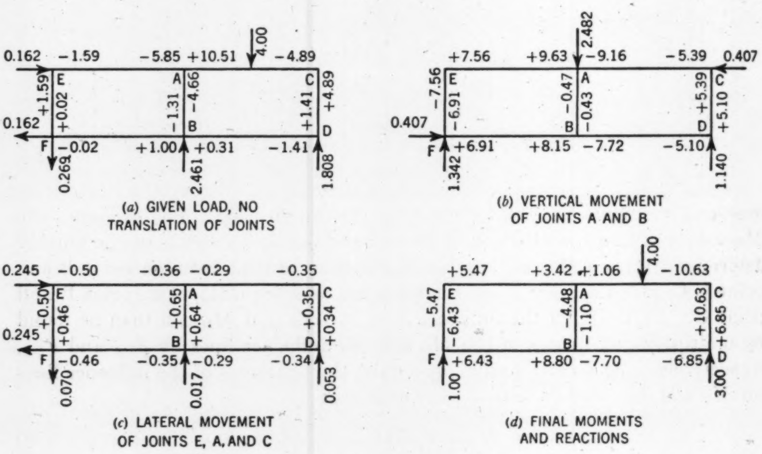
and

$$M_{CB} = 8.45\,k\,(1 - k)\,(0.406 + k) \quad \text{............. (98b)}$$

From Eqs. 98 any desired number of influence line ordinates can be easily calculated. The maximum ordinate of the BC-segment can also be determined readily. Equating to zero the first derivatives with respect to k, it is found that $k = 0.381$ and $k = 0.616$ for maximum values of M_{BC} and M_{CB}, respectively. It should be observed that this also determines maximum ordinates of the BC-segment of the influence lines for M_A and M_D, since these moments are found by multiplying M_B and M_C by the proper carry-over factors, as explained previously.

The practicing engineer prefers (and rightly so) an easily remembered, compact analysis, with a minimum number of formulas. In this discussion an attempt has been made to meet these objections to the authors' procedure. It is quite certain that most engineers will still prefer the Hardy Cross method because of its simplicity. The writer believes, however, that the application of the method to the construction of influence lines offers definite advantages.

William A. Conwell,[58] M. ASCE.—There is something of analytical beauty in the concept which the authors present in this paper. They analyze frames as they are—that is, as structures with joints, the rotation of each of which is proportional to the moments impressed upon it. Many of those who are daily engaged in analysis of continuous frames will appreciate this sophisticated outlook and will, henceforth, be inclined to treat as naive any method which does not treat structures as they actually are.


4.00
0.162
−1.59
−5.85
+10.51
−4.89
+1.59
+0.02
−1.31
−4.66
+1.41
+4.89
0.162
E
A
C
B
D
F
−0.02
+1.00
+0.31
−1.41
0.269
2.461
1.808
2.482
+7.56
+9.63
−9.16
−5.39
0.407
−7.56
−6.91
−0.47
−0.43
+5.39
+5.10
0.407
+6.91
+8.15
−7.72
−5.10
1.342
1.140
0.245
−0.50
−0.36
−0.29
−0.35
+0.50
+0.46
+0.65
+0.64
+0.35
+0.34
0.245
−0.46
−0.35
−0.29
−0.34
0.070
0.017
0.053
4.00
+5.47
+3.42
+1.06
−10.63
−5.47
−6.43
−4.48
−1.10
+10.63
+6.85
+6.43
+8.80
−7.70
−6.85
1.00
3.00

(*a*) GIVEN LOAD, NO TRANSLATION OF JOINTS

(*b*) VERTICAL MOVEMENT OF JOINTS A AND B

(*c*) LATERAL MOVEMENT OF JOINTS E, A, AND C

(*d*) FINAL MOMENTS AND REACTIONS

Fig. 32.—Steps in Solution of Moments and Reactions of Example 5 by Moment Distribution

Although the proposed method of solution involves thought that is closer to the actual condition of the structure than do accepted methods, it also contains formulas which must be remembered to be applied and entails computation which cannot be wholly written on a sketch of the structure. These elements divert the mind of the designer from the structure and are, of course, ones which moment distribution[8] was to eliminate.

---

[58] Gen. Engr., Structural Eng. and Design Dept., Duquesne Light Co., Pittsburgh, Pa.

It is believed that the relative merits of the methods of elastic supports and of moment distribution can best be compared by an application to a particular problem. With the thought of providing such a pointed comparison, the writer has solved Example 5 by moment distribution and gives the results of this analysis in Fig. 32. These values were obtained after an average of seven cycles of moment distribution, which is not a great chore. It was necessary, of course, to solve two equations simultaneously to determine the proportion of the final moments resulting from sidesway and the vertical movement of joints A and B. Aside from this disgression, it was possible to write all computations on a sketch of the frame. Assessment of the relative value of the methods can then be made by comparison of Fig. 32 with seven cycles of moment distribution visualized thereon and Tables 8 and 9 with their supplementary computations.

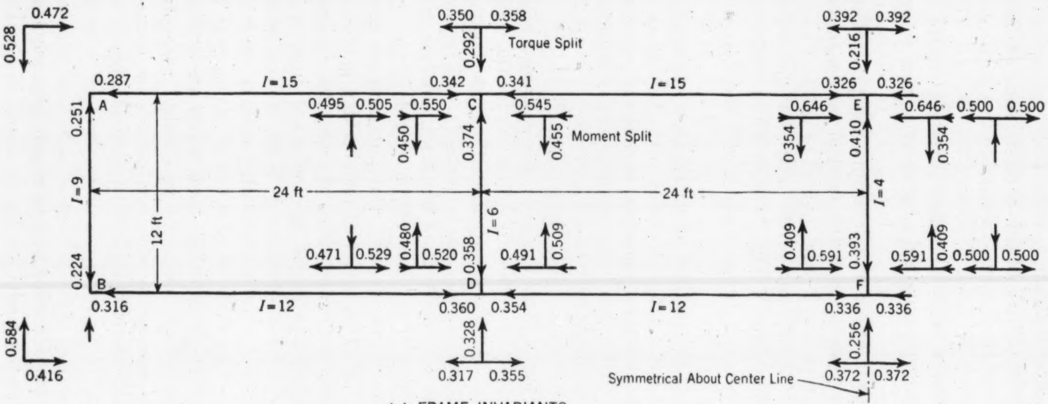
In the final analysis, the value of the method of elastic supports will be determined in design offices where it will meet competition with other present and proposed methods. One's first impression, however, is that the authors' presentation is one for the expert in continuous frame analysis, rather than for the engineer who meets such design problems only occasionally.

Harry Garfinkel,[59] Esq.—For many years the writer has examined with unabated interest the methods proposed by various investigators for the solution of stresses in continuous frames. All have concentrated on reducing the time element; many have accomplished this purpose remarkably well for the simpler types of frame. In the more complex type of structure, however, the amount of work still required for solution of an actual engineering design leaves much to be desired, both as to method and clarity. Computations for such structures can be quite lengthy and must be reviewed for checking purposes. Complex sign conventions and numerous distributions required for varying loading patterns can cause great annoyance unless the method of analysis is direct and avoids unnecessary artifices for a solution. It is therefore of great importance that any new method should avoid these difficulties as far as possible.

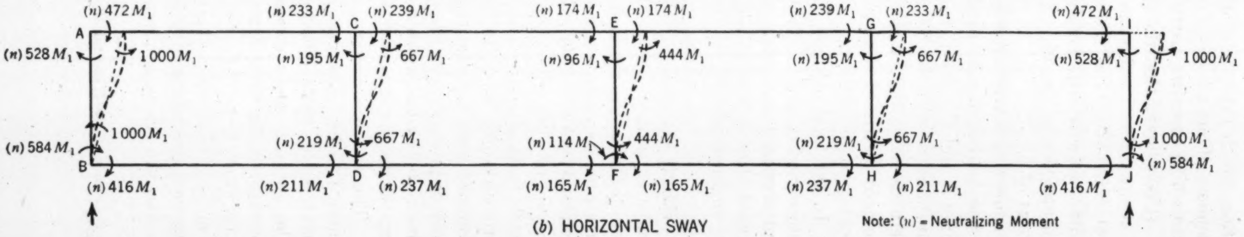
The authors propose a method of analysis of continuous frames that is claimed to be "classical" and "exact." Presumably, these terms are applied because the basic equations of the method originate in classical concepts. The authors' method and steps of operation, however, cannot be termed "exact" in the same sense that such a term is applied to methods, based on least work, slope deflection, or virtual work, which are direct and lead to exact results. Inherent in the authors' method is the artifice of fixing and relaxing one or more joints of the frame. Simultaneous equations are then required to find the releasing moments, which must additionally be distributed to the

---

[59] Civ. Engr., Board of Transportation, City of New York, New York, N. Y. (Mr. Garfinkel died on September 17, 1947, and this discussion was prepared for publication by Nathan D. Brodkin, Assoc. M. ASCE.)

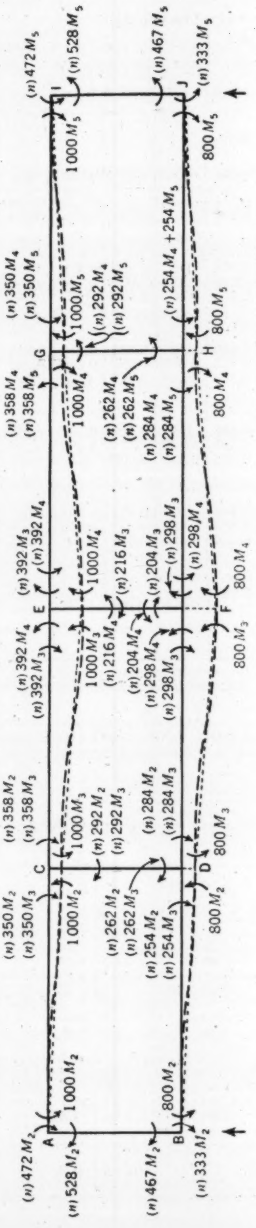

0.472
0.528
0.350 0.358
0.292
Torque Split
0.392 0.392
0.216
0.287
I = 15
0.342 0.341
I = 15
0.326 0.326
A
C
E
0.251
0.495 0.505
0.550
0.545
0.646
0.646
0.500 0.500
0.450
0.374
0.455
Moment Split
0.354
0.410
0.354
I = 9
24 ft
24 ft
I = 4
12 ft
I = 6
0.480
0.509
0.409
0.409
0.471 0.529
0.520
0.358
0.491
0.591
0.393
0.591
0.500 0.500
0.224
B
D
F
0.316
I = 12
0.360 0.354
I = 12
0.336 0.336
0.584
0.416
0.328
0.317 0.355
0.256
0.372 0.372
Symmetrical About Center Line

(*a*) FRAME INVARIANTS


A
C
E
G
I
B
D
F
H
J
(n) 472 M₁
(n) 528 M₁
1000 M₁
1000 M₁
(n) 584 M₁
(n) 416 M₁
(n) 233 M₁
(n) 239 M₁
(n) 195 M₁
667 M₁
(n) 219 M₁
667 M₁
(n) 211 M₁
(n) 237 M₁
(n) 174 M₁
(n) 174 M₁
(n) 96 M₁
444 M₁
(n) 114 M₁
444 M₁
(n) 165 M₁
(n) 165 M₁
(n) 239 M₁
(n) 233 M₁
(n) 195 M₁
667 M₁
(n) 219 M₁
667 M₁
(n) 237 M₁
(n) 211 M₁
(n) 472 M₁
(n) 528 M₁
1000 M₁
1000 M₁
(n) 584 M₁
(n) 416 M₁

(*b*) HORIZONTAL SWAY

Note: (*n*) = Neutralizing Moment


A
B
C
D
E
F
G
H
I
J
(n) 472 M_2
(n) 528 M_2
(n) 467 M_2
(n) 333 M_2
1000 M_2
800 M_2
(n) 350 M_2
(n) 350 M_3
(n) 262 M_2
(n) 262 M_3
(n) 254 M_2
(n) 254 M_3
(n) 358 M_2
(n) 358 M_3
1000 M_3
(n) 292 M_2
(n) 292 M_3
(n) 284 M_2
(n) 284 M_3
800 M_3
(n) 392 M_4
(n) 392 M_3
1000 M_3
(n) 216 M_4
(n) 204 M_4
(n) 298 M_4
(n) 298 M_3
(n) 392 M_3
(n) 392 M_4
1000 M_4
(n) 216 M_3
(n) 204 M_3
(n) 298 M_3
(n) 298 M_4
800 M_4
(n) 358 M_4
(n) 358 M_5
1000 M_4
(n) 262 M_4
(n) 262 M_5
(n) 284 M_4
(n) 284 M_5
800 M_4
(n) 350 M_4
(n) 350 M_5
1000 M_5
(n) 292 M_4
(n) 292 M_5
(n) 254 M_4 + 254 M_5
800 M_5
(n) 472 M_5
(n) 528 M_5
(n) 467 M_5
(n) 333 M_5

Note: (n) = Neutralizing Moment

(c) VERTICAL SWAY

FIG. 33.—ARTIFICIAL AND NEUTRALIZING MOMENTS FOR SWAY IN FRAME OF EXAMPLE 6

frame, resulting in little if any saving in time or energy over presently known moment-distribution methods. The authors have attempted the solution of the problem by first calculating approximate values of the frame invariants and then, with the aid of these, distributing moments caused by the loading.

Criticism must be leveled at the manner in which the authors determine these invariants. Introduction of the elastic length concept needlessly complicates the formulas and results in no actual benefit. In re-entrant or completely enclosed frames, such as Vierendeel trusses, the problem of finding the approximate carry-over factors or transmission coefficients by temporarily fixing joints against rotation is unnecessary and entails needless work in distributing moments caused by releasing the joints.

In laying the theoretical basis for their method, the authors have not carried their work through to the logical conclusion of finding all the invariants of the frame. Since these quantities are independent of the manner of loading, they can be determined completely and exactly for any given frame, and it is not necessary to resort to fixing one or more joints against rotation.

The foregoing problem has been completely solved for uniform, nonuniform, and curved members by R. C. Brumfield, M. ASCE, who has shown[56,60] how the complete invariants for a frame, whether re-entrant or not, may be obtained quickly, exactly, and simply, and then with the aid of these, how to distribute in one cycle the moments caused by the loads.

The three major invariants of a continuous frame are:

1. A transmission coefficient (carry-over factor), which is that fraction of the moment arriving at one end of a beam as

---

[60] "Restrained Beams and Rigid Frames," by R. C. Brumfield, 1941 (printed in mimeograph form by The Cooper Union, New York, N. Y.).

TABLE 25.—FRAME INVARIANTS (CHARACTERISTICS) FOR EXAMPLE 6 (FIG. 12)[a]

Joint	Member	Moment of inertia, I (in.^{4})	Length, L (ft)	K $\left(=\frac{I}{L}\right)$	$2-C$	i' $\left(=\frac{K}{2-C}\right)$	R'	COEFFICIENT		MOMENT SPLIT			Torque split
								Symbol	Value				
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
A	AB	9	12	0.750	1.776	0.422	0.377	C_{BA}	0.251	BA	1	....	0.528
	AC	15	24	0.625	1.659[b]	0.377	0.422	C_{CA}	0.287	1	CA	....	0.472
						0.799[c]							
C	CD	6	12	0.500	1.642	0.305	0.738	C_{DC}	0.374	DC	0.450	0.455	0.292
	CA	15	24	0.625	1.713	0.365	0.678	C_{AC}	0.342	0.495	AC	0.545	0.350
	CE	15	24	0.625	1.674[b]	0.373	0.670	C_{EC}	0.341	0.505	0.550	EC	0.358
						1.043[c]							
E	EF	4	12	0.333	1.607	0.207	0.754	C_{FE}	0.410	FE	0.354	0.354	0.216
	EC	15	24	0.625	1.659	0.377	0.584	C_{CE}	0.326	0.500	CE	0.646	0.392
	EG	15	24	0.625	1.659	0.377	0.584	C_{GE}	0.326	0.500	0.646	GE	0.392
						0.961[c]							
F	FD	12	24	0.500	1.646	0.304	0.514	C_{DF}	0.336	DF	0.500	0.591	0.372
	FE	4	12	0.333	1.590	0.210	0.608	C_{EF}	0.393	0.409	EF	0.409	0.256
	FH	12	24	0.500	1.646	0.304	0.514	C_{HF}	0.336	0.591	0.500	HF	0.372
						0.818[c]							
D	DB	12	24	0.500	1.684	0.297	0.641	C_{BD}	0.360	BD	0.471	0.491	0.317
	DC	6	12	0.500	1.626	0.308	0.630	C_{CD}	0.358	0.480	CD	0.509	0.328
	DF	12	24	0.500	1.666[b]	0.333	0.605	C_{FD}	0.354	0.520	0.529	FD	0.355
						0.938[c]							
B	BA	9	12	0.750	1.749	0.429	0.304	C_{AB}	0.224	AB	1	....	0.584
	BD	12	24	0.500	1.646[b]	0.304	0.429	C_{DB}	0.316	1	DB	....	0.416
						0.733[c]							

[a] Also Fig. 33(a). [b] Estimated value. [c] Value of R'_T for respective joint.

TABLE 26.—TABULATION AND DISTRIBUTION

Line	Symbol	Description	JOINT A				JOINT C					
			← M_{AB}	→	← M_{AC}	→	← M_{CA}	→	← M_{CD}	→	← M_{CE}	→
(1)	(2)	(3)	(4)		(5)		(6)		(7)		(8)	
1	$\left\{\begin{matrix}M_{CE}\\M_{EC}\end{matrix}\right\}$	Primary distribution	1.46	0.06	0.06	1.46		5.10	0.04	4.24	9.32	0.02
2	$\left\{\begin{matrix}M_{EG}\\M_{GE}\end{matrix}\right\}$	Primary distribution		0.40	0.40		1.41		1.16	0.05	0.03	2.55
3	ΣM_0	No joint translation	1.00			1.00		3.69		3.09	6.78	
4	ΣM_1	Artificial moments plus distributed neutralizing moments	392			392		298	491			193
5	ΣM_2			540	540		565			317		248
6	ΣM_3		103			103		250		314	564	
7	ΣM_4			5	5			48		58	106	
8	ΣM_5			4	4		5		11			16
9	M_1	Line 4 × − 0.002423		0.95	0.95		0.72			1.19	0.47	
10	M_2	Line 5 × 0.081047		43.77	43.77		45.79			25.69		20.10
11	M_3	Line 6 × 0.057996	5.97			5.97		14.50		18.21	32.71	
12	M_4	Line 7 × 0.045476		0.23	0.23			2.18		2.64	4.82	
13	M_5	Line 8 × 0.093567		0.37	0.37		0.47		1.03			1.50
14	ΣM	Final moment (line 3 + lines 9 to 13)		38.35	38.35		26.61			49.79	23.18	

a consequence of a unit moment applied at the other end (every member having two transmission coefficients);

2. A moment split, which is that fraction by which moment arriving at a joint through one of the radiating members of the joint is to be multiplied to give the induced moment in the member to which the moment split applies; and

3. A torque split, which is that fraction by which torque applied to a joint is to be multiplied to give the induced torque (or moment) in the member to which it applies.

Once the exact transmission coefficients have been determined, the bending moments at the ends of a loaded member may be found directly in terms of the fixed-end moments and the transmission coefficients.[61]

Comparisons are invidious, and particular problems may be solved more readily by one method than by another. Nevertheless, the writer would like to indicate how simply the invariants, especially those for closed or Vierendeel frames, may be found without arbitrarily fixing any joints, and then with the aid of the three major invariants to complete the solution of moment distribution for Example 6 as a comparison with the authors' method.

The invariants are defined as follows:

$$K = \frac{I}{L} \qquad (99)$$

C_{AB} and C_{BA} are transmission coefficients from A toward B and vice versa; m_{AB} and t_{AB} are the moment split and torque split, respectively, for member AB at joint A; i', the stiffness index, is a measure of the resistance which a member, together with the restraints applied to it at one end, offers to the rotation of

---

[61] *Transactions*, ASCE, Vol. 111, 1946, p. 838.

OF MOMENTS, EXAMPLE 6 (FIG. 33)

JOINT E						JOINT G						JOINT I			
← M_{EC} → (9)		← M_{EF} → (10)		← M_{EG} → (11)		← M_{GE} → (12)		← M_{GH} → (13)		← M_{GI} → (14)		← M_{IG} → (15)		← M_{IJ} → (16)	
0.02	8.68	3.07	0.04	5.63		1.91	0.02	0.04	0.87		1.06		0.30	0.30	
	7.51	0.06	4.10	11.58	0.03	0.03	12.43	5.66	0.05	6.79		1.94	0.08	0.08	1.94
	16.17		1.01	17.18			10.51	4.78		5.73		1.56			1.56
	178	356			178		193	491			298		392	392	
	87	36		51		16			11		5		4	4	
542			251		291		106	58		48			5	5	
291		251			542		564	314		250		103			103
	51		36	87		248		317			565		540	540	
0.43			0.86	0.43		0.47			1.19	0.72		0.95			0.95
	7.05	2.92		4.13		1.30			0.89		0.41		0.32	0.32	
31.43			14.56		16.88		6.15	3.36		2.79			0.29	0.29	
13.23		11.41			24.65		25.65	14.28		11.37		4.68			4.68
	4.77		3.37	8.14		23.20		29 66			52.86		50.53	50.53	
17.10			5.47		11.65		17.34	50.00			32.66		43.95	43.95	

TABLE 26.—

Line	Symbol	Joint B				Joint D							
		← M_{BA} →		← M_{BD} →		← M_{DB} →		← M_{DC} →		← M_{DF} →		← M_{FD} →	
(1)	(2)	(17)		(18)		(19)		(20)		(21)		(22)	
1	$\begin{Bmatrix} M_{CE} \\ M_{EC} \end{Bmatrix}$	0.33	0.26	0.26	0.33	0.82	0.12	17	1.52	0.86	0.21	0.27	0.63
2	$\begin{Bmatrix} M_{EG} \\ M_{GE} \end{Bmatrix}$	0.10	0.09	0.09	0.10		0.37	0.42	0.18	0.35	0.22	1.01	0.07
3	ΣM_0	0.08			0.08	0.33			1.11	0.78		0.58	
4	ΣM_1	362			362		281	480			199		175
5	ΣM_2		497	497		502			301		201		72
6	ΣM_3	74			74		177		298	475		457	
7	ΣM_4	7			7		31		55	86		218	
8	ΣM_5		3	3		3		8			11		35
9	M_1		0.88	0.88		0.68			1.16	0.48		0.42	
10	M_2		40.28	40.28		40.69			24.40		16.29		5.84
11	M_3	4.29			4.29		10.27		17.28	27.55		26.50	
12	M_4	0.32			0.32		1.41		2.50	3.91		9.91	
13	M_5		0.28	0.28		0.28		0.75			1.03		3.28
14	ΣM		36.75	36.75		30.30			45.70	15.40		28.29	

the joint to which it attaches at the opposite end, or

$$i' = \frac{K}{2 - C} \qquad \text{(100)}$$

R', the restraint index, is a measure of the resistance to rotation offered to a member, when it is used to rotate a joint, by the other members radiating from the joint, or

$$R' = i'_1 + i'_2 + i'_3 + \cdots + i'_n \qquad \text{(101)}$$

and R'_T, the restraint index in torsion, is a measure of the resistance to rotation offered to a member by all the members radiating from a joint, or

$$R'_T = i'_0 + i'_1 + i'_2 + i'_3 + \cdots + i'_n \qquad \text{(102)}$$

The primes used in Eqs. 100, 101, and 102 indicate the reciprocals of the values introduced by Professor Brumfield.[56,60] For beams of uniform section, the value of the transmission coefficient is given by

$$C = \frac{1}{2 + \dfrac{K}{R'}} \qquad \text{(103)}$$

for moment split, by

$$m_n = \frac{i'_n}{R'} \qquad \text{(104)}$$

and for torque split, by

$$t_n = \frac{i'_n}{R'_T} \qquad \text{(105)}$$

Calculation of the invariants is a slide rule computation and may be kept in tabular form. For Example 6, the invariants are as shown in Table 25.

(Continued)

JOINT F				JOINT H						JOINT J			
← M_{FE} → (23)		← M_{FH} → (24)		← M_{HF} → (25)		← M_{HG} → (26)		← M_{HJ} → (27)		← M_{JH} → (28)		← M_{JI} → (29)	
1.20	0.13	0.05	0.76	0.16	0.27	0.14	0.31	0.28		0.08	0.07	0.07	0.08
0.18	1.60	0.84	0.36	0.28	1.15	2.03	0.21	0.15	1.10	0.43	0.34	0.34	0.43
	0.35		0.23		0.98	1.65			0.67	0.10			0.10
350			175		199	480			281		362	362	
37		35		11			8		3		3	3	
	239		218		86	55		31		7			7
239			457		475	298		177		74			74
	37	72		201		301			502		497	497	
	0.85	0.43		0.48			1.16	0.68		0.88			0.88
3.00		2.84		0.89			0.65		0.24		0.24	0.24	
	13.86		12.64		4.99	3.19		1.80		0.41			0.41
10.87			20.78		21.60	13.55		8.05		3.37			3.37
	3.46	6.74		18.81		28.16			46.97		46.50	46.50	
	4.65		23.64		7.39	44.74			37.35		41.99	41.99	

For convenience the invariants are also placed on the frame diagram in Fig. 33(a).

If the concept of right and left arrows as shown in Table 26 is used, the chances for errors in sign are obviated. Normal bending moment, indicated by an arrow to the right, is clockwise moment applied to the beam when taken as a free body. Reverse bending moment, indicated by an arrow to the left, is a counterclockwise moment. For a downward load the moment at the right end is normal and at the left end, reverse. The moments are distributed to the frame in accordance with the frame invariants. After going through a joint, the moment changes in character. When going from one end of a beam to the other, moment remains the same in character. If only two members radiate from a joint, the moment split at that joint is unity; if more than two members occur at a joint, the moment splits would, of course, be fractional. In order to determine the transmission coefficients, it is not necessary to assume fixed joints as the authors have done. Much simpler is the method of estimating one of the coefficients and using it to compute another, which, as a result of the rapidly converging nature of Eq. 103, will be correct within one or two thousandths.

It is known that, for an elastically restrained member of uniform section, the transmission coefficient lies between 0 and 0.500, for free ends and fixed ends, respectively. For a series of equal spans having equal moments of inertia, it can be shown[62] that $C = 0.268$. Eq. 103 may be written in the form,

$$C_{AB} = \frac{1}{2 + \frac{K_{AB}}{R'_{AB}}} \qquad \text{(106)}$$

---

[62] "Restrained Beams and Rigid Frames," by R. C. Brumfield, 1941 (printed in mimeograph form by The Cooper Union, New York, N. Y.), p. 9.

and since

$$R'_{AB} = i'_{BD} = \frac{K_{BD}}{2 - C_{BD}} \tag{107}$$

for the two member joint, then

$$C_{AB} = \frac{1}{2 + \dfrac{K_{AB}}{\left(\dfrac{K_{BD}}{2 - C_{BD}}\right)}} \tag{108}$$

which is the form used in estimating C_{AB}. If member BD connects into a stiffer member, the coefficient will be greater than 0.268, and conversely.

Although the Vierendeel truss of Example 6 is symmetrical in form, it is unsymmetrically loaded. To estimate certain of the transmission coefficients (carry-over factors) of a re-entrant frame of this type, an average value must be estimated for $i' = \sum \frac{K_n}{2 - C_n}$ for each of the members radiating from the joint, except the member for which the transmission coefficient is being estimated. For example, to estimate C_{AC} (Fig. 33(*a*)), $C_{CE} > 0.268$, say, 0.300, and $C_{CD} > 0.268$, say, 0.350. Then,

$$C_{AC} = \frac{1}{2 + \dfrac{K}{R'}} = \frac{1}{2 + \dfrac{K_{AC}}{\dfrac{K_{CE}}{2 - C_{CE}} + \dfrac{K_{CD}}{2 - C_{CD}}}} = \frac{1}{2 + \dfrac{0.625}{\dfrac{0.625}{2 - 0.300} + \dfrac{0.500}{2 - 0.350}}}$$

$= 0.341$. If an average value for $2 - C_n$ is assumed, the general formula becomes

$$C_{AC} = \frac{1}{2 + \dfrac{K\,(2 - C_{avg})}{K_1 + K_2 + K_3 + \cdots + K_n}} \tag{109}$$

The value of C_{avg} is not critical. Thus, for $C_{avg} = 0.250$, $C_{AC} = 0.336$; for $C_{avg} = 0.300$, $C_{AC} + 0.340$; and for $C_{avg} = 0.350$, $C_{AC} = 0.343$. An average value of C_n based on a general mental estimate of the coefficients, in this case 0.350, is used. Therefore, $C_{CE} = \dfrac{1}{2 + \dfrac{1.650 \times 0.625}{0.625 + 0.333}} = 0.326.$ Similarly, $C_{BD} = 0.354$ and $C_{DF} = 0.334$. These coefficients give starting points, so that Table 25 becomes routine procedure, advantage being taken of frame symmetry.

The primary bending moments for the given loading without the effects of sway or vertical displacements are obtained from the following equations:[61]

$$M_R = A_R\,[(2\,M_{FR} + M_{FL}) - C_L\,(2\,M_{FL} + M_{FR})] \tag{110a}$$

$$M_L = A_L\,[(2\,M_{FL} + M_{FR}) - C_R\,(2\,M_{FR} + M_{FL})] \tag{110b}$$

$$A_R = \frac{C_R}{1 - C_L\,C_R} \tag{111a}$$

and

$$A_L = \frac{C_L}{1 - C_L C_R} \qquad \text{(111}b\text{)}$$

in which C_R and C_L are the transmission coefficients; M_{FR} and M_{FL} are the fixed-end moments; and M_R and M_L are the restraining moments at the right and left beam ends, respectively.

For span CE: $C_R = 0.326$, $C_L = 0.341$, $A_R = 0.366$, $A_L = 0.384$, $M_{EC} = M_R = 0.366 \times 0.659 \times \frac{3}{8} \times 4 \times 24 = -8.68$ ft-kips, and $M_{CE} = M_L = 0.384 \times 0.674 \times \frac{3}{8} \times 4 \times 24 = -9.32$ ft-kips. For span EG: $C_R = 0.341$, $C_L = 0.326$, $A_R = 0.384$, $A_L = 0.366$, $M_{GE} = M_R = -0.384 \times 0.674 \times \frac{1}{4} \times 8 - 24 = -12.43$ ft-kips, and $M_{EG} = M_L = -0.366 \times 0.659 \times \frac{1}{4} \times 8 \times 24 = -11.58$ ft-kips. These primary moments have been distributed in Table 26, lines 1 and 2, and totaled in line 3. Moments have been distributed through three joints from the joint of origin.

The moments caused by the horizontal sway and vertical displacements must now be determined. Since dissymmetry of loading exists, the frame will tend to lurch sideway. It is allowed to sway laterally and the artificial moments required to rotate the tangents at the column ends parallel to their original direction are shown in Fig. 33(*b*). Neutralizing moments at each joint necessary to balance the artificial moments are apportioned among the radiating members in accordance with the joint torque split obtained from Fig. 33(*a*), or Table 25, and are designated (*n*) in Fig. 33(*b*). In a similar manner, artificial and neutralizing moments are found for vertical displacement as shown in Fig. 33(*c*). The neutralizing moments are distributed to the frame in accordance with the frame invariants given in Table 25 and Fig. 33(*a*). The summations of the distributed neutralizing moments, together with the artificial moments, are given in lines 4 to 8, Table 26. Since there are five unknown moments of translation, five equations must be written and solved simultaneously. The coefficients of M_0 to M_5, forming these equations, are obtained from Table 26 as follows:

For the first equation, the sum of the horizontal shears in the posts AB, CD, EF, GH, and IJ in terms of M_0 to M'_5 is set equal to zero. Designating normal moments as plus and reverse moments as minus,

$$-4{,}156\, M_1 + 1{,}594\, M_2 + 814\, M_3 - 814\, M_4 - 1{,}594\, M_5 - 0.30 = 0 \qquad (112)$$

For the second equation, the internal shear at A and B in members AC and BD is set equal to the left reaction, $R_L = 5.5$ kips, so that

$$+1{,}333\, M_1 - 2{,}104\, M_2 + 604\, M_3 + 81\, M_4 - 15\, M_5 + 4.44 = -5.5 \times 24 \qquad (113)$$

For the third equation, the internal shear to the right of C and D in members CE and DF is set equal to the left reaction. Thus,

$$+745\, M_1 + 608\, M_2 - 2{,}038\, M_3 - 701\, M_4 + 113\, M_5 + 8.03 = (-5.5 + 2)\, 24 \qquad (114)$$

For the fourth equation, the internal shear at I and J in members GI and HJ is set equal to the right reaction, $R_R = 6.5$ kips, giving

$$+ 1{,}333\, M_1 + 15\, M_2 - 81\, M_3 - 604\, M_4 + 2{,}104\, M_5 - 6.71 = 5 \times 24 \ldots (115)$$

The fifth equation is obtained from the vertical displacement diagram (Fig. 33(*c*)). The sum of the deflections of AC and CE must be equal to the sum of the deflections of IG and GE, which can be expressed in terms of M_2 to M_5. Thus,

$$M_2 + M_3 = M_4 + M_5 \ldots\ldots\ldots\ldots\ldots\ldots (116)$$

Solving Eqs. 112 to 116, simultaneously, gives $M_1 = - 0.002423$, $M_2 = + 0.081047$, $M_3 = + 0.057996$, $M_4 = + 0.045476$, and $M_5 = + 0.093567$.

In step (*h*) of Example 6, the authors have set up four equations for vertical shear correction factors. Nowhere, however, does a similar equation or correction factor appear for the horizontal shear or horizontal movement of the top chord. The effects of both horizontal sway and vertical displacement of the joints must be considered as simultaneous actions in determining correction factors, since these factors for vertical and horizontal shear are interrelated. Such correction (for horizontal sway) is not obvious from the authors' calculations in Examples 4, 5, and 6.

The values of M_1 to M_5 found from Eqs. 112 to 116 are multiplied by the respective coefficients in lines 4 to 8, Table 26, to give lines 9 to 13, and are added to the primary moments due to loading in line 3 to give the final moments in line 14. These are to be compared with the authors' values (line 25, Table 13).

In the "Summary," the authors claim their method to be particularly adapted to the solution of Vierendeel trusses. Nevertheless, they have carried the solution of the truss of Example 6 only to the point of finding primary moments corrected for joint translation, despite the fact that a Vierendeel truss of several panels will develop secondary moments due to changes in member lengths caused by the axial stresses. These secondary moments are considerable and cannot be ignored. When the calculations for secondary moments are added to the work required by the authors' method for finding primary moments, it becomes doubtful whether any advantage is to be gained by adopting the proposed system of analysis.

It has been shown elsewhere[63] how the method as used by the writer to find the primary moments has been extended to solve the secondary stresses for this type of truss.

J. Charles Rathbun,[64] M. ASCE, and C. W. Cunningham,[65] Assoc. M. ASCE.—The large number of discussions of the paper is gratifying to the writers, as is also the fact that many discussers have made distinct contributions to the problem of frame analysis. In some of the discussions other methods of solving the same problem are presented by those who prefer them to the one presented in the paper. Although such citations are not pertinent to the subject, they do, in most cases, contribute to the problem of the solution of the rigid frame by calling attention to these methods and are, therefore, of value.

---

[63] *Transactions*, ASCE, Vol. 111, 1946, p. 848.

[64] Prof., Civ. Eng., College of the City of New York, New York, N. Y.

[65] Associate Prof., Civ. Eng., College of the City of New York, New York, N. Y.

The first paragraph of the "Synopsis" states clearly that the moment-distribution method is entirely distinct from the one presented. The words "classical" and "exact" have been used to show that the method is not one of successive approximations. The writers' use of the word classical is probably a survival of Newton's day, when the method of successive approximations was distinct from the classical methods (the Greek and Roman traditional methods).

An exact method, as used in the paper, is one which derives an exact solution from the data given. A method of successive approximations cannot be exact, in general, although it may be correct or close enough to the exact solution to satisfy the requirements. As an example, π has been computed to more than seven hundred decimal places, but this is not an exact value. Neither classical nor exact is used here in a sense complementary, or otherwise, to any method.

Of the seventeen discussions nine compare this method with the well-known moment-distribution method. These discussions are of little value because they do not apply to the method of the paper or contribute to the subject. Moment distribution is a most valuable method when the answer to a specific problem is desired, but many engineers are willing to consider that it furnishes all that is required for a solution, and they refuse to consider attempts to improve conditions.

Mr. Beaufoy furnishes an example of the trial-and-error solution in Fig. 15, wherein he compares this method with that of the paper. In Fig. 15 there is a self-consistent group of values, but moment EG is in error by 3.67%, or an amount equal to 0.425, and the error in moment GH is equal to 0.698. In his discussion, Mr. Beaufoy does not indicate a way to estimate the amount of these errors. In many problems one may obtain a solution by an approximate method with fewer computations and in less time than by an exact method, but does that constitute a solution to the general problem? The writers think not, and, although they admire this method for practical solutions, they consider it to have no place in a discussion of the paper. This same principle could have been applied in solving for the moments at the fixed joints, since the convergence of the series of trials will usually be very rapid. As this is quite obvious, the writers purposely omitted it from the paper.

The method of Otto Gottschalk is based on unloaded models. It is, therefore, useful only as a check on any analytical solution.

Messrs. Barron and Garfinkel advocate the use of transmission coefficients; this method requires a preliminary assumption for the value of a transmission coefficient. Although the assumed value produces final results that are within the limits of practical accuracy, the method is not exact, and so is not applicable to the writers' method.

Contrary to the statements of Messrs. Barron and Garfinkel, vertical and horizontal sway were treated simultaneously. If there are unbalanced shears in two mutually perpendicular directions, and if these shears are balanced in one direction, those in the other direction are automatically balanced—

provided that all joints are in equilibrium. The following is a proof of this statement as applied to Example 5:

$$M_{BA} + M_{AE} + M_{FB} + M_{BF} = 24 \qquad \text{(117a)}$$

and

$$M_{AC} + M_{CA} + M_{BD} + M_{DB} = -24 \qquad \text{(117b)}$$

Adding Eqs. 117*a* and 117*b*,

$$+ M_{BA} + M_{AE} + M_{FB} + M_{BF} + M_{AC} + M_{CA} + M_{BD} + M_{DB} = 0 \qquad \text{(117c)}$$

Substituting from the equations for joint equilibrium,

$$- M_{EF} - M_{AB} - M_{AC} - M_{FE} - M_{BA} - M_{BD} + M_{AC} - M_{CD} + M_{BD} - M_{DC} = 0 \qquad \text{(117d)}$$

—which reduces to

$$M_{EF} + M_{FE} + M_{AB} + M_{BA} + M_{CD} + M_{DC} = 0 \qquad \text{(117e)}$$

—showing that the sum of the shears in the vertical members is zero, as it should be.

Mr. Barron questions the extension of the method as stated. If the principles outlined are followed, the method can be applied to any type of continuous frame having members of constant or variable moments of inertia. The question of secondary moments in Vierendeel trusses due to change of length of the members caused by the axial stresses, as well as the effect of the width of columns and depths of beams on the analysis, is definitely outside the scope of this paper.

Mr. Oesterblom doubts the adaptability of the method to Example 7—a complicated frame with various types of loading. The authors have solved this problem, obtaining the numerically correct answer, consistent with the assumed data. This work can be duplicated by following the paper and the given examples. The validity of the method and its proof have been brought out fully in the paper. Mr. Oesterblom does not make it clear why he cannot accept it.

Professor King claims an advantage for moment distribution in computing joint and bar rotation. If there is no joint translation, Eq. 9*b* may be used to obtain joint rotation. Eq. 18 may be used to find the joint rotation if there is joint translation, noting that the shear correction factor is equal to $\frac{E\,\Delta}{L}$. Calculations of these quantities by the elastic support method are much simpler and more direct than by the method advocated by Professor King.

Messrs. Fraenkel and Janes (and others) seem to have missed the reason for presenting simple illustrative problems such as the first three. There is no doubt that the first example can be solved more readily by the three-moment theorem; however, the purpose of the examples was to familiarize the reader with the method. There has been criticism of the use of the "conjugate beam" method in deriving some of the formulas. Other accepted methods of deriving these formulas have been used, but it was thought that the conjugate beam method was the best known to the readers.[66]

---

[66] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 101, 1936, pp. 550–552.

Mr. Merritt has used the method of moment ratios to solve and check several of the examples of the paper. His expression of "one-cycle distribution" is very apt, and brings out one difference between the moment-distribution method and some newer methods. The preliminary computations should be reduced to a minimum for practical considerations, but, when a new method is being presented, clarity should not be sacrificed for brevity. In the solution of Example 4 Mr. Merritt assumes two moment ratios, which remove his method from the "theoretically exact" category. Both the methods of Mr. Merritt and R. C. Brumfield, M. ASCE, have features that contribute to the ultimate solution of the problem.

Mr. Kavanagh has made a valuable contribution by transforming Eqs. 8 and 11 into one that is well known in the literature of elastic stability of bars, thus indicating the possibility of applying the method to the calculation of buckling restraints in axially loaded members of rigid frames.

Mr. Eremin presents a graphical construction that should be very useful in checking the analytical work for both prismatic and nonprismatic members, and so is an addition to the subject.

Professor Moorman's equations for the statical moment of the moment diagram for prismatic members can be used, not only in this method, but also in any method involving these quantities. The evaluation of x usually presents no difficulty if one sketches the moment curve of the simple beam and then by moments (as in the conjugate beam method) finds its center of gravity.

Mr. Agramonte has developed an interesting but formidable set of equations for nonprismatic members. However, it should be noted that the θ used by Mr. Agramonte is measured from the rotated position of the gravity axis of the member (Fig. 22), whereas the writers measure θ from the position of the unstressed member (Fig. 5 (*b*)).

Professor Ferguson calls attention to the method presented by T. Y. Lin, Assoc. M. ASCE; he then points out the difference between the two papers, noting that the formulas differ, but that one set can be reduced to the other by algebraic transformations. This should be true if both sets of formulas are correct and if they deal with the same ideas. If, as Professor Ferguson thinks, a comparison between the two methods is valuable, it would have been a contribution if he had worked it out in his discussion. He mentions several differences, enough to make it evident that the Lin method is not the same as that of the writers, but that it is valuable.

Professor Gaylord has simplified the writers' equations in a very ingenious manner which ought to appeal to the practicing engineer. Eq. 93*h* is particularly significant since it shows how fixed-end moments are distributed about a joint that is temporarily fixed, and then allowed to assume its natural rotated position.

Mr. Conwell's remark about the analytic beauty in the concept of the method presented is appreciated. Inclusion of the properties of the connections in the analysis is, the writers feel, a must. For work to be of value in practical stress analysis there are usually other considerations besides elasticity of connections that affect the answer even more. For example, a building is stiffened 350%, or thereabouts, by the walls, partitions, and floors. From a strictly practical standpoint there is some question as to the value of any analytical method. However, this does not mean that progress should stop in the development of these methods.

The writers do not agree that the final test is in the designing office, where the designer may not have the time, energy, or desire to learn a new method. It is their experience that those who have taken time to develop skill in its use find that the analysis by elastic support action is as rapid, or more rapid, than the other methods. Its chief value is, however, the presentation of a method which will further the subject of analysis. If advancement were to stop with any of the current methods it would indeed be unfortunate.

Mr. Stewart shows considerable similarity between some of the writers' ideas and the method of traversing the elastic curve; this method is probably more popular than is generally supposed. Combining as it does the advantages of geometry and pictorial representation with the moment-area method, a very efficient system has been devised. Not only is it simple and readily understood, but problems can be solved quickly, and, because of the picture, the action is understood. As previously stated, the writers chose the conjugate beam method of deriving their equations, rather than the one suggested by Mr. Stewart, because they are of the opinion that the older method is more widely known.

Mr. Floris' statement that the writers' method is indirect, that it is primarily a relaxation method, and that conjugate beam is an incorrect name for the Mohr theorem (moment-area method or Greene's theorem) leads the writers to feel that he has deviated widely from the proper scope of discussion. That proof of the formulas and method presented has not been furnished is an unwarranted statement; Eq. 47 is the same as Eq. 10*b* with a slight, unnecessary, algebraic change, which makes it a little difficult to use. Mr. Floris' concluding paragraph is his opinion against that of the writers, whose experience has been to the contrary. Since his discussion shows some misunderstanding of the subject, he should have supported his conclusions so that they could have been of value.

In closing, the authors wish to thank those discussers who have taken the time and trouble to help develop the ideas presented, and to call attention to the writings of others along these lines.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2345

STABILITY OF THIN CYLINDRICAL SHELLS IN TORSION

By R. G. Sturm,[1] M. ASCE

With Discussion by Messrs. S. B. Batdorf and Manuel Stein, Glenn Murphy, L. H. Donnell, R. L. Moore, and R. G. Sturm

Synopsis

Beginning with previously derived general differential equations, this paper leads to the development of a single continuous expression for the critical shearing stress at buckling for all cylindrical shells or tubes whether long, medium, or short. This expression is reduced to a simple equation and a family of curves dependent only on the geometry of the shells and independent of the properties of the materials of which the shell is constructed. The simple equation introduces all factors dependent on the shell material. Comparisons between test results and theoretical values show excellent agreement. Methods for computing the strength of imperfect or incomplete shells are derived and compared with test results. A method of specifying maximum out-of-roundness or local discrepancies for a given percentage of strength is also proposed. A procedure for determining critical shearing stresses in thin-walled pipe lines, tanks, or fuselages is presented.

Introduction

The proper choice of sheet thickness in the design of stressed-skin construction for airplane fuselages, wings, and fuel tanks involves consideration of stability as well as of allowable stresses. Modern trends toward tubular construction for aircraft structures have emphasized the problem of selecting adequate wall thicknesses for very thin-walled tubes or partial tubes. Such members generally tend to buckle at loads far below that required to develop the ultimate strength of the material in tension, compression, or shear. Such buckling frequently results from shearing stresses induced by torque loads or transverse shear. Further interest in the stability of thin-walled tubes has resulted from the use of tubular members for actuating airplane controls. Such

---

Note.—Published in April, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Prof., Eng. Mechanics, and Research Prof. of Materials, Purdue Univ., Lafayette, Ind.

members may tend to "snake" and buckle laterally when subjected to torsional loads.

In the transportation of liquids by pipe lines or trucks, one of the major problems is that of selecting the proper wall thickness of the pipe or vessel containing the liquid. Although the side walls of pipe lines or tanks are by no means in simple shear, the resistance to shear buckling may be the controlling factor in the determination of adequate wall thickness. In a number of instances diagonal buckles have occurred near supports on the side of truck tanks thereby indicating the presence of buckling in shear.

In the drilling of oil wells, tubes or rods are used to rotate the drill bits. Because the relative lengths of such members are great, it has been found that a stability problem is encountered in which there is a lateral buckling of the tube or rod as a whole. This type of buckling is different from the local buckling of the walls of thin cylinders or tubes subjected to torsion. A solution to the problem of lateral buckling of a long tube or rod subject to torsion was obtained by A. G. Greenhill[2] in 1883, whereas the first solution of the buckling of the walls of long thin-walled cylinders subjected to torsion, known to the writer, was published by E. Schwerin[3] in 1924.

In 1932 Eugene E. Lundquist,[4] Assoc. M. ASCE, published the results of a great number of tests on thin-walled duralumin cylinders subjected to torsion. His work has been so carefully reported that it provides an excellent basis for comparison between theoretical and experimental results. The following year, 1933, L. H. Donnell[5] published a theoretical solution to the problem of torsion buckling of thin-walled tubes. Mr. Donnell's work is based on a number of simplifications in the general theory of buckling and leads to certain expressions for the critical shearing stress in short and moderately long thin-walled tubes and to other expressions for long tubes. An excellent evaluation of this theory has been presented by S. Timoshenko.[6] Professor Timoshenko recommends that the Donnell theory be used for short tubes but for long tubes he recommends other formulas.

Curves representing the shear buckling strength of thin-walled tubes plotted against the ratio of length to diameter show a sudden break between Mr. Donnell's theory for "short and moderately long tubes" and that of Mr. Schwerin for "long tubes" of only slightly greater length.

Mr. Donnell's experimental results indicate buckling strengths which average only about 75% of his computed values. This discrepancy was excused on the basis of initial departure from perfect circularity of tubes tested.

The confusion resulting from the use of two formulas—one for short tubes and another for long tubes—together with the difficulty encountered in justifying the discrepancies in the values obtained by existing analysis led to the study of the torsion buckling of thin-walled cylinders presented in this paper.

---

[2] "On the Strength of Shafting When Exposed Both to Torsion and to End Thrust," by A. G. Greenhill, *Proceedings*, Inst. of Mech. Engrs., London, 1883, p. 182.

[3] "Torsional Stability of Thin-Walled Tubes," by E. Schwerin, *Proceedings*, 1st International Cong. for Applied Mechanics, Delft, 1924, pp. 255–265.

[4] "Strength Tests on Thin-Walled Duralumin Cylinders in Torsion," by Eugene E. Lundquist, *Technical Note No. 427*, National Advisory Committee for Aeronautics, Washington, D. C., 1932.

[5] "Stability of Thin-Walled Tubes Under Torsion," by L. H. Donnell, *Technical Report No. 479*, National Advisory Committee for Aeronautics, Washington, D. C., 1933.

[6] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1936, p. 487.

In general this study has for its object a threefold purpose: (I) To derive, on the basis of the theory of elasticity, a general expression for the maximum torque that can be applied to a thin-walled tube without the tube buckling; (II) to compare the theoretical values of critical torque thus obtained with experimental values obtained by other investigators; and (III) to suggest ways of extending these results to practical problems involving shear buckling of thin cylinders.

PART I. DERIVATION OF FORMULAS

Procedure and Assumptions

The general procedure followed in Part I is to derive expressions for the critical torsional shearing stresses in round cylindrical shells of perfectly elastic materials and to treat departures from these conditions as corollaries of the first derivation. The argument used in determining the critical shearing stresses may be stated as follows:

> Consider the cylinder deflected into some shape such that the combined differential equations of continuity and equilibrium, together with the boundary conditions, are satisfied. If the shell is in a state of neutral equilibrium, the external forces necessary to hold the shell in the deflected position will be independent of the magnitude of the deflections as long as they are so small that they do not materially change the general shape of the shell. A slight increase in external force above this value will cause collapse. Therefore, the lowest shearing stress causing neutral equilibrium is considered as the critical shearing stress for the cylinder.

The assumptions involved in the derivation of the general equations may be enumerated as follows: (*a*) The shell is a round cylinder before buckling; (*b*) the shell is of uniform thickness throughout; (*c*) the material in the shell is homogeneous, isotropic, and perfectly elastic; and (*d*) the shell wall is so thin in proportion to the diameter that the curvature will not materially disturb the linear distribution of stress through the thickness of the shell wall—that is, the ordinary flexure formulas may be applied.

System of Coordinates.—The system of cylindrical coordinates used to define and locate any particular element of shell is indicated in Fig. 1. The entire system is based on a reference cylinder the radius of which is the average mean radius of the actual shell, and the longitudinal axis of which coincides with that of the actual shell. The origin of coordinates is taken on the cylinder of reference, at the point of maximum radial displacement of the deflected shell, and midway between the ends of the cylinder.

The y-axis is parallel to the axis of the shell, and lies on the cylinder of reference, positive toward the reader. The s-axis lies on the circumference of a right section of the cylinder of reference, positive in a clockwise direction. The coordinate is $s = r\,\theta$ in which r is the radius of the cylinder of reference, and θ is the angle subtended by s in a right section of the cylinder of reference. The z-axis coincides with the radius of the cylinder of reference, the radial displacement z being measured positive outward.

Notation.—The following letter symbols, adopted for use in the paper and for the guidance of discussers, conform essentially with American Standard Letter Symbols for Mechanics of Solid Bodies (ASA–Z10.3–1942), prepared by a sectional committee of the American Standards Association, with Society participation, and approved by the Association in 1942:

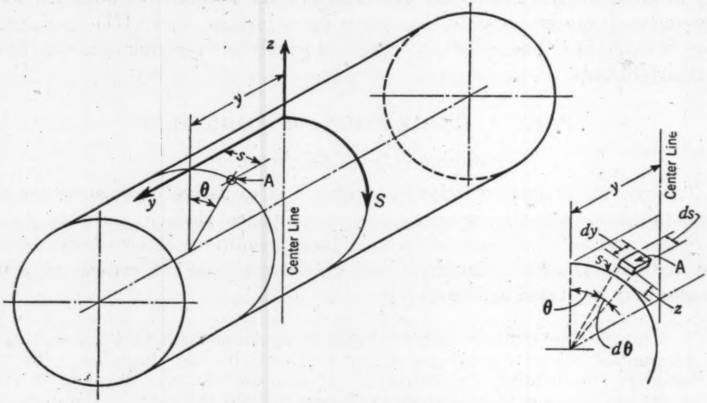


FIG. 1.—SYSTEM OF COORDINATES

a = an aribtrary numerical constant determined from boundary conditions and from the general differential equations; similar definitions apply to constants b, c, d, m, and n;

b = (see a);

c = the distance from the extreme fiber, in the cross section of a beam, to the neutral axis (see Eq. 28); also an arbitrary constant (see a);

D = mean diameter of a cylindrical shell, in inches;

d = (see a);

E = modulus of elasticity of the material in the shell, or in the stiffener, in pounds per square inch;

f = "a function of" (see Eq. 6a); similar definitions apply to functions g, h, j, and ϕ;

G = modulus of elasticity in shear;

g = "a function of" (Eqs. 7); also (see f) "a function of" as in Eq. 14b;

h = (see f);

I = moment of inertia per unit length of shell, in inches^3, $= t^3/12$;

j = (see f);

K = a numerical coefficient dependent on various dimensional ratios:

  K_D = coefficient dependent upon L/D and D/t;

  K_r = coefficient dependent upon r/L and N, introduced by the loading conditions;

L = length of the cylinder, in inches;

M = couple or moment resulting from the distribution of forces or couples on a unit length of the face of the element lying in a plane normal to a given tangent to the shell; units are in inch-pounds per inch and are positive where the force of the couple farthest away from the center of the shell is positive:

- M_s = a bending couple or moment resulting from the distribution of normal forces, in a plane normal to the circumferential (s-axis) tangent;
- M_y = a bending couple or moment resulting from the distribution of normal forces, in a plane normal to the longitudinal (y-axis) tangent;
- M_{sy} = a twisting couple or moment resulting from the distribution of shearing forces, in a plane normal to the circumferential (s-axis) tangent;
- M_{ys} = a twisting couple or moment resulting from the distribution of shearing forces, in a plane normal to the longitudinal (y-axis) tangent;

m = (see a);
N = the number of lobes into which the shell collapses;
n = (see a);
P = normal force, in pounds per linear inch, acting on a unit length of the face of an element:

- P_s = force in a plane normal to the circumferential tangent to the shell;
- P_y = force in a plane normal to the longitudinal tangent to the shell;

Q = a parameter; Q_1 and Q_2 are parameters determined by Eqs. 14 and 15;
Q' = a parameter; Q'_1 and Q'_2 are parameters determined by Eqs. 14 and 15;
q = any integer;
r = mean radius of a cylinder;
S = shearing force on the unit of length of the face of a given element, in pounds per linear inch:

- S_{sy} = force in the y-direction, with the element in a plane normal to the circumferential (s-axis) tangent to the shell;
- S_{sz} = force in the z-direction, with the element in a plane normal to the circumferential (s-axis) tangent to the shell;
- S_{ys} = force in the s-direction, with the element in a plane normal to the longitudinal (y-axis) tangent to the shell;
- S_{yz} = force in the z-direction, with the element in a plane normal to the longitudinal (y-axis) tangent to the shell;

s = an arc length; also axis s in Fig. 1;
t = thickness of shell wall, in inches;
u = longitudinal displacement of a point in the middle surface;
V = total vertical shear on a section, in pounds;
v = circumferential displacement of a point in the middle surface;

W = total weight, or total load uniformly distributed, in pounds (see Fig. 2);
X = a substitution factor (see Eqs. 22);
y = distances along the longitudinal (y-axis) of a tube;
z = radial deflection of a middle surface of a shell; z_o = initial departure from a round cylinder, in inches;
α = angle between the crest of a buckle wave and a longitudinal element of the cylinder (see Fig. 5);
γ = unit detrusion in a tangential plane;
Δ = displacement; a constant (indeterminate in magnitude at the time of collapse) which represents a maximum value of deflection:

Δ' = maximum initial departure from a round cylinder;
Δ_s = circumferential displacement of a point in the middle surface;
Δ_y = longitudinal displacement of a point in the middle surface;

ϵ = unit strain:

ϵ_r = unit strain in a radial direction;
ϵ_s = unit strain in a circumferential direction;
ϵ_y = unit strain in a longitudinal direction;

θ = angular distance;
μ = Poisson's ratio;
τ = tangential shear stress in the wall of the cylinder:

τ' = shear stress denoting combined bending and shear;
τ_c = critical buckling shear stress;

$$\tau_o = K_D \, E \, \frac{t^2}{D^2} \qquad \text{(1)}$$

τ_t = stress that causes local tension equal to the yield point stress;

ϕ = "a function of"; as used in Eq. 8; and
ψ = an infinitesimal angle at the center of curvature of the deflected shell, subtended by the circumferential element of length (see Eq. 4).

Deformed Element of Shell.—The element of shell considered is shown in its deformed state in Fig. 2. If u, v, and z are the displacements in the directions y, s, and z, respectively, of a point in the middle surface of the shell from the undeformed position, then the strains and the detrusion of the middle surface are defined as:

$$\epsilon_y = \frac{\partial u}{\partial y} \;;\quad \epsilon_s = \frac{\partial v}{\partial s} + \frac{v}{r} \;;\quad \gamma_{sy} = \frac{\partial u}{\partial s} + \frac{\partial v}{\partial y} \qquad \text{(2)}$$

By eliminating u and v, the following equation of compatibility[7,8] is obtained:

$$\frac{\partial^2 \epsilon_s}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial s^2} - \frac{\partial^2 \gamma_{sy}}{\partial s \, \partial y} - \frac{1}{r} \frac{\partial^2 z}{\partial y^2} = 0 \qquad \text{(3)}$$

---

[7] "Report on Arch Dam Investigation: Part VIII—Theoretical Analysis of the Structural Action of the Stevenson Creek Arch Dam," by H. M. Westergaard, *Proceedings*, ASCE, May, 1928, Pt. 3, Vol. 1, pp. 231–266.

[8] "Stress Functions for Shells," by H. M. Westergaard, *Technical Memorandum No. 351*, U. S. Bureau of Reclamation, July, 1933; "Die Stabilität der Kreiszylinderschale," *Ingenieur-Archiv.*, Vol. 3, p. 77.

It may be expressed as a function of θ, thus: $\partial\psi = \frac{\partial\psi}{\partial\theta} d\theta$, and consists of the following parts: (1) $d\theta$, original angle; (2) $-\frac{1}{r}\frac{\partial^2 z}{\partial\theta^2} d\theta$, due to change in slope over length ds; (3) $-\frac{z}{r} d\theta$, due to radial deflection; and (4) $\epsilon_s\, d\theta$, due to circumferential strain. Hence:

$$\frac{\partial\psi}{\partial\theta} = 1 - \frac{1}{r}\frac{\partial^2 z}{\partial\theta^2} - \frac{z}{r} + \epsilon_s \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

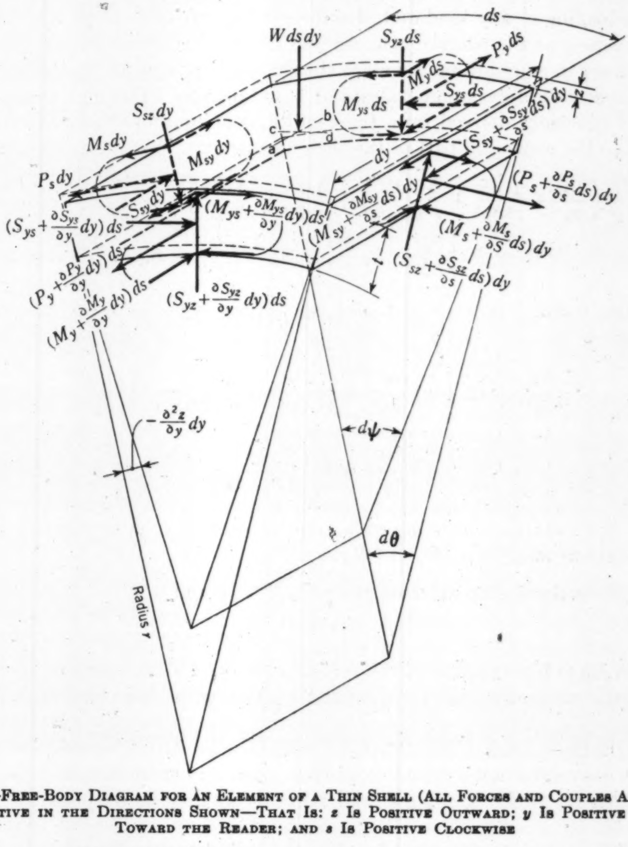

ds
$W\,ds\,dy$
$S_{yz}\,ds$
$P_y\,ds$
$M_y\,ds$
$S_{ys}\,ds$
$M_{ys}\,ds$
$(S_{sy} + \frac{\partial S_{sy}}{\partial s} ds)\,dy$
z
$S_{sz}\,dy$
$M_s\,dy$
$M_{sy}\,dy$
b
c
d
a
dy
$P_s\,dy$
$S_{sy}\,dy$
$(P_s + \frac{\partial P_s}{\partial s} ds)\,dy$
$(M_{ys} + \frac{\partial M_{ys}}{\partial y} dy)\,ds$
$(M_{sy} + \frac{\partial M_{sy}}{\partial s} ds)\,dy$
$(S_{ys} + \frac{\partial S_{ys}}{\partial y} dy)\,ds$
$(M_s + \frac{\partial M_s}{\partial S} ds)\,dy$
t
$(S_{sz} + \frac{\partial S_{sz}}{\partial s} ds)\,dy$
$(P_y + \frac{\partial P_y}{\partial y} dy)\,ds$
$(S_{yz} + \frac{\partial S_{yz}}{\partial y} dy)\,ds$
$(M_y + \frac{\partial M_y}{\partial y} dy)\,ds$
$-\frac{\partial^2 z}{\partial y} dy$
$d\psi$
$d\theta$
Radius r

Fig. 2.—Free-Body Diagram for an Element of a Thin Shell (All Forces and Couples Are Positive in the Directions Shown—That Is: z Is Positive Outward; y Is Positive Toward the Reader; and s Is Positive Clockwise

Eq. 4 defines an infinitesimal angle at the center of curvature of the deflected shell, subtended by the circumferential element of length.

Internal Forces and Couples.—Fig. 2 shows the forces and couples that hold an element of shell in equilibrium: An external pressure W, in pounds per square inch; two normal forces, P_s and P_y; four shearing forces, S_{sy}, S_{ss}, S_{ys}, and S_{ys}; and four couples, M_s, M_y, M_{sy}, and M_{ys}. These concepts are defined fully in the notation.

ADAPTATION OF GENERAL THEORY OF BUCKLING TO PROBLEM OF TORSION

The derivation of a set of general differential equations governing the behavior of thin-walled cylinders subjected to any type of external load has been presented elsewhere by the writer.[9] The equations were derived from a consideration of the forces acting on a small section of a cylinder subjected to a uniform loading of any kind and of sufficient magnitude to cause the cylinder to be precisely at the point of collapse (see Fig. 1). Since all parts of the shell in Fig. 2 are in equilibrium and since the shell wall is continuous, an interrelation between the forces and deflections is obtainable. The four general differential equations derived from the conditions of equilibrium and continuity applied to the elemental part of the shell[10] are:

$$-\frac{E\,I}{1-\mu^2}\left(\frac{\partial^4 z}{\partial y^4}+\frac{\mu}{r^2}\frac{\partial^2 z}{\partial y^2}+\frac{2}{r^2}\frac{\partial^4 z}{\partial \theta^2\,\partial y^2}+\frac{1}{r^4}\frac{\partial^4 z}{\partial \theta^4}+\frac{1}{r^4}\frac{\partial^2 z}{\partial \theta^2}\right)$$

$$= W + P_s\frac{1}{r}\frac{\partial \psi}{\partial \theta} - P_y\frac{\partial^2 z}{\partial y^2} - S_{ys}\frac{1}{r}\frac{\partial^2 z}{\partial \theta\,\partial y} - S_{sy}\frac{1}{r}\frac{\partial^2 z}{\partial \theta\,\partial y}\ldots\ldots\ldots(5a)$$

$$\frac{1}{r^2}\frac{\partial^2 P_s}{\partial \theta^2}-\frac{\partial^2 P_y}{\partial y^2}-\frac{1+\epsilon_s}{r}\frac{E\,I}{1-\mu^2}\left(\frac{1}{r^4}\frac{\partial^4 z}{\partial \theta^4}+\frac{1}{r^4}\frac{\partial^2 z}{\partial \theta^2}+\frac{2-\mu}{r^2}\frac{\partial^4 z}{\partial \theta^2\,\partial y^2}\right)=0\ldots(5b)$$

$$\frac{\partial^2 P_s}{\partial y^2}+(2+\mu)\frac{\partial^2 P_y}{\partial y^2}+\frac{1}{r^2}\frac{\partial^2 P_y}{\partial \theta^2}-\frac{\mu}{r^2}\frac{\partial^2 P_s}{\partial \theta^2}=\frac{E\,t}{r}\frac{\partial^2 z}{\partial y^2}\ldots\ldots\ldots(5c)$$

and

$$\frac{\partial^2 P_s}{\partial y^2}-\frac{\mu\partial^2 P_y}{\partial y^2}=\frac{E\,t}{r}\left(\frac{\partial^3 v}{\partial y^2\,\partial \theta}+\frac{\partial^2 z}{\partial y^2}\right)\ldots\ldots\ldots\ldots(5d)$$

For the conditions existing in a tube subjected to pure torsion, certain simplifications may be made, as follows:

(1) Since there is no external pressure, $W = 0$ and

$$P_s = 0 + f(y\,s)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6a)$$

in which $f(y\,s)$ is a function of the coordinates y and s, representing the variation of the circumferential force caused by the radial deflection of the shell. When the deflection is small, $f(y\,s)$ is small, and the quantities $\frac{1}{r}\frac{\partial^2 z}{\partial \theta^2}$, ϵ_s, and $\frac{z}{r}$ which are involved in the definition of $d\psi$ are also very small compared to unity. Consequently, products of $f(y\,s)$ and these terms may be dropped from Eqs. 5a and 5b.

(2) There is no longitudinal force on the cylinder, therefore,

$$P_y = 0 + g(y\,s)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6b)$$

---

[9] "A Study of the Collapsing Pressure of Thin-Walled Cylinders," by R. G. Sturm, *Bulletin No. 329*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., 1941.

[10] *Ibid.*, Section 11, p. 17.

in which $g(y\ s)$ is a function of the coordinates y and s representing the variation in the longitudinal forces caused by the radial deflections of the shell. When the deflection is small, $g(y\ s)$ and the term $\partial^2 z/\partial y^2$ are small. Consequently, the product of $g(y\ s)$ and $\partial^2 z/\partial y^2$ may be dropped from Eq. 5*a*.

(3) Since the tube is circular, the shearing stresses will be constant except for variations caused by radial deflections of the tube and the values S_{ys} and S_{sy} will be constant and equal to $\tau\ t$ (shearing stress times thickness) except for small amounts $j(y\ s)$ and $h(s\ y)$, respectively. Consequently, the variations in S_{ys} and S_{sy} will be very small when the radial deflection, z, is small, and the term $\partial^2 z/\partial\theta\ \partial y$ is also small. The resulting products of the terms $h(s\ y)$ and $j(y\ s)$ and $\dfrac{\partial^2 z}{\partial\theta\ \partial y}$ may be dropped from Eq. 5*a*.

(4) Because of the assumed uniformity of dimensions and material in the cylinder, the average shearing stress is considered constant throughout the length of the cylinder and the displacement v is $\dfrac{S_{ys}}{t\ G}\, y$ except for small variations of $g(y\ s)$. As in previous paragraphs, products of $g(y\ s)$ and z or its derivatives may be considered negligible.

(5) Because the circumferential forces, P_s, are zero except for small variations depending on the radial deflection, the unit strain in the circumferential direction, ϵ_s, can be neglected.

Use of these five simplifications leads to the four general differential equations:

$$\frac{E\ I}{1-\mu}\left[\frac{\partial^4 z}{\partial y^4} + \frac{\mu}{r^2}\frac{\partial^2 z}{\partial y^2} + \frac{2}{r^2}\frac{\partial^4 z}{\partial\theta^2\ \partial y^2} + \frac{1}{r^4}\left(\frac{\partial^4 z}{\partial\theta^4} + \frac{\partial^2 z}{\partial\theta^2}\right)\right] + \frac{1}{r} f(s, y) = \frac{2\ S_{ys}}{r}\frac{\partial^2 z}{\partial\theta\ \partial y} \qquad (7a)$$

$$\frac{\partial^2 f}{r^2\ \partial\theta^2} - \frac{\partial^2 g}{\partial y^2} - \frac{E\ I}{r(1-\mu^2)}\left[\frac{1}{r^4}\left(\frac{\partial^4 z}{\partial\theta^4} - \frac{\partial^2 z}{\partial\theta^2}\right) + \frac{(2-\mu)\ \partial^4 z}{r^2\ \partial\theta^2\ \partial y^2}\right] = 0 \qquad (7b)$$

$$(2+\mu)\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{r^2\ \partial\theta^2} + \frac{\partial^2 f}{\partial y^2} - \frac{\mu}{r^2}\frac{\partial^2 f}{\partial\theta^2} = \frac{E\ t}{r}\frac{\partial^2 z}{\partial y^2} \qquad (7c)$$

and

$$\frac{\partial^2 f}{\partial y^2} - \mu\frac{\partial^2 g}{\partial y^2} = \frac{E\ t}{r}\left(\frac{\partial^3 v}{\partial y^2\ \partial\theta} + \frac{\partial^2 z}{\partial y^2}\right) \qquad (7d)$$

The problem, as defined, is to find a solution of these simultaneous differential equations which will give the correct type of radial deflection of the cylinder wall and at the same time will fulfil the boundary conditions. The value of torsional shearing forces, $S_{ys} = S_{sy} = \tau\ t$, which will make the magnitude of the deflections indeterminate is the value which will produce neutral equilibrium and hence will be the critical value. The torque producing this stress is the maximum torque that can be applied to the tube without it buckling.

The boundary conditions are simply the conditions of support at the ends of the tube which depend on the fixation of the tube. The first and principal case considered is that of "simply supported ends." In this case the ends of the deflected tube are held so as to retain their original true circularity and, at the same time, to offer no external resistance to a change in slope of a longitudinal tangent.

The particular practical condition of the tube which this case represents is that encountered in an intermediate section of three or more sections of a tube subdivided by bulkheads relatively narrow in the direction of the tube axis.

The mathematical statement of these boundary conditions is as follows: When $y = +\frac{L}{2}$, $z = 0$; and, for all values of θ, $\frac{\partial z}{\partial \theta} = 0$.

The choice of the center of coordinates midway between the ends of the cylinder or tube and at the point of maximum displacement around the shell leads to the boundary condition that $\frac{\partial z}{\partial \theta} = 0$ when $y = 0$ and $\theta = 0$.

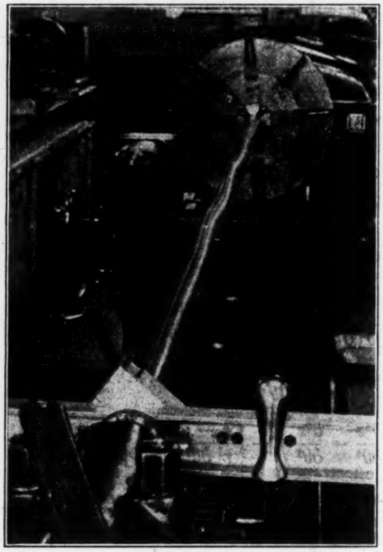


FIG. 3.—THIN-WALLED ALUMINUM ALLOY TUBE ($L/D = 44$ AND $D/t = 138$) AT NEUTRAL EQUILIBRIUM UNDER A TORSIONAL LOAD

Solution of Differential Equations.—The boundary conditions indicate that the solution of the equations is a symmetrical function with respect to both the longitudinal axis, y, and the angular axis, θ. For such high-order differential equations, solutions in the form of the product of two or more simple solutions may be sought. To obtain suggestions as to the correct type of radial deflection of the tube wall in a state of neutral equilibrium, an actual tube (Fig. 3) was twisted until it buckled and then held in its buckled state. The shape of the deflected tube clearly indicates that the general form of the deflected cross section remains essentially the same but that the points of contraflexure tend to spiral around the tube.

The general shape of the deflected tube, as shown in Fig. 3, may be represented by the equation:

$$z = \phi(y) \cos\left(N\theta + \frac{K_r \pi y}{L} \right) \qquad (8)$$

In order that $\phi(y)$ may fulfil the boundary conditions (that is, zero deflection at the end of the tube) and at the same time be symmetrical about the center line of the tube:

$$\phi(y) = \Delta \cos \frac{\pi y}{L} \qquad (9)$$

Thus, it follows that a solution to the differential equations may be expected to be in the form of:

$$z = \Delta \cos\left(N\,\theta + \frac{K_r\,\pi\,y}{L}\right)\cos\frac{\pi\,y}{L} \qquad (10a)$$

This function (Eq. 10*a*) fulfils all the boundary conditions and, by proper evaluation of the constant K_r, will satisfy the differential equations. In order that the state of neutral equilibrium may be reached (that is, in order that the coefficient of the deflection term will vanish), the various derivatives—$\frac{\partial^4 z}{\partial y^4}$, $\frac{\partial^4 z}{\partial \theta^4}$, $\frac{\partial^4 z}{\partial \theta^2\,\partial y^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial \theta^2}$, and $\frac{\partial^2 z}{\partial \theta\,\partial y}$—must all have the same general functions.

Eq. 10*a* may be expanded for the purpose of determining the values of the derivatives; thus:

$$z = \Delta\left[\cos(N\,\theta)\cos\frac{K_r\,\pi\,y}{L}\cos\frac{\pi\,y}{L} - \sin(N\,\theta)\sin\frac{K_r\,\pi\,y}{L}\cos\frac{\pi\,y}{L}\right] \qquad (10b)$$

By substituting the following quantities—

$$m = (K_r + 1)\,\frac{\pi}{L} \qquad (11a)$$

and

$$n = (K_r - 1)\,\frac{\pi}{L} \qquad (11b)$$

—it is possible to rearrange the terms in Eq. 10*a* to give:

$$z = z_1 + z_2 \qquad (12)$$

in which

$$z_1 = \frac{\Delta}{2}\left[\cos(N\,\theta)\cos(m\,y) - \sin(N\,\theta)\sin(m\,y)\right] \qquad (13a)$$

and

$$z_2 = \frac{\Delta}{2}\left[\cos(N\,\theta)\cos(n\,y) - \sin(N\,\theta)\sin(n\,y)\right] \qquad (13b)$$

When these values for z_1 and z_2 are substituted into Eq. 7*a* (which contains the value of shear, $S_{ys} = S_{sy} = \tau\,t$), it is found that they will each yield a value of τ. The function, f, may be defined in terms of K_r from Eqs. 7*b*, 7*c*, 11*a*, and 11*b*, so that the value of τ that will produce neutral equilibrium is dependent only on the constant K_r. Since there can be only one value of shearing stress which will produce neutral equilibrium, the value of τ obtained from the use of z_1 must be the same as the value of τ obtained from the use of z_2. It is possible to solve for the value of K_r that will give the same value of τ from both z_1 and z_2. This value of τ is the computed critical buckling shear stress and is designated as τ_c.

DETERMINATION OF PARAMETERS

To determine the constant parameter, K_r, it is necessary to determine the function f, but, to determine f from Eqs. 7*b* and 7*c*, it is necessary to eliminate the function, g. From Eqs. 7*b* and 7*c*, it may be noted that the functions, f and g, must have the following forms:

$$f = Q_1 z_1 + Q_2 z_2 = f_1 + f_2 \dots \tag{14a}$$

and

$$g = Q'_1 z_1 + Q'_2 z_2 = g_1 + g_2 \dots \tag{14b}$$

or

$$f_1 = Q_1 z_1 ; f_2 = Q_2 z_2 \dots \tag{15a}$$

and

$$g_1 = Q'_1 z_1 ; g_2 = Q'_2 z_2 \dots \tag{15b}$$

Since the functions, f and g, must satisfy the differential Eqs. 7*b* and 7*c*, for all values of y and θ, the parts involving z_1 and z_2 must satisfy these equations separately. The substitution of these values and the values of z_1 and z_2 from Eq. 13*b* into Eq. 7*b* gives the following relationships:

$$Q'_1 = \frac{N^2}{m^2 r^2} Q_1 + \frac{E I}{r^5 m^2 (1 - \mu^2)} [N^4 - N^2 + (2 - \mu) N^2 m^2 r^2] \dots \tag{16a}$$

and

$$Q'_2 = \frac{N^2}{n^2 r^2} Q_2 + \frac{E I}{r^5 n^2 (1 - \mu^2)} [N^4 - N^2 + (2 - \mu) N^2 n^2 r^2] \dots \tag{16b}$$

From Eq. 7*c* the values of Q'_1 and Q'_2 are found to be:

$$Q'_1 = \frac{Q_1 \left(\frac{\mu N^2}{m^2 r^2} - 1 \right) + \frac{E t}{r}}{2 + \mu + \frac{N^2}{m^2 r^2}} \dots \tag{17a}$$

and

$$Q'_2 = \frac{Q_2 \left(\frac{\mu N^2}{n^2 r^2} - 1 \right) + \frac{E t}{r}}{2 + \mu + \frac{N^2}{n^2 r^2}} \dots \tag{17b}$$

The parameter Q'_1 may be eliminated from Eqs. 16*a* and 17*a* and the value of Q_1 may be found to be:

$$Q_1 = \frac{\frac{E t}{r} - \frac{E t^3}{r^3} \frac{1}{12 (1 - \mu^2)} \cdot \left[\frac{N^4 - N^2}{m^2 r^2} + N^2 (2 - \mu) \right] \left(\frac{N^2}{m^2 r^2} + 2 + \mu \right)}{\left(\frac{N^2}{m^2 r^2} + 1 \right)^2} \dots \tag{18a}$$

and, from Eqs. 16*b* and 17*b*, the value of Q_2 is found to be:

$$Q_2 = \frac{\frac{E t}{r} - \frac{E t^3}{r^3} \frac{1}{12 (1 - \mu^2)} \left[\frac{N^4 - N^2}{n^2 r^2} + N^2 (2 - \mu) \right] \left(\frac{N^2}{n^2 r^2} + 2 + \mu \right)}{\left(\frac{N^2}{n^2 r^2} + 1 \right)^2} \dots \tag{18b}$$

With these equations f_1, f_2, and consequently f, are known in terms of m and n.

EQUATIONS FOR CRITICAL BUCKLING SHEAR STRESS

Eq. 7*a*, which contains the applied shear stress $S_{ys} = S_{sy} = \tau\, t$, may be rewritten in terms of z_1, z_2, f_1, and f_2, as follows:

$$\left\{ \frac{E\, t^3}{12\,(1-\mu^2)\, r^4} \left[(m^2 r^2 + N^2)^2 - (N^2 + \mu\, m^2 r^2) \right] + \frac{Q_1}{r} - \frac{2\, N\, m}{r} \tau\, t \right\} z_1 + \left\{ \frac{E\, t^3}{12\,(1-\mu^2)\, r^4} \left[(m^2 r^2 + N^2)^2 - (N^2 + \mu\, m^2 r^2) \right] + \frac{Q_2}{r} - \frac{2\, N\, n}{r} \tau\, t \right\} z_2 = 0 \quad (19)$$

In order that Eq. 19 may be true for all values of y and θ, the coefficients of z_1 and z_2 must each be zero. The resulting equations are:

$$\tau\, t = \frac{E}{12\,(1-\mu^2)} \frac{t^3}{r^3} \left[\frac{(m^2 r^2 + N^2)^2 - (N^2 + \mu\, m^2 r^2)}{2\, m\, N} \right] + \frac{Q_1}{2\, m\, N} \quad (20a)$$

and

$$\tau\, t = \frac{E}{12\,(1-\mu^2)} \frac{t^3}{r^3} \left[\frac{(n^2 r^2 + N^2)^2 - (N^2 + \mu\, n^2 r^2)}{2\, n\, N} \right] + \frac{Q_2}{2\, n\, N} \quad (20b)$$

The critical buckling shear stress τ_c is obtained when Eqs. 17 give the same value of τ. Used with Eqs. 11, they give sufficient conditions to solve for both K_r and τ_c. The values of Q_1, Q_2, m, and n may be substituted into Eq. 20*a*. The resulting equation is:

$$\frac{E\, t^3\, L}{12(1-\mu^2)\, r^3\, 2\, N\, \pi} \left\{ \frac{\left[(K_r+1)^2 \frac{\pi^2 r^2}{L^2} + N^2 \right]^2 - \left[N^2 + \mu (K_r+1)^2 \frac{\pi^2 r^2}{L^2} \right]}{K_r + 1} \right\} + \frac{E\, t^3\, L}{12(1-\mu^2)\, r^3\, 2\, N\, \pi} \left\{ 12(1-\mu^2) \frac{r^2}{t^2} - \left[\frac{(N^4 - N^2) L^2}{(K_r+1)^2\, \pi^2 r^2} + N^2 (2-\mu) \right] \times \left[\frac{N^2}{(K_r+1)^2} \frac{L^2}{\pi^2 r^2} + 2 + \mu \right] \div (K_r+1) \left[\frac{N^2 L^2}{(K_r+1)^2\, \pi^2 r^2} + 1 \right]^2 \right\} = \tau_c\, t \quad (21)$$

From Eq. 20*b*, a similar equation is obtained except that $K_r + 1$ is replaced by $K_r - 1$.

Now if the terms in the braces in Eq. 21 are designated by X, the resulting equation is:

$$\tau_c = E \frac{t^2}{r^2} \frac{L}{\pi\, r} \frac{X}{24(1-\mu^2) N} \quad (22a)$$

or

$$\tau_c = K_D\, E \frac{t^2}{D^2} \quad (22b)$$

in which

$$K_r = \frac{L}{\pi\, R} \frac{X}{6(1-\mu^2) N} \quad (23)$$

The two values of X obtained by using $K_r + 1$ and $K_r - 1$ must be the same since X is a single value. The particular values of $K_r + 1$ and $K_r - 1$ for which this condition is fulfilled may best be found by plotting values of X against values of K_r and locating the intersections of the two curves for X. In these computations, however, it is necessary to determine only one set of values of X because the values for $K_r - 1$ will be the same as the values for $K_r + 1$ except that the curve is moved two units to the right. The intersections of the pairs of curves thus obtained give values for both X and K_r.

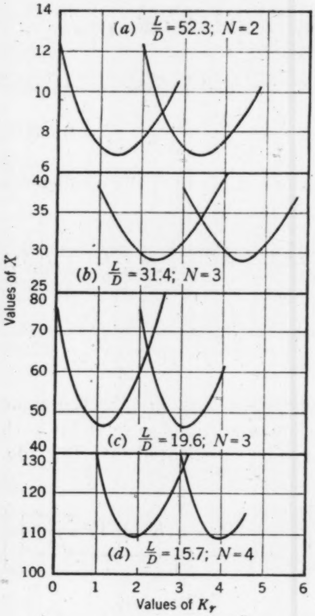


FIG. 4.—SAMPLES OF CURVES OBTAINED BY PLOTTING EQS. 18 $\left(\frac{r}{t} = 500\right)$

The values of X for any particular values of D/t and $\frac{L}{\pi\, r}$ or L/D can be computed for different integral values of N. The value of N giving the lowest resultant value of τ_e will be the number of lobes into which the tube will buckle.

In tabular computations for establishing charts it is expedient to consider given values of D/t and N and to compute values of X for a range of L/D-values in which X is a minimum or near minimum. Sample curves of X showing various types of intersections are shown in Fig. 4 and the resulting curves for K_D are shown in Fig. 5.

In subsequent considerations it will be desirable to know the angle between the crests of the buckle waves and the longitudinal elements of the cylinder. From Eq. 10a, it follows that, at the crest of a buckle wave, $N\,\theta + \frac{K_r\,\pi\,y}{L}$ must be a constant. Considering the buckle through the origin:

$$N\,\theta + \frac{K_r\,\pi\,y}{L} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (24a)$$

Hence, the angle θ will change with y so that $\frac{r\,\theta}{y}$ will be the tangent of the angle, α, between the crest and a longitudinal element of the cylinder. From Eq. 24a, it follows that:

$$\tan \alpha = \frac{r\,\theta}{y} = -\frac{K_r\,\pi\,r}{N\,L} = -\frac{K_r\,\pi\,D}{2\,N\,L} \ldots\ldots\ldots\ldots\ldots (24b)$$

To find the value of K_r from Fig. 5, the values of N and $\tan\alpha$ are first obtained and the value of K_r is found as:

$$K_r = \frac{2N}{\pi}\frac{L}{D}\tan\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots (25a)$$

Eq. 22*b* may be rewritten as:

$$\frac{\tau_e}{E} = K_D\frac{t^2}{D^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (25b)$$

All values on the right side of Eq. 25*b* are dependent on the geometry of the vessel and independent of the properties of the material in the shell wall

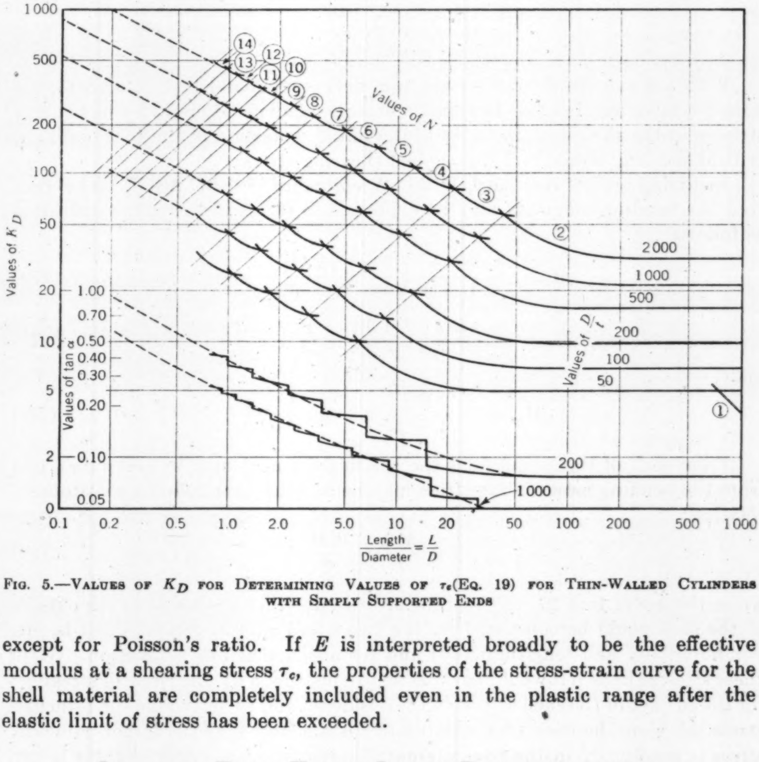


Fig. 5.—Values of K_D for Determining Values of τ_e(Eq. 19) for Thin-Walled Cylinders with Simply Supported Ends

except for Poisson's ratio. If E is interpreted broadly to be the effective modulus at a shearing stress τ_e, the properties of the stress-strain curve for the shell material are completely included even in the plastic range after the elastic limit of stress has been exceeded.

Imperfect Tubes Having Initial Departure from a True Circular Cylinder

If the tube considered is not a true circular cylinder but departs slightly from a true cylinder, it will begin to deflect as soon as torque is applied. In general, such initial departures from a true cylinder have no particular shape so that, when the tube wall begins to deflect, the initial deflection will not always be additive, in its effects, to the deflection under load. However, if the initial deflection is of the same general trend as the deflection under load, the effects may be fully additive. For all practical purposes, the greatest deflections that may occur must be considered. To obtain a measure of this greatest deflection, it may be assumed that the initial departure of the tube wall from a true cylinder corresponds to the buckle pattern of the shell at neutral equilibrium.

Under this condition, the general relationship between the magnitude of the maximum deflection under load Δ, the initial deflection Δ_o, the critical shearing stress τ_c, and the actual applied stress τ may be written[9,11] as:

$$\Delta = \frac{\Delta_o \tau}{\tau_c - \tau} \quad \text{........................} \quad (26)$$

With the magnitude of the maximum deflection Δ and Eqs. 12 and 13, it is possible to determine the bending moments, and consequently the bending stresses in the shell wall, in terms of the initial deflection of the tube wall, the critical shearing stress, and the applied shearing stress.

Assuming perfect elasticity, H. M. Westergaard,[7,8,11] M. ASCE, has shown that the bending moments may be expressed in terms of the radial deflection as follows:

$$M_y = -\frac{E\,I}{1-\mu^2}\left(\frac{\partial^2 z}{\partial y^2} + \frac{\mu}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \mu\,\frac{z}{r}\right) \quad \text{............} \quad (27a)$$

$$M_s = -\frac{E\,I}{1-\mu^2}\left(\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{z}{r^2} + \mu\,\frac{\partial^2 z}{\partial y^2}\right) \quad \text{............} \quad (27b)$$

and

$$M_{sy} = -\frac{E\,I}{1-\mu^2}(1-\mu)\,\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta\,\partial y} \quad \text{..............} \quad (27c)$$

From each of these equations the bending moment may be computed, and from the bending moment the stress may be found by the following customary relation:

$$S = \frac{M\,c}{I} = \frac{6\,M}{t^2} \quad \text{......................} \quad (28)$$

From the use of Eqs. 27, the bending stresses in either the inside or the outside of the shell could be computed for the two coordinate directions (that is, circumferentially and longitudinally) and for a diagonal (45°) direction. With these stresses known it would be possible to construct a dyadic circle of stress[12,13] for the surface of the shell at the point considered and to determine the principal stresses. Consideration of a number of cases shows that the larger principal stress is practically in the circumferential direction and is only slightly larger

---

[11] "Buckling of Elastic Structures," by H. M. Westergaard, *Transactions*, ASCE, Vol. LXXXV, 1922, pp. 576–654.

[12] "Advanced Mechanics of Materials," by F. B. Seely, John Wiley & Sons, Inc., New York, N. Y., 1932, p. 37.

[13] "The Determination of Stresses from Strains on Three Intersecting Gage Lines and Its Application to Actual Tests," by W. R. Osgood and R. G. Sturm, *Journal of Research*, U. S. Bureau of Standards, Vol. 10, May, 1933.

than the maximum circumferential stress. Consequently, the circumferential stress represents a fair approximation of the controlling bending stress. Therefore, for purposes of comparing some examples of imperfect cylinders, the circumferential stress will be used as an index of the maximum bending stress in the shell wall. In cases where the critical shearing stress, τ_c, is large as compared to the elastic limit, the combined stress resulting from shear and bending should be used.

The value of z from Eqs. 12 and 13 may be substituted into Eq. 27b to give:

$$M_s = \frac{E\,t^3}{12(1-\mu^2)}\left\{\frac{\Delta}{2}\left(\frac{N^2-1}{r^2}+\mu\,m^2\right)[\cos(N\theta)\cos(m\,y)-\sin(N\theta)\sin(m\,y)]\right.$$
$$\left.+\frac{\Delta}{2}\left(\frac{N^2-1}{r^2}+\mu\,n^2\right)[\cos(N\theta)\cos(n\,y)-\sin(N\theta)\sin(n\,y)]\right\}\,..(29a)$$

The maximum value of M_s is found at $\theta = 0$, $y = 0$ and is expressed as:

$$M_s = \frac{E\,t^3}{12(1-\mu^2)}\,\Delta\left[\frac{N^2-1}{r^2}+\mu\,(m^2+n^2)\right]\ldots\ldots\ldots\ldots(29b)$$

The values of m and n as determined from Eqs. 11 may be substituted into Eq. 29b and the resulting equation may be rearranged to give:

$$M_s = \frac{\Delta E}{12(1-\mu^2)}\frac{t^3}{r^2}\left[N^2-1+2\,\mu\frac{\pi^2 r^2}{L^2}(K_r^2+1)\right]\ldots\ldots.(30a)$$

The value for stress may be computed from Eq. 30a for the value of Δ from Eq. 26 to give:

$$S = \frac{\tau_1}{\tau_c-\tau_1}\,\frac{E}{2(1-\mu^2)}\,\frac{t^2\Delta_o}{r^2\,t}\left[N^2-1+2\,\mu\frac{\pi^2 r^2}{L^2}(K_r^2+1)\right]\ldots(30b)$$

The values of N and K_r as well as the value of τ_c may be obtained from Fig. 5.

As an example to illustrate the effect of initial deflections (out-of-roundness), Eq. 30b is applied to a long tube of steel for which the following data apply: $L/D = 150$; $D/t = 200$; $N = 2$ (from Fig. 5); $K_r = 11.5$ (from Fig. 5); $\mu = 0.3$; $\Delta_o/t = 0.5$; $E = 30{,}000{,}000$ lb per sq in.; and the tensile yield point = 40,000 lb per sq in.

The ratio of τ_1/τ_c which will give a circumferential bending stress equal to the yield point in tension is obtained from Eq. 30b as $\frac{\tau_1}{\tau_c} = 0.94$. The value of $\Delta_o/t \leqq 0.5$ is easily obtained by careful fabrication except for extremely thin tubes. Therefore, the observed values of such long thin tubes should be fairly close (within 10%) to the theoretical values.

For very thin tubes of short length such as part of an aircraft fuselage a similar comparison may be made. For this case Eq. 30b is applied to a short thin tube of steel for which the following data are known: $L/D = 0.5$; $D/t = 1{,}000$; $N = 16$ (from Fig. 5); $K_r = 1.5$; $\mu = 0.3$; $\Delta_o/t = 0.5$; $E = 30{,}000{,}000$ lb per sq in.; and the yield point = 40,000 lb per sq in. The ratio of τ_1/τ_c

which will give a circumferential bending stress equal to the yield point in tension is $\tau_1/\tau_e = 0.81$. For exceptionally thin, short tubes, the out-of-roundness may be controlled quite easily within limits; but an out-of-roundness value of $\Delta_o/t = 0.5$ is difficult to obtain. In this case the experimental results would be expected to be less than the theoretical values by about 20%.

In view of the foregoing examples, in which a reasonable initial departure from true circularity was allowed, carefully made tests should give values of from at least 80% to 90% of the theoretical values. Actual consideration of the combined stresses extant in very thin shells indicates somewhat lower ratios for τ_1/τ_e; but actual determinations are not feasible because the properties of thin sheets of metal under biaxial stress are not known.

Eq. 30*b* permits drawing specifications for maximum allowable "out-of-roundness" (departure from true circularity) to attain at least a minimum predetermined percentage of the theoretical elastic buckling strength of the shell. The reliability of the specification, however, may be questioned until more factual data have been compiled.

CYLINDERS WITH ENDS COMPLETELY FIXED

Boundary Conditions.—The boundary condition of "fixed ends" requires not only that true circularity be maintained at the ends but that the longitudinal tangent to the tube at the end be maintained parallel to the axis of the tube. The condition of absolutely fixed ends is difficult to obtain in tests and seldom, if ever, is obtained in a practical case.

The mathematical statement of these boundary conditions is as follows: When $y = \pm \frac{L}{2}$; $z = 0$; and, for all values of θ, $\frac{\partial z}{\partial y} = 0$ and $\frac{\partial z}{\partial \theta} = 0$. Again the choice of the center of coordinates midway between the ends of the tube gives longitudinal symmetry. The center of coordinates is also chosen at the point of maximum displacement as before.

Type of Solution.—Again a solution consisting of the product of two functions will be sought by the same procedure as that adopted for simply supported edges. Since the characteristic form of buckling does not change, one function is chosen as before, namely: $\cos\left(N\theta + \frac{K_r \pi y}{L}\right)$. The other function, which previously was simply $\cos \frac{\pi y}{L}$, now must be a function that has both zero deflection and zero slope at the end. The following was found to be such a function: $\cos \frac{\pi y}{b L} + c \cosh \frac{\pi y}{L d}$, in which b, c, and d are arbitrary constants to be determined by the boundary conditions and one other requirement. This other requirement is that the value of τ_e is such that the coefficients of each of these terms becomes zero. This requirement leads to extremely complicated relationships.

Reference to Fig. 3 for this case reveals that for long tubes a number of waves exist so that only the end wave would be affected materially.[14] In shorter tubes the longitudinal elements of the cylinder are affected only a short distance from the end. These facts suggest a consideration of a fixed ended tube as an equivalent shorter tube with simply supported ends.

---

[14] "Strength Tests on Thin-Walled Duralumin Cylinders in Torsion," by Eugene E. Lundquist, *Technical Note No. 427*, National Advisory Committee for Aeronautics, Washington, D. C., 1932, p. 12, Fig. 4.

Let the equivalent length be represented by $a\,L$. The critical shear then for this equivalent tube may be computed from the curves in Fig. 5 as soon as the value of a is known. To arrive at a value for a, consider a longitudinal section of the tube in its deflected position. The longitudinal half-wave length of the deflected wall may be found as the distance between points of zero deflection. In this case the deflection of the equivalent tube would be expressed by the equation:

$$z = \Delta \cos\left(N\,\theta + \frac{K_r\,\pi\,y}{a\,L}\right)\cos\frac{\pi\,y}{a\,L} \quad \text{(31a)}$$

The distance between points of zero deflection occurs when

$$N\,\theta + \frac{K_r\,\pi\,y}{a\,L} = q\,\pi + \frac{\pi}{2} \quad \text{(31b)}$$

in which q is any integer. Since θ does not vary, the difference between y_1 and y_2, two successive points of zero deflection, may be found from $(y_2 - y_1)\dfrac{K_r\,\pi}{a\,L} = \pi$; or

$$y_2 - y_1 = \frac{a\,L}{K_r} \quad \text{(31c)}$$

in which K_r is the value obtained for the "simply supported" tube whose length is $a\,L$. Now consider an end half wave where one point of zero deflection falls at the end of the tube. This curve will be very nearly a sine form. If the wave is replaced by another curve of the form, $1 - \cos 2\dfrac{\pi\,a\,L}{K_r}$, both the deflection and slope at the end would be zero. The point of contraflexure in the latter curve falls about one fourth of the way from the end to the next point of zero deflection. The tube length between these points of contraflexure is taken as the equivalent tube with simply supported ends. This length then is

$$a\,L = \left(L - \frac{2\,a\,L}{4\,K_r}\right) = \left(L - \frac{a\,L}{2\,K_r}\right) \quad \text{(32a)}$$

from which it is found that

$$a = \frac{2\,K_r}{2\,K_r + 1} \quad \text{(32b)}$$

in which K_r is based on the length $a\,L$. The value of a can be found quickly by successive approximations as follows: First, assume a value of $a\,L$; then find K_r from Fig. 5; and solve for a from Eq. 31b. With this new value of a again solve for K_r, and so on. In two or three cycles the values agree within the limits of error for reading the curves.

PART II. COMPARISON BETWEEN THEORETICAL AND EXPERIMENTAL RESULTS

Cylinders Subject to Torque

The buckling of the walls of thin-walled cylinders subjected to torque has been recognized as a problem in testing materials. In 1924, N. S. Otey[15] considered the effect of ratio of the diameter to wall thickness of specimens on the strength values obtained. In 1930 R. L. Templin, M. ASCE, and R. L. Moore, Assoc. M. ASCE, considered this phenomenon[16] from the point of view of both diameter-to-thickness ratios and length-to-diameter ratios of test specimens for determining the shearing yield strength of materials.

The first results of extensive tests to determine the buckling strength of very thin-walled cylinders were published by Mr. Lundquist[4] in 1932. The specimens for these tests were thin duralumin sheets rolled into a tube, with edges joined by bolting them to a splice plate of the same thickness as the sheets. Diameter-to-thickness ratios ranged from about 700 to about 2,800—in the range of aircraft fuselage proportions. The tubes were large in diameter (from 15 in. to 30 in.) but were relatively short (L/D-values from 0.1 to 2.5).

The following year (1933) Mr. Donnell[5] published some test results to support his theory. The specimens in these tests were also formed from thin sheets of steel and brass shim stock rolled into tubes. The edges were lapped and soldered, which gave a longitudinal seam of double thickness on one side of the tube. Mr. Donnell's specimens were mostly with small tubes (from 0.318 in. to 5.88 in. in diameter for all except one specimen). The diameter-to-thickness ratios were not as high as Mr. Lundquist's but the L/D-ratios were higher.

In 1931 Mr. Moore made some tests at the Aluminum Research Laboratories in New Kensington, Pa., on specimens machined from extruded and drawn tubing. The length-to-diameter ratios ranged from 5 to 7 and the diameter-to-thickness ratios from about 10 to 100. The results of these tests are included in a report by A. H. Stang, W. Ramberg, and G. Back.[17]

In 1939 another series of tests designed to cover the range from short to moderately long tubes was made by Mr. Moore and D. A. Paul.[18] The material used was aluminum alloy 51S–T, whose typical mechanical properties are: Yield strength, 40 kips per sq in.; tensile strength, 48 kips per sq in.; and elongation (for a thin sheet) 14% in 2 in. The tubes were extruded and drawn to size. They were very true to nominal dimensions. The maximum variation in wall thickness was less than ±5%, and the variation in diameters was less than 0.1%. The ends of the test specimens were closed with steel plugs machined to fit the inside diameter of the tubes so that, when the ends of the tubes were gripped in the jaws of the testing machine, they were quite effectively clamped as well as held to shape. The length-to-diameter ratios of these specimens ranged from 1 to 60 and the diameter-to-thickness ratios from about 75 to 140.

---

[15] "Torsional Strength of Nickel, Steel and Duralumin Tubing as Affected by the Ratio of Diameter to Gage Thickness," by N. S. Otey, *Technical Note No. 189*, National Advisory Committee for Aeronautics, Washington, D. C., 1924.

[16] "Specimens for Torsion Tests of Metals," by R. L. Templin and R. L. Moore, *Proceedings*, A.S.T.M., Vol. 30, Pt. II, 1930, pp. 534–543.

[17] "Torsion Tests of Tubes," by A. H. Stang, W. Ramberg, and G. Back, *Technical Report No. 601*, National Advisory Committee for Areonautics, Washington, D. C., 1937.

[18] "Torsional Stability of Aluminum Alloy Seamless Tubing," by R. L. Moore and D. A. Paul, *Technical Note No. 696*, National Advisory Committee for Aeronautics, Washington, D. C., 1939.

Fig. 6 shows a comparison between the measured and computed values of critical shearing stresses for two sizes of tubes. The experimental values of critical shearing stress from the Moore-Paul[18] tests generally lie above the computed curve for "simply supported ends" based on minimum measured thickness and some lie above the computed curve for "simply supported ends" based on maximum measured thickness. In all cases the measured values lie below the computed curves for fixed ends. In the range of low L/D-ratios, where the critical shearing stress in the tube approaches the shearing yield strength of the material, the measured values lie appreciably below the computed curves. This condition may be explained by a consideration of two factors, either or both of which may be responsible for this tendency.

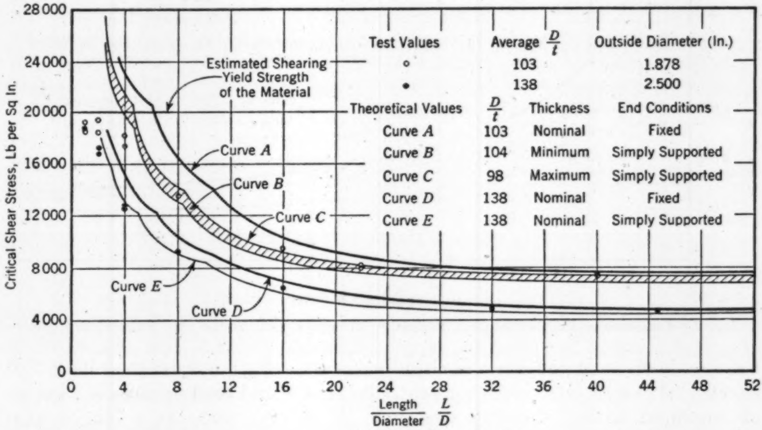


Fig 6.—Comparison of Theoretical Stress τ_c, with the Results of the Moore Tests, for Tubes 0.018 In. Thick

As the material is stressed near the yield strength in shear, plastic action takes place and the shearing deformations increase rapidly. This larger deformation would have much the same effect as a higher stress on the behavior of the tube. The shearing-stress, shearing-strain curves for most materials, including aluminum alloys, are relatively very flat at stresses above the yield strength.

The second factor influencing the critical shearing stress as determined by tests is the unavoidable initial departure from true circularity. From Eq. 30*b* it is found that, for a given value of Δ_o/t and a given maximum stress S, the ratio τ_1/τ_c will be less for short tubes in which N is large and the value of τ_c is high. For long tubes, however, the ratio of τ_1/τ_c would be expected to be much closer to unity. A comparison between the computed and measured critical shearing stresses substantiates this reasoning.

Fig. 7 shows a comparison between computed values of critical shearing stress and values taken from Mr. Lundquist's work. As previously mentioned, the specimens each contain a longitudinal seam. Such a seam may have two effects: (*a*) It may introduce an eccentricity that would tend to cause low values of measured critical shearing stress; or (*b*) it may serve as a longitudinal stiffener and may thereby tend to strengthen the tube.

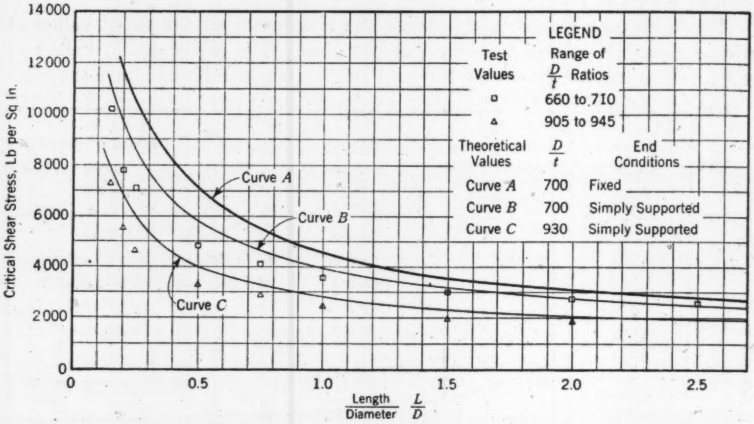


FIG. 7.—COMPARISON OF THEORETICAL STRESS τ_e WITH THE RESULTS OF THE LUNDQUIST TESTS

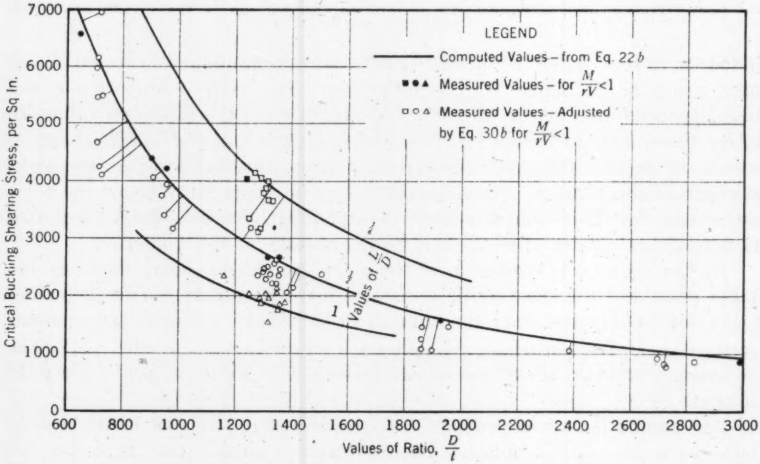


FIG. 8.—COMPARISON OF COMPUTED AND MEASURED VALUES OF CRITICAL BUCKLING SHEARING STRESS FOR TRANSVERSE SHEAR AND BENDING

The fact that Mr. Lundquist reported that the first visible indication of buckling (relatively large radial deflection) occurred at a load well below that causing failure suggests that the eccentricities were relatively large compared to the wall thickness. Fig. 8 reveals that the measured values of critical shearing stress, as reported by Mr. Lundquist,[19] are relatively lower with respect to the computed values than were the values determined from the seamless tubes used by Messrs. Moore and Paul.[18] In general, however, the agreement is quite satisfactory when allowances for out-of-roundness are considered.

Shear and Bending.—Results from tests of thin-walled duralumin cylinders in combined transverse shear and bending which Mr. Lundquist reported in 1935[19] serve as an excellent basis for a check on the extended applicability of this theory to shear failure of thin-walled cylindrical vessels subjected to shear loads other than torsion. Inasmuch as the longitudinal seam was always placed on the extreme tension side, it would not be expected to influence the shear distribution or the curvature at the sides of the cylinders. Fig. 8 shows a comparison of the measured and computed values of critical shearing stress using Eq. 33 in Part III.

PART III. APPLICABILITY OF THEORY TO OTHER PRACTICAL PROBLEMS

The analysis presented herein can be applied directly to long cylinders stiffened with circumferential rings by considering the length between stiffeners as the length of cylinder with simply supported ends. If heavy diaphragms are used, some restraint may be present; but, for design purposes, such restraint may be too indefinite to be taken into account.

Stiffened Cylinders Subject to Torque and Transverse Shear

For cylinders with both circumferential and longitudinal stiffeners, the shell may buckle, in some cases, in waves extending from one circumferential stiffener to the next without encountering a longitudinal stiffener; or it may buckle in waves that encounter a longitudinal stiffener before reaching the circumferential stiffener. If a complete buckle wave extending from one circumferential stiffener to the next is not interrupted by longitudinal stiffeners, the panel might reasonably be considered as buckling at the same stress as if the longitudinal stiffeners were not there. If the crest of a buckle wave intersects two longitudinal stiffeners, however, it may be assumed that the longitudinal distance between intersections is the effective length of the cylinder. The torsional buckling strength, then, of the stiffened cylinder would be that of an unstiffened cylinder of this shorter length. The angle α used for determining the reduced effective length should agree with the length finally computed. This computation may be made by successive approximations, because the series converges very rapidly.

Attention is called to the fact that, when such a double stiffened shell buckles, a series of trusses is formed with the buckled shell acting as a tension member similar to the web in a "tension field" girder. In this type of action both the longitudinal and circumferential stiffeners form the basic framework for carrying load.

---

[19] "Strength Tests of Thin-Walled Duralumin Cylinders in Combined Transverse Shear and Bending," by E. E. Lundquist, *Technical Note No. 523*, National Advisory Committee for Aeronautics, Washington, D. C., April, 1935.

Although no adequate method has been developed for determining, by direct analysis, the critical maximum shear for cases of variable shear, it is possible to obtain a fair indication of this critical shear by suitable assumptions. One possible application of the results of this analysis to a variable shear problem is its use in computing the critical shear in the side walls of a thin-walled cylindrical vessel acting as a beam.

In many practical cases the bending moment in the vessel is small particularly where the shear is greatest. In such cases it has been found from experience and test[19] that the small bending moment has practically no effect on the shear resistance of the vessel walls or on the character of failure obtained.

When buckling governs, the critical shearing stress for the shell wall may be closely approximated as the critical torsional shear in a cylindrical shell whose radius-to-thickness ratio is the same as that of the shell considered and whose length-to-radius ratio is such that tan α as determined from Fig. 5 is equal to r/L. This ratio can be determined quite readily from Fig. 5. In the case of a pipe or tank supported as a simple beam, if the half span is less than the length corresponding to this ratio, the half length might reasonably be used as the effective length of cylinder, since there is no shearing stress at the midspan. For cantilever beams, the full length of the cantilever should be used unless it is greater than the effective length.

A typical example of the cantilever beam action is a stressed-skin fuselage of an airplane. The structural action of such a thin sheet is simulated by the Lundquist tests.[19] The data reported by Mr. Lundquist indicate that, when the bending moment is appreciable, the resistance to buckling in combined shear and bending is less than when the bending moment is small.

A study of these data indicates that, under combined bending and torque, a reliable value of τ'_c for combined shear and bending can be found from the following relation between critical shearing stress (for pure torque) τ_c and the ratio $\frac{\bar{M}}{r\,V}$, by the equation:

$$\tau'_c = \tau_c \left[1 - 0.075 \left(\frac{\bar{M}}{r\,V} - 1 \right) \left(1 + \tfrac{1}{2} \frac{r^2}{L^2} \right) \right] \dots\dots\dots (33)$$

in which $\bar{M}$ is the total bending moment on the shell, in inch-pounds; and V is the total shear on the shell, in pounds. For values of $\frac{\bar{M}}{r\,V}$ of less than one, τ'_c may be taken equal to τ_c. For values of $\frac{\bar{M}}{r\,V}$ greater than 5, this approximation appears to be ultraconservative. Values of critical shear computed in this way and experimental values obtained by Mr. Lundquist are compared in Fig. 8. Until a more adequate theory is provided such an approximation may be very useful.

SHELLS CONTINUOUS OVER SEVERAL SUPPORTS

A specific example of continuous beam action is afforded by a pipe line filled with water and supported at intervals.[20] If the circular section of the shell is held fixed at the supports by ring girders, or by any other method, the distribution of shears in the side wall is known quite definitely and may be expressed as follows for either a single span or one of a long series of equal spans:

---

[20] *Transactions*, ASCE, Vol. 98, 1933, p. 154.

$$S_{vs} = -q\,r\,\frac{L}{2}\sin u = \tau\,t \qquad (34)$$

As would be expected, the shear is a maximum at the ends of horizontal diameters and decreases very slowly for small angular distances from this point. Hence, it may be assumed to be practically constant for about one quadrant of arc. The critical shearing stress in the side walls in this case will also depend upon the ratio $\frac{\bar{M}}{r\,V}$ at the supports and the value of τ'_c determined from Eq. 33, using a value of τ_c determined for a length L equal to $\frac{\pi r}{2}$.

CONCLUSIONS

A review of the foregoing analytical study of perfectly round cylindrical shells and shells departing slightly from perfection, the comparison of the results of this analysis with experimental results, and the adaptation of the analysis to other practical problems suggest the following conclusions:

1. The analysis has yielded a single equation (Eq. 22*b*) which together with the accompanying curve chart (Fig. 5) provides a procedure for computing a value for the critical shearing stress, the number of wrinkles, and the spiral angle of the wrinkles for thin-walled cylindrical shells of any proportions subjected to pure torsional stresses.
2. Slight departures from perfect roundness cause considerable decrease in the computed critical torsional shearing stress for thin-walled cylinders.
3. Comparison between measured and computed critical shearing stress values indicates agreement within the limits of accuracy consistent with the known variables.
4. The analytical results may be used to estimate values of critical shear for stiffened cylinders and cylinders subject to transverse shear and bending.
5. Comparisons between measured and computed values of critical shear in transverse shear and bending indicate that the computed estimates are sufficiently close to the measured values to serve as a basis for design.
6. Additional test data are needed to determine the reliability of the estimated values of critical shear for cylinders reinforced by longitudinal and transverse stiffeners.

ACKNOWLEDGMENTS

Most of the work of preparing this paper was done while the writer was an employee of the Aluminum Research Laboratories of the Aluminum Company of America and had at his disposal the original data of experimental work performed under the direction of Mr. Templin, the assistant director of research.

The writer wishes to thank Messrs. R. L. Templin and E. C. Hartmann, Members, ASCE, and R. L. Moore and Marshall Holt, Assoc. Members, ASCE, all of the Aluminum Research Laboratories, for their constructive suggestions during the development of this paper. Among the drawings and photographs in the paper, Fig. 3 is included by courtesy of the Aluminum Company of America. The essential parts of this paper were first presented as a thesis to the University of Nebraska at Lincoln, in 1938, in partial fulfilment of the requirements for the professional degree of Civil Engineer.

DISCUSSION

S. B. Batdorf,[21] Esq., and Manuel Stein,[22] Esq.—An interesting addition to the literature on buckling of cylinders in torsion is presented by Professor Sturm. As far as the writers are aware, no one else has given a solution to the problem of determination of the critical torque in cylinders of finite length by the differential equation approach, in which equations other than the simplified ones of L. H. Donnell[5] are employed. The results for very long cylinders that buckle into an elliptical cross section (two circumferential waves) are in all probability more accurate than those of Mr. Donnell, since the approximations in the Donnell differential equation were based on the assumption that the square of the number of circumferential waves is large compared to 1. This discussion is primarily for the purpose of calling attention to the situation in the limiting case of very short cylinders, and to discuss certain features of the derivation used.

Physically, if the radius of the cylinder under torsion approaches infinity while the length and thickness remain fixed, there is obtained an infinitely long flat plate of width L subjected to shear. In this case, the critical stress is given by

$$\frac{\tau_c}{E} = k_s \frac{\pi^2 t^2}{12\,(1 - \mu^2)\,L^2} \qquad (35)$$

in which $k_s = 5.34$.[23,24] This agrees with the form used by Professor Sturm (Eq. 25*b*), if

$$K_D = \frac{5.34\,\pi^2}{12\,(1 - \mu^2)} \frac{D^2}{L^2} \cong 5\,\frac{D^2}{L^2} \qquad (36)$$

This value of K_D can be plotted in Fig. 5 as a straight line whose slope is -2. Since the critical stress of a curved sheet can never be below that of the corresponding flat sheet, Fig. 5 is unduly conservative when it predicts stresses below this value—that is, no curve should pass to the left of this straight line. That each of the curves of Fig. 5 should be faired into this line, rather than join it abruptly, is apparent from other studies of this problem.[5,25]

The problem of buckling of a finite cylinder in torsion defied solution for many years because of the difficulty of satisfying simultaneously the differential equation and the boundary conditions. Professor Sturm's solution does indeed satisfy the differential equations, but not the condition of simple support. That it satisfies the differential equation and one of the simple support conditions, $z = 0$ when $y = \pm \frac{L}{2}$, has been shown in the paper. That it does not, however, satisfy the other simple support condition, $\frac{\partial^2 z}{\partial y^2} = 0$ at $y = \pm \frac{L}{2}$ can be seen by differentiation of the buckle form assumed (Eq. 31*a*).

---

[21] Physicist, Langley Memorial Aeronautical Laboratory, National Advisory Committee for Aeronautics, Langley Field, Va.

[22] Aeronautical Engr., Langley Memorial Aeronautical Laboratory, National Advisory Committee for Aeronautics, Langley Field, Va.

[23] "Critical Shear Stress of an Infinitely Long Flat Plate with Equal Elastic Restraints Against Rotation Along the Parallel Edges," by Elbridge Z. Stowell, *Advance Restricted Report No. 3K12* (Wartime Report L476), National Advisory Committee for Aeronautics, Washington, D. C., 1943

[24] "On the Stability Under Shearing Forces of a Flat Elastic Strip," by R. V. Southwell and Silvia W. Skan, *Proceedings*, Royal Soc. of London, Series A, May 1, 1924, pp. 582-607.

[25] "Critical Stress of Thin-Walled Cylinders in Torsion," by S. B. Batdorf, Manuel Stein, and Murry Schildcrout, *Technical Note No. 1344*, National Advisory Committee for Aeronautics, Washington, D. C. 1947.

Some interesting light is thrown on this deflection function and the Sturm method of solution by solving the simple problem of an infinitely long flat plate in shear by this method, and comparing the results with those obtained by the energy method using the same deflection function, as well as with results obtained by exact methods for simply supported and clamped edges. These results [23, 24, 26] are summarized in Table 1.

TABLE 1.—COMPARISON OF RESULTS OF DIFFERENT COMPUTATION METHODS

Method of Computation (1)	k_s (2)	Wave length, λ/L (3)	Wave angle, $\tan \alpha$ (4)
Simple support (exact).....	5.34	1.26	....
Energy method...........	5.7	1.224	0.707
Sturm method............	6.7	0.95	1.1
Clamped edges (exact).....	8.98	0.84	....

Nodal angles are not given for the exact solutions because these solutions give curved rather than straight nodal lines. It appears that Professor Sturm's method is an exact solution for a type of restraint intermediate between simple support and clamped edges, but in which the moment applied to the edge is out of phase with the slope by a quarter wave length.

For other than very short cylinders the difference in buckling load for clamped and simply supported edges is quite small, so that it is not anticipated that any considerable error in critical stresses will result from the peculiar boundary conditions implied. However, from the comparison in Table 1 it appears that there may be considerable error in the buckling configuration.

An ingenious suggestion is made by Professor Sturm on how to use results for complete cylinders to obtain approximate values for the shear buckling stress of the curved panels between adjacent stiffeners in stiffened cylinders. It may be of interest that calculations have been recently published for the critical shear stresses of simply supported and clamped curved panels.[27]

GLENN MURPHY,[28] M. ASCE.—This is an interesting approach to a timely problem, in that the author's analysis leads to results which not only are in reasonable agreement with representative tests, but which also tend to substantiate the theories of previous investigators. A number of conclusions may be drawn from Fig. 5, which gives values of K_D for use in Eq. 25*b*. It is apparent from Fig. 5 that there are two separate phases of behavior—one, in

---

[26] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1936, p. 361.

[27] "Critical Shear Stress of Curved Rectangular Panels," by S. B. Batdorf, Manuel Stein, and Murry Schildcrout, *Technical Note No. 1348*, National Advisory Committee for Aeronautics, Washington, D. C., 1947.

[28] Prof. of Theoretical and Applied Mechanics, Iowa State College, Ames, Iowa.

which $\left(\text{with constant } \frac{D}{t}\right)$ K_D, and consequently τ_c, decreases with increasing values of $\frac{L}{D}$; and the other, in which K_D and τ_c are independent of $\frac{L}{D}$. These two phases, between which there is shown smooth transition, confirm previous conclusions that one expression may be used for short cylindrical shells and a different expression for long shells.

In the range in which values of $\frac{L}{D}$ are large and the curves of Fig. 5 are horizontal, the equation for K_D is readily determined as:

$$K_D = 0.70 \sqrt{\frac{D}{t}} \quad \text{.......................(37)}$$

If this value of K_D is substituted in Eq. 25*b*, it becomes:

$$\frac{\tau_c}{E} = 0.70 \left(\frac{t}{D}\right)^{\frac{3}{2}} \quad \text{.....................(38a)}$$

which is exactly equivalent to the equation given by S. Timoshenko for long cylindrical shells with Poisson's ratio of 0.33. L. H. Donnell's equation reduces to

$$\frac{\tau_c}{E} = 0.80 \left(\frac{t}{D}\right)^{\frac{3}{2}} \quad \text{.....................(38b)}$$

In the short-tube range an approximate equation may be developed for K_D by replacing the author's curves (formed from a series of loops) by a family of straight lines tangent to the curves. Whereas the slopes of these straight lines vary somewhat, none deviates appreciably from 0.55, and the equation of the family of lines may be determined as:

$$K_D = \left(\frac{D}{L}\right)^{0.55} \left(\frac{D}{t}\right)^{0.80} \quad \text{...................(39)}$$

If this value of K_D is substituted in Eq. 25*b*, the latter becomes:

$$\frac{\tau_c}{E} = \left(\frac{D}{L}\right)^{0.55} \left(\frac{t}{D}\right)^{1.20} \quad \text{...................(40)}$$

The Donnell equation for short tubes in torsion may be reduced to the form:

$$\frac{\tau_c}{E} = 0.75 \left(\frac{D}{L}\right)^{0.50} \left(\frac{t}{D}\right)^{1.25} \quad \text{..................(41)}$$

It is apparent that Eqs. 40 and 41 are similar although not identical.

The approximate boundary between the short-tube range and the long-tube range may be established from Fig. 5 by noting that the locus of the points of intersection of the family of lines for the short cylinders with the corresponding $\left(\text{same } \frac{D}{t}\right)$ lines for the long cylinders is a straight line sloping upward to the right. The equation of this line may be determined as:

$$\frac{L}{D} = 2.45\sqrt{\frac{D}{t}} \qquad (42)$$

If $\frac{L}{D}$ is less than $2.45\sqrt{\frac{D}{t}}$, the shell may be considered short and Eq. 40 applies; whereas if $\frac{L}{D}$ is greater than $2.45\sqrt{\frac{D}{t}}$, the shell may be considered long and Eq. 38*a* applies.

An interesting comparison may be made between Eq. 42 and the Donnell equation defining the boundary between the short tube and the long tube. For simply supported edges, Mr. Donnell establishes the boundary by

$$\frac{1}{\sqrt{1-\mu^2}}\frac{L^2 t}{D^3} = 5.5 \qquad (43)$$

or

$$\frac{L^2 t}{D^3} = 5.19 \qquad (44)$$

for Poisson's ratio equal to 0.33. However, Eq. 42 may be in the form:

$$\frac{L^2 t}{D^3} = 2.45^2 = 6 \qquad (45)$$

which compares favorably with Eq. 44. It should be noted that Eqs. 40 and 42, developed by the writer from Fig. 5, are approximate and are presented primarily to indicate qualitative agreement between the results of the author and other investigators.

L. H. DONNELL,[29] ESQ.—A welcome addition to the already considerable literature on the subject has been made by this paper. Like previous solutions of this problem, those cited in the paper are based on approximations, but these approximations are different from those used in previous solutions. It is reassuring, in a field in which no exact solution has been obtained, that this solution, in the region to which it has been applied, never differs by more than 13% or 14% from previous solutions.

Unfortunately, the paper may be misleading in a number of respects to those not familiar with the subject. For instance, unlike some other treatments[5] that cover all possible proportions, the author's treatment covers only the range of proportions which might be described as "moderately long" cylinders. This point is not even discussed; thus readers are not warned against applying the results to very short or very long cylinders, where they may be quite inaccurate and on the unsafe side.

Moreover, the treatment of boundary conditions is certainly confusing and incomplete. These conditions, for the simply supported cylinder, are stated (under the heading, "Adaptation of General Theory of Buckling to Problem of Torsion") as:

"* * * When $y = +\frac{L}{2}$, $z = 0$; and, for all values of θ, $\frac{\partial z}{\partial \theta} = 0$.

---

[29] Research Prof. of Mechanics and Director, Fundamental Mechanics Research Laboratory, Illinois Inst. of Tech., Chicago, Ill.

"The choice of the center of coordinates midway between the ends of the cylinder or tube and at the point of maximum displacement around the shell leads to the boundary condition that $\frac{\partial z}{\partial \theta} = 0$ when $y = 0$ and $\theta = 0$."

The complete boundary conditions for this case, as given by the writer[5] in 1933 and quoted by S. Timoshenko,[6] are (using the symbols in the paper): When $y = \pm \frac{L}{2}$, $u = 0$, $v = 0$, $z = 0$, and $\frac{\partial^2 z}{\partial y^2} + \frac{\mu}{r^2}\frac{\partial^2 z}{\partial \theta^2} = 0$. The first two of these conditions ($u = 0$ and $v = 0$) are not very important, but the third and fourth are both essential. The fourth condition states that the edge moments are zero; that is, that the edges are freely hinged. Without this condition the boundary conditions would be indeterminate, and might cover an infinite variety of conditions, including the case of fixed edges or even a case of moments opposite to those present when the edges are fixed.

The solution given in this paper for simply supported edges does not satisfy any of these conditions except the third. The treatment of the fixed edge condition in this paper is even more confusing, but is apparently a still further approximation, based on the solution for simply supported edges. On the other hand, in the writer's paper[5] the boundary conditions for both cases were completely satisfied except for the very unimportant condition, $u = 0$. However, other approximations were made, in the form of omitting certain minor terms which are considered by the present author.

In discussing the writer's paper[5] in the "Introduction," the author cites its approximations and its differences between theory and experiments as the chief justifications for his own paper. On the other hand, the approximations of the new solution are not even mentioned, and its agreement with experiments is described as "quite satisfactory." The reader is left with the impression that the new theory eliminates the approximations of the old and is in good agreement with experiments, whereas the old theory was not. However, no evidence is given to support such implications, even for the range to which the new theory applies. No comparison is made between the two theories or between their experimental checks.

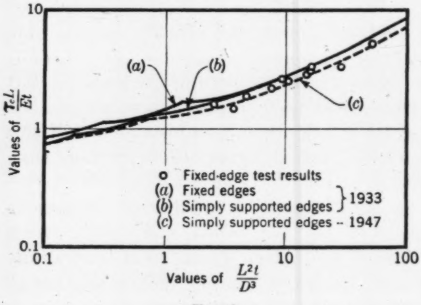


FIG. 9

In Fig. 9 the full lines show the results from the 1933 paper[5] for the "moderately long" range. For simplicity, only the most interesting part of this range is shown and Poisson's ratio is taken as 0.3. When the theoretical values are plotted with the variables shown, a single curve is obtained which covers all materials and proportions in this range for each edge condition. One such curve gives all the information offered by a family of curves such as that shown in Fig. 5. There is no "sudden break" or "confusion resulting from the use of two formulas," as might be inferred from the author's "Introduction." To be sure, such a curve covers only the moderately long range, and different formulas or curves are needed both for very short cylinders—where the conditions approach those found by R. V. Southwell and S. W. Skan,[24] in 1924, for flat strips under shear—and for very long cylinders—where the number of circumferential waves may become one and the conditions are as found by A. G. Greenhill[2] in 1883. However, such "complications" cannot satisfactorily be avoided by simply ignoring the phenomena that occur in such ranges of dimensions, as the author has done.

Charts such as Fig. 9 are easy to use for numerical computation. When the computer knows the specifications of the cylinder he can compute the ordinate and select the abscissa from the curve; the value of the buckling stress, τ_e, is then readily calculated from this value by reference to the specifications. Such calculations are no more difficult than for the family of curves given by the author, and the difficulty of interpolating between, or extrapolating beyond, this family of curves is avoided. In the writer's paper[5] the effects of variations in Poisson's ratio (which may be considerable but which are neglected in Professor Sturm's paper) are also considered.

The method of plotting Fig. 9 is convenient for comparing the two theories and their relation to experiments. If the results given in Fig. 5 are plotted in Fig. 9 they all fall, within 1% or 2%, on the dotted curve (Fig. 9, curve (c)). This means that the results of experiments cannot "line up" better or show appreciably less scatter when plotted in Fig. 5 than they do when plotted against the single curves in Fig. 9. (However, a method of plotting which does show some promise of reducing the scatter is shown elsewhere.[30])

The maximum difference between the two theories in this range is about 13% (no comparison can be made for the extremes of length-to-diameter ratios, since the new theory does not cover them and would show very large discrepancies if so extended).

The question now arises as to which theory is the most accurate in this range. As has been stated, the older theory satisfied edge conditions but had other simplifications, whereas the newer theory does not satisfy edge conditions and avoids these simplifications. It would be difficult if not impossible to state, without further evidence, which of these approximations is most important in the present case. To be sure, Professor Timoshenko, in discussing this problem,[6] seems to feel that edge conditions become unimportant for long tubes; but he gives no reasons or evidence for this feeling. In cases where the deformations consist of many waves in the direction of the length, as in cylinders under axial load or struts with elastic support, such a conclusion is certainly justifiable. In the case of torsion buckling, however, there is only one wave in the longitudinal direction. This spirals around the cylinder and a longitudinal section may seem to indicate a number of waves, but there is actually only one continuous wave in this direction. In this respect therefore this case is similar to the case of a simple strut. For such a strut, of course, the buckling strengths for fixed and hinged ends always have a ratio of 4 to 1, no matter how great the length. In the case of tubes in torsion where any fixity is in the wall rather than in the entire end, there is certainly not a 400% effect; but there is no reason whatever for not expecting a possible 13% effect due to the end conditions.

---

[30] "The Stability of Isotropic or Orthotropic Cylinders or Flat or Curved Panels, Between or Across Stiffeners, with Any Edge Conditions Between Hinged and Fixed, Under Any Combination of Compression and Shear," by L. H. Donnell, *Technical Note No. 918*, National Advisory Committee for Aeronautics, Washington, D. C., 1943, Fig. 19.

There is only one real item of evidence which indicates that end conditions are unimportant, say in the range of $\frac{L^2 t}{D^3} > 5$. This is that the writer's theory[5] gives practically the same result for the hinged end and fixed end conditions in this range. However, this is the only region in which the Sturm theory gives a value appreciably different from the old theory and it is perhaps not entirely logical to reject one theory and choose another on the sole evidence of the theory which is rejected.

As to the experimental evidence, all the available test results reported in both papers, in the range where they differ appreciably, are shown in Fig. 9. If it be assumed that the author's theory would give approximately the same results in this range for the fixed edge condition (all experiments have been for this condition) then at first glance the evidence seems to favor that theory. The Sturm curve seems to pass through about the average of the experiments, whereas the experiments all lie below the curve representing the writer's theory. However, this is actually evidence against the author's theory and not in favor of it—paradoxical as this may seem. It should be emphasized that all these theories are for "perfect" specimens. Actual specimens always involve some imperfections, so that, in general, the buckling strengths determined in good experiments are always below the strengths predicted by a good theory.

In comparing two approximate theories, the one that happens to give the lower strengths will usually be closer to the median of the experiments and so will apparently "check" the experiments better; but this does not mean that it is the better approximation. On the contrary, when experimental points lie above a theoretical curve for buckling of ideal specimens it is an indication that something is wrong. If an approximation in a theory happens substantially to balance the effect of specimen imperfections, this may be convenient; but it would be safer to have a better theory and use an empirical or other factor to allow for the imperfections of actual specimens. Both the theoretical and experimental evidence is insufficient for deciding which of the two theories is the better in this range. The principal conclusion to be drawn is that both theories are evidently excellent approximations, which would seem to be the chief value of the author's solution.

The author also discusses the effects of imperfections in actual cylinders. Although this approach is interesting, it seems to be much too approximate to be quantitatively significant, especially for studying the rather small effects that appear to be present. The discussion is based on the assumption that the ultimate strength is just about reached when the maximum stresses reach the yield point of the material. This is a reasonable assumption, but in calculating the maximum stress only the bending stresses are considered, whereas both ordinary and large deflection membrane stresses are probably of the same or greater importance. Also, the expression used for calculating the deflection under load is probably not at all accurate in the large deflection range, and most such specimens are probably in this range long before ultimate strength is reached.

R. L. MOORE,[31] Assoc. M. ASCE.—Although consideration of the buckling characteristics of thin-walled cylinders in torsion has been limited largely to aircraft design, the solutions presented by Professor Sturm are of interest wherever thin curved plates are required to resist shear. The application of the theory within the range of proportions covered by Fig. 5 is relatively simple and a reasonably good agreement between observed and computed shear buckling stresses for tubes and thin cylinders has been shown. The purpose of this discussion is to present a few additional data and thus to make some evaluation of the method proposed for predicting shear buckling in stiffened cylinders or curved panels.

TABLE 2.—COMPARISON OF OBSERVED AND COMPUTED SHEAR BUCKLING CHARACTERISTICS OF ALUMINUM ALLOY CYLINDERS HAVING RING STIFFENERS ONLY

Spacing of rings L_r, in inches	Ratio[a] $\frac{L_r}{D}$	SHEAR BUCKLING STRESS, IN POUNDS PER SQUARE INCH		WAVE ANGLE, α, IN DEGREES	
		Observed[b]	Computed[c]	Observed[b]	Computed[c]
35.62	1.18	1,110	1,500	14	11
27.00	0.90	1,280	1,750	18	12.5
13.50	0.45	1,710	2,560	25	17.5
9.00	0.30	2,570	3,170	30	21.5
8.81	0.29	2,310	3,220	30	22

[a] Mean diameter $D = 30.08$ in. and wall thickness $t = 0.020$ in. [b] The observed values are averages of three to five determinations for each ratio of L_r/D. [c] The computed values are based on the Sturm solution for simply supported ends.

Tables 2 and 3 show comparisons between observed and computed shear buckling characteristics for aluminum alloy, circular cylinders having both circumferential and longitudinal stiffeners. These results, obtained in tests made during 1943 and 1944[32] at the Aluminum Research Laboratories (New Kensington, Pa.), are of interest because they involve ratios of diameter-to-wall thickness (D/t) greater than those considered by Professor Sturm in his comparisons, and because they show directly the influence of longitudinal stiffeners on shear buckling resistance.

Figs. 10 and 11 show the type of specimens used and indicate the manner of loading. The ends of the cylinders were fitted with steel bulkheads, which in turn were bolted to frames through which torque could be applied. One frame was fixed while the other (visible in Fig. 10) could be rotated about the central axis of the specimen. The cylinders were 30 in. in diameter and were formed of sheets 0.020 in. by 36 in. by 96 in., which were wrapped around the end bulkheads and riveted together in a simple lap joint. The ring stiffeners used on the inside were formed of ½-in. square bars, whereas the longitudinal stiffeners used on the outside were of **U**-section, formed of 0.032-in. sheet aluminum. All sheet and stiffeners were of 24S-T3 and T4 aluminum alloy having the following typical properties: Tensile yield strength, 46,000 lb per sq in. and tensile ultimate strength, 68,000 lb per sq in. It is evident from the magnitude of the stresses shown that buckling in the tests occurred well within the elastic range of the material.

---

[31] Research Structural Engr., Aluminum Research Laboratories, Aluminum Co. of America, New Kensington, Pa.

[32] "Torsion Tests of Stiffened Circular Cylinders," by R. L. Moore and C. Wescoat, *Advance Restricted Report No. 4E31*, National Advisory Committee for Aeronautics, Washington, D. C., May, 1944.

TABLE 3.—COMPARISON OF OBSERVED AND COMPUTED SHEAR BUCKLING CHARACTERISTICS OF ALUMINUM ALLOY CIRCULAR CYLINDERS HAVING LONGITUDINAL STIFFENERS AS WELL AS RING STIFFENERS

Spacing of rings L_r, in inches	Ratio[a] $\frac{L_r}{D}$	Wall thickness t, in inches	OBSERVED VALUES[b] FOR DIFFERENT COMBINATIONS OF LONGITUDINAL STIFFENERS						COMPUTED[c] FOR NO LONGITUDINALS		
			No Longitudinals		Eight Longitudinals, 11.81 in. apart		Sixteen Longitudinals, 5.90 in. apart				
			Buckling stress	Wave angle, α	Buckling stress[b]	Wave angle, α	Buckling stress[b]	Wave angle, α	Wave angle, α	$L_r \tan \alpha$	Buckling stress
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
13.50	0.45	0.020	1,710	25	2,110	18	2,870	19	17.5	4.25	2.560
9.00	0.30	0.020	2,570	30	2,950	26	3,860	24	21.5	3.55	3,170
6.75	0.23	0.0205	....	....	....	....	4,740	28	25	3.14	3,860
4.31	0.14	0.020	....	....	4,130	35	6,750	....	32	2.69	4,660

[a] Mean diameter $D = 30.08$ in. [b] For specimens having longitudinal stiffeners, where tension field action prevented, the determination of single, well-defined values of buckling torque, observed stresses (in pounds per square inch) were based on the mean torques required to produce visible buckling in all panels of any one size. [c] Based on the Sturm solution for simply supported ends.

A study of Table 2 reveals two interesting results: (1) The observed shear buckling stresses for cylinders having a D/t ratio of 1,500, with ring stiffeners only, ranged from 67% to 81% of the computed values; and (2) the wave angles α of the buckles, such as shown in Fig. 10, were in every case greater than those computed. The comparison from the standpoint of buckling stresses was only slightly less satisfactory than that shown in Fig. 7 for cylinders having a D/t-ratio of approximately 930. However, it was definitely not as favorable as that shown in Fig. 6 for relatively much longer and thicker tubes, having a D/t-ratio of only 138. Local out-of-roundness or initial buckles in the thinnest cylinders undoubtedly accounted for some of the discrepancies indicated between buckling stresses; but it is not so evident that imperfections of this kind were also responsible for the differences shown between measured and computed wave angles. In view of these observations it is significant that a considerably closer agreement between observed and computed buckling stresses might have been shown in Table 2 had the torsion theories of L. H.

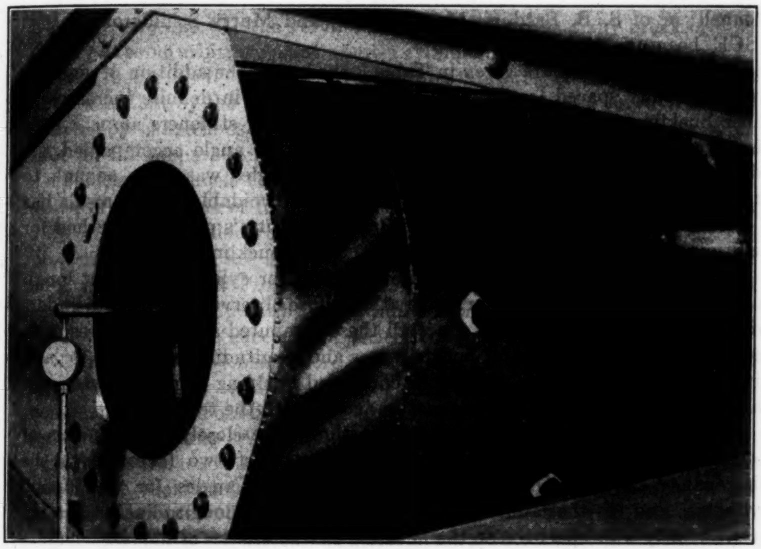


FIG. 10.—SHEAR-BUCKLING FAILURE OF CYLINDER HAVING RING STIFFENERS ONLY (y_r/D-RATIO OF BUCKLED PANEL = 0.29)

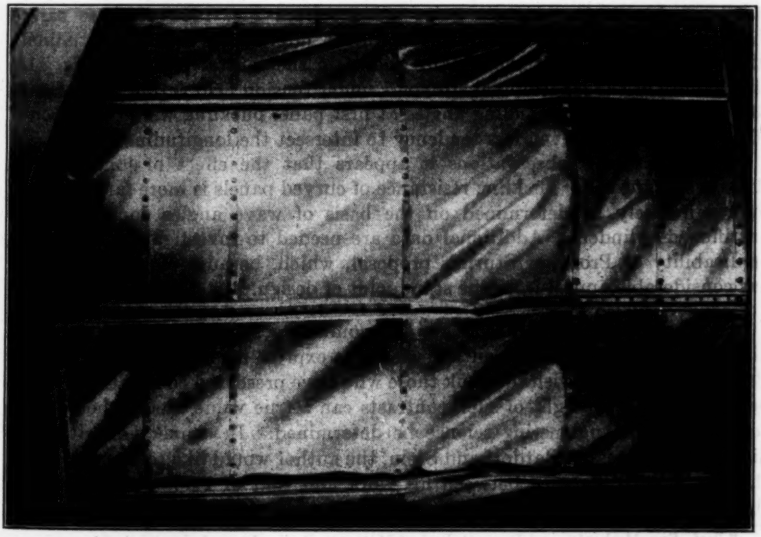


FIG. 11.—FAILURE OF STIFFENED CYLINDER UNDER TORSION LOADING CARRIED BEYOND THE POINT OF PANEL BUCKLING (y_r/D-RATIOS = 0.30 AT CENTER AND 0.14 AT ENDS)

Donnell[5] or of S. B. Batdorf, Manuel Stein, and Murry Schildcrout,[25] Jun. ASCE, been used.

The experimental data given in Table 3 show that the addition of eight or sixteen equally spaced longitudinal stiffeners resulted in definite increases in shear buckling resistance over that observed for ring stiffeners alone. It is also evident in a few cases that a decrease in wave angle accompanied the addition of stiffeners, even though none of the angles was large enough to make a buckle intersection with such stiffeners unavoidable. This result has a direct bearing on the applicability of Professor Sturm's proposed method for determining the effect of longitudinal stiffeners on buckling resistance.

The computed wave angles shown in Table 3 for cylinders without longitudinal stiffeners were in every case less than those observed, even for cylinders with longitudinals. Moreover, since all these computed angles were less than required for an intersection of buckle waves and longitudinal stiffeners for any given spacing of rings, no increase in computed buckling resistance was indicated. This result was obviously not in accord with the tests.

The failure of the author's method to show more closely the influence of longitudinal stiffeners may be attributed to at least two factors—(1) the discrepancy between observed and computed wave angles for unstiffened cylinders; and (2) the failure to recognize the effect of longitudinal stiffeners, even at fairly wide spacings, on the buckling characteristics of the intermediate curved panels. Table 3 shows, for example, that for the tests in which the length-to-diameter ratio (L_r/D) was only 0.30 the wave angles were about 30° for the case of no longitudinals, 26° for eight longitudinals, and 24° for sixteen longitudinals. It would seem quite evident from Fig. 11 that these early shifts in angle were not the result of direct interference with longitudinal stiffeners. The angle of the buckles shown in the central panels in Fig. 11, after failure of the longitudinal stiffeners under tension field action, was about 33°, whereas the corresponding angles at first panel buckling were only about 26° and would show even less tendency to intersect the longitudinals.

From these few observations it appears that the effect of longitudinal stiffeners on the shear buckling resistance of curved panels is more far-reaching than can safely be determined on the basis of wave angles computed for unstiffened cylinders. Additional data are needed to investigate further the applicability of Professor Sturm's proposal, which, because of its simplicity, has considerable appeal from the standpoint of design.

R. G. **Sturm**,[33] M. ASCE.—For the comments presented by the various discussers of this paper, the author wishes to express his sincere appreciation. Particularly does he wish to thank those who have presented contrasting views, because only in the light of such contrasts can a true value of any theoretical contribution to engineering science be determined. In connection with the discussion by Messrs. Batdorf and Stein, the author would like to mention that the significant difference between his work and that of Professor Donnell is that one single expression is used to cover the entire gamut of cylinder lengths from the very short to the very long, thereby eliminating any break or discontinuity at some arbitrary point where a cylinder is considered short or very long.

---

[33] Prof., Eng. Mechanics, and Research Prof. of Materials, Purdue Univ., Lafayette, Ind.

The discussion by Professor Murphy presents another very interesting point of view in that by the use of the algebraic equivalent to the curves some rather simple approximations are evolved. Such approximations are extremely valuable for the preparation of design charts.

Professor Donnell has presented a basic thought in connection with the value of discussions of technical papers—the significance of various points of view. Every solution requires certain assumptions and, in most cases, approximations which are usually different for different types of analysis. The author believes that Professor Donnell has made a real contribution to the general problem by pointing out a number of comparisons, and especially by his discussion of the effects of imperfections in actual cylinders.

The discussion by Mr. Moore is a welcome contribution to the general problem of the stability of thin cylindrical shells in that it gives for the first time a comparison of data obtained on complicated shells with the values computed by analytical methods. Although the discrepancies between measured and computed buckling loads are considerable, it is of interest to note that in all cases the actual buckling load is between 70% and 80% of the computed value, whereas the observed wave angle ranged from 1.26 times to 1.45 times that computed. Such consistency suggests more than a random relationship and indicates that perhaps some factor or factors are always present in the tests or should be included in the theory. The writer agrees with Mr. Moore in his statement about the effectiveness of longitudinal stiffeners and hopes that more experimental data will be forthcoming to serve as a basis for either establishing or modifying the theory to make it reliable for design.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2346

CONSOLIDATION OF FINE-GRAINED SOILS BY DRAIN WELLS

BY REGINALD A. BARRON,[1] JUN. ASCE

WITH DISCUSSION BY MESSRS. KENNETH S. LANE, PHILIP KEENE, WALTER KJELLMAN, AND REGINALD A. BARRON

SYNOPSIS

Drain wells have been used to accelerate the consolidation of fine-grained compressible soils subjected to new loads. However, mathematical analyses of such wells have become available only recently, and are confined to wells having infinitely pervious well backfill and no peripheral smear. Complete formulas for consolidation by vertical and radial flow to wells, for cases with or without peripheral smear and drain well resistance, are presented in this paper; necessary derivations are given in Appendix II. Because soils are not homogeneous, and because of incomplete knowledge of stress-strain consolidation characteristics of soils, these solutions should be regarded as approximate when applied to practical problems.

INTRODUCTION

When a soil mass is subjected to a new load, compressive stresses cause a reduction of pore space and, if saturated, the excess water is expelled. For slightly pervious soil, time is required to accomplish this, and, initially, before any reduction of pore space has occurred, the new compressive stresses are carried by the water (called hydrostatic excess or excess pore-water pressure). As the excess water drains, the new compressive stresses are transferred to the soil grains increasing the intergranular pressures. This process is known, in soil mechanics terminology, as consolidation.

New loads applied to foundation soils create shearing stresses; to maintain stability the soil must not be overstressed. For slightly pervious compressible soils, the factors affecting the shearing strength are not fully understood at present; however, it is known that the shearing strength increases with a reduction of the soil pore water.

Most compressible soils are alluvial deposits and are more pervious in the direction of the bedding plane than in a perpendicular direction. This is very

---

NOTE.—Published in June, 1947, *Proceedings*. Positions and titles are those in effect when the paper or discussion was received for publication.

[1] Engr., Soil Mechanics, Embankment and Foundation Branch, U. S. Waterways Experiment Station, Vicksburg, Miss.

pronounced in varved deposits[2] laid down in lakes during the wasting away of the continental glaciers. When such soils are loaded, horizontal flow accelerates the consolidation of the soil mass as compared with strictly vertical flow. Because of the horizontal migration of water, however, this process imposes a dangerous condition on the stability of the foundation due to increased pore-water pressure[3] reducing the intergranular stresses at the toes of fills.

The limitation that the shearing strength shall not be exceeded may be met by widening the loading area (thus reducing the shearing stresses in the foundation), or by stage construction, thereby permitting a gain in shearing strength from the consolidation process. These methods are costly. Sometimes a more economical method may be to accelerate the consolidation process, and thus hasten a gain in shearing strength, by the use of vertical sand-filled holes known as drain wells. Drain wells are especially efficient in stratified soils because of the greater perviousness parallel to the bedding. Furthermore, drain wells permit control over the water migrating in a horizontal direction, thus reducing the excess pore-water pressures that might be built up in the toe area of an earth fill.

Drain wells have been used in this country mainly by the California State Highway Department,[4,5] and also at an eastern shipyard.[6] They have also been used by the Providence (R. I.) District, United States Engineer Department, as a repair measure to a dike failure. Further use is planned by the latter organization to permit construction of a moderately high dam on a foundation containing a large deposit of soft varved silt.

The theory of consolidation by drain wells was developed by the author during the design of the two Providence District projects previously noted. The theory for ideal wells having no resistance to flow up the well bore and having no smear at the well periphery has been presented elsewhere,[7,8,9,10,11] but consolidation theory which includes the effect of well smear and well resistance is presented here publicly for the first time.

Notation.—The letter symbols in this paper are defined where they first appear, in the text or by diagrams, and are assembled for convenience in Appendix I. In so far as possible the symbols conform with those given in "Soil Mechanics Nomenclature," *Manual of Engineering Practice No. 22.*

---

[2] "The Recession of the Last Ice Sheet in New England," by Ernst Antevs, *Research Series No. 11*, Am. Geographical Soc., Washington, D. C., 1922.

[3] "Stability of Fills Above Horizontal Clay Strata," by Karl Terzaghi, *Proceedings*, 6th Texas Conference on Soil Mechanics and Foundation Eng., Univ. of Texas, Austin, Tex., 1943.

[4] "Studies of Fill Construction Over Mud Flats Including a Description of Experimental Construction Using Vertical Sand Drains to Hasten Stabilization," by O. J. Porter, *Proceedings*, International Conference on Soil Mechanics and Foundation Eng., Harvard Univ., Cambridge, Mass., Vol. I, 1936, pp. 229–235.

[5] "Studies of Fill Construction Over Mud Flats Including a Description of Experimental Construction Using Vertical Sand Drains to Hasten Stabilization," by O. J. Porter, *Proceedings*, Highway Research Board, Vol. 18, Pt. VI, pp. 129–141.

[6] "Submerged Shipways with Steel Sheeting Walls," by Adolph J. Ackerman and C. B. Jansen, *Civil Engineering*, July, 1943, p. 309.

[7] "The Influence of Drain Wells on the Consolidation of Fine-Grained Soils," by R. A. Barron, Providence (R. I.) Dist., U. S. Engr. Office, 1944.

[8] "Der Hydrodynamische Spannungsausgleich in Zentral Entwasserten Tonzylindern," by L. Rendulic, *Wasserwirtsch, u, Technik*, Vol. 2, Jahigang, 1935, pp. 250–253 and pp. 269–273.

[9] "Consolidation of a Soil Stratum Drained by Wells," by N. Carrillo, Harvard Univ., Cambridge, Mass., 1941 (mimeographed).

[10] "Simple Two and Three Dimensional Cases in the Theory of Consolidation of Soils," by N. Carrillo, *Journal of Mathematics and Physics*, March, 1942.

[11] "Theoretical Soil Mechanics," by Karl Terzaghi, John Wiley & Sons, Inc., New York, N. Y., 1943, pp. 290–296.

Basic Considerations

The basic theory of consolidation has been developed by Karl Terzaghi, M. ASCE.[12] For the purpose of this paper the following basic assumptions are made:

(1) All vertical loads are initially carried by excess pore-water pressure u.

(2) All compressive strains within the soil mass occur in a vertical direction. Thus, the basic partial differential equation for consolidation by three-directional flow in stratified soils is:

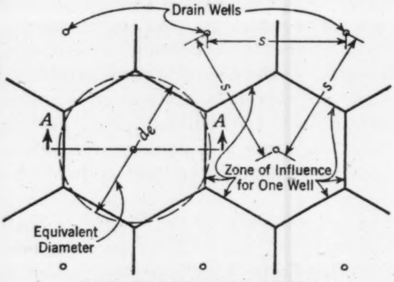
$$\frac{k_h}{\gamma_w}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{k_v}{\gamma_w}\left(\frac{\partial^2 u}{\partial z^2}\right) = \frac{a_v}{1+e}\left(\frac{\partial u}{\partial t}\right) \qquad (1a)$$

in which k_h is the coefficient of permeability of the soil in a horizontal direction; k_v is the coefficient of permeability in a vertical direction, γ_w is the unit weight of water; x, y, and z are rectangular coordinates; e is the void ratio of the soil; t is time; and a_v, the coefficient of compressibility of the soil, equals $-\frac{de}{dp}$. For symmetrical flow to a central drain well Eq. 1a becomes:

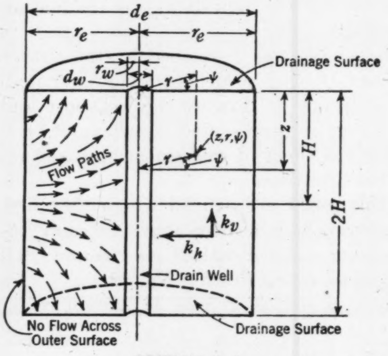
$$\frac{k_h}{\gamma_w}\left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}\right) + \frac{k_v}{\gamma_w}\left(\frac{\partial^2 u}{\partial z^2}\right) = \frac{a_v}{1+e}\left(\frac{\partial u}{\partial t}\right) \qquad (1b)$$

in which r and z are cylindrical coordinates as defined by Fig. 1.

(3) The most economical pattern of drain wells is that shown in Fig. 1. An exact analysis should include the load distribution and the effect of each well on the rate of consolidation at any point in the foundation. However, it is believed that such a solution would be extremely cumbersome and rather academic when consideration is given to the heterogeneity of the foundation, and the present incomplete knowledge of the three-dimensional stress-strain consolidation properties of fine-grained soils.



PLAN OF DRAIN WELL PATTERN



SECTION A-A

Fig. 1.—Plan of Drain Well Pattern and Fundamental Concepts of Flow Within Zone of Influence of Each Well

(4) The zone of influence of each well is a circle (see Fig. 1).

(5) Load distribution is uniform over this area.

---

[12] "Erdbaumechanik auf Bodenphysikalischer Grundlage," by Karl Terzaghi, F. Deutiche, Vienna, 1925.

IDEAL DRAIN WELLS

Wherever drain wells are installed construction operations generally cause some remolding or smear of the fine-grain soil adjacent to the well periphery, especially in varved soils. Also, the backfilling of the well, even with pervious material, results in some resistance to flow of the expelled water up the well bore. For ideal drain wells these effects are ignored, although solutions for these effects are discussed subsequently.

Case of Free Strain.—The initial solution for ideal drain wells is made on the assumptions that the load is uniform over the circular zone of influence for each well and that differential settlements occurring over such a zone, as consolidation progresses, have no effect on redistribution of stresses by arching of the fill. Furthermore, it is assumed that shear strains developed in the foundation by differential settlements will have no effect on the consolidation process. The boundary conditions are:

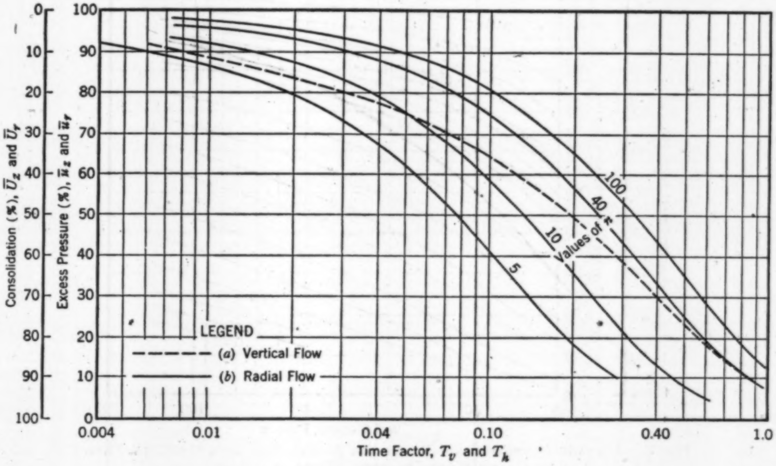


FIG. 2.—AVERAGE CONSOLIDATION RATES
(a) For Vertical Flow in a Clay Stratum of Thickness 2 H Drained on Both Upper and Lower Surfaces
(b) For Radial Flow to Axial Drain Wells in Clay Cylinders Having Various Values of n

(1) Initial excess pore-water pressure, u_0, is uniform throughout the soil mass when $t = 0$.

(2) Excess pore-water pressure at the drain well surface (r_w) is zero when $t > 0$.

(3) The external radius, r_e, is impervious or, because of symmetry, no flow occurs across this boundary; that is, $\frac{\partial u}{\partial r} = 0$ when $r = r_e$.

(4) Excess pore-water pressure at the upper horizontal boundary of the soil mass ($z = 0$) is zero when $t > 0$.

(5) The lower horizontal boundary of the soil mass ($z = H$) is impervious or, because of symmetry, no flow occurs across this boundary; that is, $\frac{\partial u}{\partial z} = 0$ at $z = H$.

Subject to the foregoing conditions a solution of Eq. 1*b* for excess pore-water pressure, $u_{r,z}$, at any point due to consolidation by both radial and vertical flow is:

$$u_{r,z} = \frac{u_r\, u_z}{u_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which u_r and u_z are, respectively, the excess pore-water pressures due to radial flow only and vertical flow only. Similarly, the average excess pore-water pressure throughout the entire soil mass is:

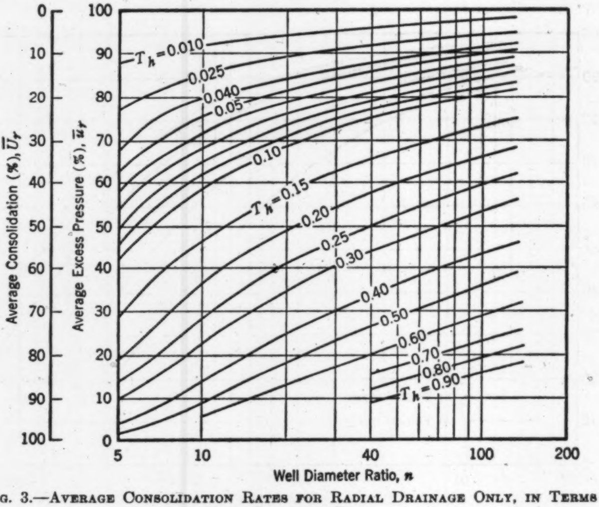


FIG. 3.—AVERAGE CONSOLIDATION RATES FOR RADIAL DRAINAGE ONLY, IN TERMS OF WELL DIAMETER RATIO, *n*

$$\bar{u}_{r,z} = \frac{\bar{u}_r\, \bar{u}_z}{u_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

in which $\bar{u}_r$ and $\bar{u}_z$ are the average excess pore-water pressures throughout the entire soil mass due to radial flow and vertical flow, respectively. Equations for u_r and $\bar{u}_r$ were developed by R. E. Glover;[13] those for u_z and $\bar{u}_z$ were developed by Professor Terzaghi.[11,12] A more general solution may be obtained when the initial excess pore-water pressure is a function of z, provided no expansion of the soil occurs. Equations for u_z and $\bar{u}_z$ for this case were developed by G. Gilboy,[14] Assoc. M. ASCE.

---

[13] *Technical Memorandum No. 158*, by R. E. Glover, Bureau of Reclamation, Denver, Colo., 1930.

[14] "Soil Mechanics," by G. Gilboy, Mass. Inst. Tech., Cambridge, Mass., 1930 (mimeographed notes).

Curves of average excess pore-water pressures $\bar{u}_r$ due to purely radial flow, versus time factor $T_h = \dfrac{k_h\,(1 + e)\,t}{\gamma_0\, a_v\, d^2_e}$ for various values of n are shown in Fig. 2, in which n is the ratio of the zone of well influence to well diameter. Also given in Fig. 2 is a curve of average excess pore-water pressure $\bar{u}_z$ due to vertical flow only, versus time factor $T_v = \dfrac{k_v\,(1 + e)\,t}{\gamma_0\, a_v\, H^2}$. Curves of $\bar{u}_r$ versus well diameter n for various time factors T_h are shown in Fig. 3. These curves permit the construction of average pore-water pressures versus time factor curves similar to those shown in Fig. 2 for any value of n between 5 and 100. Variations of excess pore-water pressures u_r at various points, due to radial flow, for a well diameter ratio $n = 40$, are shown in Fig. 4, indicating a much faster rate of consolidation nearer to the well as compared to that nearer the outer zone of well influence. Shown in Fig. 5 is excess pore-water pressure, u_z, due to vertical flow only, versus relative depth, $Z\%$, of clay layer for various time factors T_v. The relative depth, $Z\%$, is defined as the ratio of the depth z to the point to the total thickness H of the clay deposit. Use of the curves shown in Fig. 2 and Fig. 5 is explained in the following example.

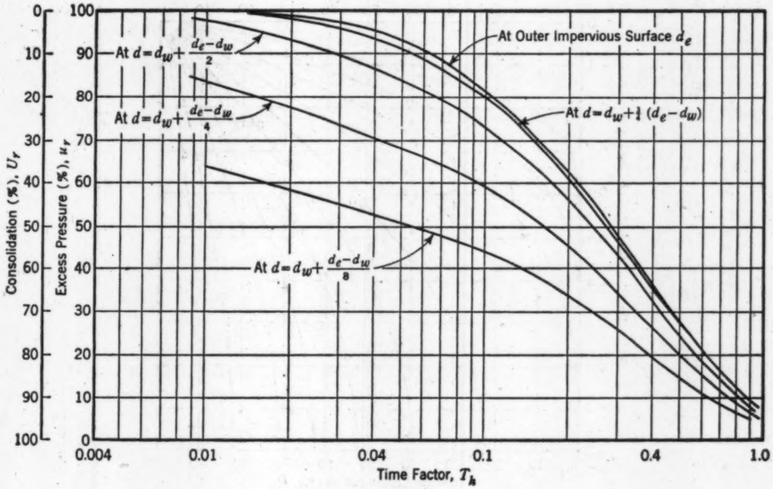


FIG. 4.—RADIAL CONSOLIDATION RATES AT VARIOUS RADIAL SURFACES FOR CLAY CYLINDER HAVING WELL DIAMETER RATIO, $n = 40$

Example.—A bed of varved clay, 40 ft thick, resting on impervious rock, and having free drainage at its upper surface is quickly loaded with a very wide fill to 1 ton per sq ft. Tests on undisturbed soil samples furnish the following average data: $a_v = 13 \times 10^{-5}$ cm² per g; $e = 1.500$; $k_v = 100 \times 10^{-8}$ cm per sec; $k_h = 500 \times 10^{-8}$ cm per sec (assumed); and $\Delta e = 0.125$. Settlement under the 1-ton loading will be:

$$\Delta H = \frac{H\,\Delta e}{1+e} = \frac{(40 \text{ ft})\,(0.125)}{1+1.500} = 2.0 \text{ ft}$$

Fig. 6(*a*) shows a time-average consolidation curve for vertical flow only. (The effect of horizontal flow has been neglected.) For time in months,

$$t_v = \frac{T_v\,\gamma_w\,a_v\,H^2}{k_v\,(1+e)} = \frac{T_v\,(1)\,(13\times10^{-5})\,(40\times30.5)^2}{(100\times10^{-8})\,(1+1.500)\,(60\times60\times24\times30)} = 30\;T_v$$

Values for T_v at various stages of consolidation are obtained from time factor consolidation curve for vertical flow only, as given in Fig. 2. After the average consolidation has reached 50%, sand-filled drain wells 1 ft in diameter are installed 40 ft apart in an equilateral pattern. For time in months,

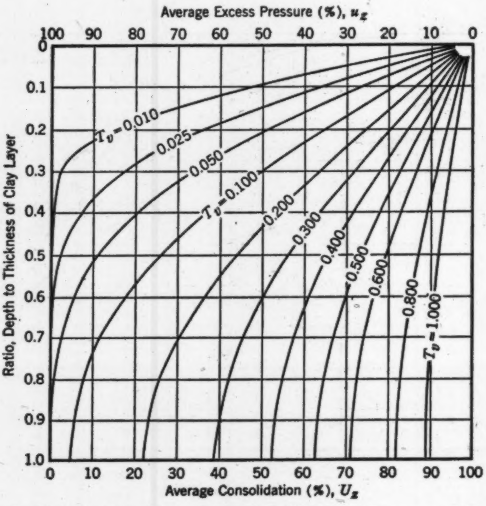


FIG. 5.—VARIATION IN CONSOLIDATION BY VERTICAL FLOW FOR VARIOUS TIME FACTORS AND DEPTH RATIOS

$$t_h = \frac{T_h\,\gamma_w\,a_v\,d^2_e}{k_h\,(1+e)} = \frac{T_h\,(1)\,(13\times10^{-5})\,(40\times30.5)^2}{(500\times10^{-8})\,(1+1.500)\,(60\times60\times24\times30)} = 6\;T_h$$

Values for T_h are obtained from the time factor consolidation curve for $n = 40$ in Fig. 2. The time for 50% consolidation by vertical flow only is

$$t_v = 30\times0.195 = 5.85 \text{ months}$$

which is, therefore, the origin for the radial-average consolidation curve. At any time, *t*, after installations of the wells, $\bar{u}_{r,s}$ is obtained by use of Eq. 3.

For example, at nine months

$$\bar{u}_{r,z} = \frac{\bar{u}_r \times \bar{u}_z}{u_0} = \frac{24\% \times 39\%}{100\%} = 9.4\%$$

The average consolidation is 100 − 9.4 = 90.6%.

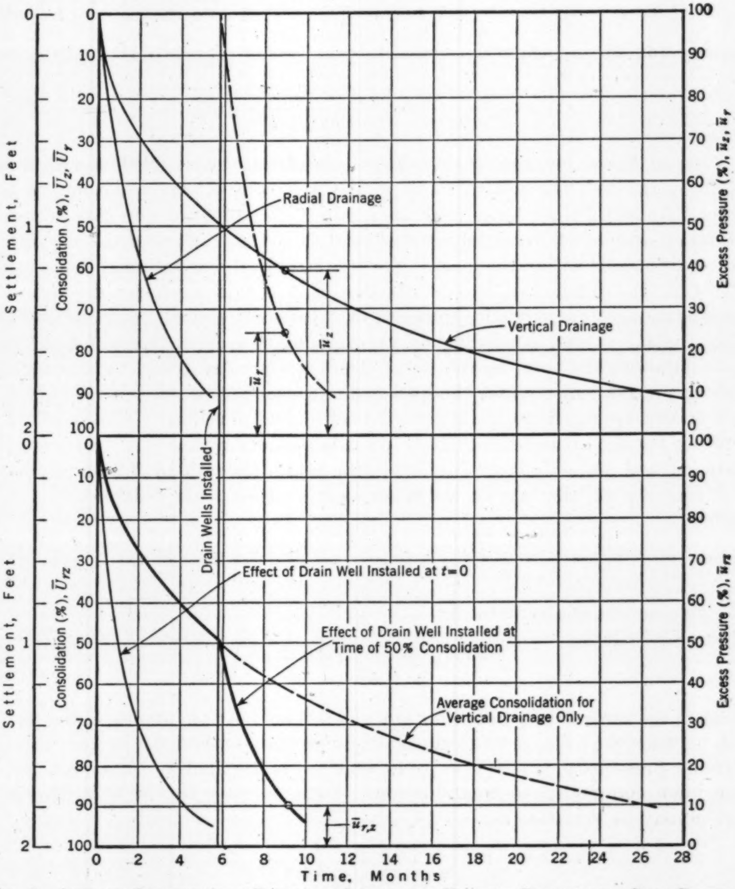


FIG. 6.—AVERAGE CONSOLIDATION RATES FOR RADIAL AND VERTICAL DRAINAGE IN CLAY CYLINDER IN WHICH $D = H$, $n = 40$, AND $k_h = 5\,k_v$, SHOWING EFFECT OF DRAIN WELLS INSTALLED WHEN LOAD IS APPLIED AND WHEN 50% CONSOLIDATION HAS TAKEN PLACE

For wells installed before the load has been added, the solid curve for $\bar{u}_r$ shown in Fig. 6 may be used. The effect of drain wells installed when 50% consolidation had been obtained and before application of load is shown in Fig. 6. Where $k_h \geqq 5\,k_v$, the effect of wells is obvious.

The determination of excess pressure at a point on the bottom of the clay layer and at the outer edge of the influence zone of a well nine months after load is applied is as follows: $Z\% = 40/40 = 1.0$, and $T_v = 9/30 = 0.3$. From Fig. 5, u_z for a relative depth of $Z\% = 1$ and $T_v = 0.3$ is 61%. The value of u_r at the outer surface of the zone of influence (for $n = 40$) is obtained from Fig. 4 at $T_h = \frac{9 - 5.85}{6} = 0.525$, and is 26%. Therefore the excess pressure at this point, which, incidently, consolidates at the slowest rate of all, by Eq. 2, is

$$u_{r,z} = \frac{u_r \times u_z}{u_0} = \frac{26\% \times 61\%}{100\%} = 16\%$$

Case of Equal Vertical Strain.—For consolidation by radial flow with free strains permitted, the soil adjacent to the well consolidates and compresses faster than soil farther away from the drain. This difference in rate of consolidation develops differential settlement of the upper surface of the soil mass and shear strains within the mass. For free strains, it was assumed that these effects did not influence the redistribution of load to the soil nor the rate of consolidation. It is obvious, however, that these effects will redistribute the load to some extent, depending on the amount of arching developed in the material above the compressible soil. The extreme case would be where the arching process redistributes the load to the consolidating soil so that all vertical strains are equal and no differential settlement develops. This condition is rather severe. It can be obtained in the laboratory by use of a rigid loading platform, and probably is approached in the field if the ratio of H to d_e is large.

Using Eq. 1*b*, the rate at which water is expelled from a differential soil cube is:

$$\frac{\partial Q}{\partial t} = -\left[\frac{k_h}{\gamma_w}\left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}\right) + \frac{k_v}{\gamma_w}\frac{\partial^2 u}{\partial z^2}\right] 2\,\pi\,r\,dr\,dz \ldots\ldots\ldots (4)$$

The volume lost at any point is:

$$-\frac{\partial V}{\partial t} = \frac{-\,\partial e}{1 + e}\left(\frac{2\,\pi\,r\,dr\,dz}{\partial t}\right) \ldots\ldots\ldots\ldots\ldots\ldots (5a)$$

which is the same at any depth for any given time. At any depth, z, the relation between the average hydrostatic excess pressure, $\bar{u}$, and the average intergranular pressure, $\bar{p}$, is $\bar{u} + \bar{p} = p_f$, in which p_f is the final intergranular pressure upon completion of consolidation. Thus, $-\,\partial\bar{u} = \partial\bar{p}$, $\partial e = a_v\,\partial\bar{u}$, and Eq. 5*a* may be rewritten as:

$$\frac{-\,\partial V}{\partial t} = \frac{-\,a_v\,\partial\bar{u}}{(1 + e)\,\partial t}\,2\,\pi\,r\,dr\,dz \ldots\ldots\ldots\ldots\ldots\ldots (5b)$$

Equating Eqs. 4 and 5*b* and simplifying,

$$c_h\left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}\right) + c_v\frac{\partial^2 u}{\partial z^2} = \frac{\partial\bar{u}}{\partial t} \ldots\ldots\ldots\ldots\ldots (6a)$$

in which c_h is a coefficient of consolidation for horizontal flow and is equal to

$\frac{k_h (1 + e)}{a_v \gamma_w}$. For radial flow only, this becomes:

$$c_h \left(\frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} \right) = \frac{\partial \bar{u}}{\partial t} \quad \text{.....................} (6b)$$

A solution for this expression is—

$$u_r = \frac{4 \bar{u}}{d^2_e F(n)} \left[r^2_e \log_e \left(\frac{r}{r_w} \right) - \frac{r^2 - r^2_w}{2} \right] \quad \text{.............} (7)$$

—in which

$$\bar{u} = u_0 \epsilon^{\lambda} \quad \text{..........................} (8)$$

ϵ is the base of natural logarithms,

$$\lambda = \frac{-8 T_h}{F(n)} \quad \text{.........................} (9)$$

and

$$F(n) = \frac{n^2}{n^2 - 1} \log_e (n) - \frac{3 n^2 - 1}{4 n^2} \quad \text{...............} (10)$$

The initial distribution of the hydrostatic excess pressure is not uniform (see Fig. 7); but with passage of time the distribution approaches that for the free strain case (see Fig. 8).

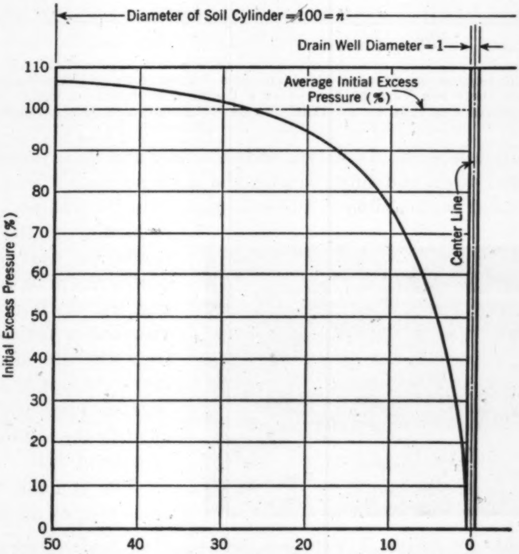


FIG. 7.—DISTRIBUTION OF INITIAL EXCESS PORE-WATER PRESSURE IN SOIL CYLINDER WITH $n = 100$

Effect of Peripheral Smear

If drain wells are installed by driving cased holes and then backfilling as the casing is withdrawn, the driving and pulling of the casing will distort and remold the adjacent soil. In varved soils the finer and more impervious layers will be dragged down and smeared over the more pervious layers, resulting in a zone of reduced permeability in the soil adjacent to the well periphery. This distortion of the soil is very similar to that which occurs during sampling operations (see Fig. 9). It is not difficult to visualize a far greater smear from the use of a heavy casing or hollow mandrel with a bottom plug which displaces entirely the soil at the location of a drain well.

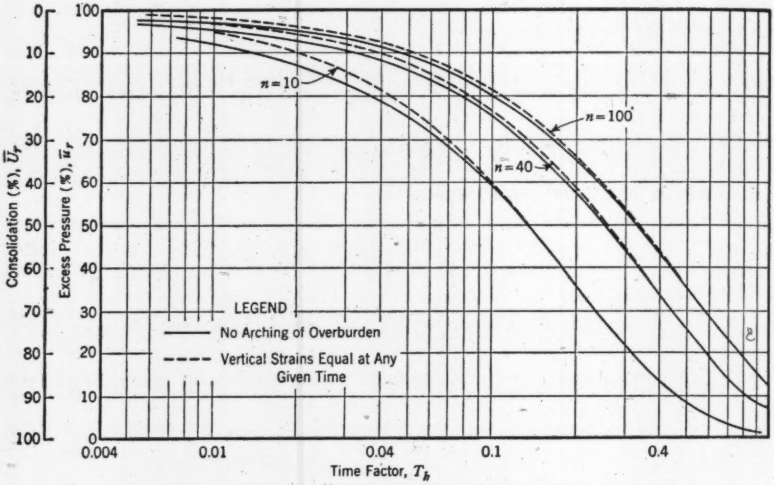


Fig. 8.—Comparison of Average Consolidation Rates in Clay Cylinders by Radial Drainage Only for Various Values of *n* Under Conditions of Equal Vertical Strains at Any Given Time and of No Arching of Overburden

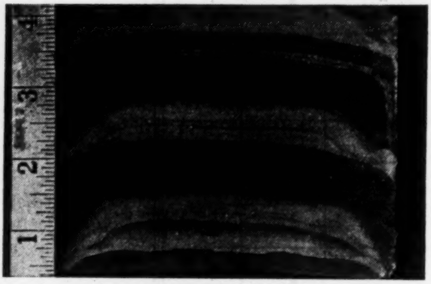


Fig. 9.—Distorted or Smeared Varved Clay Sample Taken with Early Massachusetts Institute of Technology Type 4½-In.-Diameter Sampler

The remolded or smeared zone creates additional resistance which must be overcome by the excess water being expelled. This, in turn, will retard the rate of consolidation. The smeared zone will not be of constant thickness nor will it be homogeneous as far as soil properties are concerned. For this analysis, however, to obtain solutions for smear effect it is assumed that the thickness of the smeared zone is constant and homogeneous. It is further assumed that, because the remolded zone is adjacent to the drain well and will consolidate very fast, its consolidation can be ignored and the zone treated as an incompressible material.

In the smeared zone Eq. 1*b* then becomes:

$$\frac{-k_s}{\gamma_w}\left(\frac{1}{r}\frac{\partial u'}{\partial r}+\frac{\partial^2 u'}{\partial r^2}\right)=0 \qquad (11)$$

in which k_s is the coefficient of permeability of the smeared soil.

It follows that variation of excess water pressure between r_w and r_s, the outer limit of the smeared zone, is:

$$u'_r = u'_{rs}\left[\frac{\log_e\left(\frac{r}{r_w}\right)}{\log_e(s)}\right] \qquad (12)$$

in which u'_{rs} is the excess pressure at r_s, and $s = r_s/r_w$. Added boundary conditions required because of the presence of the smeared zone are: First, that the excess water pressure at the boundary of the smeared zone be the same in the undisturbed zone as in the smeared zone ($u_{rs} = u'_{rs}$); and, second, that the rate of flow out of the undisturbed zone be equal to that into the smeared zone, so that at r_s:

$$k_h\frac{\partial u}{\partial r} = k_s\frac{\partial u'}{\partial r} \qquad (13)$$

Case of Free Strain.—In consolidation by radial flow to a central drain well with a smear zone at its periphery, Eq. 1*b* reduces to—

$$c_h\left(\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}\right)=\frac{\partial u}{\partial t} \qquad (14)$$

—from which a solution for u_r, for uniform initial excess pore-water pressure distribution, subject to the previously noted boundary conditions, is:

$$u_r = u_0\sum_{\alpha_1,\alpha_2,\alpha_3\cdots}^{\alpha=\infty}\frac{-\frac{2}{s\,\alpha}U_1(\alpha\,s)\,U_0\left(\frac{\alpha\,r}{r_w}\right)\epsilon^{\mu}}{\frac{4}{\pi^2\alpha^2 s^2}-U^2_0(\alpha\,s)-U^2_1(\alpha\,s)} \qquad (15)$$

in which

$$U_0(\alpha\,s) = J_0(\alpha\,s)\,Y_1(\alpha\,n) - J_1(\alpha\,n)\,Y_0(\alpha\,s) \qquad (16a)$$

$$U_1(\alpha\,s) = J_1(\alpha\,s)\,Y_1(\alpha\,n) - J_1(\alpha\,n)\,Y_1(\alpha\,s) \qquad (16b)$$

$$U_0\left(\frac{\alpha\,r}{r_w}\right) = J_0\left(\frac{\alpha\,r}{r_w}\right)Y_1(\alpha\,n) - J_1(\alpha\,n)\,Y_0\left(\frac{\alpha\,r}{r_w}\right) \qquad (16c)$$

The functions $J_0(\;)$ and $J_1(\;)$ are Bessel functions of the first kind of zero and first order, respectively, and $Y_0(\;)$ and $Y_1(\;)$ are Bessel functions of the second kind of zero and first order, respectively, and α, α_1, α_2, α_3, $\cdots$, are the roots of

$$\frac{k_s\, U_0\,(\alpha\, s)}{k_h\, \alpha\, s \log_e (s)} + U_1\,(\alpha\, s) = 0 \qquad (17)$$

and

$$\mu = -\,4\, n^2\, \alpha^2\, T_h \qquad (18)$$

The average excess pore-water pressure between r_s and r_e is:

$$\bar{u}_r = u_0 \sum_{\alpha_1,\, \alpha_2,\, \alpha_3 \cdots}^{\alpha=\infty} \frac{U^2{}_1\,(\alpha\, s)\, \epsilon^{\mu}}{\alpha^2\,(n^2 - s^2)\left[\dfrac{4}{\pi^2\,\alpha^2\, s^2} - U^2{}_0\,(\alpha\, s) - U^2{}_1\,(\alpha\, s)\right]} \qquad (19)$$

If consolidation occurs by both radial and vertical flow, an approximate solution for the excess water pressure is:

$$u_{r,z} = \frac{u_r\, u_z}{u_0} \qquad (20)$$

in which u_r is found by Eq. 15; the average excess pressure for the entire body between r_s and r_e is:

$$\bar{u}_{r,z} = \frac{\bar{u}_r\, \bar{u}_z}{u_0} \qquad (21)$$

in which $\bar{u}_r$ is obtained from Eq. 19, and values of u_z and $\bar{u}_z$ are obtained from Figs. 2 and 5. Eqs. 20 and 21 are approximate because the effect of vertical flow in the smeared zone has been neglected. However, for reasonable values of k_h and k_v this effect is believed to be negligible.

Case of Equal Vertical Strain.—Solution of Eq. 6*b* for consolidation by radial flow to a central drain well with a smeared zone at its periphery is—

$$u_r = \bar{u}_r\, \frac{\left[\log_e\left(\dfrac{r}{r_s}\right) - \dfrac{r^2 - r^2{}_s}{2\, r^2{}_e} + \dfrac{k_h}{k_s}\left(\dfrac{n^2 - s^2}{n^2}\right) \log_e (s)\right]}{\nu} \qquad (22)$$

in which

$$\nu = F\,(n, s, k_h, k_s)$$

$$= \left[\frac{n^2}{n^2 - s^2} \log_e \left(\frac{n}{s}\right) - \frac{3}{4} + \frac{s^2}{4\, n^2} + \frac{k_h}{k_s}\left(\frac{n^2 - s^2}{n^2}\right) \log_e (s)\right] \qquad (23)$$

and $\bar{u}_r$ can be expressed as—

$$\bar{u}_r = u_0\, \epsilon^{\xi} \qquad (24)$$

—in which

$$\xi = \frac{-\,8\, T_h}{\nu} \qquad (25)$$

For combined radial and vertical flow, approximate solutions may be obtained using Eqs. 22 and 24 with u_z and $\bar{u}_z$ obtained as noted previously.

For purposes of studying the effect of smear in retarding the average rate of consolidation, Eq. 24 is preferred to Eq. 19 because of its simplicity. The effect of smear is indicated in Fig. 10. For a greater ratio of k_h/k_v the difference is still larger.

EFFECT OF WELL RESISTANCE

The foregoing developments are for ideal drain wells of infinite permeability offering no resistance to flow up the well. Actually, head losses will occur due to resistance of well backfill material to flow. If the flow is large, or if the well area is small, then the back pressure due to well resistance will be high. On the other hand, if the flow is small, as for shallow deposits and tight deposits, or, if the well area is large, then the resistance of the well to flow will be small. A very pervious filling would be ideal from the standpoint of minimizing well resistance but filter action is needed with sufficiently small voids in the backfill to prevent "inwash" of the surrounding soil from clogging the well. This need is especially required for cohesionless silts and for varved soils in which the more pervious part of the varves is fine sand or coarse silt. For clays the filter requirements are less certain.

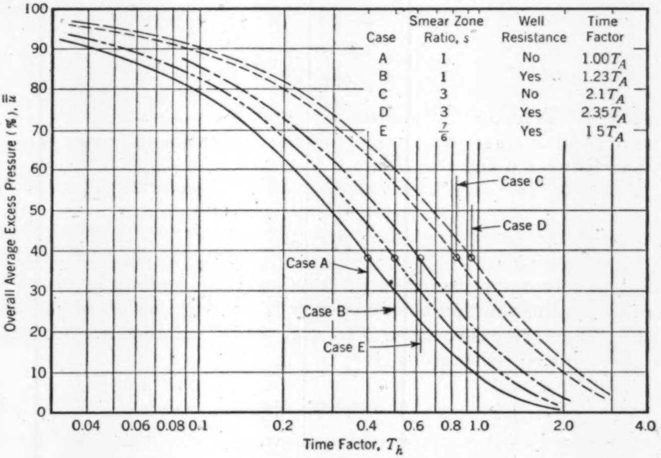


FIG. 10.—EFFECT OF SMEAR AND WELL RESISTANCE ON EQUAL STRAIN CONSOLIDATION BY RADIAL FLOW TO DRAIN WELLS IN WHICH $n = 60$, $k_w/k_h = 20{,}000$, $d_e/H = \frac{60}{80}$, AND $k_h/k_s = 10$

A solution for the effect of well resistance on the rate of consolidation for free strain has not been developed. A solution has been developed for equal vertical strain, with or without smear, where k_v of the consolidating soil is equal to zero; that is, $\frac{\partial u}{\partial z} \neq 0$, but because $k_v = 0$ no vertical flow exists.

A solution of Eq. 6*b* for $u_{r,z}$ at any point where smear and well resistance are involved is—

$$u_{r,z} = \bar{\bar{u}}_z \left\{ \frac{f(z)}{\nu} \left[\log_e \frac{r}{r_s} - \frac{r^2 - r^2_s}{2\, r^2_e} + \frac{k_h}{k_s} \left(\frac{n^2 - s^2}{n^2} \right) \log_e (s) \right] + 1 - f(z) \right\} \quad (26)$$

—in which $\bar{\bar{u}}_z$, the average excess pressure between r_e and r_s at depth z, is:

$$\bar{\bar{u}}_z = u_0\, \epsilon^{\xi f(z)} \quad (27)$$

$$f(z) = \frac{\epsilon^{\beta(z-2H)} + \epsilon^{-\beta z}}{1 + \epsilon^{-2\beta H}} \quad (28)$$

and

$$\beta = \left[\frac{2\, k_h\, (n^2 - s^2)}{k_w\, r^2_e\, \nu} \right]^{\frac{1}{2}} \quad (29)$$

in which k_w is the permeability of the well backfill.

The over-all average excess pressure over the entire soil mass between r_e and r_s and between 0 and $z = H$ is:

$$\bar{\bar{u}} = \int_0^H \bar{\bar{u}}_z \, dz \quad (30)$$

This may be solved approximately by graphical integration or by use of Simpson's rule. In this solution horizontal flow in the drain well backfill has been ignored. Where there is no smear $F\,(n, s, k_h, k_s)$ reduces to $F(n)$. An example of the effect of well resistance is given in Fig. 10.

ACKNOWLEDGMENTS

Mathematical solutions presented in this paper, except as otherwise noted, were developed by the writer for use in special design studies made by the Providence District, U. S. Engineer Department. The writer wishes to express his appreciation to Prof. W. Prager, Brown University, Providence, who checked the derivation of Eqs. 2 and 3, and to Kenneth S. Lane, Assoc. M. ASCE, formerly head, Providence District Soils Laboratory, and now head, Garrison (N. Dak.) District Soils Laboratory, United States Engineer Department, for his valuable aid and encouragement.

APPENDIX I. NOTATION

The following letter symbols, adopted for use in this paper and for the guidance of discussers, conform essentially to "Soil Mechanics Nomenclature," ASCE *Manual of Engineering Practice No. 22*, prepared by a Committee of the Soil Mechanics and Foundations Division, and adopted by the Society in April, 1941. In addition, the symbols of linear concepts are defined by Fig. 1.

- A = constant of integration;
- a_v = coefficient of compressibility $= -\dfrac{de}{dp}$;
- B = constant of integration;
- C = constant of integration;
- c = coefficient of consolidation:
  - c_h = for horizontal flow $= \dfrac{k_h\,(1+e)}{a_v\,\gamma_w}$;
  - c_v = for vertical flow $= \dfrac{k_v\,(1+e)}{a_v\,\gamma_w}$;
- d = diameter:
  - d_e = of zone of influence;
  - d_s = of remolded zone;
  - d_w = of drain well;
- e = void ratio of soil;
- H = depth of clay mass from a free-draining horizontal surface to an impervious one;
- i = hydraulic gradient;
- k = coefficient of permeability:
  - k_h = in horizontal direction;
  - k_s = of remolded zone;
  - k_v = in vertical direction;
  - k_w = of well backfill;
- n = a ratio $= r_e/r_w = d_e/d_r$;
- p = intergranular pressure:
  - p_f = final;
  - $\bar{p}$ = average;
  - $\bar{p}_z$ = average, at depth z;
- Q = rate of flow of water;
- r = radius, a coordinate in the cylindrical system:
  - r_e = of zone of influence;
  - r_s = of remolded zone;
  - r_w = of drain well;
- S = spacing of drain wells;
- s = a ratio $= r_s/r_w$;
- T = time factor:
  - T_h = for radial flow $= \dfrac{c_h\,t}{d^2_e}$;
  - T_v = for vertical flow $= \dfrac{c_v\,t}{H^2}$;
- t = time of consolidation:
  - t_h = due to horizontal flow;
  - t_v = due to vertical flow;
- u = excess pore-water pressure:
  - u_0 = initial uniform;
  - u_r = due to radial flow;
  - u_{rs} = due to radial flow at outer limit of remolded zone;

$u_{rs,z}$ = due to radial flow at outer limit of remolded zone and depth z;
u_{rz} = due to radial flow only at depth z;
$u_{r,z}$ = due to combined radial and vertical flow;
$u_{w,z}$ = in drain well at depth z;
u_z = due to vertical flow;
u'_r = due to radial flow in the smeared zone at point r;
u'_{rs} = due to radial flow on remolded side of remolded zone boundary;
$\bar{u}$ = average excess pore-water pressure:
$\bar{u}_r$ = due to radial flow;
$\bar{u}_{r,z}$ = throughout entire mass, due to combined radial and vertical flow;
$\bar{u}_z$ = due to vertical flow;
$\bar{\bar{u}}_z$ = in undisturbed zone at depth z, due to radial flow with well resistance;
$\bar{\bar{u}}$ = in entire undisturbed soil mass, due to radial flow with well resistance;
V = volume of soil mass;
x = a coordinate in the rectangular system;
y = a coordinate in the rectangular system;
$Z\%$ = relative depth $= \frac{z}{H} = \frac{\text{depth } z}{\text{depth of clay mass}}$;
z = a coordinate in both the rectangular and the cylindrical systems;
α = root of Eq. 17;
β = a substitution factor defined by Eq. 29;
γ_w = unit weight of water;
ϵ = base of natural logarithms;
θ = a substitution factor defined by Eq. 39;
λ = a substitution factor defined by Eq. 9;
μ = a substitution factor defined by Eq. 18;
ν = a substitution factor defined by Eq. 23;
ξ = a substitution factor defined by Eq. 25;
$\bar{\sigma}_z$ = total average stress at depth z; and
ψ = a coordinate in the cylindrical system.

APPENDIX II. DEVELOPMENT OF SOLUTIONS

CASE OF FREE STRAIN—NO WELL RESISTANCE NOR PERIPHERAL SMEAR

Derivations of Eqs. 2 and 3 are given elsewhere,[11,12,13,14] as is also proof of their validity.[10] Therefore, these derivations and proofs are not presented here.

CASE OF FREE STRAIN WITH PERIPHERAL SMEAR

Assume a cylindrical body of fine-grained, saturated compressible soil with an outer impervious boundary and a central drain having no resistance to flow

($k_w = \infty$). Also, assume that the horizontal permeability of the soil mass is k_h while that in a vertical direction, k_v, is zero. Further assume that the central drain is surrounded by a zone of remolded soil whose permeability is k_s, and which is bounded by the well surface, r_w, and by the outer radius of the remolded zone, r_s. The outer radius of the cylindrical body of undisturbed soil is r_e. It is further assumed that the cylindrical body is loaded so that an excess pore-water pressure $u_r = f(r)$ is induced at time zero and that during the process of consolidation vertical strains do not redistribute the load in the soil mass. The remolded soil is assumed to undergo no consolidation.

The boundary conditions are:

(1) When $t = 0$, $u_r = f(r)$ for $r_e > r > r_s$.

(2) No flow occurs accross the impervious surface at the outer boundary r_e of the soil cylinder $\left(\dfrac{\partial u_r}{\partial r} = 0\right)$.

(3) No vertical flow takes place because $k_v = 0$.

(4) The flow of water through the remolded zone surrounding the drain, for $r_s > r > r_w$, is given by Eq. 12.

(5) The seepage to the interface between the undisturbed and remolded soils at r_s is equal to the seepage away (Eq. 13).

The basic partial differential equation in the undisturbed zone ($r_e > r > r_s$) is expressed by Eq. 6*b*, a solution for which is

$$u_r = \left[A\, J_0\left(\frac{\alpha\, r}{r_w}\right) + B\, Y_0\left(\frac{\alpha\, r}{r_w}\right)\right] \epsilon^{\mu} \quad \text{.............} (31)$$

—in which A and B are constants of integration. At r_e the second boundary condition is operative and

$$\frac{\partial u_r}{\partial r} = \frac{-\alpha}{r_w}\left[A\, J_1(\alpha\, n) + B\, Y_1(\alpha\, n)\right] \epsilon^{\mu} = 0 \quad \text{..........} (32)$$

Therefore,

$$A\, J_1(\alpha\, n) + B\, Y_1(\alpha\, n) = 0 \quad \text{.................} (33)$$

and

$$B = -A\left[\frac{J_1(\alpha\, n)}{Y_1(\alpha\, n)}\right] \quad \text{.....................} (34)$$

Eq. 31 may be written, therefore, as

$$u_r = A\left[J_0\left(\frac{\alpha\, r}{r_w}\right) - \frac{J_1(\alpha\, n)}{Y_1(\alpha\, n)}\, Y_0\left(\frac{\alpha\, r}{r_w}\right)\right] \epsilon^{\mu} = A\, V_0\left(\frac{\alpha\, r}{r_w}\right) \epsilon^{\mu} \quad \text{..} (35)$$

The excess pressure in the remolded zone ($r_s > r > r_w$) may be expressed as

$$u'_r = \frac{\log_e\left(\dfrac{r}{r_w}\right)}{\log_e(s)}\, V_0(\alpha\, s)\, A\, \epsilon^{\mu} \quad \text{................} (36)$$

Also, at r_s,

$$k_s \frac{\partial u'_{rs}}{\partial r} = \frac{k_s}{r_s \log_e (s)} V_0 (\alpha s) A \epsilon^{\mu} \ldots\ldots\ldots\ldots\ldots (37)$$

and

$$k_h \frac{\partial u_{r,s}}{\partial r} = - k_h \frac{\alpha}{r_w} \left[J_1 (\alpha s) - \frac{J_1 (\alpha n)}{Y_1 (\alpha n)} Y_1 (\alpha s) \right] A \epsilon^{\mu}$$

$$= - k_h \frac{\alpha}{r_w} V_1 (\alpha s) A \epsilon^{\mu} \ldots (38)$$

Equating Eqs. 37 and 38, simplifying, and substituting,

$$\frac{k_s}{\alpha k_h s \log_e s} = \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

$$\theta V_0 (\alpha s) + V_1 (\alpha s) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

If α is a root of this equation, the fifth boundary condition is satisfied and the roots are infinite. Therefore,

$$u_r = \sum_{\alpha_1, \alpha_2, \alpha_3, \cdots}^{\alpha = \infty} A V_0 \left(\frac{\alpha r}{r_w} \right) \epsilon^{\mu} \ldots\ldots\ldots\ldots\ldots\ldots (41)$$

is also a solution of Eq. 6*b*, provided A can be so determined that, when $t = 0$,

$$f(r) = \sum_{\alpha_1, \alpha_2, \alpha_3, \cdots}^{\alpha = \infty} A V_0 \left(\frac{\alpha r}{r_w} \right) \ldots\ldots\ldots\ldots\ldots\ldots (42)$$

to satisfy the first boundary condition.

To solve for A, multiply both sides of Eq. 42 by $r V_0 \left(\frac{\alpha r}{r_w} \right) dr$ and integrate from r_s to r_e. Noting that $V_1 (\alpha n) = 0$, and if $\alpha \neq \alpha'$,

$$\int_{r}^{r_e} A V_0 \left(\frac{\alpha r}{r_w} \right) V_0 \left(\frac{\alpha' r}{r_w} \right) r\, dr$$

$$= \frac{+ A r^2_w}{\alpha^2 - (\alpha')^2} \left[- \frac{\alpha' r}{r_w} V_0 \left(\frac{\alpha r}{r_w} \right) V_1 \left(\frac{\alpha' r}{r_w} \right) + \frac{\alpha r}{r_w} V_1 \left(\frac{\alpha r}{r_w} \right) V_0 \left(\frac{\alpha' r}{r_w} \right) \right]_{r_s}^{r_e}$$

$$= \frac{+ A r^2_w}{\alpha^2 - (\alpha')^2} [\alpha' s V_0 (\alpha s) V_1 (\alpha' s) - \alpha s V_1 (\alpha s) V_0 (\alpha' s)]$$

$$= \frac{- A r^2_w s}{\alpha^2 - (\alpha')^2} [\alpha' \theta' V_0 (\alpha s) V_0 (\alpha' s) - \alpha \theta V_0 (\alpha s) V_0 (\alpha' s)] = 0 \ldots\ldots (43)$$

Therefore, all the right-hand terms of Eq. 42 when multiplied and integrated as previously noted vanish except where $\alpha = \alpha'$, and

$$\int_{r_s}^{r_e} f(r) V_0 \left(\frac{\alpha r}{r_w} \right) r\, dr = \int_{r_s}^{r_e} A V^2_0 \left(\frac{\alpha r}{r_w} \right) r\, dr \ldots\ldots\ldots (44)$$

Thus,

$$A = \frac{\int_{r_s}^{r_e} f(r)\, V_0\left(\frac{\alpha\, r}{r_w}\right) r\, dr}{\int_{r_s}^{r_e} V^2{}_0\left(\frac{\alpha\, r}{r_w}\right) r\, dr} \dots\dots\dots\dots\dots\dots (45)$$

If $f(r) = u_0$,

$$u_0 \int_{r_s}^{r_e} V_0\left(\frac{\alpha\, r}{r_w}\right) r\, dr = u_0 \left[\frac{r\, r_w}{\alpha} V_1\left(\frac{\alpha\, r}{r_w}\right)\right]_{r_s}^{r_e} = \frac{-\, u_0\, r_s\, r_w\, V_1\,(\alpha\, s)}{\alpha} \dots (46)$$

and

$$\int_{r_s}^{r_e} V^2{}_0\left(\frac{\alpha\, r}{r_w}\right) r\, dr = \left\{\frac{r^2}{2}\left[V^2{}_0\left(\frac{\alpha\, r}{r_w}\right) + V^2{}_1\left(\frac{\alpha\, r}{r_w}\right)\right]\right\}_{r_w}^{r_e}$$

$$= \frac{r^2{}_e}{2} V^2{}_0\,(\alpha\, n) - \frac{r^2{}_s}{2} V_0\,(\alpha\, s) - \frac{r^2{}_s}{2} V^2{}_1\,(\alpha\, s) \dots\dots\dots\dots (47)$$

Therefore,

$$A = \frac{-\,\frac{2\, u_0}{s\, \alpha}\, U_1\,(\alpha\, s)\; Y_1\,(\alpha\, n)}{\frac{4}{\pi^2\, \alpha^2\, s^2} - U^2{}_0\,(\alpha\, s) - U^2{}_1\,(\alpha\, s)} \dots\dots\dots\dots\dots (48)$$

in which

$$U_0\,(\alpha\, n) = \frac{-\,2}{\pi\, \alpha\, n} \dots\dots\dots\dots\dots\dots\dots\dots\dots (49a)$$

$$U_0\left(\frac{\alpha\, r}{r_w}\right) = V_0\left(\frac{\alpha\, r}{r_w}\right) Y_1\,(\alpha\, n) \dots\dots\dots\dots\dots (49b)$$

$$U_1\left(\frac{\alpha\, r}{r_w}\right) = V_1\left(\frac{\alpha\, r}{r_w}\right) Y_1\,(\alpha\, n) \dots\dots\dots\dots\dots (49c)$$

Substituting Eq. 48 in Eq. 41 for the initial uniform excess pressure, $u_0 = f(r)$:

$$u_r = u_0 \sum_{\alpha_1,\, \alpha_2,\, \alpha_3 \cdots}^{\alpha = \infty} \frac{-\,\frac{2}{s\, \alpha}\, U_1\,(\alpha\, s)\; U_0\left(\frac{\alpha\, r}{r_w}\right) \epsilon^{\mu}}{\frac{4}{\pi^2\, \alpha^2\, s^2} - U^2{}_0\,(\alpha\, s) - U^2{}_1\,(\alpha\, s)} \dots\dots\dots (50)$$

which is Eq. 15.

The average excess pressure in the cylinder of compressible soil between r_e and r_s is found by integrating Eq. 50 and dividing by the area. Thus,

$$\bar{u}_r = \int_{r_s}^{r_e} \frac{2\, \pi\, r\, u_r\, dr}{\pi\,(r^2{}_e - r^2{}_s)} \dots\dots\dots\dots\dots\dots (51)$$

Substitution of Eq. 50 in Eq. 51, followed by integration and simplification produces the following, which is Eq. 19 of the main paper:

$$\bar{u}_r = u_0 \sum_{\alpha_1, \alpha_2, \alpha_3 \cdots}^{\alpha = \infty} \frac{U^2{}_1 (\alpha\, s)\, \epsilon^{\mu}}{\alpha^2 (n^2 - s^2) \left[\frac{4}{\pi^2 \alpha^2 s^2} - U^2{}_0 (\alpha\, s) - U^2{}_1 (\alpha\, s) \right]} \ldots (52)$$

CASE OF EQUAL VERTICAL STRAIN WITH OR WITHOUT WELL RESISTANCE OR PERIPHERAL SMEAR

Assume a cylinder of soil with conditions similar to those of the previous case except that the well backfill has a finite permeability, k_w, which furnishes resistance to flow of expelled water up the well bore. Further assume that all vertical strains at any depth z are equal. This means that there are no shearing strains and consequently no differential settlement of the upper surface. Because of this last condition the same amount of water is expelled from all differential soil cubes at a given depth for any given time. All flow in the well bore is assumed to be vertical.

The boundary conditions are then as follows:

(1) Initial uniform load placed on upper surface of cylinder.

(2) No flow across the impervious surface at outer boundary of soil cylinder $\left(\frac{\partial u}{\partial r} = 0 \right)$.

(3) No vertical flow in soil cylinder because $k_v = 0$.

(4) Flow of water through the remolded zone surrounding the drains is, for $r_s > r > r_w$,

$$\frac{1}{r} \frac{\partial u'}{\partial r} + \frac{\partial^2 u'}{\partial r^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (53)$$

for conditions of no consolidation in this zone.

(5) Seepage to the interface between the remolded soil and undisturbed soil is equal to the seepage away $\left(k_h \frac{\partial u}{\partial r} = k_s \frac{\partial u'}{\partial r} \right)$.

(6) At r_s the excess pressures on both sides of the interface are equal; that is, $u_{rs,z} = u'_{rs,z}$.

(7) Excess pressure in the drain well is zero at $z = 0$.

(8) At base of drain well vertical flow ceases; that is, $\partial u_{w,z} / \partial z = 0$ at $z = H$.

(9) The radial flow from the soil into the well at depth, z, is equal to the increase in flow up the well:

$$2\, \pi\, r_w\, k_s \frac{\partial u'_{rw,z}}{\partial r}\, dz = -\, \pi\, r^2{}_w\, k_w \frac{\partial^2 u_{w,z}}{\partial z^2}\, dz \ldots\ldots\ldots\ldots (54)$$

Because the vertical strains are all equal at any given depth, z, the amount of water expelled from any small soil mass is the same throughout the entire mass at any given time. Therefore the amount of water passing any radial surface in the zone $r_e > r > r_s$ is proportional to the volume of the cylinder between that radial surface and the external boundary. Thus,

$$\frac{\partial Q}{\partial t} = k\, i\, A' = \frac{-k_h}{\gamma_w} \frac{\partial u}{\partial r}\, 2\, \pi\, r\, dz = C_1\, (r^2_e - r^2)\, \pi \frac{dz}{\partial t} \quad \text{.......} (55)$$

in which i is the hydraulic gradient; A' is the area of flow; and C, with appropriate subscripts, represents integration constants. Therefore,

$$\frac{\partial u}{\partial r} = \frac{C_1\, \gamma_w}{2\, k_h\, \partial t} \frac{r^2_e - r^2}{r} = C_2 \left(\frac{r^2_e - r^2}{r} \right) \quad \text{.............} (56)$$

and the pressure distribution at any depth, z, at time t is:

$$u_{r,z} = C_2 \int (r^2_e - r^2) \frac{dr}{r} = C_2 \left[r^2_e \log_e (r) - \frac{r^2}{2} \right] + C_3 \quad \text{......} (57)$$

The excess pressure, $u'_{r,z}$, in the remolded zone by condition (4) is:

$$u'_{r,z} = (u'_{rs,z} - u_{w,z}) \frac{\log_e \left(\dfrac{r}{r_w} \right)}{\log_e (s)} + u_{w,z} \quad \text{..............} (58)$$

in which $u_{rs,z}$ is the excess pressure at r_s and depth z and $u_{w,z}$ is the excess pressure in the drain well at depth z and condition (6) requires that $u_{rs,z} = u'_{rs,z}$ at the interface. Equating Eq. 57 with Eq. 58, at r_s,

$$C_2 \left[r^2_e \log_e (r_s) - \frac{r^2_s}{2} \right] + C_3 = u'_{rs,z} \quad \text{..............} (59)$$

In addition, condition (5) requires that, at r_s, $k_h \dfrac{\partial u}{\partial r} = k_s \dfrac{\partial u'}{\partial r}$. Therefore,

$$k_h\, C_2 \left(\frac{r^2_e - r^2_s}{r_s} \right) = k_s \frac{(u'_{rs,z} - u_{w,z})}{r_s \log_e (s)} \quad \text{...............} (60)$$

so that

$$u'_{rs,z} = u_{rs,z} = \frac{k_h}{k_s} C_2\, (n^2 - s^2)\, r^2_w \log_e (s) + u_{w,z}$$

$$= C_2 \left[r^2_e \log_e (r_s) - \frac{r^2_s}{2} \right] + C_3 \quad \text{..} (61)$$

This makes

$$C_3 = C_2 \left[\frac{k_h}{k_s} (n^2 - s^2)\, r^2_w \log_e (s) - r^2_e \log_e (r_s) + \frac{r^2_s}{2} \right] + u_{w,z} \quad \text{....} (62)$$

so that substitution in Eq. 61 gives:

$$u_{r,z} = C_2 \left[r^2_e \log_e \left(\frac{r}{r_s} \right) - \left(\frac{r^2 - r^2_s}{2} \right) + \frac{k_h}{k_s} (n^2 - s^2)\, r^2_w \log_e (s) \right] + u_{w,z} \quad \text{..} (63)$$

The average excess pressure between r_e and r_s is:

$$\bar{\bar{u}}_z = \frac{1}{\pi\,(r^2_e - r^2_s)} \int_{r_s}^{r_e} u_{r,z}\, 2\,\pi\, r\, dr$$

$$= C_2\, r^2_e \left[\frac{n^2}{n^2 - s^2} \log_e \left(\frac{n}{s} \right) - \frac{3}{4} + \frac{s^2}{4\,n^2} + \frac{k_h}{k_s} \left(\frac{n^2 - s^2}{n^2} \right) \log_e (s) \right] + u_{w,z}$$

$$= C_2\, \frac{d^2_e}{4}\, F\,(n, s, k_h, k_s) + u_{w,z} \ldots\ldots (64)$$

in which $F\,(n, s, k_h, k_s) = \nu$, which is defined by Eq. 23.

In accordance with condition (9) and Eq. 60, at r_w:

$$\frac{2\,k_s}{r_w\,k_w}\, \frac{\partial u'_{r,z}}{\partial r} = -\,\frac{\partial^2 u_{w,z}}{\partial z^2} = \frac{2\,k_s}{r^2_w\,k_w}\, \frac{(u'_{rs,z} - u_{w,z})}{\log_e (s)} = 2\,C_2\, \frac{k_h}{k_w}\,(n^2 - s^2) \ldots (65)$$

However, from Eq. 64:

$$C_2 = \frac{4\,(\bar{\bar{u}}_z - u_{w,z})}{d^2_e\, F(n, s, k_h, k_s)} \ldots\ldots\ldots\ldots\ldots\ldots (66)$$

and Eq. 65 may be written as,

$$\frac{-\,\partial^2 u_{w,z}}{\partial z^2} = \frac{8\,k_h\,(n^2 - s^2)\,(\bar{\bar{u}}_z - u_{w,z})}{k_w\, d^2_e\, F(n, s, k_h, k_s)} = \beta^2\,(\bar{\bar{u}}_z - u_{w,z}) \ldots\ldots (67)$$

When $t = 0$, $\bar{u}_z = u_0$ and Eq. 67 becomes—

$$\frac{-\,\partial^2 u_{w,z}}{\partial z^2} = \beta^2\,(u_0 - u_{w,z}) \ldots\ldots\ldots\ldots\ldots\ldots (68)$$

—a solution for which is:

$$u_0 - u_{w,z} = C_4\, \epsilon^{\beta z} + C_5\, \epsilon^{-\beta z} \ldots\ldots\ldots\ldots\ldots (69)$$

At $z = 0$, $u_{w,z} = 0$ (condition (7)) so that $u_0 = C_4 + C_5$; and, at $z = H$, $\frac{\partial u_{w,z}}{\partial z} = 0$ (condition (8)), so that:

$$\frac{\partial u_{w,z}}{\partial z} = +\,\beta\, C_4\, \epsilon^{\beta H} - \beta\, C_5\, \epsilon^{-\beta H} = 0 \ldots\ldots\ldots\ldots (70)$$

Substituting these in Eq. 69 gives:

$$u_0 - u_{w,z} = u_0\, \frac{\epsilon^{\beta(z-2H)} + \epsilon^{-\beta z}}{1 + \epsilon^{-2\beta H}} = u_0\, f(z) \ldots\ldots\ldots\ldots (71)$$

Thus, at $t = 0$,

$$u_{w,z} = u_0\, [1 - f(z)] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (72)$$

Also, at $t = 0$,

$$\bar{\bar{u}}_z = u_0 = \frac{C_2\, d^2_e}{4} F(n, s, k_h, k_s) \qquad (73)$$

therefore,

$$C_2 = \frac{4\, u_0\, f(z)}{d^2_e\, F(n, s, k_h, k_s)} \qquad (74)$$

The relationship between average excess pressure $\bar{\bar{u}}_z$ and average soil pressure $\bar{p}_z$ is:

$$\bar{\bar{u}}_z + \bar{p}_z = \bar{\sigma}_z \qquad (75)$$

in which $\bar{\sigma}_z$, the total average stress, is a constant for any fixed loading. Therefore,

$$\partial \bar{\bar{u}}_z = -\, \partial \bar{p}_z \qquad (76)$$

and, because all strains are equal,

$$\partial e = -\, a_v\, \partial \bar{p}_z = \partial e_z = a_v\, \partial \bar{\bar{u}}_z \qquad (77)$$

The volume change then at any depth z between r and r_e is:

$$-\frac{\Delta \partial V}{\partial t} = -\frac{\partial e}{1 + e} (r^2_e - r) \frac{\pi\, dz}{\partial t} = -\frac{a_v\, \partial \bar{\bar{u}}_z}{1 + e} (r^2_e - r^2) \frac{\pi\, dz}{\partial t} \qquad (78)$$

and the total rate of volume change at depth z for time t is:

$$\frac{-\,\partial V}{\partial t} = \frac{-\, a_v\, \partial \bar{\bar{u}}_z}{1 + e} (r^2_e - r^2_s)\, \pi \frac{dz}{dt} \qquad (79)$$

From Eqs. 55 and 56, the total water flowing into the well at depth z is:

$$\frac{\partial Q}{\partial t} = \frac{k_h}{\gamma_w} C_2 \frac{r^2_e - r^2_s}{r_s}\, 2\, \pi\, r_s\, dz \qquad (80)$$

which is equal to Eq. 79; and, thus, when $t = 0$,

$$C_2 = \frac{-\, a_v\, \partial \bar{\bar{u}}_z\, \gamma_w}{2\,(1 + e)\, k_h\, \partial t} = \frac{4\, u_0\, f(z)}{d^2_e\, F(n, s, k_h, k_s)} \qquad (81)$$

and

$$\frac{\partial \bar{u}_z}{\partial t} = -\frac{u_0\, 8\, f(z)}{F(n, s, k_h, k_s)} \left[\frac{(1 + e)\, k_h}{a_v\, \gamma_w\, d^2_e} \right] = -\frac{u_0\, 8\, f(z)\, T_h}{F(n, s, k_h, k_s)\, t} = \frac{\xi\, u_0\, f(z)}{t} \qquad (82)$$

If it is assumed that

$$\bar{\bar{u}}_z = u_0\, \epsilon^{\xi f(z)} \qquad (83)$$

then,

$$\frac{\partial \bar{\bar{u}}_z}{\partial t} = \frac{\xi\, u_0\, f(z)}{t}\, \epsilon^{\xi f(z)} \qquad (84)$$

and, at $t = 0$,

$$\frac{\partial \bar{\bar{u}}_z}{\partial t} = -\frac{u_0\, 8\, f(z)}{F(n, s, k_h, k_s)} \left[\frac{(1 + e)\, k_h}{a_v\, \gamma_w\, d^2{}_e} \right] \qquad (85)$$

which will be recognized as Eq. 82. From Eq. 64 excess pressure in the drain well is:

$$\begin{aligned} u_{w,z} &= \bar{\bar{u}} - \frac{C_2\, d^2{}_e\, \nu}{4} = \bar{\bar{u}}_z + \frac{a_v\, \gamma_w}{2\,(1 + e)\, k_h}\, \frac{d^2{}_e\, \nu}{4}\, \frac{\partial \bar{\bar{u}}_z}{\partial t} \\ &= \bar{\bar{u}}_z - \frac{a_v\, \gamma_w}{(1 + e)\, k_h}\, \frac{d^2{}_e\, \nu}{8\, \nu}\, \frac{u_0\, 8\, f(z)\, (1 + e)\, k_h}{a_v\, \gamma_w\, d^2{}_e}\, \epsilon^{\xi f(z)} \\ &= \bar{\bar{u}}_z - u_0\, \epsilon^{\xi f(z)} = \bar{\bar{u}}_z\, [1 - f(z)] \qquad (86) \end{aligned}$$

From Eqs. 37 and 38 the excess pressure at any point within the undisturbed soil is:

$$\begin{aligned} u_{r,z} = \bar{\bar{u}}_z \Bigg\{ & \frac{f(z)}{F(n, s, k_h, k_s)} \\ & \times \left[\log_e \left(\frac{r}{r_s} \right) - \frac{r^2 - r^2{}_s}{2\, r^2{}_e} + \frac{k_h}{k_s} \left(\frac{n^2 - s^2}{n^2} \right) \log_e (s) \right] + 1 - f(z) \Bigg\} \qquad (87) \end{aligned}$$

The average excess pressure, $\bar{\bar{\bar{u}}}$, throughout the entire undisturbed soil mass is:

$$\bar{\bar{\bar{u}}} = \frac{1}{H} \int^{H} \bar{\bar{u}}_z\, dz \qquad (88)$$

which may be solved by such approximate method as Simpson's rule. For no smear but with well resistance, $F(n, s, k_h, k_s)$ reduces to $F(n)$ as given in Eq. 10. For smear but no well resistance, $f(z)$ becomes equal to 1.

DISCUSSION

Kenneth S. Lane,[15] Assoc. M. ASCE.—As the drain well promises to become a very useful tool in foundation engineering, the author's development of the theory of consolidation with drain wells is particularly timely. The process of consolidation involves the expulsion of water from the soil voids, thus permitting the grains to attain closer contact under the influence of applied loads. In a homogeneous material the time required for consolidation varies as the square of the distance the water must travel. Insertion of sand-filled drain wells in a foundation shortens the water escape path and thus greatly accelerates the rate of consolidation. With water escaping radially to the wells, in addition to escaping in the normal vertical direction, the time of consolidation can be reduced to a fraction of that for vertical flow only (see Fig. 6)—governed mainly by the choice of well spacing. This benefit from drain wells is far greater in a horizontally stratified material (as the author has noted) where the horizontal or radial permeability is generally much larger than that in a vertical direction.

Benefits of Drain Wells.—There are three major benefits from using drain wells in a weak compressible foundation: (1) Acceleration in the rate of settlement; (2) acceleration in the rate of gain of shear strength; and (3) reduction in the lateral transmission of excess pressure.

1. *Acceleration in Rate of Settlement.*—The percentage of ultimate settlement at a given time is the same as the percentage of consolidation.

2. *Acceleration in Rate of Gain of Shear Strength.*—Let σ_t be the total unit stress on an element of soil; σ_o be the initial stress; $\Delta\sigma$ be the increment of stress added to σ_o; $\bar{\sigma}$ be the intergranular pressure; and σ_f be the final intergranular pressure. In the notation of the paper $z\,\gamma_w$ is the hydrostatic pressure and u is the pore water pressure in excess of hydrostatic pressure. The total stress on an element of soil consists of intergranular pressure plus water pressure. With the addition of a new load to a soil which has reached full consolidation under its former loads, this results in:

$$\begin{array}{lcccccl}
\sigma_t = & \sigma_o & + & \Delta\sigma & + & z\,\gamma_w & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(89a)\\
 & \downarrow & & \downarrow & & \downarrow & \\
\sigma_t = & \bar{\sigma}_o & + & u & + & z\,\gamma_w & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(89b)\\
 & \multicolumn{3}{c}{\underbrace{\hphantom{\bar{\sigma}_o + u}}_{\downarrow}} & & \downarrow & \\
\sigma_t = & & \bar{\sigma}_f & & + & z\,\gamma_w & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(89c)
\end{array}$$

Under a common hypothesis for the shear strength of clays, only the intergranular pressure, $\bar{\sigma}$, is effective in creating shear strength from internal friction. The process of consolidation involves the transfer of added stress, $\Delta\sigma$, initially carried in the pore water as excess pressure, u, to intergranular stress with resulting gain in shear strength. Eqs. 89*b* and 89*c* represent the beginning and end of consolidation, the hydrostatic portion of the pore-water pressure, $z\,\gamma_w$, remaining unchanged.

---

[15] Chf., Soils and Geology Branch, Garrison Dist., Corps of Engrs., Bismarck, N. Dak.

As the author has indicated, the factors governing the shear strength of clays are not well understood. The other common approach is based on empirical discoveries that shear strength varies inversely with water content. Under either approach shear strength increases as consolidation progresses by decreasing both pore pressure and water content.

3. *Reduction in Lateral Transmission of Excess Pressure, u, in the More Pervious Layers of Stratified Soil.*—Beneath the center of an embankment the added stress, $\Delta\sigma_c$, and the resulting excess pressure, u_c, reach their maximum values. If the horizontal permeability is much greater than the vertical permeability, as in the varved clay pictured in Fig. 9, a portion of the center line excess pressure, u_c, will be transmitted laterally to the zone beneath the embankment toe. For such conditions at the toe, Eqs. 89 become:

$$\begin{array}{rcccl}
\sigma_t = & \sigma_{ot} & + \Delta\sigma_t & + z\,\gamma_w & \ldots\ldots\ldots\ldots\ldots(90a) \\
 & \downarrow & \downarrow & \downarrow & \\
\sigma_t = & \overbrace{\bar{\sigma}_{ot} + f(u_c)} & + \; u_t & + z\,\gamma_w & \ldots\ldots\ldots\ldots\ldots(90b) \\
 & \multicolumn{2}{c}{\underbrace{\qquad\qquad\qquad\qquad}} & & \\
 & \multicolumn{2}{c}{\downarrow} & \downarrow & \\
\sigma_t = & \multicolumn{2}{c}{\bar{\sigma}_{ft}} & + z\,\gamma_w & \ldots\ldots\ldots\ldots\ldots(90c)
\end{array}$$

in which $f(u_c)$ is the part of u_c transmitted to beneath the toe. The effective intergranular pressure is reduced below its initial value to

$$\bar{\sigma}_{ot} = \sigma_{ot} - f(u_c) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(91)$$

The shear strength is correspondingly reduced, and may be dangerously so, since an embankment induces high shear stresses in the zone beneath its toe.

By intercepting such laterally transmitted excess pressure, drain wells serve to minimize reduction in shear strength beneath an embankment toe. Although the danger from this lateral transmission of excess pressure becomes greater as the ratio of horizontal to vertical permeability increases, fortunately the same condition operates to improve the efficacy of the drain wells in attracting radial drainage.

Usage of Drain Wells.—Sand piles have been used to strengthen soft foundations for many years in the Low Countries of Europe and such sand columns undoubtedly accelerated consolidation when loads were added; although it is questionable if constructors of the sand piles contemplated this added benefit or relied on it. To the writer's knowledge the first intended application as true drain wells occurred on the "SVIR 3" power dam constructed from 1930 to 1935 in Russia.[16] Here 12-in. sand-filled wells were installed 40 ft deep in a bed of interstratified clay and sand of Devonian age and appear to have served several purposes. Although they were initially installed for water control during excavation, they seem to have been used later for seepage control as relief wells for reducing seepage pressure in the more pervious buried strata. They also served to accelerate settlement of this concrete dam, as well as rate of gain of shear strength and corresponding resistance against sliding.

In American practice it appears that O. J. Porter, M. ASCE, and his associates at that time in the California Highway Department were the first to design and install sand-filled drain wells for the expressed purpose of accelerating consolidation—an experimental installation in 1934 followed by several highway embankments over marshes, peats, and harbor silts.[4,5] Early work was characterized by relatively large wells, up to about 30 in. in diameter. Subsequently, drain wells have been used considerably on the west coast of the United States on several important war construction and other projects. Recent large installations are on the Terminal Island Freeway near Los Angeles, Calif.,[17] and on the Bayshore Freeway near San Francisco, Calif.,[18] employing drain wells from 18 in. to 22 in. in diameter and from 30 ft to 60 ft deep. The objective of these installations has been largely for accelerating settlement and, to a lesser degree, for increasing shear strength to reduce toe displacements.

---

[16] "Some Features in Connection with the Foundation of SVIR 3 Hydro-electric Power Development," by H. Graftio, *Proceedings*, International Conference on Soil Mechanics and Foundation Eng., Harvard Univ., Cambridge, Mass., Vol. 1, 1936, pp. 284–290.

For similar purposes, the Connecticut Highway Department has employed drain wells of 12 in. in diameter and 25 ft in depth on 15-ft centers in crossing a marsh near Old Lyme, Conn.[19] An extensive installation is contemplated on a very wide or "dual-dual" highway near Elizabeth, N. J., using wells 18 in. in diameter on 10-ft centers up to 24 ft deep across a marsh.[20] It is particularly interesting that alternate bids on this New Jersey project showed a saving of more than 30% in favor of drain wells over excavation of the marsh deposit. An application almost solely for accelerating settlement is the experimental installation at LaGuardia Field, New York, N. Y.[21]—17-in. wells about 80 ft deep in silt with spacing varied from 8 ft to 14 ft.

Drain wells for the major purpose of increasing rate of gain in shear strength were employed during the construction of cellular cofferdam walls for submerged shipways.[6] Here 12-in. wells were placed inside the cofferdam cells to stiffen the filling of clay-like material formed from dredged-in marl. As an interesting innovation to speed further the consolidation of the cell fill, the central wells were pumped.

For the prime purpose of reducing lateral transmission of excess pressure, the first installation to the writer's knowledge was employed on the reconstruction of a slide on Riverfront Dike in Hartford, Conn.[22] As a matter of interest, this particular slide is one of the few that was analyzed and predicted in advance. A highway of dredged fill was being added landward soon after substantial completion of the dike, and the highway engineers decided to take the risk in the face of warnings on the probability of a slide, which occurred substantially as predicted. In designing the earthwork phases of the reconstruction for the Providence (R. I.) District of the Corps of Engineers, the writer and the author concentrated three lines of drain wells just inside the toe of the rebuilt embankment to insure against loss of shear strength at this critical zone.

Some ninety drain wells were constructed, in February and March, 1942, through 40 ft of sand and 40 ft of soft, varved clay (see Fig. 9). Wells were spaced in a triangular pattern on 20-ft centers and, to facilitate construction by hired labor equipment, were 6 in. in diameter. A 6-in. temporary casing was used through the sand, advancing the hole with a churn drill. To minimize smear, the clay was excavated by water jets of a special clay auger designed by F. E. Fahlquist, M. ASCE. Construction cost was about $1.00 per ft of well.

---

[17] "Vertical Sand Drains," *Roads and Streets*, August, 1946, p. 72.

[18] "Vertical Sand Drains Speed Consolidation of Soft Water-Bearing Subsoil," *Engineering News-Record*, March 6, 1947, p. 88.

[19] "Sand Drainage Wells Provide Possibilities for Economy," Highway Research Abstracts, Highway Research Board, October, 1947, p. 6.

[20] "Plan Use of Vertical Sand Drains Under New Jersey Superhighway," *Engineering News-Record*, October 23, 1947, p. 7.

[21] "Sand Drains for LaGuardia Field," *ibid.*, August 28, 1947, p. 7.

[22] "Foundation Failure Causes Slump in Big Dike at Hartford, Conn.," *ibid.*, July 31, 1941, p. 142.

A group of closed-system piezometers was in place, with which it was hoped to measure the effect of these drain wells. However, the effect of daily temperature variations from darkness to daylight made the closed-system piezometers very unsatisfactory and, by removing the Bourdon gages, they were converted to the open-system type (open pipe with filter at the tip). After this change the piezometers also failed to give dependable readings because of the effect of dampening on changing pressures, from the flow into an open-system piezometer from a highly impervious soil. Accordingly, no dependable measurements were obtained on the performance of these drain wells except that, from general observations, it appeared satisfactory.

One question that arose in the design of these Riverfront Dike drain wells concerned a possible tendency of the well to act as a sand pile or hard point in the foundation, attracting load and somewhat relieving stresses in the surrounding compressible deposit. Although this tendency would be less for wells of small diameter than for larger wells with a smaller slenderness ratio, it was decided to place two cushion layers of cinders on the Riverfront Dike wells, with the thought that this material would be more compressible than the sand backfill, and substantially as pervious. However, the cinders obtainable proved to be a poor filter against intrusion of the silt strata in the surrounding varved clay and were difficult to install, tending to sink through water slowly and to mix with the sand. Accordingly, use of the cinder cushion layers was abandoned and most of the wells were backfilled entirely with sand, poured in from the surface.

If a drain well should act as a stiff point in attracting load in spite of the loose state of its sand backfill, the effect on arching of load would apparently be the opposite of the author's solution for equal vertical strain. The effect on consolidation rate would probably not be large, judging from the small difference in rates for free strain and for equal vertical strain shown in Fig. 8. Of more concern is the action of the sand column in deforming under a load concentration as the surrounding, and probably more compressible, soil settles. This deformation may be accomplished either by a bulging of the well column of sand with a small increase in density, or by a local shearing with possibly a relative displacement of the sand column. Bulging is not objectionable and is believed to be the more probable action in soft foundation soils where most drain wells have been installed to date, except on the SVIR 3 dam.[16] For example, to accommodate about 4 ft of settlement, an 18-in. drain well 80 ft long would have to increase uniformly in diameter only about 0.4 in., and considerably less than this by allowing for the densification of the sand backfill from its initially loose state. A definite shearing of the well column could be very undesirable because, if continuity of the drainage path were broken, the break might even nullify the effect on the consolidation contributed by the region of the well below the blocked zone. Such shearing of the sand column would seem more likely to occur in brittle, but still quite compressible, clays and, pending further knowledge, is probably best handled by conservatism in selecting well spacing and size.

In what is believed to be the first application of drain wells to the design of an earth dam, from 1943 to 1945, the writer and the author designed a system of about 240 drain wells to be placed in a triangular pattern on 60-ft centers over substantially the entire valley bottom of Claremont Dam, near Claremont, N. H. Although the design of this dam was essentially completed in 1945 by the Providence District, Corps of Engineers, its construction has been deferred pending further negotiation with local interests. The dam is intended to be about 120 ft high over a foundation of approximately 15 ft of sand and gravel lying over from 40 ft to 80 ft of soft, varved silt (silt, clay, and fine sand interstratified, as shown in Fig. 9, with silt predominant). The weak silt is underlain by impervious glacial till which is likely to create a case of single drainage where excess pore water can escape only in an upward direction. Drain wells 18 in. in diameter, extended to the bottom of the silt with methods to minimize smear, were included in the design because flat slopes to reduce shear stresses and an extended construction period of gradual load application were not in themselves considered sufficient for stability with this very weak silt foundation. The wells are intended both to reduce lateral transmission of excess pressure and to accelerate rate of gain of shear strength.

Conclusions.—For the design of a drain well installation, the author's solutions are excellent examples of the case where a mathematical solution is invaluable in indicating the relative importance of the different variables—ratio of horizontal to vertical permeability, well spacing, well resistance, peripheral smear, and well size. In general, these can be evaluated from the curves which the author has laboriously computed and presented in this paper and in his earlier paper.[7] If the author has other similar curves available, it is felt they would be a worthwhile inclusion in the closing discussion, particularly where they would aid in showing the effect of single variables and comparing the relative importance of the different variables.

The author has properly indicated that certain solutions were developed by L. Rendulic[8] in 1934 and by N. Carillo[9,10] in 1940.[23] For the record, it is worthy of mention that the author also developed substantially the same solutions independently from 1940 to 1942 before learning of the solutions of Messrs. Rendulic and Carillo.

Philip Keene,[24] Assoc. M. ASCE.—A valuable contribution to the subject of drainage wells for hastening consolidation of fine-grained soils, a subject which is receiving increased attention in the field of soil mechanics, has been made by the author. After a study of the paper, the graphs become useful tools in analyzing actual problems and in designing installations, as mentioned by the author. Mr. Barron is also to be congratulated for his solutions showing the effects of smear and well resistance which were not presented before. When the wells are installed without casing or mandrel, the smear zone is probably less than ½ in. thick and can be ignored for wells larger than 6 in. in diameter. For other cases, the author's solution indicates that the smear zone materially affects the time factor and rate of settlement.

---

[23] "Drainage of Clay Strata by Filter Wells," by K. Terzaghi, *Civil Engineering*, October, 1945, p. 463.

[24] Associate Highway Engr. (Soils and Foundations), State Highway Dept., Hartford, Conn.

During 1947 the writer designed and supervised the installation of about 400 sand drainage wells on a highway project in Old Lyme, Conn.[25] A fill from 15 ft to 20 ft high was placed for a four-lane divided highway on a foundation of approximately 30 ft of soft organic clayey silt. Final settlements will average about 5.4 ft; 3.5 ft (or 65%) consolidation occurred during the six weeks necessary to place the fill. Assuming the average date of load application to be three weeks after commencement of placing the fill, 4.0 ft (or 75%) consolidation occurred five and one-half weeks after this average loading date. This is about 6% of the time estimated for 75% consolidation if no wells had been installed. Calculations made with the aid of Fig. 3 indicate that the actual time-settlement curve for that project will roughly approximate a theoretical time-settlement curve, if it is assumed that horizontal permeability is ten times as great as vertical permeability. Well resistance was not included in these calculations, although it was actually present in approximately the same value used in Fig. 10.

WALTER KJELLMAN,[26] ESQ.—The consolidation of a fine-grained soil subjected to a load can be accelerated by vertical drains inserted in the soil. In the United States such drains consist of circular sand-filled wells, having normally about a 20-in. diameter and a spacing of from 10 ft to 15 ft. In Sweden, band shaped cardboard wicks of ⅛-in. by 4-in. cross section (Fig. 11) are used. These are furnished with inner longitudinal channels and spaced about 4 ft apart. It may perhaps interest readers to learn how the problems, so clearly dealt with in this paper, are looked upon in Sweden.

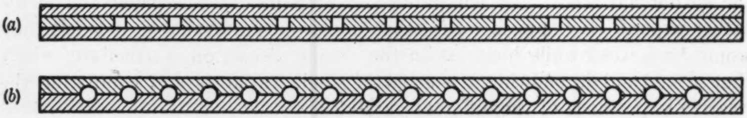


FIG. 11.—CROSS SECTION OF BAND SHAPED CARDBOARD DRAIN
(a) Present Type
(b) Prospective Type

Swedish investigations concerning "deep drainage" of soils were started in 1936. Owing to arching of ground and overburden in all practical cases, the momentary strains were deemed to be nearly equal everywhere in the ground. Therefore, all Swedish calculations refer to this very simple case of equal strain. In the United States, on the contrary, all calculations of this kind prior to Mr. Barron's paper seem to refer to the very complicated case of free strain. Fig. 8 shows that both cases give nearly the same result. Consequently, there is no reason for anyone to become further involved with free strain.

When determining the spacing of the drains in the United States, great reliance is placed on the horizontal draining effectuated by coarse-grained layers.

---

[25] "Sand Wells Speed Marsh Crossing," by Philip Keene, *Better Roads*, September, 1947.

[26] Head, Statens Geotekniska Institut (The Royal Swedish Geotechnical Institute), Narvavägen 25 Stockholm, Sweden.

In Sweden such layers exist in the lower part of the glacial clay deposits, but they are of little use since the spacing must be determined with regard to the upper part of the deposits (the upper part of the glacial clay and, on top of it, the postglacial clay), which as a rule contains no coarse-grained layers. Furthermore, the fine-grained soils in Sweden seem on the whole to be considerably less pervious than those in America. For these reasons, and because the equivalent radius of the wicks used in Sweden is much smaller than the radius of the American wells, the spacing is considerably smaller in Sweden.

By laboratory tests it has been found that the permeability of a clay without coarse-grained layers is not appreciably reduced by remolding. The unfavorable effect of peripheral smear, discussed by the author, must therefore refer to coarse-grained layers only. No reliance is placed on such layers in Sweden, so that peripheral smear need not be considered.

In Sweden the spacing of the drains is rather small. Thus, the vertical water flow is unimportant when compared to the radial flow and is disregarded. This approximation, which simplifies the calculation and leads to safe results, is justified by the uncertainty of the calculation.

In principle the triangular drain pattern is the most economical one, as stated by Mr. Barron. However, certain considerations have shown that in this respect the difference between the triangular and square patterns is quite unimportant. Therefore, in order to attain certain small advantages when driving the wicks, no fixed pattern is used, other than to make the spacing of the wicks in each row equal to the distance between rows.

The longitudinal channels in the cardboard wick easily can be made numerous enough and wide enough to let through (without appreciable resistance) any water flow that the one drain may be called on to carry. Furthermore, the permeability of the cardboard is very great when compared to that of the fine-grained soils in question. Therefore, there is no need to complicate the calculations by taking into account the flow resistance in the drain. Because the cardboard serves as a perfect filter, there is, of course, no risk of the channels becoming clogged as may occur in sand-filled wells. Laboratory tests have shown that the channels do not collapse even under a very high clay pressure on the wick. This is probably due to arching in the clay, causing the pressure to be much lower on the covers of the channels than between them.

The question has been raised as to whether a well (like a pile) attracts a great part of the load, and whether it can be damaged when the soil settles. A cardboard wick cannot, of course, affect the load distribution. Tests have shown that it may crease, if the vertical compression is great, but that it is not damaged.

In Swedish practice (as follows from the foregoing statements), the time-settlement curve of a deep-drained soil is calculated in the simplest possible manner—assuming equal strain, no coarse-grained layers, radial flow only, no peripheral smear, and no drain resistance. The calculation, made in 1937 but never published, is contained in principle in Mr. Barron's paper. As a result of the calculation the following procedure is used for determining the spacing of the drains (square pattern):

(*a*) Having assumed a time, t_h, in months, in which a certain percentage of the final settlement is to occur, Fig. 12 is used to find the value of g, which is a quantity of calculation with no physical meaning.

(*b*) Knowing the value of g, and also the coefficient of consolidation of the soil, c_h (in square centimeters per second), the value of m, which is a quantity of calculation with no physical meaning, is computed from

$$m = g\,c_h \qquad (92)$$

(*c*) Knowing the value of m, Fig. 13 is used to find a convenient combination of drain radius r_w and spacing S.

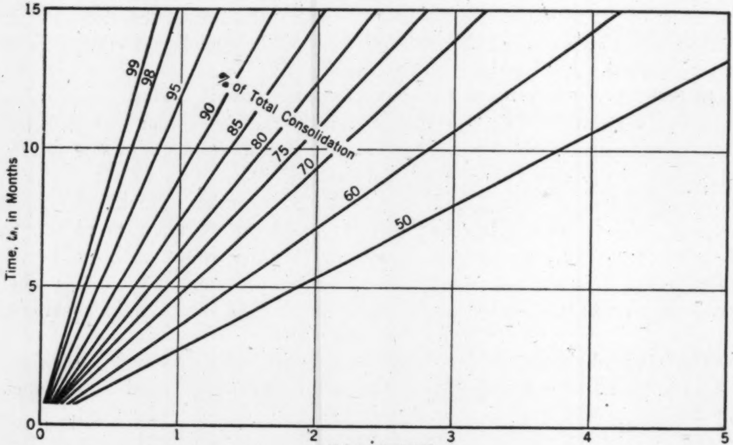


FIG. 12.—DIAGRAM FOR FINDING VALUES OF g

This procedure may be used in cases not too divergent from Swedish conditions.

Obviously, the draining effect of a drain depends to a great extent upon the circumference of its cross section, but very little upon its cross-sectional area. Therefore, the circular cross section has been abandoned in Sweden in favor of the band shaped drain. Certain considerations show that the cardboard wick is as effective as a circular drain with a 1-in. radius. (The circumference of the latter is slightly greater than that of the former.) Thus, the curve in Fig. 13 for a 1-in. radius can be used for the wicks. The curve for a 10-in. radius is valid for the sand wells with a 10-in. radius frequently used in the United States. Comparing these two curves, it appears that wells with 10-ft spacing are equivalent to wicks spaced 6.2 ft apart. Thus, in this case, one well is equivalent to 2.5 wicks.

The wick has several advantages over the well. First, very little material is consumed in its manufacture, and, second, as both its weight and its volume per unit of length are small and as it can be rolled on a drum, it is easy to handle. This means that production can be concentrated in a factory, whence the wicks can be transported inexpensively to the different sites and driven into the ground by a machine working at high speed in about the same manner as a sewing machine.

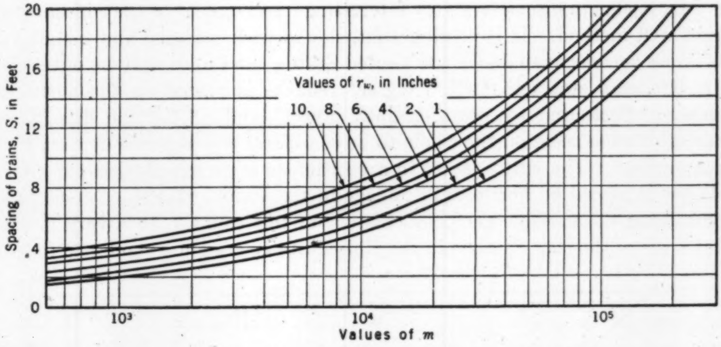


FIG. 13.—DIAGRAM FOR FINDING THE SPACING OF DRAINS

At the Halmsjön Airport outside Stockholm (Sweden) some 3,000,000 ft of cardboard wicks are to be driven in 1947–1949. The cost is between 10¢ per ft and 15¢ per ft. Further information about the wick method will appear in the *Proceedings* of the Second International Conference on Soil Mechanics to be held in Rotterdam, Holland, in 1948.

REGINALD A. BARRON,[27] ASSOC. M. ASCE.—The benefits of drain wells have been brought out by Mr. Lane's discussion. Although, probably, they are not quantitatively correct, Eqs. 89, 90, and 91 are of great use qualitatively. The writer believes that Mr. Lane is correct in his statement that column action by the well backfill would redistribute the soil stresses and would result in a slower rate of consolidation. However, Fig. 8 indicates that the difference in consolidation rates is very minor and of no practical importance. In fact, such column action may be much less than the possible compressive effect of the horizontal seepage gradient. This possibility was not considered by the writer but its effect would be to compress the soil radially in a horizontal plane adjacent to the well where the seepage gradients are high. Because of this radial compression, the soil farther out will probably cause vertical settlement as it deforms radially in extension.

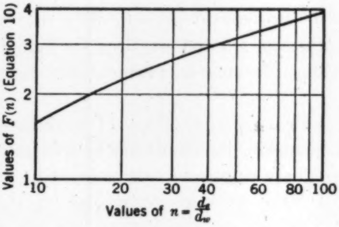


FIG. 14.—CURVE FOR THE DETERMINATION OF $F(n)$

Mr. Keene has discussed the use of consolidation drain wells on a highway project. The use of drain wells to accelerate the consolidation of foundations for highway fills is valuable, especially considering the effect of foundation consolidation on the finished quality of the pavement. On the other hand, such measures are not always necessary for earth dams, and it may be more economical to obtain stability by using flatter embankment slopes.

---

[27] Head, Special Studies Section, Embankment and Foundation Branch, Soils Div., U. S. Waterways Experiment Station, Vicksburg, Miss.

Mr. Kjellman discusses a very interesting development of the consolidation drain well method. It is possible that, should wick material and installation machines become available in the United States, sand wells may be outmoded. Mr. Kjellman states that "* * * great reliance is placed on the horizontal draining effectuated by coarse-grained layers" and "* * * the fine-grained soils in Sweden seem on the whole to be considerably less pervious than those in America." The writer is not in a position to state exactly what degree of reliance is placed on horizontal drainage in the United States. Furthermore, considering the variable geological history of this country, it is very possible that some of the clay deposits in the United States are quite similar to those of Sweden. The writer considered the effect of horizontal permeability in the theoretical developments because such conditions existed at sites for which the studies were made. For other cases this difference in permeability may not be so marked.

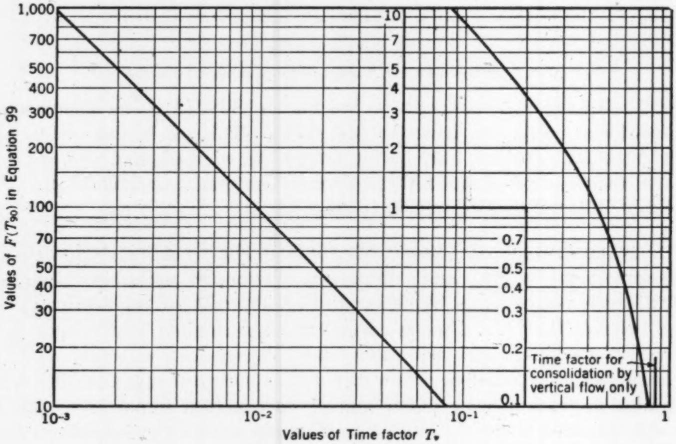


FIG. 15.—CURVE FOR THE DETERMINATION OF TIME FACTOR OF 90% CONSOLIDATION DUE TO COMBINED RADIAL AND VERTICAL FLOW

Mr. Lane requests additional curves indicating the effect of variables. Fig. 14 shows the variation of $F(n)$ as defined in Eq. 10 versus n which is used in the following development for instantaneously applied loads using Eq. 21 in which $\bar{u}_r$ is obtained from Eq. 8. For 90% average consolidation the average hydrostatic excess pressure is 10%. Thus:

$$\bar{u}_{r,z} = \bar{u}_z \, \epsilon^{\lambda} = 0.1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (93)$$

Expressions for the time factors are

$$T_h = \frac{k_h \, (1 + e) \, t}{\gamma_o \, a_v \, d^2_e} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (94)$$

and

$$T_v = \frac{k_v\,(1 + e)\,t}{\gamma_o\,a_v\,H^2} \qquad (95)$$

and, therefore,

$$T_h = T_v\,\frac{H^2}{d^2_e}\,\frac{k_h}{k_v} \qquad (96)$$

and

$$\lambda = \frac{-\,8\,T_h}{F(n)} = \frac{-\,8\,T_v\,H^2\,k_n}{k_v\,d^2_e\,F(n)} \qquad (97)$$

Consequently,

$$2.3026\,(1 + \log_{10}\bar{u}_z) = \frac{8\,T_v\,H^2\,k_n}{k_v\,d^2_e\,F(n)} \qquad (98)$$

or

$$\frac{(1 + \log_{10}\bar{u}_z)}{T_v} = \frac{8\,k_h\,H^2}{2.3026\,k_v\,d^2_e\,F(n)} = F(T_{90}) \qquad (99)$$

A curve of $F(T_{90})$ versus T_v is given in Fig. 15.

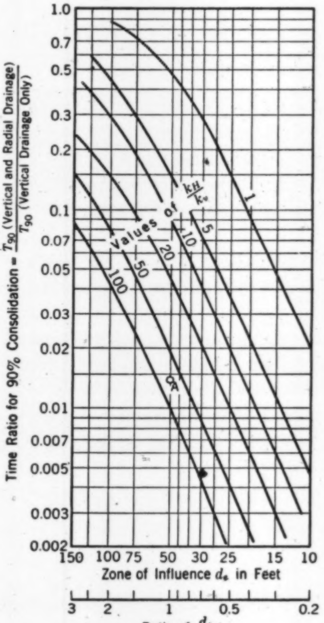


FIG. 16.—EFFECT OF WELL SPACING AND PERMEABILITY RATIO ON TIME REQUIRED FOR 90% CONSOLIDATION

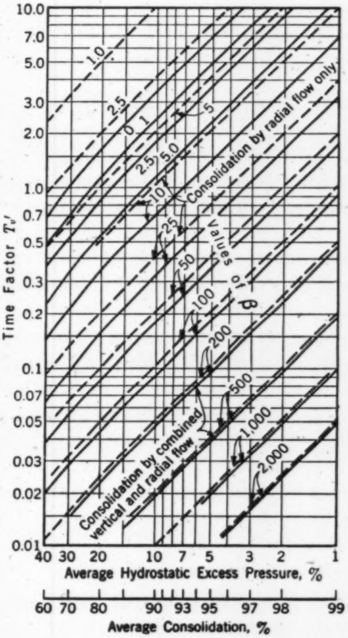


FIG. 17.—AVERAGE PERCENTAGE OF HYDROSTATIC EXCESS PRESSURE AND CONSOLIDATION AT END OF UNIFORM RATE OF LOADING PERIOD

Using the curve in Fig. 15, the curves in Fig. 16 for various permeability ratios were determined for the conditions $d_w = 1$ ft and $H = 50$ ft, with no smear or well resistance. The influence of permeability ratio k_h/k_v and well spacing is evident.

Point A in Fig. 16 was determined for the case of no smear but with well resistance for the condition that $k_w/k_v = 2{,}000{,}000$ and $k_h/k_v = 100$. Well resistance thus requires about 40% more time to reach 90% consolidation; for small ratios of k_h/k_v the effect would be less.

The curves in Fig. 17 for several values of

$$\beta = \frac{8\, k_h\, H^2}{k_v\, d_e^2\, F(n)} \qquad (100)$$

indicate the effects of k_h, k_v, H, d_e, and n on the average percentage of consolidation and average hydrostatic excess pore water pressure at the end of uniform rate of load application.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

TRANSACTIONS

Paper No. 2347

INFLUENCE LINES FOR CONTINUOUS STRUCTURES BY GEOMETRICAL COMPUTATION

BY DEAN F. PETERSON, JR.,[1] ASSOC. M. ASCE

WITH DISCUSSION BY MESSRS. RALPH E. SPAULDING, RALPH W. STEWART, YI-MAI YAO, OTTO GOTTSCHALK, AND DEAN F. PETERSON, JR.

SYNOPSIS

As the practical span length of continuous girder bridges and rigid frame bridges becomes progressively greater, the use of influence lines or coefficients, for the purpose of investigating the effect of more numerous possible combinations of loading, and for placing loads for maximum effect, becomes correspondingly important. Although such influence lines may be constructed in the usual manner by loading the structure at a number of different points and computing the resulting stress or moment, this is an arduous task and amounts to finding the desired load points by trial and error. The equality of the elastic curve of a structure loaded with a unit deformation at the point for which the stresses are desired, and the influence line for stress at that point, has long been used for determining the influence line for trussed structures. The principle was applied mechanically to all types of indeterminate structures by the late George Beggs,[2] M. ASCE, Otto Gottschalk,[3] and others. This relationship may easily be extended to the analytical computation of influence lines for continuous structures subjected principally to flexure, accomplishing analytically what the use of models accomplishes mechanically. Examples of influence lines for several types of continuous structures have been computed by this procedure using various types of statical analysis. Formulas have also been developed for locating the position and value of the maximum influence ordinate and for the area under the influence line for members of constant cross section.

NOTE.—Published in September, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Associate Prof. of Civ. Eng., Utah State Agri. College, Logan, Utah.

[2] *Transactions*, ASCE, Vol. 88, 1925, p. 1208.

[3] *Ibid.*, Vol. 103, 1938, p. 1019.

BASIC RELATIONSHIP

The basic relationship involved in the development of this paper—the Müller-Breslau relation— is well known. Its validity follows from a consideration of Maxwell's principle of reciprocal deflections, and the principle of superposition which is valid as long as the structures are built of materials that act elastically.

FUNDAMENTAL EQUATIONS OF FLEXURAL GEOMETRY

This paper is concerned with flexural equations both as a means of (1) producing a statical solution and (2) computing elastic curve ordinates. The fundamental differential equation of flexural geometry, $\frac{d^2y}{dx^2} = \frac{M}{E\,I}$ (neglecting the distortions due to shear and direct stresses), is a special case of the general equations of elasticity and has the following first and second integrals:

$$\left.\frac{dy}{dx}\right|_x = \int_0^x \frac{M_\xi}{E_\xi\, I_\xi}\, d\xi + C_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

and

$$y\,\Big|_x = \int_0^x \frac{M_\xi}{E_\xi\, I_\xi}\,(x - \xi)\, d\xi + C_1\, x + C_2 \ldots\ldots\ldots\ldots (1b)$$

in which, in addition to the variable coordinates x and y, M_ξ is the bending moment at ξ; E_ξ is the modulus of elasticity at ξ; I_ξ is the moment of inertia at ξ; C_1 is a constant of integration; and C_2 is a constant of integration. Eqs. 1 may sometimes be solved by integral calculus, but are usually solved by finite integration. However, in the field of indeterminate structures, the imposition of more than the required boundary conditions makes necessary the solution of a set of simultaneous equations, or its mathematical equivalent. The many short cuts which the modern structural engineer uses in his analysis are simply means of setting up and solving Eqs. 1.

PROCEDURE FOR SOLUTION

Influence lines for moments, shears, thrusts, and reactions are obtained by determining the elastic curve of the structure when it is loaded with the corresponding unit axial rotation, transverse axial displacement, longitudinal axial displacement, or deflection in the direction of reaction. It is necessary first to make a statical analysis of the structure bearing only the required displacement loading using any method—moment distribution, slope deflection, elastic curve traverse, etc.—that the designer desires. In general, since all the members are unloaded (in the sense that they do not bear any gravity, wind, hydrostatic, or other usual load), the analysis is simpler than when conventional loads are involved.

After statical solution of the structure has been completed, the end rotations and deflections for each of the members are computed. Using some methods of analysis, such as slope deflection or elastic curve traverse, these will be known directly from the process of solution. Other methods of solution will require the use of the flexural-geometry equations, or some derivative of them (such as the moment-area principle), to arrive at the required rotations and deflections. The elastic curve can be computed by solving Eq. 2, or by various modifications of Eq. 2.

INDIRECT METHOD

The statical solution for the structure loaded with the unit deflection or rotation may also be made using the method, developed in connection with moment distribution,[4] of (*a*) considering separately an unbalanced moment at each joint, (*b*) solving for the required moments in each case, and then (*c*) combining the result of the several fixed-end moments due to the loading in proportion to the original distributions. Instead of choosing fixed-end moments resulting from conventional loads, fixed-end moments compatible with the imposed displacement are proportioned and combined. This method is illustrated, subsequently, by Example 9 using moment distribution and may also be applied using the elastic curve traverse method.[5]

CALCULATING ELASTIC CURVES OF MEMBERS

Throughout the mathematical analysis the following sign convention will be adopted: Ordinate and abscissa as usual. Counterclockwise rotation in the direction of $+x$ will be plus and will produce a positive deflection from the tangent in the direction of $+x$. Moment causing concave bending upward will be considered positive. Positive moment will always give positive rotations and positive deflections in the direction of positive x-values. For a member whose axis can be considered straight, if y_A is the deflection from the original axis of point A and θ_A is the slope of the tangent to the elastic curve at point A, then, for any point, x_n:

$$y_n = y_A + \theta_A\, x_n + \sum_{\xi=1}^{\xi=n} (x_n - x_\xi)\, \frac{M_\xi}{E_\xi\, I_\xi} \Delta\xi \ldots\ldots\ldots\ldots (2)$$

Generally, however, it is simpler to compute the slope and deflection point by point; thus:

$$y_n = y_{n-1} + \theta_{n-1} \frac{\Delta x_{n-1} + \Delta x_n}{2} \ldots\ldots\ldots\ldots (3a)$$

in which

$$\theta_n = \theta_{n-1} + \frac{M_n}{E\, I_n} (\Delta x_n) \ldots\ldots\ldots\ldots (3b)$$

Analytical Solution.—Referring to the haunched beam in Fig. 1, suppose that:

$$\theta_A = -\frac{1{,}000}{E\, I_d}\,; \quad \theta_B = \frac{300}{E\, I_d} \ldots\ldots\ldots\ldots (4)$$

$$y_A = \frac{20{,}000}{E\, I_d}\,; \quad y_B = 0 \ldots\ldots\ldots\ldots (5)$$

---

[4] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, p. 105.

[5] "Relative Flexure Factors for Analyzing Continuous Structures," by Ralph W. Stewart, *Transactions*, ASCE, Vol. 104, 1939, p. 521.

in which I_d equals the moment of inertia, in feet^4, at the center section of depth d; and E is the modulus of elasticity in kips per square foot.

The end moments may be obtained by any desired method. For purposes of generalization the basic formulas (Eqs. 1) will be applied; thus, after simplifying and substituting finite summation for the integrals:

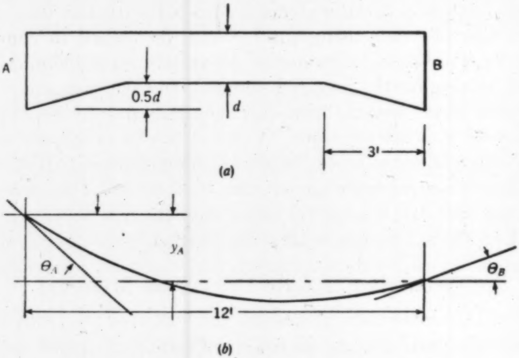


FIG. 1.—EXAMPLE OF A HAUNCHED BEAM

$$1{,}300 = M_A \sum_1^n K_n \left(1 - \frac{x_n}{12}\right) \Delta x_n + M_B \sum_1^n K_n \frac{x_n}{12} \Delta x_n \ldots\ldots (6a)$$

and

$$-\,8{,}000 = M_A \sum_1^n K_n \left(1 - \frac{x_n}{12}\right)(12 - x_n)\,\Delta x_n$$

$$+\, M_B \sum_1^n K_n \frac{x_n}{12}(12 - x_n)\,\Delta x_n \ldots\ldots\ldots\ldots\ldots\ldots (6b)$$

TABLE 1.—COMPUTATION OF M_A AND M_B IN THE SOLUTION OF EQS. 6

Segment (1)	x_n (ft) (2)	K_n (Eq. 7) (3)	$\Delta x_n\, K_n$ (4)	$\frac{x_n}{12}$ (5)	$\frac{x_n}{12}\,\Delta x_n\, K_n$ (6)	$1-\frac{x_n}{12}$ (7)	$\left(1-\frac{x_n}{12}\right) \times \Delta x_n\, K_n$ (8)	$12-x_n$ (9)	$\frac{x_n}{12}(\Delta x_n\, K_n) \times (12-x_n)$ (10)	$\left(1-\frac{x_n}{12}\right)(\Delta x_n\, K_n) \times (12-x_n)$ (11)
1...	0.5	0.353	0.353	0.0416	0.0147	0.9583	0.338	11.5	0.169	3.885
2...	1.5	0.510	0.510	0.1250	0.0638	0.8750	0.446	10.5	0.670	4.480
3...	2.5	0.787	0.787	0.2083	0.1640	0.7916	0.623	9.5	1.557	5.920
4...	3.5	1.000	1.000	0.2916	0.2916	0.7083	0.7083	8.5	2.478	6.015
5...	4.5	1.000	1.000	0.3750	0.3750	0.6250	0.6250	7.5	2.815	4.685
6...	5.5	1.000	1.000	0.4583	0.4583	0.5416	0.5416	6.5	2.979	3.520
7...	6.5	1.000	1.000	0.5416	0.5416	0.4583	0.4583	5.5	2.975	2.520
8...	7.5	1.000	1.000	0.6250	0.6250	0.3750	0.3750	4.5	2.810	1.688
9...	8.5	1.000	1.000	0.7083	0.7083	0.2916	0.2916	3.5	2.480	1.020
10...	9.5	0.787	0.787	0.7916	0.623	0.2083	0.1640	2.5	1.558	0.410
11...	10.5	0.510	0.510	0.8750	0.446	0.1250	0.0638	1.5	0.669	0.096
12...	11.5	0.353	0.353	0.9583	0.338	0.0416	0.0147	0.5	0.169	0.007
Σ....	....	....	9.300	....	4.6493	....	4.6493	....	21.329	34.446

in which

$$K_n = \frac{E\,I_d}{E\,I_n} \qquad (7)$$

The coefficients under the sigma signs in Eqs. 6 are calculated by Table 1, and

TABLE 2.—COMPUTATION OF DEFLECTION CURVE FOR THE BEAM IN FIG. 1

x_n (ft) (1)	$M_A\left(1-\frac{x_n}{12}\right)$ (2)	$M_B\,\frac{x_n}{12}$ (3)	M_n (4)	$K_n\,M_n\,\Delta x_n$ (5)	θ_n (6)	$\frac{\Delta x_{n-1}+\Delta x_n}{2}$ (7)	Δy (8)	y_n (9)
0.0....	−1,061	0	−1,061	....	−1,000	....	....	20,000
0.5....	−1,018	56	− 962	−340	−1,340	0.5	− 500	19,500
1.5....	− 930	168	− 762	−389	−1,729	1.0	−1,340	18,160
2.5....	− 841	279	− 562	−442	−2,171	1.0	−1,729	16,431
3.5....	− 752	391	− 361	−361	−2,532	1.0	−2,171	14,260
4.5....	− 664	503	− 161	−161	−2,693	1.0	−2,532	11,728
5.5....	− 576	614	38	38	−2,655	1.0	−2,693	9,035
6.5....	− 487	726	239	239	−2,416	1.0	−2,655	6,380
7.5....	− 398	839	440	440	−1,976	1.0	−2,416	3,964
8.5....	− 310	950	640	640	−1,336	1.0	−1,976	1,988
9.5....	− 221	1,060	839	660	− 676	1.0	−1,336	652
10.5....	− 134	1,173	1,039	529	− 147	1.0	− 676	− 24
11.5....	− 44	1,283	1,239	443	300	1.0	− 147	−171
12.0....	0	1,341	1,341	....	....	0.5	150	− 21
Total...	....	....	....	....	....	....	....	121,882

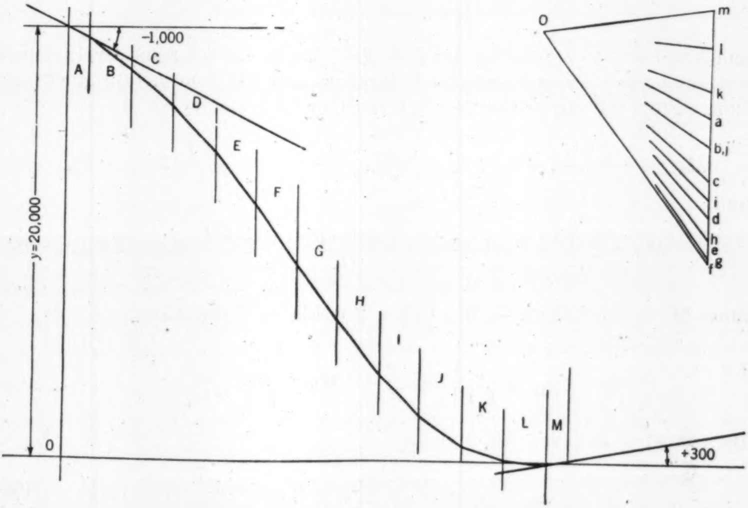


FIG. 2.—CONSTRUCTION OF DEFLECTION CURVE FOR THE BEAM IN FIG. 1

the equations thus solved yield: $M_A = -1{,}061$ ft-kips; and $M_B = 1{,}341$ ft-kips. Using Eqs. 3 the elastic curve is calculated by means of Table 2.

Graphical Solution.—Instead of using Table 2, Eq. 3*a* may be solved graphically by the use of the funicular polygon in Fig. 2. The chords passing through the end points of the beam must have a slope equal to the known slope of the end tangents.

Prismatic Members.—Although beams of constant section may be handled by either the analytical or graphical method of finite integration, it is possible to pass to the integral in the summation of Eq. 2 and derive general equations for the elastic curves when the end rotations and deflections are known. Consider a length AB of a straight, unloaded member (Fig. 3). Let the rotation at end A be equal to θ_A; and, at end B, to θ_B. Let y_A and y_B, respectively, represent the perpendicular displacement of points A and B from their original position. From the slope-deflection equations, or by geometry:

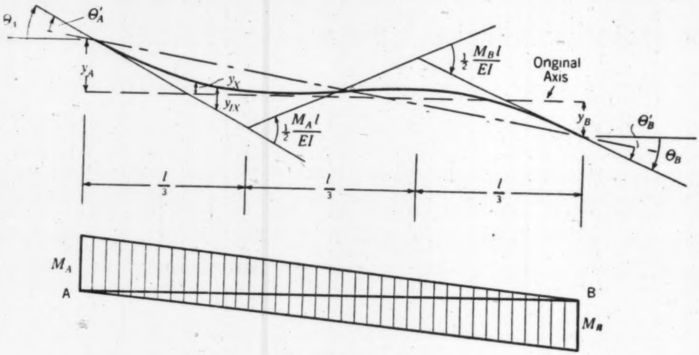


FIG. 3.—EXAMPLE OF A PRISMATIC MEMBER

$$M_A = -\frac{2\,E\,I}{l}\left(2\,\theta_A + \theta_B - 3\,\frac{y_B - y_A}{l}\right) = \frac{-2\,E\,I}{l}\,(2\,\theta'_A + \theta'_B)\,..\,(8a)$$

and

$$M_B = \frac{2\,E\,I}{l}\left(\theta_A + 2\,\theta_B - 3\,\frac{y_B - y_A}{l}\right) = \frac{2\,E\,I}{l}\,(\theta'_A + 2\,\theta'_B)\,....\,(8b)$$

Since $M_\xi = M_A + (M_B - M_A)\,\frac{\xi}{l}$, Eq. 2 yields, by integration,

$$y_{1x} = \frac{1}{E\,I}\left[\,M_A\,\frac{x^2}{2} + (M_B - M_A)\,\frac{x^3}{6\,l}\right]\,..............\,(9a)$$

By geometry, however, Eq. 2 yields,

$$y_x = y_A + \frac{y_B - y_A}{l}\,x + \theta'_A\,x + y_{1x}\,..............\,(9b)$$

Substituting Eq. 9*a* in Eq. 9*b* as well as the values of M_A and M_B from Eqs. 8, simplifying and collecting terms:[6]

$$y_x = y_A + (y_B - y_A)(k_1)(1 - k^2_1 + k_1\,k_2) + l\,(\theta_A\,k_1\,k^2_2 - \theta_B\,k^2_1\,k_2)\,..\,(10)$$

---

[6] *Transactions*, ASCE, Vol. 104, 1939, p. 555, Eq. 23.

in which $k_1 = \frac{x}{l}$, and $k_2 = 1 - \frac{x}{l}$.

Location of Maximum Load Point

When concentrated loads are being considered, it is important to know where the load should be placed to produce the maximum effect on the section being investigated. Eq. 10 may be rewritten since $k_2 = 1 - k_1$; thus:

$$\begin{aligned} y_x = y_A &+ [l\,\theta_A + l\,\theta_B - 2\,(y_B - y_A)]\,k_1^3 \\ &+ [3\,(y_B - y_A) - 2\,l\,\theta_A - l\,\theta_B]\,k_1^2 + l\,\theta_A\,k_1 \ldots\ldots\ldots\ldots (11) \end{aligned}$$

Differentiating with respect to k_1 and equating the result to zero a quadratic equation results, which may be solved to yield:

$$\begin{aligned} k_{1m} = &\frac{1}{3\,[l\,(\theta_A + \theta_B) - 2\,(y_B - y_A)]}\,[l\,(2\,\theta_A + \theta_B) - 3(y_A - y_B) \\ &\pm \sqrt{9\,(y_B - y_A)^2 - 6\,l\,(y_B - y_A)(\theta_A + \theta_B) + (\theta_A^2 + \theta_A\,\theta_B + \theta_B^2)\,l^2}\,] \ldots (12a) \end{aligned}$$

If $y_B - y_A = 0$, then:

$$k_{1m} = \frac{1}{3\,(\theta_A + \theta_B)}\,[(2\,\theta_A + \theta_B) \pm \sqrt{\theta_A^2 + \theta_B^2 + \theta_A\,\theta_B}\,] \ldots\ldots (12b)$$

Value of Maximum Influence Ordinate

It is possible to derive an expression for the value of the maximum influence ordinate for a member of constant section when $y_B - y_A = 0$. Substituting the value of k_{1m} from Eq. 12b for k_1 in Eq. 11 and solving:

$$\begin{aligned} y_m = &\frac{l}{27\,(\theta_A + \theta_B)^2}\,[(\theta_A^2 + 4\,\theta_A\,\theta_B + \theta_B^2) \mp (\theta_A - \theta_B)\,\sqrt{\theta_A^2 + \theta_B^2 + \theta_A\,\theta_B}\,] \\ &\times [(\theta_A - \theta_B) \mp \sqrt{\theta_A^2 + \theta_B^2 + \theta_A\,\theta_B}\,] \ldots\ldots\ldots\ldots (13) \end{aligned}$$

Area Under the Influence Line

An equation for the area under the influence line for a prismatic member may be obtained by integrating Eq. 11 between the limits $k_1 = 0$ and $k_1 = 1$. This operation yields:

$$Q = \frac{l}{2}\,(y_A + y_B) + \frac{1}{12}\,l^2\,(\theta_A - \theta_B) \ldots\ldots\ldots\ldots (14)$$

in which Q is the area under the influence line. Eq. 14 is of value when the entire span carries a uniformly distributed load. It cannot be used directly when the deflection curve is discontinuous or contains a slope discontinuity within the span. For members with varying cross section the area under the influence line may be determined by a summation of the influence line ordinates (see Col. 9, Table 2).

Examples

A fairly large number of examples have been included. This was done in order to furnish an illustration for each of several different methods of statical analysis of the structure bearing the deformation load. Although fairly simple structures have been used in the examples, the principles can be extended easily to more complex structures.

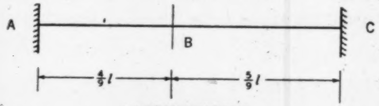
Example 1. Draw the Influence Line for Moment at a Given Section of a Fixed-Ended Beam with Constant Cross Section.—The deflection y_B of point B in Fig. 4(a) can be determined by Mr. Stewart's traverse method,[7] imposing a rotation, θ, at that section. Designating the traverse angle associated with M_B as 4, the corresponding angle in length BC must be 5 since it also is associated with M_B. By equating the deflection at point B:

$$\Delta_1 = \frac{5}{4}\Delta_2 + \frac{9}{8} \quad \ldots\ldots (15)$$

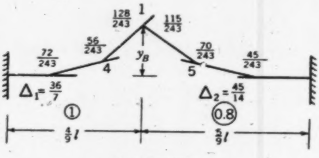
in which Δ (see Fig. 4(b)) is the relative curvature in the given length resulting from an end moment. From the moment diagram,

$$M_B = M_A + \frac{4}{9}(M_C - M_A) \quad . (16)$$

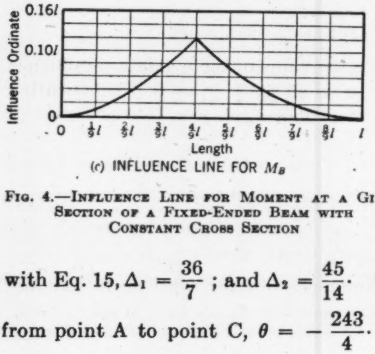
but, in relative terms, $M_A = (1)\,\Delta_1 = \Delta_1$; $M_B = (1)\,4 = 4$; and $M_C = 0.8\,\Delta_2$. Substituting these values in Eq. 16 and solving simultaneously with Eq. 15, $\Delta_1 = \frac{36}{7}$; and $\Delta_2 = \frac{45}{14}$. Solving for θ, by summation of rotations from point A to point C, $\theta = -\frac{243}{4}$. For $\theta = 1$ the traverse angles must then be, in order, $\frac{72}{243}$, $\frac{56}{243}$, $\frac{70}{243}$, and $\frac{45}{243}$, respectively. From geometry, $y_B = \left(\frac{72}{243}\right) \times \left(\frac{8}{27}\,l\right) + \left(\frac{56}{243}\right)\left(\frac{4}{27}\,l\right) = 0.122\,l$. Other ordinates for the influence line (Fig. 4(c)) may be computed as desired using Eq. 10.



(a) BEAM DIAGRAM



(b) TRAVERSE



(c) INFLUENCE LINE FOR M_B

FIG. 4.—INFLUENCE LINE FOR MOMENT AT A GIVEN SECTION OF A FIXED-ENDED BEAM WITH CONSTANT CROSS SECTION

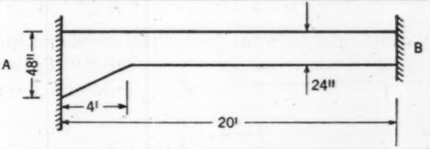
Example 2. Draw the Influence Line for the Fixed-End Moment at the Haunched End of a Fixed Beam.—Imposing the unit rotation at end A of the beam in Table 3 solve for the statical condition of the beam using column analogy.[8] With the results listed in Table 3, assuming a unit angle load at end A, the end moments are, respectively: $M_A = \frac{1}{26.15} + \frac{11.21 \times 11.21}{697.7} = +\,0.2189$; and $M_B = \frac{1}{26.15} - \frac{11.21 \times 8.79}{697.7} = -\,0.1031$. The final computations of influence ordinates are arranged in a manner similar to Table 2.

---

[7] *Transactions*, ASCE, Vol. 101, 1936, p. 105.

[8] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, Chapter III.

TABLE 3.—COMPUTATION OF END MOMENTS BY COLUMN ANALOGY[a]



Section	L	d	x	a	$a\,x$	$a\,x^2 + i_x$
1.........	1	3.75	−9.5	0.228	−2.16	20.5
2.........	1	3.25	−8.5	0.349	−2.96	25.2
3.........	1	2.75	−7.5	0.574	−4.38	32.9
4.........	1	2.25	−6.5	1.050	−6.82	49.3
5.........	16	2.00	2.0	24.000	48.00	608.0
....	....	....	....	26.151	31.68	735.9
Correction.	($\bar{x} = +1.21$)		....	....	....	−38.2
Total......	....	....	....	....	....	697.7

[a] L = length AB; d = depth of the uniform section; x = distance from center of beam to centroid of each segment; $a = 12\,L/d^3$; and $i_x = 1/12\;a\,L^2$.

Example 3. Shear at the Support of a Fixed-Ended Beam.—Assume a beam of uniform cross section, fixed as shown in Fig. 5, with a unit deflection imposed at end A. In this instance the boundary conditions for the differential equations of flexure are: $y_A = 1$; $y_B = 0$; $\theta_A = 0$; and $\theta_B = 0$; and the general solution is to be found in Eq. 10 which reduces to

$$y_x = 1 - k_1 \, (1 - k^2_2 + k_1\,k_2) \quad (17)$$

and which is used to compute the influence line (see Fig. 5(*b*)).

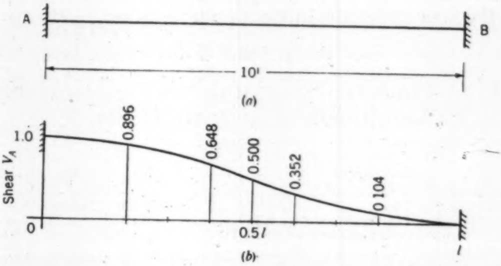


FIG. 5.—INFLUENCE LINE FOR SHEAR AT END A

Example 4. Find the Maximum Value of M_C on the Frame of Uniform Cross Section in Fig. 6 Due to a Moving Load of 10 Kips. (*Also Find the Moment at Joint* C *Due to a Uniform Load of 1.5 Kips per Ft*).—From the slope-deflection equations,

$$\theta_{DC} = 0 = \frac{10\,M_{DC}}{3\,E\,I} + \frac{10\,M_{CD}}{6\,E\,I} + \frac{y}{10} \quad \text{(18a)}$$

$$\theta_{CD} = -\,\frac{10\,M_{DC}}{6\,E\,I} - \frac{10\,M_{CD}}{3\,E\,I} + \frac{y}{10} \quad \text{(18b)}$$

$$\theta_{CB} = \frac{10\,M_{BC}}{6\,E\,I} + \frac{10\,M_{CB}}{3\,E\,I} \qquad (18c)$$

$$\theta_{BC} = -\frac{10\,M_{BC}}{3\,E\,I} - \frac{10\,M_{CB}}{6\,E\,I} \qquad (18d)$$

$$\theta_{BA} = \frac{6\,M_{BA}}{3\,E\,I} + \frac{y}{6} \qquad (18e)$$

$$\theta_{AB} = -\frac{6\,M_{BA}}{6\,E\,I} + \frac{y}{6} \qquad (18f)$$

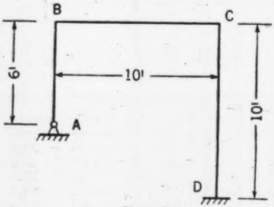


Fig. 6

in which y equals the horizontal displacement of points B and C due to side lurch. From geometry, remembering that a unit rotation is imposed at joint C:

$$\theta_{CD} - \theta_{CB} = 1; \quad \theta_{BC} - \theta_{BA} = 0 \qquad (19)$$

From statics,

$$M_{CD} = M_{CB}; \quad M_{BC} = M_{BA} \qquad (20)$$

and if H and V are the horizontal and vertical components of the reaction at point A:

$$M_{DC} = 4\,H + 10\,V; M_{CD} = 10\,V - 6\,H; M_{BC} = 6\,H;$$
$$\text{and } M_{CD} - M_{BC} = 10\,V \qquad (21)$$

The simultaneous solution of Eqs. 18 to 21 yields: $M_{BA} = M_{BC} = -0.0444\,E\,I$; $M_{CD} = M_{CB} = 0.0972\,E\,I$; $\theta_{CD} = 0.603$; $\theta_{CB} = -0.398$; $\theta_{BC} = 0.310$; $\theta_{BA} = 0.311$; and $\theta_{AB} = 0.444$. From Eq. 12b: $k_{1m} = \frac{1}{3\,(0.310 - 0.398)}$ $(2 \times 0.310 - 0.398 \pm \sqrt{0.310^2 + 0.398^2 - 0.398 \times 0.310}) = 0.531$; and from Eq. 10: $y_m = 10\,[0.310 \times 0.531 \times 0.469^2 + 0.398 \times 0.531^2 \times 0.469] = 0.888$. Eq. 14 yields the area under the influence curve, thus: $\frac{1}{12}\,(10)^2(0.310 + 0.398) = 5.90$. The maximum moment at joint C due to a moving load of 10 kips, therefore, will be 8.88 ft-kips when the load is placed 5.31 ft from joint B (Fig. 5). The moment at joint C due to the uniform load will be 5.90 × 1.5, or 8.85 ft-kips.

TABLE 4.—COMPUTATION OF MOMENTS CORRESPONDING TO A UNIT ROTATION AT JOINT B, FIG. 6

Line	Description	M_B	M_C	M_D	Sidesway force
1	Moments from Fig. 7(*c*)	+2.50	−8.00	−4.50	....
2	Moments from Fig. 7(*d*) multiplied by 1.389	+9.72	−2.78	+1.39	....
3	Undistributed fixed-ended moment at point D	....	....	+17.00	....
4	....	+12.22	+10.78	13.89	4.504
5	Multiply line 4 by 1.467/4.504	+3.98	+3.51	+4.53	1.467
6	Moments Fig. 7(*a*)	−7.00	−2.00	−1.00	−1.467
7	Summation, lines 5 and 6	−3.02	+1.51	+3.53	0
8	Moment for unit rotation (divide line 7 by 6.8)	0.442	0.224	0.520	....

Example 5. Find the Maximum Value of Moment at Joint B, Fig. 6, Using an Elastic Curve Traverse for Statical Analysis.—Impose the rotation θ at joint B. Beginning with a value of 1 for the lower elastic angle in column CD complete the traverse (Fig. 7), remembering that $M_{BC} = M_{BA}$. From Fig. 7(*a*), $\theta = 2.8 + 4.0 = 6.8$. Since sidesway has not been considered, these values must be corrected. The value of the unbalanced horizontal force is 1.467 EI. Assume the frame to be deflected sidewise without rotation of joints B and C (Fig. 7(*b*)). Assign the value Δ to the traverse angles and M to the end moments in CD. Since the "sidelurch" is equal for the vertical members the traverse angle in column AB must be 0.833 Δ and $M_{BA} = \frac{5}{6} \times \frac{5}{3} \times M = 1.389\,M$ (see line 2, Table 4).

Figs. 7(*c*) and 7(*d*) are solved for unbalanced moments at joints B and C. The unbalanced moment in each case is 17. If the value M in Fig. 7(*b*) is made equal to 17 then the moments for Fig. 7(*b*) may be computed and the sidesway correction made. The final moments must be divided by 6.8 to reduce to the condition of unit deflection at joint B. Starting at point D with these moments and changing to the equivalent angle units the corrected traverse may be constructed as shown in Fig. 7(*e*). From Eq. 12*b*, $k_{1m} = 0.474$ and, from Eq. 10, $y_m = 0.838$.

Example 6.—The influence line for vertical deflection at the center of span BC, Fig. 6, may be calculated by placing a unit vertical load at the center of the span and computing the resulting deflection curve. The values of M_B and M_C for this loading can be obtained from Eq. 10 using the values of θ_{BC} and θ_{CB} from Examples 4 and 5:

$$M_B = 10\,(0.369 \times 0.5^3 + 0.295 \times 0.5^3) = 0.830$$
$$M_C = 10\,(0.310 \times 0.5^3 + 0.398 \times 0.5^3) = 0.885$$

From statics, $V_B = 0.4945$, and $V_C = 0.5055$, and the moment diagram, Fig. 8(*a*) can be drawn. Applying Eq. 2 between points B and C, $\theta_{BC} = -\frac{2.02}{EI}$ and $\theta_{CB} = \frac{1.92}{EI}$ for the struc-

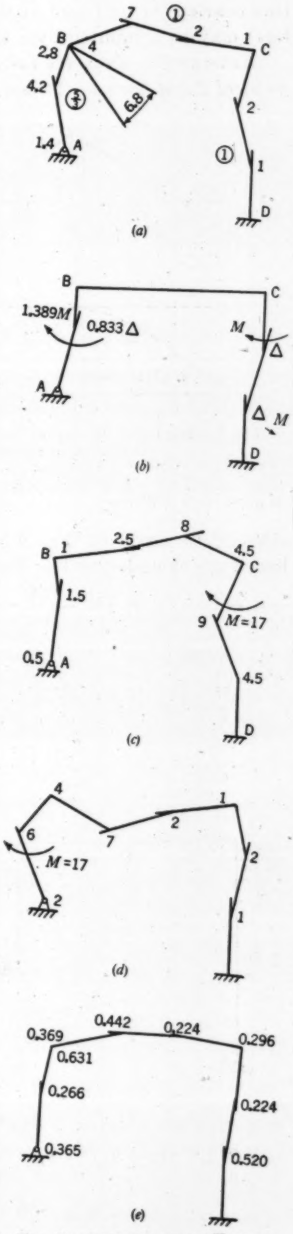


FIG. 7.—ANALYSIS OF AN UNSYMMETRICAL BENT

ture bearing the unit load at the center. The elastic curve (required influence line) may be computed by a tabulation using Eq. 3*a* (see Fig. 8(*b*)).

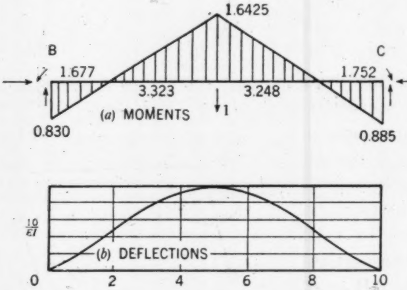

1.6425
B
C
1.677
1.752
3.323
3.248
1
(a) MOMENTS
0.830
0.885
$\frac{10}{EI}$
(b) DEFLECTIONS
0
2
4
6
8
10

FIG. 8.—INFLUENCE LINE FOR DEFLECTION AT THE CENTER OF THE SPAN

Example 7. Draw the Influence Line for M_{BC} for a Vertical Load on a Symmetrical Rigid Frame. (What Is the Moment at Joint B (Fig. 9) Due to a Horizontal, Uniformly Distributed Load of 500 Lb per Ft on Member AB?).—The method of moment distribution will be used in exploring this question. Impose a unit rotation at joint B. The fixed-end moments in member BC are then $M_{BC} = \frac{4\,E\,I}{l}$ and $M_{CB} = \frac{2\,E\,I}{l}$, respectively. The solution of Fig. 9(*a*) yields $M_B = -\,1.615\,\frac{E\,I}{l}$; $M_C = 0.461\,\frac{E\,I}{l}$; and $M_D = -\,0.230\,\frac{E\,I}{l}$. The unbalanced shear is $0.231\,\frac{E\,I}{l}$. Correction for sidesway is made by Fig. 9(*b*) and the corrected moments with sidesway allowed are encircled in Fig. 9(*a*). From the relation between end rotations and end moments, $\theta_{BC} = \frac{1.090}{3} + \frac{0.273}{6} = 0.409$; and $\theta_{CB} = \frac{-\,1.090}{6} - \frac{0.273}{3} = -\,0.273$. Since $\theta_{BA} = \theta_{BC} - 1 = -\,0.591$, Eqs. 8 yield, for member AB:

$$M_{AB} = 0 = -\,2\left(2\,\theta_{AB} - 0.591 - \frac{3\,y}{l}\right) \qquad (22a)$$

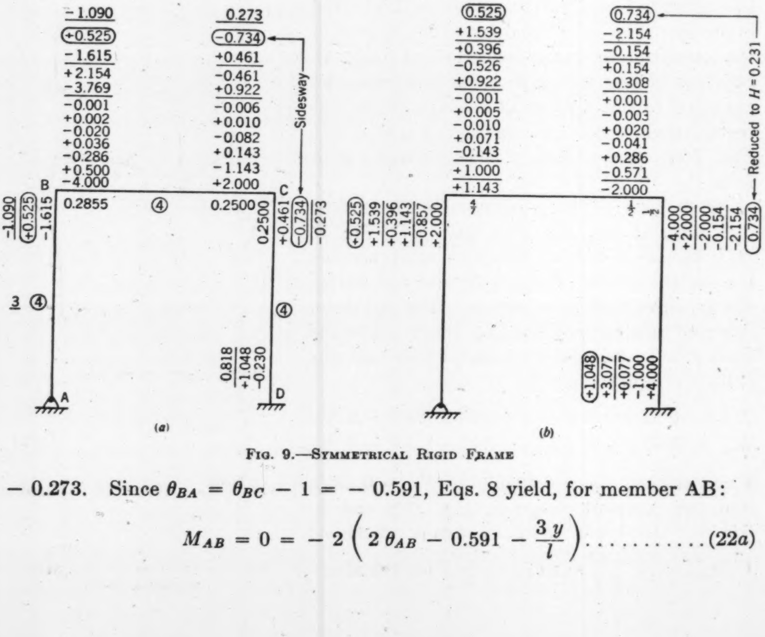

-1.090
+0.525
-1.615
+2.154
-3.769
-0.001
+0.002
-0.020
+0.036
-0.286
+0.500
-4.000
0.273
-0.734
+0.461
-0.461
+0.922
-0.006
+0.010
-0.082
+0.143
-1.143
+2.000
Sidesway
B
C
0.2855
④
0.2500
-1.090
+0.525
-1.615
0.2500
+0.461
-0.734
-0.273
3 ④
④
0.818
+1.048
-0.230
A
D
(a)
0.525
+1.539
+0.396
-0.526
+0.922
-0.001
+0.005
-0.010
+0.071
-0.143
+1.000
+1.143
0.734
-2.154
-0.154
+0.154
-0.308
+0.001
-0.003
+0.020
-0.041
+0.286
-0.571
-2.000
Reduced to H = 0.231
4/7
1/2
+0.525
+1.539
+0.396
+1.143
-0.857
+2.000
-4.000
+2.000
-2.000
-0.154
-2.154
0.734
+1.048
+3.077
+0.077
-1.000
+4.000
(b)

FIG. 9.—SYMMETRICAL RIGID FRAME

and

$$M_{BA} = -\ 1.090 = 2\left[\ \theta_{AB} - 2\,(0.591) - \frac{3\,y}{l}\right]\ldots\ldots\ldots(22b)$$

in which y is the displacement due to sidesway. Solving simultaneously, $\theta_{AB} = -\ 0.046$ and $y = -\ 2.277$. From Eq. 12*b* for member BC: $k_{1m} = 0.456$ and, from Eq. 10, $y_m = 0.862$. From Eq. 14 for member AB: $Q = \frac{10}{2}\,(-\ 2.277) + \frac{1}{12}\,10^2\,(-\ 0.046 + 0.591) = 6.848$. Consequently, the moment at joint B due to a uniformly distributed horizontal load of 500 lb per ft on member AB is $M_B = 6.848 \times 500 = 3{,}424$ ft-lb.

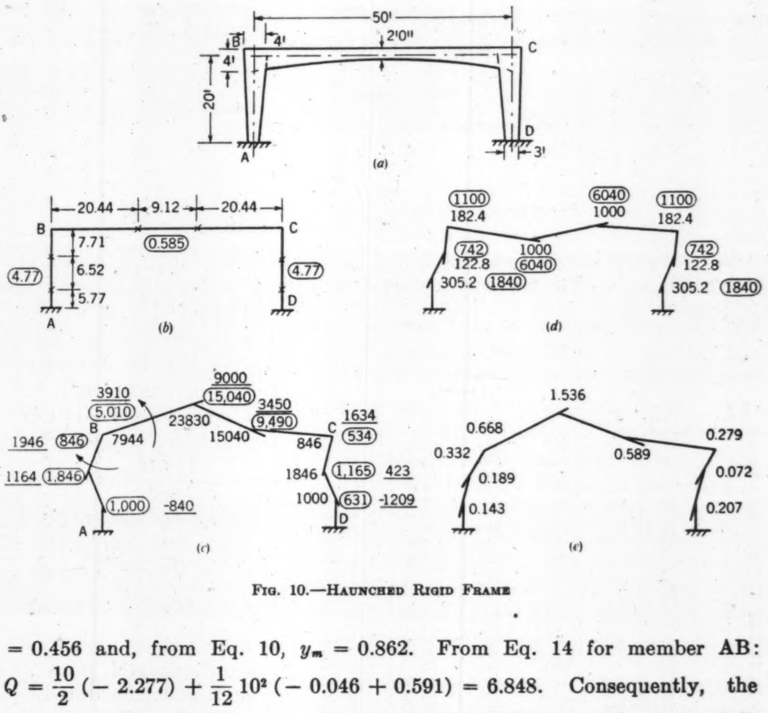

50'
4'
2'0"
B
C
4'
20'
A
D
3'
(a)
20.44
9.12
20.44
B
C
7.71
0.585
4.77
6.52
4.77
5.77
A
D
(b)
1100
182.4
6040
1000
1100
182.4
742
122.8
1000
6040
742
122.8
305.2
1840
305.2
1840
(d)
9000
15,040
3910
5,010
3450
9,490
23830
B
1634
C
534
1946
846
7944
15040
846
1164
1,846
1846
1,165
423
1,000
-840
1000
631
-1209
A
D
(c)
1.536
0.668
0.279
0.332
0.589
0.189
0.072
0.143
0.207
(e)

FIG. 10.—HAUNCHED RIGID FRAME

Example 8. Haunched Rigid Frame.—Using the traverse method trace influence line for M_B for the frame of Fig. 10 due to vertical load on beam BC. What is the value of M_B for a uniform load of 1.5 kips per ft? Using this method, the traverse angle occurs at the center of gravity of the $\frac{M}{E\,I}$-diagram considering one end of the member to be loaded with a moment and the other end to be hinged. The distance to the center of gravity point is designated by $\bar{x}$. Solution procedures for these points and for the stiffnesses of members are demonstrated in Tables 5 and 6. In Table 5 a moment of $1{,}000\,\frac{E}{12}$ is applied at joint B to member BC, Fig. 10. From this table, $\bar{x}_{BC} = \bar{x}_{CB} = \dfrac{\sum \frac{M}{EI}(\Delta x)(x_n)}{\sum \frac{M}{EI}(\Delta x)} = \dfrac{35{,}075.4}{1{,}711.18} = 20.44$ ft and the stiffness (the moment required to produce unit rotation in beam BC) is calculated by $\dfrac{1{,}000\,\frac{E}{12}}{1{,}711.18} = 0.585\,\dfrac{E}{12}$.

From Table 6, $\bar{x}_{BA} = \dfrac{1{,}613.8}{209.8} = 7.71$ ft; $\bar{x}_{AB} = \dfrac{1{,}611.6}{279.6} = 5.77$; and

$$\text{Stiffness BA} = \frac{1{,}000\,\frac{E}{12}}{209.8} = 4.77\,\frac{E}{12}$$

$$\text{Stiffness AB} = \frac{1{,}000\,\frac{E}{12}}{279.6} = 3.58\,\frac{E}{12}$$

A rotation is imposed at joint B and the traverse, Fig. 10(c), is drawn ignoring sidesway. The traverse is started at point A and carried to point B, and independently at point D and carried to B. Since $M_{BA} = M_{BC}$, the right-hand side is correlated with the left-hand side by multiplying the right-hand side rotations by $\dfrac{15{,}040}{23{,}830}$. Corrected rotations are shown encircled.

The total unbalanced shear in the vertical members is $1{,}010\,\dfrac{E}{12}$. The solution procedure for an imposed horizontal force at joint B is demonstrated by Fig. 10(d). From symmetry, the traverse angles in beam BC are equal. Assuming them to be equal to 1,000 the traverse is completed as shown. The value of the horizontal force, Fig. 10(d), is $167.5\,\dfrac{E}{12}$. For a horizontal force corresponding to that of Fig. 10(c) the angles must be multiplied by $\dfrac{1{,}010}{167.5}$. Corrected angles are encircled in Fig. 10(d). These angles are then added algebraically to the corresponding angles of Fig. 10(c) to arrive at the correction

TABLE 5.—COMPUTATION OF PROPERTIES OF BEAM BC

Segment	x_n (ft)	d_x	d^3	$\frac{M}{E/12}$	$\frac{M}{EI}\Delta x$	$\frac{M}{EI}\Delta x\, x_n$
1......	1	3.84	56.7	980	34.56	34.6
2......	3	3.55	44.2	940	42.44	127.4
3......	5	3.28	35.1	900	51.24	256.2
4......	7	3.03	28.2	860	61.02	433.6
5......	9	2.82	22.3	820	73.50	663.0
6......	11	2.63	18.1	780	86.16	949.0
7......	13	2.46	14.8	740	99.90	1,297.0
8......	15	2.32	12.4	700	112.90	1,692.0
9......	17	2.20	10.6	660	124.50	2,116.0
10......	19	2.12	9.39	620	132.10	2,508.0
11......	21	2.05	8.59	580	135.10	2,838.0
12......	23	2.01	8.07	540	133.90	3,079.0
13......	25	2.00	8.00	500	125.00	3,125.0
14......	27	2.01	8.07	460	114.10	3,082.0
15......	29	2.05	8.59	420	96.80	2,808.0
16......	31	2.12	9.39	380	80.90	2,504.0
17......	33	2.20	10.6	340	64.10	2,016.0
18......	35	2.32	12.4	300	48.34	1,691.0
19......	37	2.46	14.8	260	35.12	1,301.0
20......	39	2.63	18.1	220	24.30	948.0
21......	41	2.82	22.3	180	16.14	662.0
22......	43	3.03	28.2	140	9.94	427.8
23......	45	3.28	35.1	100	5.70	254.6
24......	47	3.55	44.2	60	2.72	127.8
25......	49	3.84	56.7	20	0.70	34.4
Total...	..	....	....	....	1,711.18	35,075.4

due to sidesway. The resulting angles are shown underlined in Fig. 10(*c*). The value of the imposed rotation at joint B is 1,946 + 3,910 = 5,856. To reduce the rotation to unity it is necessary to multiply all the angles by $\frac{1}{5,856}$. These angles are shown in Fig. 10(*e*). Using Eq. 3*a* the elastic curve for beam BC (influence line) is computed in Table 7. The area under the influence line is given by $\Sigma y_n \Delta x$ or $122.58 \times 2 = 245.16$ and the moment due to uniform load of 1.5 kips per ft is 367.74 kip-ft.

TABLE 6.—COMPUTATION OF PROPERTIES OF MEMBER AB

x_n (ft)	d_x	d^3_x	$\frac{M_B}{E/12}$	$\frac{M_B}{EI}\Delta x$	$\frac{M_B\,\Delta x}{EI}x_n$	$\frac{M_A}{E/12}$	$\frac{M_A}{EI}\Delta x$	$\frac{M_A\,\Delta x}{EI}(20-x_n)$
1.......	3.95	61.5	950	30.86	30.8	50	1.62	30.8
3.......	3.85	56.8	850	29.92	89.8	150	5.28	87.6
5.......	3.75	52.3	750	28.66	143.2	250	9.56	143.4
7.......	3.65	48.5	650	26.60	187.6	350	14.22	187.6
9.......	3.55	44.2	550	24.88	223.8	450	20.38	224.0
11.......	3.45	40.8	450	22.06	242.8	550	26.96	242.4
13.......	3.35	37.4	350	18.72	243.4	650	34.74	243.0
15.......	3.25	34.0	250	14.70	220.4	750	44.10	220.4
17.......	3.15	31.0	150	9.68	164.4	850	54.84	164.8
19.......	3.05	28.1	50	2.56	67.6	950	67.70	67.6
Σ.......	....	....	....	209.84	1,613.8	....	279.62	1,611.6

Example 9. Simply Supported Continuous Beam.—Using the indirect method and moment-distribution method draw influence lines for the continuous beam in Fig. 11. The analysis for the effect of undistributed moments of +100 at joint B and joint C is made in Fig. 11. (In this illustration the previous sign convention for moments has been temporarily discarded for convenience and moments acting clockwise on the joint are considered positive. For other computations involved in Example 9, the previous sign convention is used.)

For moment at support B a unit rotation is considered to be imposed at point B. Assuming this to be imposed on span AB alone the resulting fixed-end moment, M_{BA}, is $3\,E\,K$ and, since the stiffness, $E\,K$ is equal to 2, $M_{BA} = 3 \times 2 = 6$. Distributing this fixed-end moment in accordance with the distribution of Fig. 11 the moments are found to be $M_B = 4.120$, $M_C = -0.729$, and $M_D = 0.121$; and the end rotations are calculated as follows:

$$\theta_{AB} = -\frac{4.120}{6}\,\frac{1}{2} = -0.34$$

$$\theta_{BA} = +\frac{4.120}{3}\,\frac{1}{2} = +0.69$$

$$\theta_{BC} = -\frac{4.120}{3}\,\frac{1}{4} + \frac{0.729}{6}\,\frac{1}{4} = -0.31$$

$$\theta_{CB} = \theta_{CD} = \frac{4.120}{6}\,\frac{1}{4} - \frac{0.729}{3}\,\frac{1}{4} = +0.11$$

$$\theta_{DC} = \theta_{DE} = -\frac{0.121}{3}\,1 = -0.04$$

$$\theta_{ED} = +\frac{0.121}{6}\,1 = +0.02$$

For the reaction at support B a unit deflection is imposed at point B, Fig. 11. The corresponding fixed-end moments must be applied:

$$M_{BA} = \frac{3\,E\,K}{l} = +\,\frac{3 \times 2}{30} = +\,0.200$$

$$M_{BC} = \frac{6\,E\,K}{l} = +\,\frac{6 \times 4}{30} = +\,0.800$$

$$M_{CB} = \frac{6\,E\,K}{l} = -\,\frac{6 \times 4}{30} = -\,0.800$$

Using the distribution of Fig. 11 the resulting moments are computed to be

TABLE 7.—Computation of Deflection Curve

x_n (ft)	$\dfrac{M_B\left(1-\frac{x_n}{50}\right)}{E/12}$	$\dfrac{M_c\left(1-\frac{x_n}{50}\right)}{E/12}$	$\dfrac{M_n}{E/12}$	$\dfrac{M_n\,\Delta x}{E\,I}$	θ_n	$\dfrac{\Delta x_{n-1}+x_n}{2}$	Δy	y_n
0.......	−0.899	0	−0.899	....	0.668	....	....	0.000
1.......	−0.881	0.007	−0.874	−0.031	0.637	1	0.668	0.668
3.......	−0.845	0.020	−0.825	−0.037	0.600	2	1.274	1.942
5.......	−0.809	0.304	−0.775	−0.044	0.556	2	1.200	3.142
7.......	−0.733	0.049	−0.724	−0.052	0.504	2	1.112	4.254
9.......	−0.737	0.062	−0.675	−0.060	0.444	2	1.008	5.262
11.......	−0.701	0.076	−0.625	−0.069	0.375	2	0.888	6.150
13.......	−0.665	0.089	−0.576	−0.078	0.297	2	0.750	6.900
15.......	−0.630	0.104	−0.526	−0.085	0.212	2	0.594	7.494
17.......	−0.594	0.117	−0.477	−0.090	0.122	2	0.424	7.918
19.......	−0.554	0.131	−0.426	−0.091	0.031	2	0.244	8.162
21.......	−0.521	0.145	−0.376	−0.089	−0.058	2	0.062	8.224
23.......	−0.486	0.159	−0.327	−0.081	−0.138	2	−0.116	8.108
25.......	−0.450	0.172	−0.278	−0.069	−0.208	2	−0.278	7.830
27.......	−0.414	0.186	−0.228	−0.057	−0.265	2	−0.416	7.414
29.......	−0.378	0.200	−0.178	−0.040	−0.305	2	−0.530	6.884
31.......	−0.342	0.213	−0.128	−0.027	−0.332	2	−0.610	6.274
33.......	−0.306	0.227	−0.078	−0.015	−0.347	2	−0.664	5.610
35.......	−0.270	0.241	−0.029	−0.006	−0.353	2	−0.694	4.916
37.......	−0.234	0.255	0.021	0.002	−0.351	2	−0.706	4.210
39.......	−0.198	0.268	0.070	0.008	−0.343	2	−0.702	3.508
41.......	−0.162	0.282	0.121	0.011	−0.332	2	−0.686	2.822
43.......	−0.126	0.296	0.170	0.012	−0.320	2	−0.664	2.158
45.......	−0.090	0.310	0.221	0.012	−0.308	2	−0.640	1.518
47.......	−0.054	0.324	0.270	0.013	−0.295	2	−0.616	0.902
49.......	−0.018	0.338	0.320	0.011	−0.284	2	−0.590	0.312
50.......	0	0.345	0.345	....	....	1	−0.284	0.000

$M_B = +\,0.298$, $M_C = -\,0.194$, and $M_D = 0.032$; and the corresponding rotations, remembering that $\delta_B = -\,1$, are:

$$\theta_{AB} = -\,\frac{1}{30} - \frac{1}{6} \times 0.298 \times \frac{1}{2} = -\,0.0583$$

$$\theta_{BA} = \theta_{BC} = -\,\frac{1}{30} + \frac{1}{3} \times 0.298 \times \frac{1}{2} = +\,0.0164$$

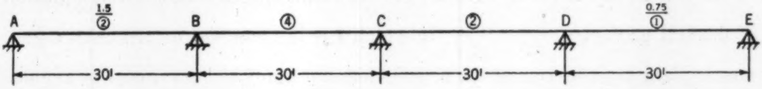
$$\theta_{CB} = \theta_{CD} = +\,\frac{1}{3} \times 0.194 \times \frac{1}{2} - \frac{1}{6} \times 0.032 \times \frac{1}{2} = +\,0.0296$$

$$\theta_{DC} = \theta_{DE} = -\,\frac{1}{3} \times 0.032 = -\,0.0107$$

$$\theta_{ED} = +\,\frac{1}{6} \times 0.032 = +\,0.0053$$

Conclusions

Influence lines may be computed exactly as the elastic curve due to a unit displacement corresponding to the "action" being studied and placed at the section under consideration. The computation of such an influence line involves (1) a statical analysis of the structure bearing only the displacement and (2) computation of as many points on the elastic curve as may be necessary. This is somewhat more work than a single statical analysis of the structure bearing a conventional load although the additional labor of calculating deflections is offset somewhat by the fact that the statical analysis is generally simplified since it involves only the unloaded structure. The statical analysis may be made using whatever method the designer prefers. It may be made indirectly by first distributing an unbalanced moment at each of the joints and then summing up proportionately a set of fixed-end moments corresponding to the required deflection. The elastic curve is calculated by geometry, or by equations or methods based on geometry. (Methods of analysis, as such, do not come within the scope of discussion.)



+100					
−27.25	−72.75	−36.38			
	+12.13	+24.26	+12.13	+6.06	
−3.31	−8.82	−4.41	−2.21	−4.41	−1.65
	+2.20	+4.41	+2.21	+1.10	
−0.60	−1.60	−0.80	−0.40	−0.80	−0.30
	+0.40	+0.80	+0.40	+0.20	
−0.11	−0.29	−0.14	−0.07	−0.15	−0.05
	+0.07	+0.14	+0.07	+0.03	
−0.02	−0.05	−0.02	−0.01	−0.02	−0.01
	+0.01	+0.02	+0.01		
−31.29	−68.70	−12.13	+12.13	+2.01	−2.01
		+100			
	−33.33	−66.67	−33.33	−16.67	
+9.07	+24.26	+12.13	+6.07	+12.13	+4.53
	−6.07	−12.13	−6.07	−3.03	
+1.67	+4.40	+2.20	+1.10	+2.20	+0.83
	+1.10	−2.20	−1.10	−0.55	
+0.30	+0.80	+0.40	+0.20	+0.40	+0.15
	−0.20	−0.40	−0.20	−0.10	
+0.05	+0.15	+0.07	+0.03	+0.07	+0.03
	−0.03	−0.07	−0.03	−0.01	
+0.01	+0.02				
+11.10	−11.10	−66.67	−33.33	−5.55	+5.55

Fig. 11.—Simply Supported Continuous Beam

In many cases complete elastic curves are unnecessary. For prismatic members the location of the load for maximum effect, and the amount of the effect, are given by Eqs. 12. For uniform load on prismatic members the area under the influence line is given by Eq. 14.

DISCUSSION

Ralph E. Spaulding,[9] M. ASCE.—The fundamental differential equation of flexural geometry, $\frac{d^2y}{dx^2} = \frac{M}{E\,I}$ (in which I is the moment of inertia of a geometric plane cross section of the beam), does not apply to curvalinear or angular beams. The centroidal axis, in bending, of a curvalinear or angular beam is not at the geometric centroid of a plane cross section of the member. The formula is accurate only for beams (including continuous beams), whose geometric axis is a straight line. Its relative inapplicability depends on the relative deviation from a straight line, which may be caused by the angularity of curvature of the member under study—whether this deviation is caused by a curved knee, an angular knee, or a haunch.

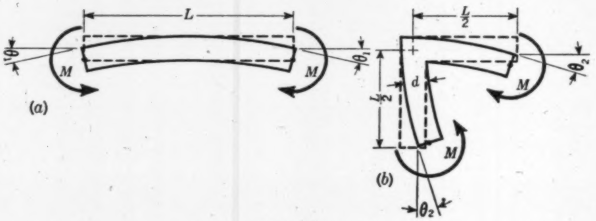


Fig. 12.

It would seem advisable to emphasize the fact that the foregoing basic formula is not applicable to homogeneous rigid frames such as shown in Figs. 6 and 10. Before it may be so used, it is necessary to build up the knee in each case to the point where its stiffness as an angular beam is equal to that of the straight beam adjoining it.

For example, Fig. 12(*a*) represents a homogeneous beam having an uninterrupted straight axis and Fig. 12(*b*) represents a similar beam of the same length except that it has a right-angle knee. Equal moments being applied to the ends of both beams, the rotation θ_2, in the beam containing the knee will be substantially greater than θ_1 in the straight beam; whereas $\theta_1 = \frac{M\,L}{2\,E\,I}$ and $\theta_2 = \frac{M}{2\,E\,I}(L + d\sqrt{2})$, approximately.

Ralph W. Stewart,[10] M. ASCE.—The various examples demonstrated by the author all relate to the influence line for some one specific point, or to the influence lines for supports only.

In the design of reinforced concrete rigid frame bridges, it is desirable and entirely practicable to construct the envelope that passes through the maximum influence line ordinates for all points of the structure. This will enable the computer to plan the reinforcing bars so that they will conform accurately to the requirement at each point.

---

[9] Cons. Engr., Jacksonville, Fla.

[10] (*Retired*), Los Angeles, Calif.

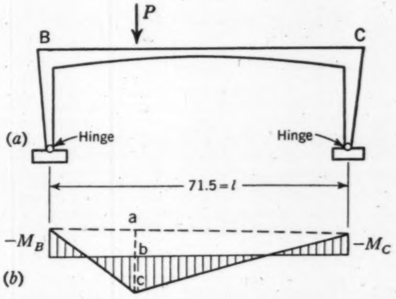


FIG. 13

Fig. 13(*a*) indicates a single-span rigid frame bridge. Modern design specifications require the decks of such bridges to be designed as if the bases were hinged.[11]

The constants of flexure for the deck and abutments are determined and a traverse constructed[12] as shown in Fig. 14(*a*). This shows the effect of a fixed-end moment of 7.28 applied at the left end only of the deck with sidesway prevented. To determine the properties of the structure for any symmetrical loading simply reverse Fig. 14(*a*) on itself and combine the moments. The result, Fig. 14(*b*), shows the haunch moments (5.69) produced by symmetrical fixed-end moments of 7.28. To determine the effect of a single fixed-end moment corrected for sidesway, divide the haunch moments in Fig. 14(*b*) by 2 and write them in Fig. 14(*d*); also, write the fixed-end moment 7.28 at the left end of the deck and Fig. 14(*d*) will then show the effect of this fixed-end moment with sidesway permitted.

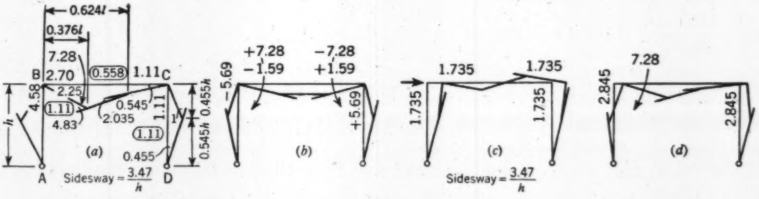


FIG. 14

A more easily understood but longer method of correcting for sidesway is to note that the sidesway force in Fig. 14(*a*) is $\frac{3.47}{h}$. Since the structure is symmetrical, one half of this goes to each abutment, thus yielding Fig. 14(*c*). Now combine Fig. 14(*a*) with Fig. 14(*c*), and Fig. 14(*d*) is again obtained.

Figs. 14(*b*), 14(*c*), and 14(*d*) now cover all conditions of flexure. Fig. 14(*a*) is simply a means to this end and can be abandoned thereafter. For this very simple structure Fig. 14(*a*) need not have been constructed. A traverse for symmetrical loading, similar to Fig. 14(*b*), can be constructed directly by two slide rule settings. A figure similar to Fig. 14(*d*), will then, as already shown, follow directly from it. For more general cases, however, a diagram corresponding to Fig. 14(*a*) must be drawn.

---

[11] "Specifications for Highway Bridges," Am. Assn. of State Highway Officials, Washington, D. C., 1944.

[12] "Relative Flexure Factors for Analyzing Continuous Structures," by Ralph W. Stewart, *Transactions*, ASCE, Vol. 104, 1939, p. 527, Fig. 4.

Cols. 1 and 2, Table 8, show the fixed-end moment coefficients by which ($P\,l$)-values are multiplied to compute the fixed-end moments for concentrated loads at each of the $\frac{1}{12}$ length points along the span. There are several available tables and charts giving these coefficients. With structural symmetry and hinged bases the haunch moments of this frame will always be equalized by the sidesway; therefore, the computer can combine the items in each line of Cols. 1 and 2 and obtain Col. 3.

TABLE 8.—FIXED-END MOMENT COEFFICIENTS FOR CONCENTRATED LOAD (FIG. 13)

Load of unity at point[a]	COEFFICIENTS FOR: (FEM)$_L$ (1)	COEFFICIENTS FOR: (FEM)$_R$ (2)	Col. 1 plus Col. 2 (3)	$M_{BC} = M_{BA}$ (4)	Simple beam moment[b] (5)	Moment under load[b] (6)
1	0.0744	0.0053	0.0797	2.22	5.46	3.24
2	0.1290	0.0210	0.1500	4.19	9.94	5.75
3	0.1628	0.0463	0.2091	5.84	13.40	7.56
4	0.1750	0.0785	0.2535	7.08	15.90	8.82
5	0.1679	0.1136	0.2815	7.85	17.38	9.53
6	0.1455	0.1455	0.2910	8.12	17.88	9.76

[a] Point 6 is the center line of the span. [b] See Fig. 13(*b*): The moment under the load (bc) equals the simple beam moment (ac) minus ab.

Each haunch moment for a unit load recorded in Col. 4, Table 8, will then be $\frac{2.845}{7.28}$ (Fig. 14(*d*)) multiplied by the adjoining item in Col. 3 and the length of the deck member. The simple moments shown in Col. 5 are then computed. By deducting the moments in Col. 4 from the simple moments the moments under the load are obtained as shown in Col. 6, Table 8. Fig. 13(*b*) illustrates this computation for the general condition. The ordinate ac will be recognized as the maximum simple beam moment and the ordinate bc of the shaded area ($\overline{ac} - \overline{ab}$) is the moment under the load. For symmetrical structures with hinged bases it will not be necessary to interpolate to obtain the ordinate ab as M_B and M_C are equal for all positions of the load.

To construct Table 9 first write in the values under point 0 which show the moments at ends B and C. These are taken from Col. 4, Table 8. The underscored items running diagonally across the upper half of Table 9 are the moments under the load taken from Col. 6, Table 8. Each is written where its line number intersects its column number. All other items in Table 9 are computed by horizontal interpolation. A property of this tabulation for a symmetrical frame is that it is symmetrical in all directions about the point where line 6 (Table 9) intersects the last column, point 6; in other words, about the center of the structure. Therefore, Table 9 may be constructed for only half the length of the deck as shown. The vertical columns can now be plotted and will give the influence lines for a unit load shown in Fig. 15.

The summation of the positive and negative influence line ordinates in Table 9 serves two purposes—(1) placing discontinuous sections of uniform live load on the span to get maximum moments, and (2) checking the accuracy of the table. The check is as follows: The simple moment for a load of 1 kip

TABLE 9.—INFLUENCE ORDINATES FOR A UNIT CONCENTRATED LOAD[a]

Line	0	1	2	3	4	5	6
1	−2.22	+3.24	+2.73	+2.23	+1.72	+1.22	+0.71
2	−4.19	+0.78	+5.75	+4.76	+3.77	+2.79	+1.78
3	−5.84	−1.37	+3.09	+7.56	+5.07	+4.58	+3.09
4	−7.08	−3.10	+0.87	+4.85	+8.82	+6.33	+4.84
5	−7.85	−4.38	−0.90	+2.58	+6.05	+9.53	+7.05
6	−8.12	−5.14	−2.16	+0.82	+3.80	+6.78	+9.76
7	−7.85	−5.87	−2.89	−0.40	+2.08	+4.57	+7.05
8	−7.08	−5.09	−3.11	−1.12	+0.87	+2.85	+4.84
9	−5.84	−4.35	−2.86	−1.37	+0.11	+1.60	+3.09
10	−4.19	−3.19	−2.20	−1.20	−0.21	+0.78	+1.78
11	−2.22	−1.73	−1.24	−0.75	−0.26	+0.22	+0.71
12	−62.48	−33.72	−15.36	−4.84	−0.47	....	....
13	....	+4.02	+12.44	+22.80	+32.29	+41.25	+44.70

[a] Point 0 is at end B, Fig. 13(*a*), and the same values, by symmetry, apply to end C (point 12).

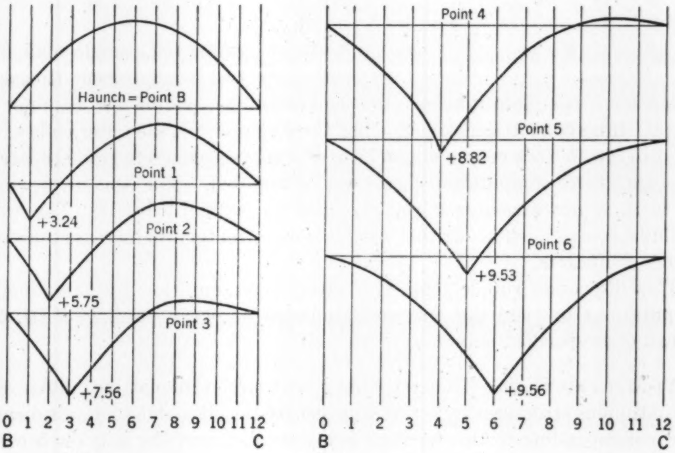


FIG. 15

per ft on the 71.5-ft span is $(71.5)^2 = 638$ ft-kips. From the first column of Table 9 (point 0, line 12), $M_B = 62.48 \times \frac{71.5}{12} = 373$. Subtracting M_B from the simple moment gives 265 ft-kips as the moment at the center of the span.

From the last column of Table 9 (point 6, line 13) the moment at the center of the span is $44.7 \times \frac{71.5}{12} = 266$ ft-kips, which checks.

After this check is accomplished Fig. 16 may be drawn. This shows maximum positive and maximum negative moments all along the span due to a concentrated moving load of unity. From Table 9 or the influence line graphs, the moments caused by any live load are readily computed and combined with the dead load, temperature, and other moments so that the required reinforcement for every cross section along the member can be determined. It is not necessary to plot the influence lines (Fig. 15) except to detect errors in the computations which will be disclosed if the points on the curves are obviously out of line.

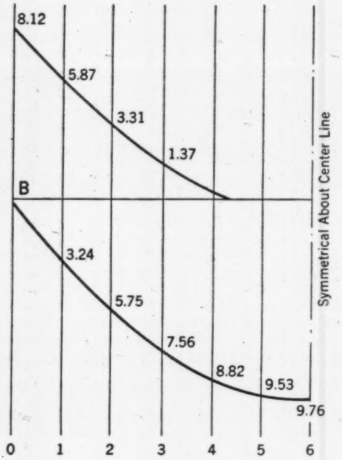


FIG. 16

Many engineers and writers whose opinions carry weight have stated that the computation of influence lines is seldom desirable and that the time and expense of such computations can be saved.

For a symmetrical, single-span structure only two computation sheets are needed for a complete influence line computation covering the bending moments. If a symmetrical structure has three spans with the outer ends simply supported (which is a very common form of reinforced concrete grade-separation structure), three computation sheets will be required. The time or expense of this work is not excessive if the computer is familiar with it. The author should be commended for writing a paper which will tend to increase the supply of such computers.

This discussion supplements the author's presentation by extending the computations to yield the maximum moment at every point in a structure instead of at specified points.

YI-MAI YAO,[13] ESQ.—The author has given a method of plotting influence lines for continuous structures by computing the elastic curve due to a unit displacement corresponding to the "action" being studied, and the unit displacement being located at the section under consideration. The formula for computing the ordinates of an elastic curve, such as Eq. 10, involves the joint rotation θ. The θ-value can be obtained directly and quickly by balancing the angle changes[14] produced by the "unit displacement," a method presented by L. E. Grinter, M. ASCE.

---

[13] Junior Engr., TangKu New Harbor Construction Bureau, TangKu, Tientsin, China.

[14] "Analysis of Continuous Frames by Balancing Angle Changes," by L. E. Grinter, *Transactions*, ASCE, Vol. 102, 1937, pp. 1020–1067.

According to equations derived by Professor Grinter[15] the modified stiffness κ_{ABm} and the modified carry-over factor γ_{ABm} for the direct distribution process are as follows:

$$\kappa_{ABm} = \frac{4\,\kappa_{AB}}{3 + \dfrac{\kappa_{BA}}{\Sigma\kappa'_{Bm}}} \quad \ldots\ldots\ldots\ldots \quad (23)$$

and

$$\gamma_{ABm} = -\frac{2\,\dfrac{\kappa_{AB}}{\Sigma\kappa'_{Bm}}}{3 + \dfrac{\kappa_{BA}}{\Sigma\kappa'_{Bm}}} \quad \ldots\ldots\ldots\ldots \quad (24)$$

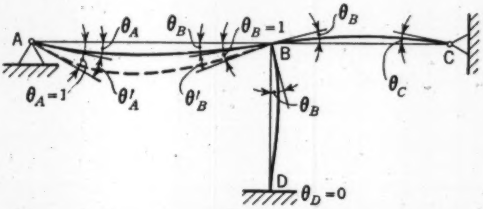


FIG. 17

in which $\Sigma\kappa'_{Bm}$ indicates the summation of the modifier stiffness of the member tributary to joint B except BA plus the unmodified stiffness of BA.

The simply supported continuous girder in Example 9 can be solved easily by direct distribution, as shown in Table 10, for plotting the influence lines for moment M_B and reaction R_B.

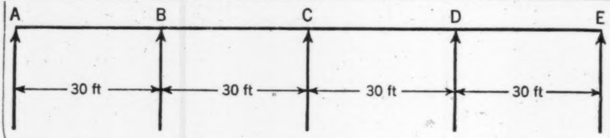
Line 3, Table 10, shows the unmodified stiffness of member, κ. Line 4 shows the modified stiffness κ_m by Eq. 23; for example, $\kappa_{CBm} = \dfrac{4 \times 4}{3 + \dfrac{4}{4+2}} = 4.36$ and $\kappa_{DCm} = \dfrac{4 \times 2}{3 + \dfrac{2}{2 + 4.36}} = 2.41$. Line 5 shows the $\dfrac{\kappa_m}{\Sigma\kappa_m}$-value, and line 6 shows the modified carry-over factor γ_m; for example, $\gamma_{CBm} = -\dfrac{2\,\dfrac{4}{4+2}}{3 + \dfrac{4}{4+2}} = -0.363$ and $\gamma_{DCm} = -\dfrac{2\,\dfrac{2}{2+4.36}}{3 + \dfrac{2}{2+4.36}} = -0.187$. Line 7, Table 10, shows the "unit displacement" corresponding to the influence line for M_B; line 8 shows the direct distribution—for example, $\theta_{BA} = +1.00 \times 0.687 = +0.69$, which, carried over, yields $+0.69 \times -0.500 = -0.34$ to joint A. The rotation, $\theta_1 = 1.00$, must remain at joint B, and therefore, in line 6, -0.31 must remain in member BC. Consequently, $-0.31 \times -0.355 = +0.11$ is car-

---

[15] "Analysis of Continuous Frames by Balancing Angle Changes," by L. E. Grinter, *Transactions*, ASCE, Vol. 102, 1937, p. 1035, Eqs. 6 and 7.

TABLE 10.—MODIFIED SOLUTION OF EXAMPLE 9

(a) DISTRIBUTION

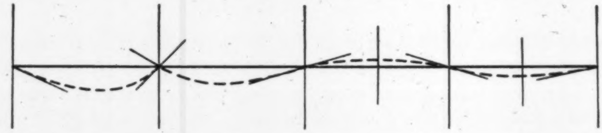

A
B
C
D
E
30 ft
30 ft
30 ft
30 ft

Line No.	Symbol								
1	Joint	A	B		C		D		E
2	Member	AB	BA	BC	CB	CD	DC	DE	ED
3	κ	2.00	2.00	4.00	4.00	2.00	2.00	1.00	1.00
4	κ_m		2.00	4.39	4.36	2.18	2.41	1.00	
5	$\kappa_m/\Sigma\kappa_m$	1.000	0.313	0.687	0.667	0.333	0.706	0.294	1.000
6	γ_m	−0.188	−0.500	−0.355	−0.363	−0.363	−0.189	−0.500	−0.178

(b) INFLUENCE LINE FOR MOMENT M_B

Line No.	Symbol	AB	BA	BC	CB	CD	DC	DE	ED
7	θ_1			+1.00					
8	θ	−0.34	+0.69	−0.31	+0.11	+0.11	−0.04	−0.04	+0.02

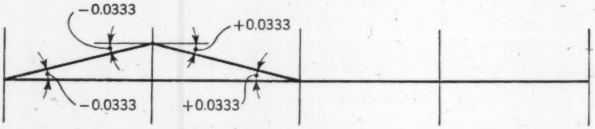
(b)



(c) DISTRIBUTION OF ANGLE CHANGES

Line No.	Symbol	AB	BA	BC	CB	CD	DC	DE	ED
9	θ_1	−0.0333(1)	−0.0333(2)	+0.0333(3)	+0.0333(4)				
10	Bal. θ(AB) ..	−0.0333	0	0	0	0	0	0	0
11	Bal. θ(BA) ..	−0.0115	−0.0104	−0.0104	+0.0037	+0.0037	−0.0014	−0.0014	+0.0007
12	Bal. θ(BC) ..	−0.0115	+0.0229	+0.0229	+0.0037	+0.0037	−0.0014	−0.0014	+0.0007
13	Bal. θ(CB) ..	−0.0020	+0.0040	+0.0040	+0.0222	+0.0222	−0.0080	−0.0080	+0.0040

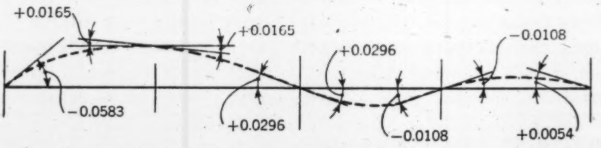
(c)


−0.0333
+0.0333
−0.0333
+0.0333

(d) INFLUENCE LINE FOR REACTION, R_B

Line No.	Symbol	AB	BA	BC	CB	CD	DC	DE	ED
14	θ	−0.0583	+0.0165	+0.0165	+0.0296	+0.0296	−0.0108	−0.0108	+0.0054

(d)


+0.0165
+0.0165
+0.0296
−0.0108
−0.0583
+0.0296
−0.0108
+0.0054

ried to joint C; + 0.11 × − 0.363 = − 0.04 is carried to joint D; and −0.04 × − 0.500 = + 0.02 is carried to joint E. The influence line for M_B can be plotted with θ-values, −0.34, +0.69, −0.31, +0.11, −0.04, and +0.02, as shown in Table 10(*b*). Line 9, Table 10, shows the angle changes $-\theta = 1/30 = 0.0333$ due to a "unit displacement" of support B when joints A, B, and C are hinged (see Table 10(*c*)); line 10 shows the results of balancing the angle change −0.0333 at AB, which member keeps the entire amount since joint A remains hinged. Line 11 shows the results of balancing the quantity −0.0333 at BA. The end BA keeps −0.0333 × 0.313 = − 0.0104. The quantities carried over are, for example, −0.500 [−0.0104 − (− 0.0333)] = − 0.0115 from BA to AB; −0.335 × − 0.0104 = + 0.0037 from BC to CB; and − 0.363 × + 0.0037 = − 0.0014 from CD to DC. Lines 12 and 13, Table 10(*c*), show the balancing process of angle changes +0.0333 at BC and CB. Line 14, Table 10(*d*), shows the final joint rotation θ, the sum of lines 10 to 13 in Table 10(*c*). The influence line for R_B can be plotted by means of Eq. 10 using the unit rise at support B, and the joint rotations listed in the various columns of Table 10(*d*).

The theory supporting Professor Peterson's method is not new; it is merely the modified form of the Müller-Breslau principle. The fundamental Eq. 10 can be derived by the classical moment-area method without any knowledge of "geometry." The author gives a rapid and accurate procedure for computation. Although so many different methods of statical analysis can be used, the writer has considered Professor Grinter's method to be the best, especially for the prismatic member, which avoids the tedious moment computation. Its utilization will increase the value of the author's paper.

OTTO GOTTSCHALK,[16] ESQ.—The neat arrangement of this paper, conforming with current practice, is commendable. The thing that the writer would like to see corrected is the abstract symbolism used in most of the recent papers on structural design, including this one. At an alarming rate, this field is accumulating barriers of learned thought instead of simple methods based on elemental geometrical analysis. Although the author implies that geometrical methods are used, he applies abstract symbols of geometrical units as loads, which is not the same.

A characteristic of the geometrical method of analysis is that the several steps are visible and, thereby, they are an aid to a designer's intuition. Abscissas, ordinates, radii, curvatures, linear or angular displacements or deformations, and the separation or the breaking of originally continuous parts are all visible in a truly geometrical method of analysis. Stresses and stress functions and moment and other stress areas are not visible and therefore are not used.

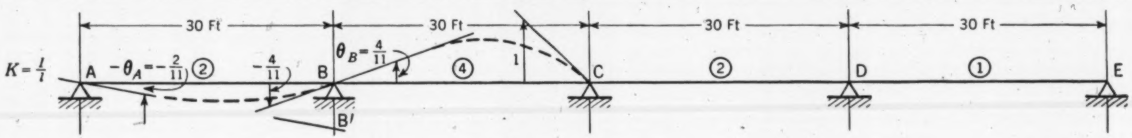
For purposes of analysis, rigid structures are geometrical bodies and must be analyzed by geometrical methods. Every step away from this principle leads to complications. If a given structure is said to be rigid, any geometrical disturbance at one end of one of its members is resisted by the stiffness of adjacent members, so that a unit geometrical displacement at any given section causes deformation all over the structure, bending the originally straight members into curves which are the influence lines. (Generally, the designer is not

---

[16] Engr., Central Argentine Ry., Villa Ballester, Argentina.

TABLE 11.—GEOMETRICAL ANALYSIS OF A CONTINUOUS BEAM

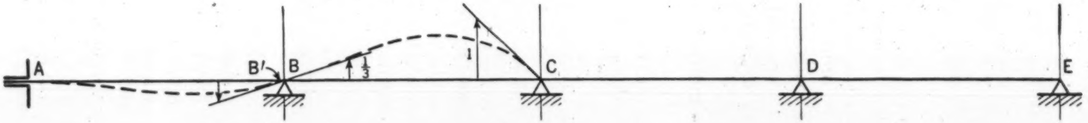
(a) BEAM ABCDE, SIMPLY SUPPORTED AS IN FIG. 11


30 Ft
30 Ft
30 Ft
30 Ft
$K=\frac{I}{l}$
A
$-\theta_A=-\frac{2}{11}$
②
$-\frac{4}{11}$
B
$\theta_B=\frac{4}{11}$
④
1
C
②
D
①
E
B′

Quantity	Supports: A	B		C		D		E
$m=\frac{\theta'}{\theta}=0.5\frac{k}{k+S'}$			$\frac{11}{31}$		$\frac{4}{11}$	←	$\frac{1}{2}$	
$m'=\frac{\theta}{\theta'}=0.5\frac{k}{k+S}$		$\frac{1}{2}$	→	$\frac{0.5\times 4}{4+\frac{3}{2}}=\frac{4}{11}$		$\frac{0.5\times 2}{2+\frac{36}{11}}=\frac{11}{58}$		
$S=k(1-0.5\,m)$			$\frac{102}{31}$		$\frac{18}{11}$	←	$\frac{3}{4}$	
$S'=k(1-0.5\,m')$		$2\left(1-\frac{0.5}{2}\right)=\frac{3}{2}$	→	$4\left(1-\frac{0.5\times 4}{11}\right)=\frac{36}{11}$		$2\left(1-\frac{0.5\times 11}{58}\right)=\frac{105}{58}$		
$S'_n+S_{n+1}=K$		$\frac{297}{62}$		$\frac{54}{11}$		$\frac{594}{232}$		
$\frac{S'_n}{K};\frac{S_{n+1}}{K}$		$\frac{3\times 31}{297}=0.3131$	$\frac{102\times 2}{297}=0.6869$	$\frac{36}{54}=0.6667$	$\frac{18}{54}=0.3333$	$\frac{420}{594}=0.7071$	$\frac{174}{594}=0.2929$	

TABLE 11.—(*Continued*)

(*b*) Beam ABCDE, Fixed at Support A (Values of *m* and *S* Are the Same as in Table 11(*a*))



Quantity	Supports: A		B		C		D	E
$m'\frac{\theta}{\theta'} = 0.5\frac{k}{k+S}$		0	$\longrightarrow$	$\frac{0.5 \times 4}{4+2} = \frac{1}{3}$		$\frac{0.5 \times 2}{2+\frac{10}{3}} = \frac{3}{16}$		
$S'k(1-0.5m')$		$2(1-0) = 2$	$\longrightarrow$	$4\left(1-\frac{0.5}{3}\right) = \frac{10}{3}$		$2\left(1-\frac{0.5 \times 3}{16}\right) = \frac{29}{16}$		
$S'_n + S_{n+1} = K$		$2+\frac{102}{31} = \frac{164}{31}$		$\frac{10}{3}+\frac{18}{11} = \frac{164}{33}$		$\frac{29}{16}+\frac{3}{4} = \frac{41}{16}$		
$\frac{S'_n}{K};\ \frac{S_{n+1}}{K}$		$\frac{62}{164} = 0.378$	$\frac{102}{164} = 0.622$	$\frac{110}{164} = 0.6708$	$\frac{54}{164} = 0.3292$	$\frac{29}{41} = 0.7073$	$\frac{12}{41} = 0.2927$	

interested in the true shape of the curve at every point, but in the value of selected ordinates.)

The cross sections at either end of every straight beam or column are perpendicular to the axis before loading. When the structure is deformed these sections are either rotated or translated, or both. When such beams are prismatic, they can be analyzed easily by purely geometrical methods because their curvature changes in proportion to their abscissas, so that, graphically, if the curvature is represented by the ordinate, the curvature at any point is represented by the straight line[17] that joins the ordinates representing curvature at the ends. In true geometrical analysis loads are considered external to the structure. They are not a part of it and do not enter into the analysis. They are only factors with which to multiply ordinate values determined by the geometry of the problem.

By applying equations of influence lines such as those developed by the writer (which the author has cited[3]), the laborious work of computing the tables required by the author is made unnecessary. For prismatic members (span AB, Fig. 11) the equation of the influence line[3] is

$$y = \frac{x\,x'\,(x\,\theta' + \theta\,x')}{l^2} \quad \text{.....................(25)}$$

in which, conforming to the notation of the paper, l is the span length AB; x and x' are the abscissas; and θ and θ' are the rotations of the points A and B, respectively. As the author indicates (see "Synopsis"), the methods in current use require the computation of a considerable number of points to obtain an influence line. By contrast, the geometrical formula, Eq. 25, requires only one point in each span plus one more point in the span where the deformation occurs. This simple formula, therefore, makes the influence line a most handy tool for all designs involving the laws of statics. For beams with haunches, the results obtained for prismatic members are modified by values selected from precalculated tables.

A demonstration of the reduction of work made possible by a true geometrical analysis of influence lines is presented in Table 11, for comparison with Fig. 11. Let $k = i/L$ equal the stiffness; S and S', the relative stiffness (resistance to rotation) at the left and right ends of the various spans; and $S'_n + S_n + 1 = K_j$, the joint stiffness at a support. Also, m is the ratio of end rotations. The computation of m and S begins at point D of span DE and progresses toward the left; and the computation of m' and S' begins at point B in span BA, progressing to the right. Table 11(a) includes a computation for support D, which is not included in Fig. 11. By moment-distribution methods this would require the same labor as that for support B.

It is thus shown that, by geometrical analysis, the stiffness percentages at both sides of supports can be obtained directly, and accurately, with less than one fifth of the work required by the distribution method, and with about one tenth the chance of error.

In Table 11(b) the same continuous beam has been analyzed, but with support A fixed. Values of m and S are the same as for the simply supported condition (Table 11(*a*)). Incidentally, these results show that, by fixing point A, the stiffness distribution is changed considerably at point B but scarcely at points C and D.

---

[17] "Calculo Estático de Estructuras Rigidas sobre Bases Naturales," by Otto Gottschalk, Librería Hachette, Buenos Aires, Argentina, p. 14, Fig. 13.

The foregoing shows that geometrical analysis is not made more complex by the fixing of exterior supports. Furthermore, it shows that the concepts of statical indeterminacy and redundancy are man-made inventions which are uncalled for in design procedure and therefore obsolete. All in all, the writer agrees with the author that, in future design of rigid, or continuous, structures, the influence line will be a most valuable tool, but only when it is applied in the fundamental geometrical sense, as presented herein.[18,19,20,21]

DEAN F. PETERSON, JR.,[22] Assoc. M. ASCE.—Mr. Spaulding's criticism that the fundamental equation of flexure, $\frac{d^2y}{dx^2} = \frac{M}{E\,I}$, does not apply to curved members is a criticism of all the modern methods for analyzing stresses in frames—from the classical theorems of strain energy to moment distribution and the elastic curve traverse. All common methods of frame analysis are based on Eq. 1 and give identical results. A discussion of the shortcomings of the basic assumptions of these methods is beyond the scope of this paper. Part of the writer's purpose was to demonstrate how to utilize these methods of solving for the end rotations and deflections of the members of unloaded frames subjected to the proper relative deflections to produce the desired influence line. The basic thesis of the paper will not be amended by greater refinement in the application or the premises of these methods.

Considerable progress has already been made toward understanding the action of knees and curved members subjected to flexure. The work of Bruce Johnston, M. ASCE and Edward H. Mount[23] and a paper by Mr. Stewart[24] are valuable contributions toward explaining the action of structural joints, including knees in framed structures. Such amendment in procedure as may be necessary is easily accomplished by interpretation and clarification of Eq. 1. The action of curved beams is treated by standard texts on strength of materials and unless the curvature is relatively great—that is, unless the radius of curvature approaches the depth of the beam—the corrections to Eq. 1 are immaterial.

Eq. 1 is also used by the writer for determining the ordinates to the elastic curves of the various members of the frame. Curvature of the members of frames is so small that corrections to Eq. 1 when thus used become unnecessary.

The contribution by Mr. Yao is valuable in that it demonstrates an additional method for finding the end rotations of the members of the unloaded structure subjected to the desired strain conditions.

---

[18] *Transactions*, ASCE, Vol. 105, 1940, p. 1019.

[19] *Ibid.*, p. 65.

[20] *Ibid.*, Vol. 107, 1942, p. 944.

[21] *Journal of the Franklin Institute*, December, 1941, p. 553.

[22] Associate Prof. of Civ. Eng., Utah State Agri. College, Logan, Utah.

[23] "Analysis of Building Frames with Semi-Rigid Connections," by Bruce Johnston and Edward H. Mount, *Transactions*, ASCE, Vol. 107, 1942, p. 993.

[24] "Analysis of Frames with Elastic Joints," by Ralph W. Stewart, *Proceedings*, ASCE, December, 1947, p. 1499.

Mr. Gottschalk criticizes the writer's use of nongeometrical methods for determining the end rotations of the various members of the structure bearing the imposed strain conditions. The writer claims to have used geometrical methods only for computing the deflection curve after the end rotations have been determined. Calculation of the end rotations of the members can be made by any method the designer desires. Whether he uses methods based principally on geometry, such as the elastic curve traverse, or the method proposed by Mr. Gottschalk (Table 11) is immaterial to this paper. Examples are given using several methods, although the writer's personal preference is for the geometrical methods.

The writer agrees with Mr. Gottschalk that modern structural theory is cluttered with unnecessary symbolism. It is more fundamental and just as easy, for example, to consider a change in the inclination of the tangents at two points of a beam as an angle (which it really is), instead of as the shear or weight borne by an imaginary beam. On the other hand, the writer believes a realization, for instance, of the relationship between bending moments and beam curvature to be essential. Use of real bending moments to express curvature is not believed to be "abstract symbolism."

Mr. Gottschalk states that, for prismatic beams, "* * * the laborious work of computing the tables required by the author is made unnecessary," and suggests Eq. 25. Of the seven tables, six concern computation for nonprismatic members. In all examples for prismatic members, the writer used Eq. 10 which is a generalized statement of Eq. 25.

Table 11 shows a method for obtaining the distribution factors on both sides of the support of a continuous beam. This method is given as a substitute for moment distribution and is therefore interesting. Mr. Gottschalk does not make clear, however, the method for finding the distribution of an unbalanced rotation at a remote joint—for example, the distribution at point D resulting from a rotation at point C. Moment distribution is only one of the many methods by which rotations over the supports of the continuous beam with unit rotational strain imposed at point B (Fig. 11) may be deduced. These may be found with great rapidity, for example, using the elastic curve traverse.[25] The method to be used depends somewhat on the personal preference and training of the designer.

In conclusion, the writer believes that the influence line computed as the deflection curve of the unloaded structure subjected to the proper strain conditions furnishes the simplest method of analyzing the stress condition of a structure if more than one loading is to be considered. The work involved in solving for the influence line in the manner proposed is about equal to that involved in a stress solution for any single set of loads.

---

[25] Discussion by Dean F. Peterson, Jr., of "Flexure Factors for Analyzing Continuous Structures," by Ralph W. Stewart, *Transactions* ASCE ,Vol. 104, 1939, p. 533.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2348

DRAINAGE OF AIRPORT SURFACES—SOME BASIC DESIGN CONSIDERATIONS

By Stifel W. Jens,[1] M. ASCE

With Discussion by Messrs. H. M. Williams, W. I. Hicks, E. J. A. Gain, E. F. Brater, Carl F. Izzard, and Stifel W. Jens

Synopsis

For determination of runoff from the flat turf surfaces encountered on the inter-runway areas of airports, a methodology is presented employing the newer knowledge concerning retention, infiltration, and overland flow. Since determination of supply rates is required, a detailed procedure is given that takes into account not only the fact that in nature antecedent precipitation within the storm reduces infiltration capacity values, but also the fact that in many storms rainfall rates are less at times than the related infiltration capacities.

Within most turfed areas a peak-reducing pondage is available and practical procedures are developed for obtaining the outflow rates from the ponded areas. Two examples of design are presented. These illustrate that, once pipe sizes are selected and a hydraulic balance is achieved for runoff from rainfall of a selected frequency and duration, the capacity of a system varies but little for other frequencies or durations due to the relation of large changes in mass pondage to very small changes in depths.

Introduction

To permit safe and satisfactory landing and take-off of airplanes, the removal of surface runoff is generally the primary problem in airport drainage. At properly selected airfields, only isolated and relatively small parts of the area ordinarily will require subdrainage. For scheduled airline operation or any other use that does not require all-over landing, a basic drainage requirement has been established by government regulatory authorities, airport managers, and engineers that surface runoff from rains occurring once in two years be removed to the extent necessary to prevent pondage on runways, taxiways, or aprons—or within about 50 ft of their edges. This requirement

---

Note.—Published in September, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Partner, Horner and Shifrin, St. Louis, Mo.

offers grading and drainage engineers a remarkable opportunity to minimize the cost of the drainage facilities.

In setting the pavement and inter-runway grades, consideration should be given to the possibilities for providing maximum pondage capacity below a horizontal plane surface whose nearest edge approaches within 50 ft of the pavement edges. Thus, practically the entire field can be served by inlets at low points in turf areas surrounded by paved runways and taxiways. The dominant character of this type of drainage is illustrated by the fact that, for a major transport field, such as the Lambert-Municipal Airport at St. Louis, Mo. (Fig. 1), about 80% of the total area requiring artificial drainage within the airport boundaries can be handled in this manner. This paper is confined to a discussion of the basic requirements of design for drainage of such inter-runway pockets, in most of which it is possible to develop useful pondage.

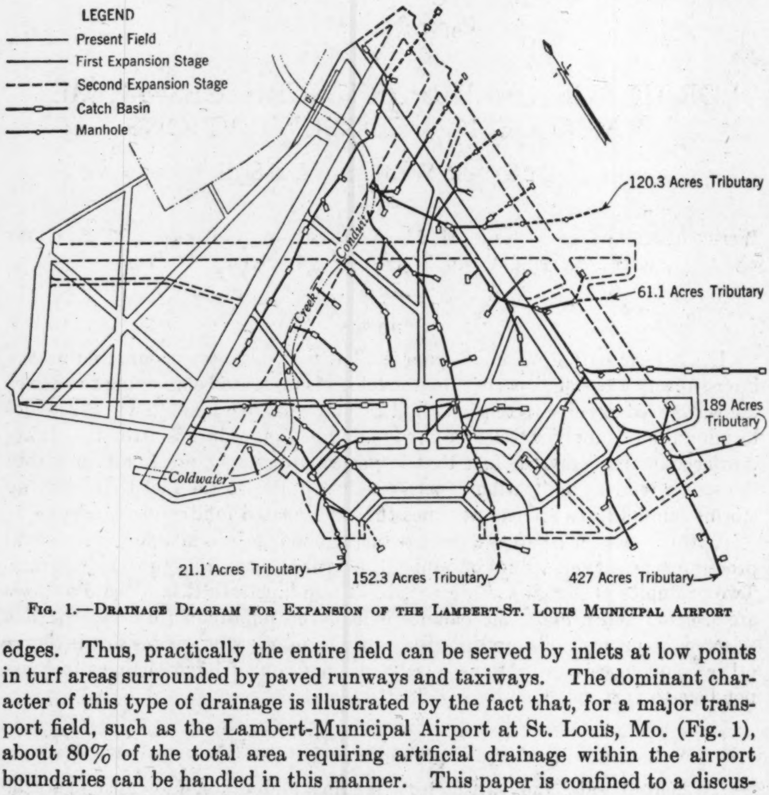

LEGEND
Present Field
First Expansion Stage
Second Expansion Stage
Catch Basin
Manhole
120.3 Acres Tributary
61.1 Acres Tributary
189 Acres Tributary
Creek
Conduit
Coldwater
21.1 Acres Tributary
152.3 Acres Tributary
427 Acres Tributary

FIG. 1.—DRAINAGE DIAGRAM FOR EXPANSION OF THE LAMBERT-ST. LOUIS MUNICIPAL AIRPORT

GENERAL DISCUSSION OF METHOD

Operational requirements limit the rate of grade for the turf off the paving; consequently, the shaping of each pocket is possible within a relatively narrow range of plane slopes. Thus, for each pocket the designer has a tributary area of accurately determinable drainage characteristics such as length and slope of sheet flow, and depth and extent of available pondage, as well as actual amounts of pervious and impervious areas. Determination of flow rates to and from each ponded area lends itself especially well to methods employing the newer knowledge concerning retention, infiltration, and overland flow.

The so-called rational method is inapplicable in view of the availability of better hydrologic knowledge and the lack of data or experience on which to base selection of the runoff coefficient C of the rational formula—

$$Q = C\,I\,A \qquad (1)$$

—in which Q is the rate of runoff, in cubic feet per second; I is the intensity of rainfall, in inches per hour; and A is the drainage area, in acres.

Using only accepted hydrologic engineering principles and generally available data, it is possible to develop a procedure whereby values of q, the discharge per unit width, may be directly determined. Such procedures involve: (1) Choice of gross rainfall patterns or critical rates for various durations for the agreed frequency (in this case two years); (2) selection of an infiltration capacity curve representative of soil and cover conditions under average antecedent circumstances; (3) correlation of the infiltration capacity curve with the gross rainfall, to secure a net supply pattern; (4) further correction of the net supply pattern by abstraction of retention from the first excess rainfall; (5) determination of the effect of surface detention in altering the shape and peak of the net supply pattern through the requirements of overland or sheet flow; and (6) determinations, required for the ponded areas in the airport, of the effects of pond storage on the shape and peak of the inflow hydrograph in its translation to an outflow hydrograph. For urban areas, similar procedures have been developed as in Los Angeles,[2] Calif., and St. Louis.[3] Also, the United States Engineer Department has adopted a general procedure for the design of airport drainage,[4] and is making systematic observations of rainfall and runoff at selected military airports for the purpose of checking and improving design procedure.

Organization of Basic Rainfall Data

The customary rainfall-duration curve for any specific frequency, such as is shown in Fig. 2, gives only the average precipitation intensity for any particular duration and supplies no information as to how the intensity varies from the average during the duration period—and the intensity practically always does vary in nature. Neither does the curve suggest how these isolated periods are involved in storms as a whole.

Since detailed knowledge of both these matters is necessary to the methodology presented, a special analysis is carried out in the following manner: For the St. Louis area,[5] detailed rainfall data forming the basis of the statistical studies for the development of the two-year frequency rainfall curve shown in Fig. 2 are available. For each duration a sample of about twelve

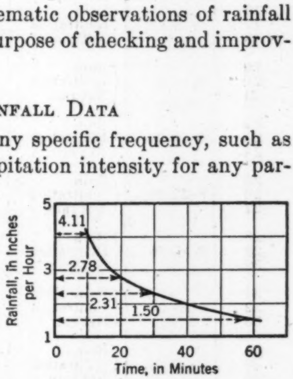


Fig. 2.—Rainfall-Duration Relation for Two-Year Frequency in the St. Louis Area

individual storms was chosen, well distributed throughout the series, with the same average precipitation rate for the critical period as that shown on the two-year frequency curve.

---

[2] "A Method of Computing Urban Runoff," by W. I. Hicks, *Transactions*, ASCE, Vol. 109, 1944, p. 1217.

[3] "Surface Runoff Determination from Rainfall Without Using Coefficients," by W. W. Horner and S. W. Jens, *ibid.*, Vol. 107, 1942, p. 1039.

[4] "Design of Drainage Facilities," by Gail A. Hathaway, in "Military Airfields: A Symposium," *ibid.*, Vol. 110, 1945, p. 697.

[5] "Relation Between Rainfall and Run-Off from Small Urban Areas," by W. W. Horner and F. L. Flynt, *ibid.*, Vol. 101, 1936, p. 140.

Table 1 tabulates these storms for the maximum 20-min intensities. For each storm the rainfall rate in each 5-min or 10-min period is so arranged that the critical 20-min values are entered one below the other. In Table 1, Col. 4 denotes the 10-min period beginning 70 min before start of the maximum 20-min period; Col. 10, that 10-min period immediately preceding the maximum period. From Table 1 and similar tables, the average precipitation in each 10 min was determined and the resulting patterns appear in Fig. 3 for the critical

TABLE 1.—DEVELOPMENT OF NET RAINFALL PATTERN, INCLUDING IMPERVIOUS AREA AND

Line No.	Date of storm	Item	PRECEDING 10-MIN PERIODS						
			First	Second	Third	Fourth	Fifth	Sixth	Seventh
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	8- 2-15...	Rain	....	....	....	....	....	....	....
2		Infiltration	....	....	....	....	....	....	....
3		Net rain	....	....	....	....	....	....	....
4	8-12-16...	Rain	....	....	....	....	....	....	0.26
5		Infiltration	....	....	....	....	....	....	2.94
6		Net rain	....	....	....	....	....	....	0
7	4-25-21...	Rain	....	....	....	....	....	....	....
8		Infiltration	....	....	....	....	....	....	....
9		Net rain	....	....	....	....	....	....	....
10	8-27-21...	Rain	....	....	....	....	....	0.42	1.82
11		Infiltration	....	....	....	....	....	2.92	2.34
12		Net rain	....	....	....	....	....	0	0
13	8-22-22...	Rain	....	....	....	....	....	....	....
14		Infiltration	....	....	....	....	....	....	....
15		Net rain	....	....	....	....	....	....	....
16	8- 8-23...	Rain	....	....	....	....	....	....	....
17		Infiltration	....	....	....	....	....	....	....
18		Net rain	....	....	....	....	....	....	....
19	8-11-25...	Rain	....	....	....	....	....	....	2.36
20		Infiltration	....	....	....	....	....	....	2.38
21		Net rain	....	....	....	....	....	....	0
22	9-12-25...	Rain	....	....	....	0.71	0.72	2.64	1.52
23		Infiltration	....	....	....	2.84	2.44	1.66	1.04
24		Net rain	....	....	....	0	0	0.38	0.48
25	8-23-26...	Rain	....	....	....	....	....	....	0.09
26		Infiltration	....	....	....	....	....	....	2.98
27		Net rain	....	....	....	....	....	....	0
28	9- 8-26...	Rain	....	....	....	....	....	....	....
29		Infiltration	....	....	....	....	....	....	....
30		Net rain	....	....	....	....	....	....	....
31	6-19-28...	Rain	1.14	2.20	0.76	0.55	0.25	0.23	0.62
32		Infiltration	2.72	1.91	1.17	0.93	0.83	0.78	0.72
33		Net rain	0	0	0	0	0	0	0
34	7- 5-28...	Rain	....	....	....	....	....	....	0.43
35		Infiltration	....	....	....	....	....	....	2.91
36		Net rain	....	....	....	....	....	....	0
37	Average...	Rainfall	(0.08)	(0.18)	(0.12)	(0.11)	(0.08)	(0.27)	(0.59)
38	Average...	Supply rate	(0)	(0)	(0)	(0)	(0)	(0.03)	(0.04)

two-year, 10-min, 20-min, 30-min, and 60-min duration periods. For each of these average storm patterns, there was appreciable precipitation both before and after the critical period. Study of these patterns shows clearly that the idea of isolated critical periods is quite unrealistic.

THE MAXIMUM 20-MIN RAIN, IN INCHES PER HOUR, FOR 0% TWO-YEAR FREQUENCY

Maximum 20 Min				Ensuing 10-Min Periods				Line No.
5 Min (11)	5 Min (12)	10 Min (13)	Total (14)	First (15)	Second (16)	Third (17)	Fourth (18)	(19)
1.20	5.16	1.49	2.33	0.06	....	....	....	1
2.86	2.42	1.62	1.94	1.05	0.80	0.68	0.61	2
0	1.54	0	0.39	−0.99	−0.80	−0.68	−0.61	3
1.44	2.41	2.28	2.10	0.82	0.35	0.12	....	4
2.70	2.24	1.32	1.87	0.92	0.74	0.67	0.60	5
0	0.00	0.45	0.23	−0.10	−0.39	−0.55	−0.60	6
1.80	2.34	2.61	2.34	1.04	0.84	0.35	0.02	7
2.77	2.27	1.54	2.09	0.98	0.78	0.68	0.61	8
0	0	0.50	0.25	0.06	0.06	−0.33	−0.59	9
2.64	3.04	4.35	3.59	0.01	....	....	....	10
1.66	1.26	0.96	1.50	0.77	0.68	0.61	0.56	11
0	1.56	3.39	2.09	−0.76	−0.68	−0.61	−0.56	12
3.34	5.40	0.30	2.34	0	....	....	....	13
2.65	2.06	1.20	1.63	0.89	0.73	0.66	0.59	14
0	2.83	0	0.71	−0.89	−0.73	−0.66	−0.59	15
3.72	7.23	4.33	4.90	3.80	0.74	0.65	0.45	16
2.65	2.06	1.20	2.07	0.89	0.73	0.66	0.59	17
0	5.04	3.13	2.83	2.91	0.01	−0.01	−0.14	18
0.44	0.20	2.48	1.40	0.06	....	....	....	19
1.79	1.68	1.26	1.09	0.90	0.74	0.66	0.60	20
0	0	0.62	0.31	−0.84	−0.74	−0.66	−0.60	21
2.52	3.72	1.68	2.40	0.49	0.03	....	....	22
0.86	0.77	0.69	0.75	0.63	0.57	0.53	0.52	23
1.66	2.95	0.99	1.65	−0.14	−0.54	−0.53	−0.52	24
1.26	2.26	4.32	3.04	0.36	....	....	....	25
2.81	2.40	1.62	1.99	1.02	0.80	0.69	0.62	26
0	0	2.10	1.05	−0.66	−0.80	−0.69	−0.62	27
5.89	2.90	1.80	3.09	1.55	1.12	0.63	....	28
2.65	2.06	1.20	2.07	0.89	0.73	0.66	0.59	29
2.04	0.84	0.60	1.02	0.66	0.39	−0.03	−0.59	30
1.44	3.00	2.52	2.37	0.99	0.45	....	....	31
0.68	0.65	0.60	0.79	0.55	0.52	0.51	0.51	32
0.14	2.35	1.92	1.58	0.44	−0.07	−0.51	−0.51	33
4.38	4.68	2.31	3.42	0.74	1.40	0.07	....	34
2.50	1.94	1.27	2.04	0.92	0.74	0.66	0.60	35
0.68	2.74	1.04	1.38	−0.18	0.66	−0.59	−0.60	36
(2.506)	(3.529)	(2.539)	(2.778)	(0.827)	(0.411)	(0.152)	(0.039)	37
(0.377)	(1.654)	(1.228)	(1.122)	(−0.041)	(−0.304)	(−0.488)	(−0.544)	38

INFILTRATION CAPACITY OF TURFED AREAS

The United States Department of Agriculture, through measurements of flow from small watersheds and from infiltrometer applications, has produced a large number of infiltration capacity curves for various soils with grass cover, generally for agricultural areas used as meadow or pasture. At least two studies have been made of the infiltration capacity of such turfed areas as would be normal for airfields. These areas would be similar in surface condition to city lawns. One such series was carried out at St. Louis for lawn grass on a yellow clay subsoil; the other in Anacostia Park, at Washington, D. C., for lawn grass on hydraulic fill subsoil consisting of mud dredged from the Potomac River. For similar antecedent conditions the values for these two areas are very nearly identical. Curve (*a*), Fig. 4, is representative of the values at either location for generally wet seasonal conditions and heavy rainfall within the preceding 24 hours. Curve (*b*) is representative of St. Louis conditions under average summer monthly rainfall with no considerable application of precipitation for the preceding three days. Undoubtedly, for very dry summer conditions with considerable soil checking, a much higher series of values would prevail.

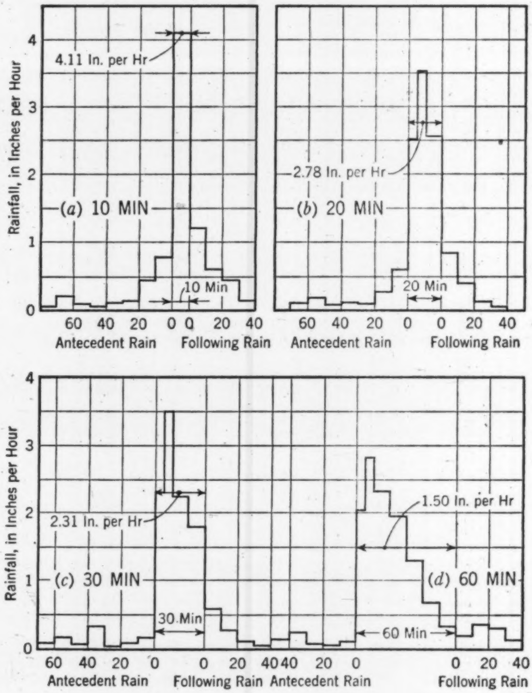


FIG. 3.—AVERAGE RAINFALL PATTERNS FOR STORMS OF VARIOUS DURATION PERIODS IN THE TWO-YEAR FREQUENCY SERIES

Infiltration capacity values shown in Fig. 4 may be accepted as reasonably representative of the values for a turfed cover for a rather wide range of soils. There is little artificial compaction such as would occur from trampling of the surface under intensive recreational use or from walking over city lawns. In all probability, representative curves for city lawns would be slightly lower than those shown. For sandy loams or sands, values would be materially higher.

RELATING INFILTRATION CAPACITY TO RAINFALL OCCURRENCE

Since the objective is to develop runoff values which on the average will be equaled or exceeded once in two years, the sample critical rainfalls previously discussed should be matched by an infiltration capacity curve which is normal for the turf areas at the time of occurrence. Also, the curves should be properly related on the basis of the rainfall preceding the critical period. For the sample rains, the average elapsed time to the preceding rain was 3 days, and the average mass of the preceding rain was about 0.4 in. These conditions compare closely to normal summer conditions for the area and result in the conclusion that, on the average, curve (*b*), Fig. 4, is properly applicable during the occurrence of these storms. Applying an initial value of 3 in. per hr as the probable infiltration capacity rate at the beginning of rainfall, curve (*b*), Fig. 4, was matched to the rainfall pattern in each of the representative sample storms.

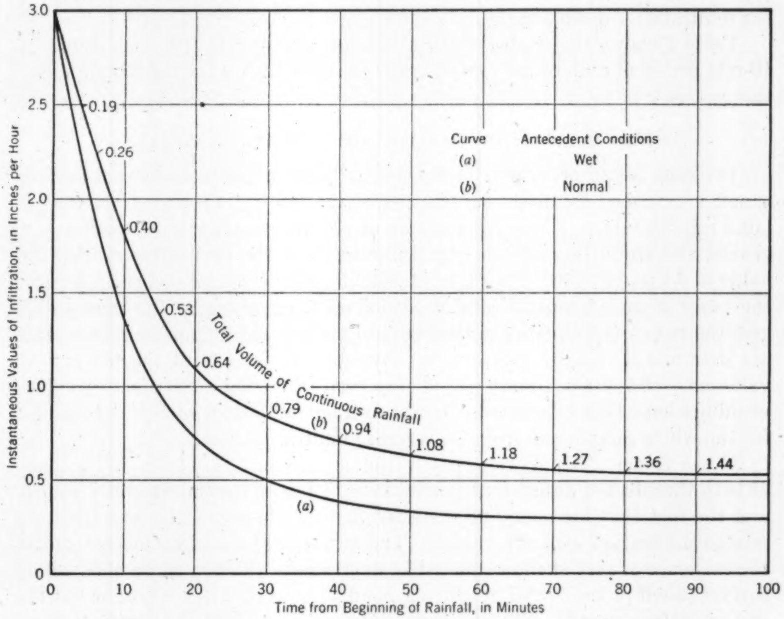


FIG. 4.—INFILTRATION CAPACITY CURVES FOR WET AND NORMAL ANTECEDENT CONDITIONS OF TURF AREAS

To obtain infiltration values (such as are given in Table 1) that would apply to each 5-min or 10-min rainfall period, the following principle was used: On curve (*b*), Fig. 4, are indorsed, at various points, the mass infiltration that could occur up to each time if the rainfall at all times until then equaled or exceeded the rates shown on the infiltration capacity curve. For example, at capacity rates, the total possible mass infiltration between 15 min (0.53 in.) and 20 min (0.64 in.) is 0.11 in. Whenever the rainfall rate exceeds the infiltration capacity rates throughout any one time period, the drop in average infiltration capacity from one period to another follows curve (*b*), Fig. 4. Whenever the rainfall rate falls below the infiltration capacity values for normal antecedent conditions, the position of the actual infiltration rate at the end of such a time period is assumed to be at the value shown by the infiltration capacity curve corresponding to the same actual incremental mass infiltration interpolated on the curve. This relationship between infiltration capacity, actual rate, and mass is only approximate, but it is sufficiently well indicated to justify its use in positioning the infiltration capacity curve.

Table 1 shows the related infiltration and rainfall rates for each 5-min or 10-min period of each of the representative sample storms for the 20-min duration period.

NET SURFACE RUNOFF FOR 100% PERVIOUS AREAS

For each 5-min or 10-min period of the rains in Table 1, the net surface runoff is obtained by direct subtraction of the infiltration rates from the related rainfall rates. These rates represent production of surface runoff at the ground and are further adjusted by deducting from the first excess rainfall the value of 0.1 in. to allow for surface retention. The results so secured represent the series of surface supply rates produced on the average by the rains which give the two-year frequency critical 20-min period. More particularly, values are determined rain by rain and no averaging is done until the net supply values in each 5-min or 10-min period have been calculated. Satisfactory values could not have been calculated from the average rainfall for each 10-min period for the whole sample and from the average infiltration rate.

These values of supply rates result from a procedure that takes full account of both the effect of antecedent rainfall in reducing infiltration capacity values, and the fact that for many storms rainfall rates are sometimes less than the related infiltration capacity values. The adjustment for retention recognizes the existence of very shallow saucer-like depressions which must be filled before surface runoff proceeds and in which a small amount of water is retained at the end of surface runoff. The procedure also makes evident the infiltration capacity rates that will exist after the end of production of net supply, and therefore permits evaluation of the additional infiltration that will result from overland flow during the recession of the hydrograph.

RUNOFF FROM PARTLY IMPERVIOUS AIRPORT AREAS

The procedure thus far gives fully realistic values of runoff from wholly pervious surfaces where infiltration is proceeding throughout. However, in the majority of the ponded interpavement areas, water is discharged from half-runways or half-taxiways on to the turf. These mixed areas are designated by the percentage of impervious surface.

It should be possible to evaluate such mixed areas satisfactorily by assuming that the discharge from the pavement, after a slight deduction (0.05 in.) for retention on the pavement, can be considered as resulting from a proportionate increase of rainfall on the turfed area. For example, for a 30% impervious area, the impervious part is three sevenths of the turfed part, and the rainfall pattern accordingly can be increased item by item to ten sevenths of the original value. This procedure has been followed for each of the selected sample rains in each of the duration series and for imperviousnesses of 10%, 30%, and 50%—the range of values encountered in airport drainage.

With these modified rainfall patterns, infiltration capacities were recomputed for each successive time period for each sample rain, and new patterns of supply were prepared in the same manner as in Table 1. When the antecedent rainfall on the turf is assumed increased in this manner, infiltration capacities at the beginning of the critical duration are successively reduced below those originally computed, and hence the resulting supply rate patterns are increased not only by higher rainfall rates but also by reduction of infiltration capacity. Fig. 5 shows patterns for one rainfall duration and four percentages of imperviousness.

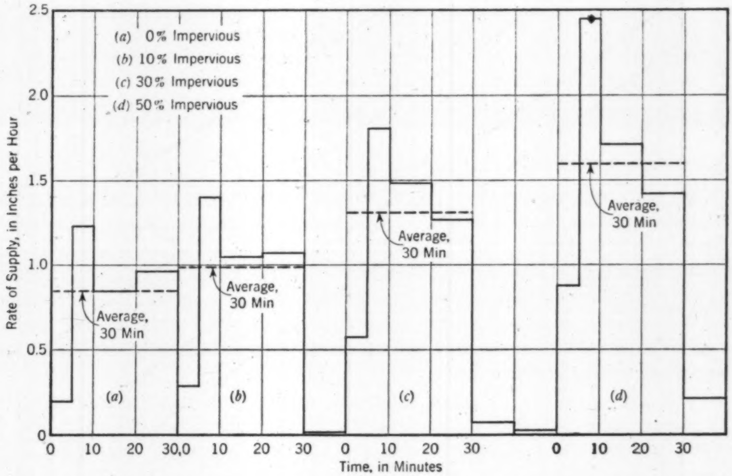


FIG. 5.—PATTERNS OF SURFACE RUNOFF SUPPLY RATES FOR 30-MIN RAINFALL PERIOD AND VARIOUS PERCENTAGES OF IMPERVIOUSNESS

The question naturally arises as to the physical propriety of treating water poured on the upper margin of the turf from the pavement as the equivalent of additional distributed rainfall over the turf. This matter has been subjected to rough analytical checks. Generally the velocities of flow over the turf are sufficiently high so that some of the water entering the upper margin near the beginning of the rain will arrive at the outlet by the end of the critical period. The water from the pavement should become distributed over the turf strip within a critical duration. No doubt, the actual distribution is not the same as that resulting from the assumption, but this difference does not seem great enough to alter either the average detention on the turf or the critical rate of outflow from the lower margin of the turf.

Fig. 6 shows the entire series of supply rates for the various durations and percentages of imperviousness considered. These two-year frequency values are related to an all-turf area and must be adjusted by multiplying by 0.5, 0.7, 0.9, and 1.0 to obtain the supply rates applicable to the entire area for 50%, 30%, 10%, and 0% imperviousnesses, respectively. The encircled points are those computed by averaging representative rains (properly related to the two-year frequency average intensities for various durations).

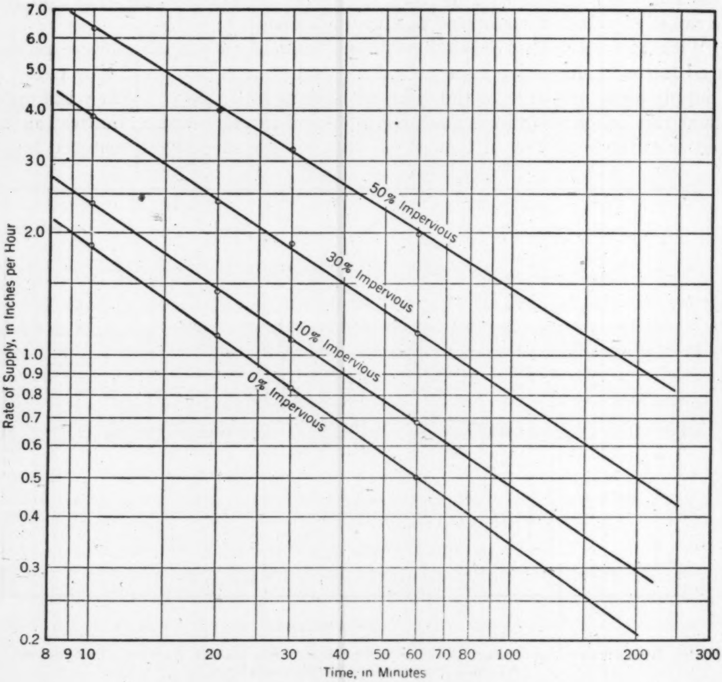


FIG. 6.—TWO-YEAR FREQUENCY SUPPLY RATES FOR VARIOUS PERCENTAGES OF IMPERVIOUSNESS AND DURATIONS OF MAXIMUM PERIOD OF SUPPLY

RATES OF INFLOW TO PONDS OR INLETS

With the supply patterns available for various durations and various percentages of impervious area, it becomes necessary to determine the hydrographs of inflow to the ponds or inlets. Thus, an evaluation of the effects of overland flow is required.

On the impervious areas, the supply for surface runoff quickly develops a thin film through which flow takes place down the slope to the upper edge of the adjacent turf. For the lengths and slopes involved in half-runways and half-taxiways, flow is quite rapid and reaches equilibrium in from about 2 min to 5 min; thereafter, for a constant supply rate, discharge from the pavement on to the turf continues equal to the supply rate. As discussed previously, the assumption is made that the effects of runoff from the pavement may be treated by adjustment of the rainfall on the turf.

Characteristics of flow over turf are quite different. As the rainfall builds up supply within the grass, flow takes place down the slope between the grass blades at comparatively low velocities. For determining the outflow rate at the lower margin of a turf strip and the related depth of surface detention on the turf, the following formula developed by the late R. E. Horton,[6] M. ASCE, have been used:

$$q = \sigma \tanh^2 \left(\frac{K \sqrt{3{,}520}\, S^{0.25} \sigma^{0.50} t}{60\, l^{0.50}} \right) \dots\dots\dots\dots\dots (2)$$

and

$$\delta = \frac{l^{0.5} \sigma^{0.5}}{S^{0.25} \sqrt{3{,}520}} \tanh \left(\frac{K \sqrt{3{,}520}\, \sigma^{0.5} S^{0.25} t}{60\, l^{0.5}} \right) \dots\dots\dots\dots (3)$$

in which q is the outflow rate, in inches per hour, at the lower margin of a turf strip; σ is the supply rate, in inches per hour; tanh is the hyperbolic tangent; t is the time of supply, in minutes; δ is the depth of flow, in inches, at the lower margin of the turf; l is the length of overland flow, in feet; S is the absolute slope of the turf strip; and K is the reciprocal of the ratio of the average depth to that at the margin. The numerical coefficient 3,520 was selected from pertinent experimental data for good bluegrass turf—closely checking the n-value of 0.400 used by the United States Army Engineers.[7] This n-value is not strictly comparable to the Manning n. An average overland flow slope of 1.75% has been used for all pervious areas.

Mr. Horton found that, for overland flow over turf, the profile is essentially parabolic with an average depth two thirds of that at the margin. Where additional supply is poured on to the upper margin from pavement, the profile will involve greater depths in the upper part of the turf strip and the average depth will be somewhat greater than two thirds. For 50% impervious areas the average depth has been assumed as fourth fifths and intermediate values were used for the lesser amounts of imperviousness.

Eqs. 2 and 3 are for strips of uniform width and are, therefore, rigorously applicable only to a rectangular area where the flow arrives along one margin. In practically all airfield grading, the pockets are essentially inverted pyramids with the flow converging toward a central inlet. The flow, therefore, may be visualized as occurring on triangular strips with an average width of 1 ft. For this situation, the overland flow hydrograph is so complex that it cannot be reduced to usable mathematical equations. A rough analysis indicates that the total detention on the triangular strip may be slightly less than that of the equivalent rectangular strip, and the discharge at the lower margin may be slightly greater. The differences, however, are not great and the basic formulas have been used to determine the rates of flow into the pond or inlet.

---

[6] "The Interpretation and Application of Runoff Plat Experiments," by R. E. Horton, *Proceedings*, Soils Science Soc. of America, Vol. 3, 1938, p. 340.

[7] "Design of Drainage Facilities for Airfields," *Engineering Manual*, Office of the Chf. of Engrs., War Dept., Washington, D. C., September 1942, Chapter XXI.

Table 2 and Fig. 7 illustrate application of Eqs. 2 and 3 to the supply rates. For the actual varying supply rates the rising side of the hydrograph of flow to the inlet is shown as curve A; the corresponding curve B arises out of the aver-

TABLE 2.—COMPUTATION OF OVERLAND FLOW PEAKS AND NET MASS RUNOFFS FOR TURF AREAS 1 FT WIDE, 100 FT LONG, AND 0% AND 30% IMPERVIOUS

Impervious area (%)	Time period (min)	σ (in. per hr)	Discharge, q[a] (in. per hr)	Mass rain (in.)	HYDROGRAPH COMPONENT[b]				NET MASS RUNOFF (IN.)		Corrected hydrograph base (min)
					Estimate of base (min)	Rising side (min)	Estimate of recession (min)	Recession mass infiltration (in.)	On all-pervious area	On actual area	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
0	10	0.08	0.00159	0.013	....	....	....	....	....	....	....
	10	0.10	0.00323	0.017	....	....	....	....	....	....	....
	10[c]	1.88	0.835	0.314	51.9	11.0	40.9	0.069	0.292	0.292	42.0
	10	0.10	....	0.017	....	....	....	....	....	....	....
	Total	....	....	0.361	....	....	....	....	....	....	....
	10	0.03	0.000276	0.005	....	....	....	....	....	....	....
	10	0.04	0.001384	0.007	....	....	....	....	....	....	....
	20[c]	1.12	0.762	0.373	60.6	20.6	40.0	0.050	0.335	0.335	52.7
	Total	....	....	0.385	....	....	....	....	....	....	....
	30[c]	0.848	0.690	0.425	73.3	30.0	43.3	0.071	0.354	0.354	61.6
	45[c]	0.622	0.571	0.467	98.1	45.0	53.1	0.082	0.385	0.385	81.0
	60[c]	0.503	0.483	0.503	125.0	60,0	65.0	0.086	0.417	0.417	103.8
	90[c]	0.378	0.374	0.566	181.5	90.0	91.5	0.115	0.451	0.451	145.0
30	10	0.22	0.0108	0.037	....	....	....	....	....	....	....
	10	0.46	0.0922	0.077	....	....	....	....	....	....	....
	10[c]	3.84	2.465	0.640	42.7	11.8	30.9	0.060	0.818	0.572	38.4
	10	0.67	(1.727)	0.112	....	....	....	....	....	....	....
	10	0.07	...	0.012	....	....	....	....	....	....	....
	Total	....	....	0.878	....	....	....	....	....	....	....
	10	0.16	0.00577	0.027	....	....	....	....	....	....	....
	10	0.18	0.0240	0.030	....	....	....	....	....	....	....
	20[c]	2.388	2.035	0.796	53.6	21.4	32.2	0.065	0.845	0.59	49.9
	10	0.34	(1.424)	0.057	....	....	....	....	....	....	....
	Total	....	....	0.910	....	....	....	....	....	....	....
	10	0.02	....[d]	0.003	....	....	....	....	....	....	....
	30[c]	1.888	1.745	0.944	66.4	30	36.4	0.071	0.894	0.626	61.5
	10	0.11	(1.222)	0.018	....	....	....	....	....	....	...
	Total	....	....	0.965	....	....	....	....	....	....	...
	45[c]	1.375	1.343 (0.941)	1.031	92.3	45	47.3	0.088	0.943	0.66	84.0
	60[c]	1.13	1.12 (0.784)	1.13	121.0	60	61.0	0.071	1.049	0.733	112.5
	90[c]	0.86	0.86 (0.602)	1.29	180.0	90	90	0.122	1.168	0.819	163.0

[a] Col. 4 based on Eq. 2, with $K = 1.5$ for 0% imperviousness and 1.35 for 30% imperviousness. In Col. 4, for 30% impervious, numerals in parentheses are 0.7 of the discharge immediately above and hence represent actual runoffs from areas of 30% imperviousness. [b] Cols. 6 through 12 based on Eqs. 4 and 5; f taken from tables similar to Table 1. [c] Maximum. [d] Negligible.

age supply rate for the critical period. The differences are not great and, in all computations for peak rate to the inlet, the average supply rate has been used. Examination of a number of such hydrographs indicates that a straight line, C, can be adopted without appreciable error, with the shaded area representing water in detention storage at the end of the critical period. This water, reduced by some further infiltration, supplies the recession side of the hydrograph marked A before correction for additional infiltration and, after such correction, hydrograph R. A number of recession curves indicate again the propriety of a straight line for design purposes and in all cases the overland flow hydrograph has been assumed to be a triangle as shown in Fig. 7, with the peak rate q_p constituting the altitude and with the area of the triangle equal to the mass runoff, m.

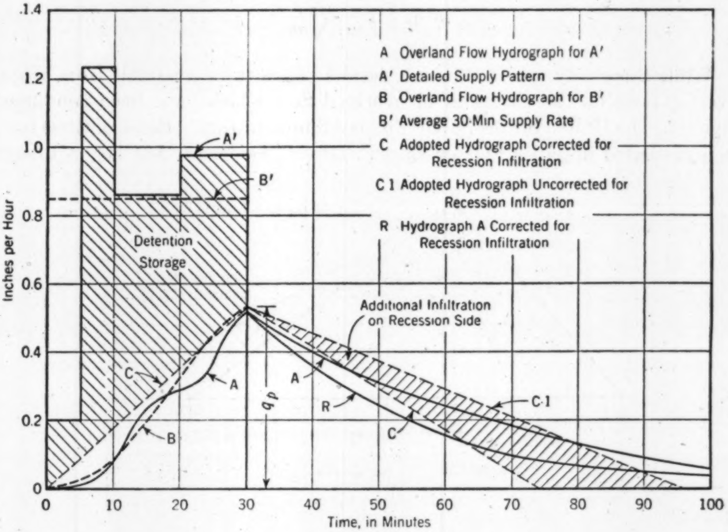


FIG. 7.—SUPPLY AND RUNOFF CURVES FOR 30-MIN SUPPLY ON 0% IMPERVIOUS AREA WITH OVERLAND FLOW LENGTH OF 185 FT AND SLOPE OF 1.75%

Likewise, the method of correcting for recession infiltration is shown in Fig. 8 and Table 2. In Fig. 8, area ABC represents the corrected mass runoff, m (Col. 10, Table 2); area A′BC′ represents p, the available mass pondage; and area CBD represents F, the mass infiltration, on the recession side of the hydrograph. All mass units are in inches of depth over the entire tributary area. In these determinations it has been assumed that the average infiltration capacity rate f is fully effective during only one third of the entire recession time. This assumption is necessary because the receding sheet of overland flow after cessation of supply covers a diminishing part of the total area and consequently the infiltration opportunity progressively becomes less. Thus,

$$F = \frac{q_p X}{2 \times 60} = \frac{(t_r - X) f}{3 \times 60} \qquad \text{(4)}$$

and

$$X = \frac{t_r f}{1.5\, q_p + f} \quad\text{.........................}(5)$$

Also, from similar triangles,

$$q_o = q_p \left(1 - \sqrt{\frac{p}{m}} \right) \quad\text{.....................}(6)$$

While this study was in progress, results began to be available from actual investigations on the character of overland flow which have been conducted largely by the United States Public Roads Administration. Some of these have been presented in papers[8,9,10] by Carl F. Izzard, Assoc. M. ASCE. Although the studies were primarily on pavement surfaces, including aprons and runways, a number of runs were made on turfed surfaces.

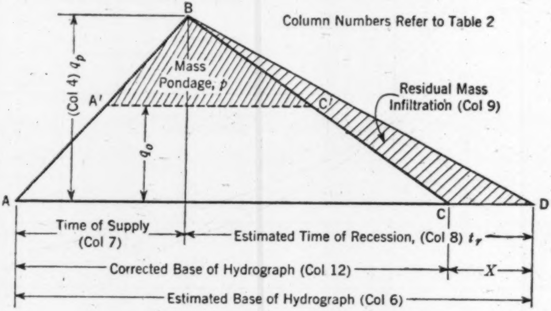

B
Column Numbers Refer to Table 2
(Col 4) q_p
Mass Pondage, p
A'
C'
Residual Mass Infiltration (Col 9)
q_o
A
D
C
Time of Supply (Col 7)
Estimated Time of Recession, (Col 8) t_r
Corrected Base of Hydrograph (Col 12)
X
Estimated Base of Hydrograph (Col 6)

FIG. 8.—SCHEMATIC REPRESENTATION OF TRIANGULAR HYDROGRAPH SHOWING ADJUSTMENT FOR INFILTRATION ON RECESSION SIDE

Several comparisons have been made between the data developed by Mr. Izzard and those computed by the Horton formulas. For 30-min rains and overland flow lengths of from 150 ft to 200 ft, the two sets of values are very closely in accord. However, on 10-min rains, Mr. Izzard's values are only about two thirds of those computed by the Horton equation. There is a question, therefore, whether the application of the Horton equation to periods of short duration may not indicate excessive values of inflow to the ponds.

In Tables 3 and 4, design values are prepared for a three-inlet system for duration periods of from 10 min to 60 min. If Mr. Izzard's data are applicable, the 10-min flows to the ponds will be reduced to about two thirds of those shown and the 20-min inflow rates will be slightly reduced. Thus, for this particular system, the 20-min or 30-min rain would be critical and the correct design rate for the outlet pipe would be about 12 cu ft per sec instead of 14 cu ft per sec. If the outlet were actually designed for a flow of 14 cu ft per sec, there would

---

[8] "Runoff from Flight Strips," by Carl F. Izzard, *Proceedings*, Highway Research Board, National Research Council, Vol. 22, 1942, p. 94.

[9] "Preliminary Report on Analysis of Runoff Resulting from Simulated Rainfall on a Paved Plot," by C. F. Izzard and M. T. Augustine, *Transactions*, Am. Geophysical Union, 1943, Pt. II, pp. 500–511.

[10] "The Surface Profile of Overland Flow," by C. F. Izzard, *ibid.*, 1944, Pt. VI, pp. 959–968.

be less utilization of storage in the ponds, so that the ponds would stand at slightly lower levels than those indicated in Table 3 and Fig. 9.

Since the overdesign is apparently not great (in fact in this instance it would have been impossible to use a smaller commercial pipe size), and since all the basic design data are predicated upon the Horton equations, the illustrative examples will be calculated from those data.

TABLE 3.—COMPUTATION OF OUTFLOW RATES FROM PONDS, AND PRELIMINARY PIPE SIZE SELECTION, THREE-POND SYSTEM

Pond or pipe No. (1)	Flow quantity, (cu ft per sec) (2)	DURATION (MIN)					Preliminary slope[a] (ft per ft) (8)	Diameter (in.) (9)	Preliminary velocity (ft per sec) (10)	$1.5 \frac{V^2}{2g}$ (11)	Adjusted slope (ft per ft) (12)
		10 (3)	20 (4)	30 (5)	45 (6)	60 (7)					
1......	Q_o	7.1	6.1	5.4	4.3	3.0	....	....	....	....	....
1–2......	Q_o	7.1	6.1	5.4	4.3	3.0	0.0039	19	3.9	0.35	0.0044
2......	Q_o	1.7	1.7	1.7	1.6	1.5	....	....	....	....	....
2–3......	Q_o	8.8	7.8	7.1	5.9	4.5	0.0071	18	5.0	0.58	0.0104
3......	q_p[b]	1.16	1.11	1.04	0.86	0.75	....	....	....	....	....
	m[c]	0.54	0.57	0.61	0.67	0.73	....	....	....	....	....
	Q_p	8.0	7.7	7.2	5.9	5.2	....	....	....	....	....
	Q_o	5.2	5.0	4.8	4.1	3.6	....	....	....	....	....
3-outlet..	Q_o	14.0	12.8	11.9	10.0	8.1	0.0256	17	9.0	1.89	0.020

[a] Cols. 8 through 12 based on quantities of Col. 3. [b] Inches per hour. [c] Inches.

BASIC DESIGN DATA

For the two-year frequency rainfall, the values of the peak rate of runoff, q_p, in inches per hour or in cubic feet per second per acre, and the values of the related mass runoff m, in inches, have been determined. These values have been calculated directly from the supply rates for the 10-min, 20-min, 30-min, and 60-min durations; for other durations, such as 45 min and 90 min, corresponding supply rates have been extrapolated from Fig. 6. The values have been determined separately for areas of 0%, 10%, 30%, and 50% imperviousnesses, and for turf lengths of overland flow of from 100 ft to 500 ft. Different plottings have been prepared to make available the best facility for interpolation. Figs. 10 and 11 show typical sheets relating q_p and m to length of overland flow for 30% imperviousness and for various times of supply.

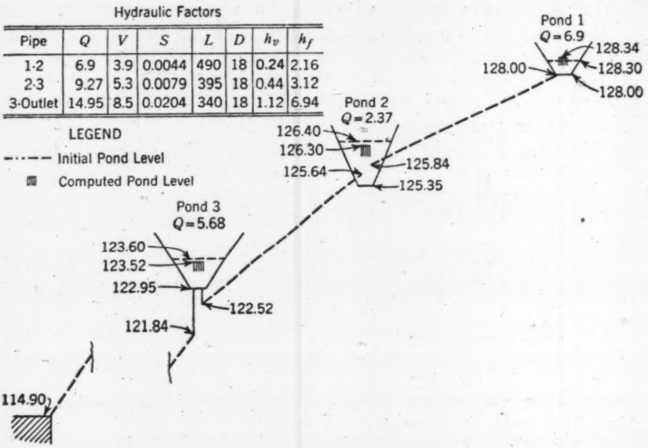
TABLE 4.—CHARACTERISTICS OF PONDS IN THREE-POND SYSTEM

Pond No.	Area (acres)	Imperviousness (%)	Flow length (ft)	Pondage (in.)
1......	4.04	50	150	0.0082
2......	5.84	30	200	0.31
3......	6.90	30	200	0.068

USE OF HYDROGRAPHS IN DESIGN

Where no deliberate storage ponding is possible, the inflow hydrograph to the inlet may be assumed to be triangular with its peak flow rate and mass runoff determined by selection from the basic design data. Such hydrographs may be offset by the time of flow from one inlet to another, and the peak rate of flow below any inlet may be determined by the addition of ordinates.

Where pondage is practicable, the triangular hydrograph represents inflow to the pond from which a simple determination can be made for the outflow rate as reduced by detention storage in the pond. In Fig. 8, from the properties of a triangle, the relation between inflow, outflow, pondage, and mass runoff can be expressed as Q_o (in cubic feet per second) = Q_p (in cubic feet per second) $\times \left(1 - \sqrt{\frac{p}{m}}\right)$, with both p, the mass pondage, and m, the mass runoff, in inches of depth over the entire tributary area. For any particular inlet area, the values of Q_p (q_p from design graphs times the total area), m (from the design diagrams), and p (the permissible pondage developed from the field grading plans) are available. Then Q_o is the minimum outflow capacity that must be provided if the pondage is not to exceed that computed as the desirable maximum. However, it should be remembered that the pyramidal shape of the ponds permits an appreciable change in volume without any large change either in depth, or in the distance from margin of pond to pavement. To accommodate commercial pipe sizes, the initial or optimum values of p obtained from the grading plan may, therefore, be varied somewhat in the course of design. Changes in pond depths are assumed to vary with the cube roots of the related pond volumes.



Hydraulic Factors

Pipe	Q	V	S	L	D	h_v	h_f
1-2	6.9	3.9	0.0044	490	18	0.24	2.16
2-3	9.27	5.3	0.0079	395	18	0.44	3.12
3-Outlet	14.95	8.5	0.0204	340	18	1.12	6.94

LEGEND

–·–– Initial Pond Level

▥ Computed Pond Level

FIG. 9.—PROFILE AND HYDRAULIC FACTORS FOR THREE-POND SYSTEM

APPLICATION OF BASIC DATA AND HYDROGRAPHS IN ACTUAL DESIGN

The principle of designing the field drainage system, which takes advantage of permissible ponding in the intra-runway pockets, requires that the outlet capacity from any one pocket be the same as the calculated outflow rate from that pocket when the inflow to it has been reduced by the available detention storage. Thus, the objective of the design of the pipe system is to develop pipe capacities which will throttle the outflow from the ponds to such an extent that the permissible pondage will be utilized.

If the grading plan provides a series of pockets with the water surface elevations descending in order toward the outlet of the system, it is entirely feasible to provide pipes of such capacity that the ponds will "float on the line." Where this can be done, inlet manholes with adequate gratings can be placed at the low point in each pond. In actual operation a water column will be descending through the manholes and the acceleration of the water from zero velocity in the pond to the velocity in the outgoing pipe will utilize a velocity head equal to $\frac{V^2}{2g}$. In addition, there will be entrance losses through the grating and at the bottom of the manhole; and, to provide for this, the total entrance head including velocity head and entrance losses has been treated as $1.5\,\frac{V^2}{2g}$. Thus, the hydraulic gradient in the pipe line at any particular inlet manhole must be lower than the level of the pond in the amount indicated by this expression, and pipe capacities must be based on the hydraulic gradient so determined.

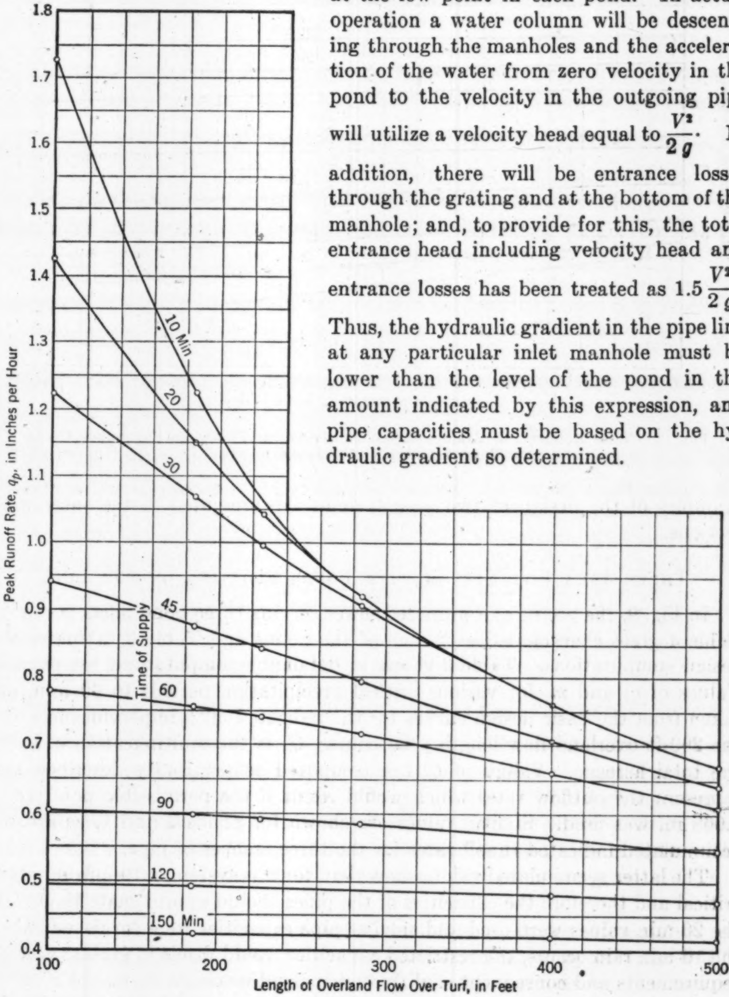


FIG. 10.—PEAK RUNOFF RATES FOR VARIOUS DURATIONS OF RAINFALL IN TWO-YEAR FREQUENCY CYCLE ON TURF AREA WITH 30% IMPERVIOUSNESS

Obviously, when the nearest commercial size pipes have been selected, both the flow rates in the pipes and the concordant surface elevations of the ponds will, in the state of equilibrium, become stabilized with slightly different values than those indicated by permissible storage from the ponds alone. Actual examples of the design of two pipe systems are presented in the succeeding sections.

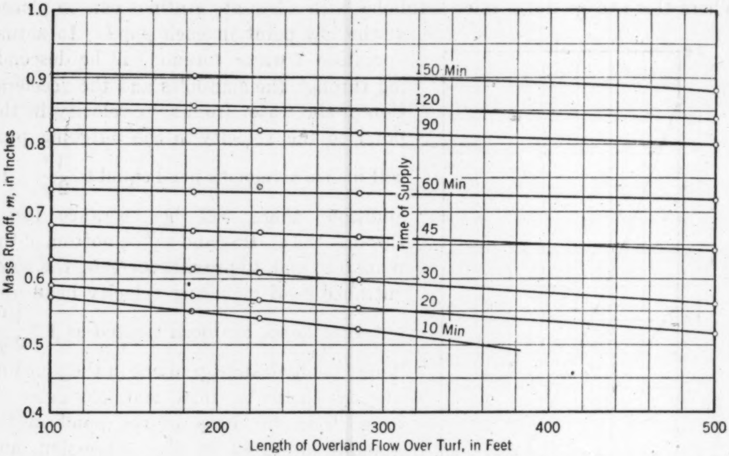


FIG. 11.—MASS RUNOFF FOR VARIOUS STORM DURATIONS IN TWO-YEAR FREQUENCY CYCLE ON TURF AREA WITH 30% IMPERVIOUSNESS

THREE-AREA PIPE SYSTEM WITH PONDS FLOATING ON THE LINE

In Fig. 9, the profile of a proposed sewer serving three-pond inlets is shown. Table 4 gives characteristics of each of the ponds and Table 3 tabulates the design computations. Table 3 shows in detail the computations for pond 3. Values of q_p and m, for various critical precipitation periods to 60 min, are taken from the basic design curves for the indicated 30% imperviousness and the 200-ft overland flow length. Of course, Q_p is the multiplication of q_p by the total acreage. Values of Q_o are computed as previously described and represent the outflow rates which would result if the permissible pondage of 0.068 in. was used. Similar values are shown for ponds 2 and 1, as are the accumulated indicated runoff rates for the three reaches of pipe.

The latter accumulated values show that, for this system, a 10-min supply is critical and therefore the capacities of the pipes should approximate them. If the 20-min values were used and smaller pipe capacities were furnished, when the 10-min rain occurs, the restricted capacities would result in greater storage requirements and consequent slightly higher pond levels.

Using the 10-min runoffs an initial selection of pipe sizes can be made based on differences in permissible pond levels. This procedure gives the hydraulic grades shown in Col. 8, Table 3; and, for $n = 0.013$, pipe sizes of 19 in., 18 in., and 17 in. shown in Col. 9; obviously the commercial 18-in. pipe is indicated.

As will be noted later, design is practically complete. The pipes may be laid at satisfactory construction grades and will operate under pressure with the ponds floating on the system at slightly different elevations than those originally chosen. However, the actual outlet discharge rate must be accurately checked; and it is also desirable to demonstrate as a matter of experience just how the ponds will adjust themselves.

Accordingly, the chosen pipe system is re-analyzed in detail. The first step involves recognition of the entrance head and other minor losses required for the individual ponds. Col. 11, Table 3, shows the approximate values of $1.5 \frac{V^2}{2g}$, which, when plotted down from the chosen pond levels, give the revised hydraulic grades shown in Col. 12. At this point it is possible to visualize very nearly the values that will enter into an equilibrium condition in the system. For line 1–2, the final grade will probably be a little greater and the quantity a little smaller, since an 18-in. pipe is being used instead of the 19-in. pipe ideally indicated. Line 2–3 will involve a somewhat greater quantity because of a slightly higher gradient. Since the quantity coming in from line 1–2 will be somewhat less than the original value, it is clear that the discharge from pond 2 will be appreciably more than it was originally.

The 18-in. pipe is somewhat large for the line 3-outlet. However, some of this disadvantage is offset by the lower available grade; and, consequently, there should be a slight increase in discharge to the outlet.

From the foregoing it seems probable that the equilibrium flow rates with ponds floating on the line would be about as shown in Fig. 9. The equilibrium conditions determined from the use of these revised discharges are also shown in Fig. 9 and indicate that pond 3 will stand 0.08 ft below the originally chosen value; pond 2, 0.10 ft below; and pond 1, 0.04 ft above. Ponds 2 and 3 are now in exact balance and pond 1 is within 0.01 ft of an exact balance.

Any check to within about 0.02 ft is considered satisfactory since the theoretical changes in pipe flows and pond levels accompanying such very small corrections would not affect the design in a practical way.

THREE-POND SYSTEM UNDER 30-MIN, TWO-YEAR FREQUENCY SUPPLY

Since the critical storm for the main trunk sewer into which the three-pond system discharges will be much longer than 10 min, it is necessary to investigate the rate of discharge that will occur for longer storms. Table 5 gives an analysis of the 30-min values for Q_p and m, with Q_o equal to the pipe capacities which were found to represent an equilibrium condition for the 10-min supply. The mass pondages and depths related to these outflow rates are given in Cols. 6 and 7.

From Table 5, it can be concluded, therefore, that, for storms of longer duration, the discharge from the chosen commercial-sized pipe system will be slightly less than the discharges for short storms, but less by so small an amount that the original value can be used in the design of the trunk system over a wide range of durations.

THREE-POND SYSTEM UNDER 30-MIN, TEN-YEAR FREQUENCY SUPPLY

Since the trunk sewer will carry an "off-field" flow based upon a ten-year frequency rainfall it is important to determine not only the rate of discharge from the three-pond system, but also the pond levels under the less frequent but greater rains. A preliminary calculation indicates that, for the higher supply rates of the ten-year rains, the 30-min duration produces the worst condition of ponding. This condition results because the longer rain permits overland flow more nearly to reach equilibrium, and the peak rates at the end of the rain are actually higher for the 30-min than for the 10-min duration. Also, the 30-min mass runoff is appreciably greater and hence uses more pondage volume.

TABLE 5.—ANALYSIS OF THREE-POND LINE FOR 30-MIN RAINFALL OF TWO-YEAR FREQUENCY

Pond No. (1)	Inflow, Q_p (cu ft per sec) (2)	Mass runoff, m (in.) (3)	Outflow, Q_o (cu ft per sec) (4)	Ratio, $\frac{Q_o}{Q_p}$ (5)	Mass pondage, p (in.) (6)	Pond depth (ft) (7)	Equilibrium outflow (cu ft per sec) (8)	Outflow below pond (cu ft per sec) (9)	Final pond depth (ft) (10)
1......	6.1	0.83	6.90	1.13	0	0	6.0	6.0	0
2......	6.0	0.61	2.37	0.39	0.23	0.95	3.10	9.10	0.81
3......	7.2	0.61	5.68	0.79	0.027	0.477	5.80	14.90	0.45
Total..	19.2	....	14.95	....	....	....	14.90	....	....

The values in Table 6 are closely representative of an equilibrium condition under the occurrence of a ten-year frequency, 30-min rain, and it may be concluded that the discharge under the more severe conditions would increase a negligible amount (about 2%). Therefore, in the design of the main trunk sewers to carry the runoff from a ten-year frequency rain on the "off-field" area, the contributions of the ponded laterals can be taken to be the same as those obtained from the two-year frequency analysis.

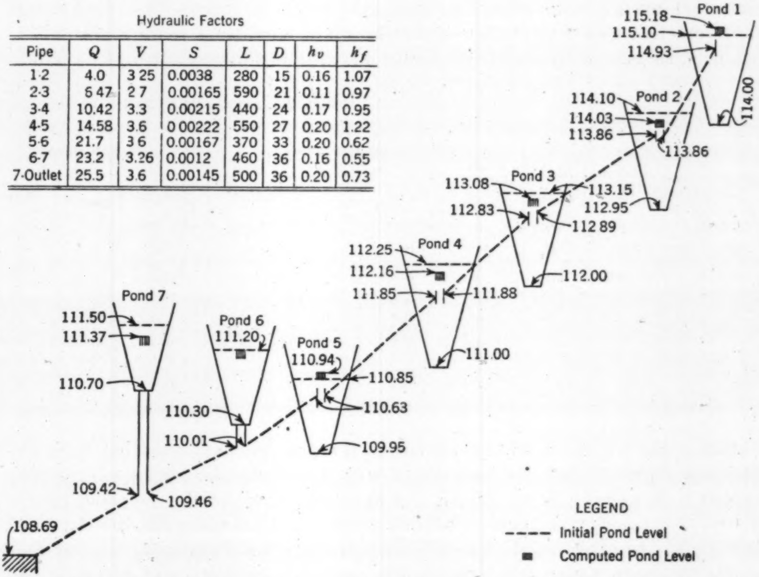
TABLE 6.—ANALYSIS OF THREE-POND LINE FOR 30-MIN RAINFALL OF TEN-YEAR FREQUENCY

Pond No. (1)	Supply rate (in. per hr) (2)	Area (acres) (3)	Peak inflow, q_p (in. per hr) (4)	Mass runoff, m (in.) (5)	Peak inflow, Q_p (cu ft per sec) (6)	Estimated outflow, Q_o (cu ft per sec) (7)	Ratio, $\frac{Q_o}{Q_p}$ (8)	Pondage (in.) (9)	Pond depth (ft) (10)
1........	2.60	4.04	2.6	1.23	10.50	6.9	0.65	0.15	0.80
2....	2.25	5.84	2.05	1.09	11.97	2.4	0.20	0.70	1.38
3........	2.25	6.90	2.05	1.09	14.14	6.0	0.43	0.35	1.10
3-outlet...	....	....	....	....	36.61	15.30	....	....	....

For the half-foot higher pond elevations due to a ten-year storm, the edge of the ponds will be about 25 ft from the nearest pavement edges. Since this condition will occur on the average only once in ten years, and then will persist for only one or two hours, it is clear that no appreciable intrusion of soil moisture under the pavement subgrade will result, and the condition is one which may be safely accepted.

SEVEN-POND SYSTEM WITH "NONFLOATING" PONDS

Fig. 12 is a condensed profile in which the upper five ponds descend in order of elevation, but ponds 6 and 7 ascend. Obviously, no hydraulic grade could permit these two ponds to float on the line. In actual design the pipes connecting the upper five ponds are treated in the same manner as the previously discussed three-pond system; open grated manholes can be installed at the low points of these ponding areas.



Hydraulic Factors

Pipe	Q	V	S	L	D	h_v	h_f
1-2	4.0	3.25	0.0038	280	15	0.16	1.07
2-3	6.47	2.7	0.00165	590	21	0.11	0.97
3-4	10.42	3.3	0.00215	440	24	0.17	0.95
4-5	14.58	3.6	0.00222	550	27	0.20	1.22
5-6	21.7	3.6	0.00167	370	33	0.20	0.62
6-7	23.2	3.26	0.0012	460	36	0.16	0.55
7-Outlet	25.5	3.6	0.00145	500	36	0.20	0.73

Pond 1
115.18
115.10
114.93
114.00
Pond 2
114.10
114.03
113.86
113.86
Pond 3
113.08
112.83
113.15
112.95
112.89
112.00
Pond 4
112.25
112.16
111.85
111.88
111.00
Pond 5
110.94
110.85
110.63
109.95
Pond 6
111.20
110.30
110.01
Pond 7
111.50
111.37
110.70
109.42
109.46
108.69
LEGEND
---- Initial Pond Level
■ Computed Pond Level

FIG. 12.—PROFILE AND HYDRAULIC FACTORS FOR SEVEN-POND SYSTEM

For ponds 6 and 7, similar inlet manholes will be constructed but they will be connected to the main system by small laterals so chosen in size and length as satisfactorily to throttle discharge from the ponds. Discharge from the ponds will then be determined in no direct way by the capacity of the pipes in the main system.

Determination of outlet rates as related to the permissible pondage for each of the areas is given in Table 7. It is interesting to note that, for pond 7, Q_o is given as 0 for the 10-min time—because the permissible pondage is greater than the whole of the mass rainfall. However, an outlet pipe is required to drain the pond and a specific discharge rate will exist during the rain. In

Table 8, it will be noted that a nominal value of 1 cu ft per sec has been initially assumed.

The totals at the outlet as shown in Table 7 indicate that the 10-min critical rainfall produces the largest runoff, and the design of the pipe system is worked out to handle these rates adequately (Col. 8, Table 7) without appreciably exceeding the desirable pond depths.

Table 8 summarizes the design information. Final equilibrium gradients and pond levels are shown in Fig. 12.

It must be understood that equilibrium flow conditions require not only a balanced scheme of hydraulic gradients but also outflow rates from the ponds (which make up the discharge values of the pipe system) resulting from the pondage depths which are in balance with the hydraulic grades of the pipe system. For example, at pond 5, the hydraulic grade of the outgoing pipe is 110.63, the velocity in the pipe is 3.6 ft per sec, and 1.5 times the velocity head is 0.31 ft. In other words, the elevation of the pond floating on the line at a static elevation which will provide the necessary velocity and entrance loss heads is 110.63 plus 0.31 or 110.94. From Table 7, it is noted that the peak flow rate to this pond is 13.5 cu ft per sec; and the outflow rate is 7.12 cu per sec (Table 8). For the pond to reduce the runoff rate by this amount, the volume of pondage must be 0.20 in., equivalent to a 0.99-ft depth. When this value is added to the elevation of the inlet grating, the surface of the pond checks the 110.94 elevation obtained, indicating that the pipe system and the detention pondage are in accord. As previously stated, any check to within about 0.02 ft is ordinarily considered satisfactory.

TABLE 7.—REDUCTION OF DISCHARGE

Pond No. (1)	Area (acres) (2)	Impervious (%) (3)	Flow length, L (ft) (4)	Mass pondage, p (in.) (5)	10-MIN SUPPLY			20-MIN SUPPLY		
					Inflow, Q_p (cu ft per sec) (6)	Runoff, m (in.) (7)	Outflow, Q_o (cu ft per sec) (8)	Inflow, Q_p (cu ft per sec) (9)	Runoff, m (in.) (10)	Outflow, Q_o (cu ft per sec) (11)
1......	7.51	46	180	0.27	12.5	0.73	4.9	11.3	0.75	4.5
2......	7.55	34	180	0.41	10.1	0.59	1.7	9.3	0.62	1.7
3......	9.42	32	220	0.28	11.0	0.57	3.2	10.3	0.59	3.2
4......	10.30	40	210	0.42	14.5	0.66	2.9	13.8	0.68	3.0
5......	5.60	60	100	0.15	13.5	0.89	8.0	12.0	0.91	7.1
6......	8.68	32	150	0.45	12.8	0.58	1.5	10.7	0.60	1.5
7......	11.2	27	265	0.51	10.3	0.50	0	10.0	0.53	0.2
Total..	....	....	....	....	84.7	....	22.2	77.4	....	21.2

To complete the design of this system, the connecting pipes for ponds 6 and 7 must be selected. For pond 6, the desired outflow of 1.5 cu ft per sec occurs when the pond level is 111.20 and the available fall from this water surface to the hydraulic gradient of the pipe line is 1.19 ft. If an 8-in. pipe is used, the velocity head plus entrance loss is 0.42 ft. Thus, 0.77 ft is available for friction loss; and, since the 8-in. pipe requires a gradient of 0.0152, the necessary resistance would accordingly be provided by a pipe connection 50.6 ft long. A similar analysis for a flow of 2.3 cu ft per sec from pond 7 results in an 8-in.

pipe connection 26.5 ft long. The computed connection lengths are quite important and must actually be used in the plan if outflows from these ponds are to be throttled to the amounts indicated.

RELATION OF FLOW TIME THROUGH PIPES TO OUTLET DISCHARGE RATE

In the foregoing discussion the flow through any reach of pipe has been taken as the sum of the inflows at all points upstream. This procedure is correct since all hydrographs resulting from restricted pond outflow will have flat tops of such considerable duration as to overlap. Fig. 13 shows this characteristic of the outflow hydrographs for the seven-pond system.

BY PONDAGE IN SEVEN-POND SYSTEM

30-MIN SUPPLY			45-MIN SUPPLY			60-MIN SUPPLY		
Inflow, Q_p (cu ft per sec) (12)	Runoff, m (in.) (13)	Outflow, Q_o (cu ft per sec) (14)	Inflow, Q_p (cu ft per sec) (15)	Runoff, m (in.) (16)	Outflow, Q_o (cu ft per sec) (17)	Inflow, Q_p (cu ft per sec) (18)	Runoff, m (in.) (19)	Outflow, Q_o (cu ft per sec) (20)
10.3	0.79	4.3	8.4	0.87	3.7	7.1	0.94	3.3
8.6	0.66	1.8	7.1	0.71	1.7	6.1	0.78	1.7
9.7	0.63	3.2	8.2	0.70	3.0	7.25	0.76	2.8
12.7	0.72	3.0	10.4	0.79	2.8	8.9	0.86	2.7
9.75	0.95	5.9	7.5	1.05	4.7	6.1	1.16	3.9
10.0	0.64	1.6	8.0	0.69	1.5	6.8	0.76	1.6
9.75	0.57	0.5	8.6	0.63	0.9	7.7	0.69	1.1
70.8	....	20.3	58.2	....	18.3	49.95	....	17.1

Table 9 gives pertinent data required for the preparation of the hydrographs. Cols. 2, 3, and 4 are taken from Tables 7 and 8. Col. 5 represents the time length of the rising side of the hydrograph, obtained by multiplying the 10-min time at which Q_p would have occurred if no pondage were available by the ratio of Q_o to Q_p. This relationship arises out of the properties of the assumed triangular hydrograph.

Col. 6 is the average time length of the hydrograph from the center of the rising side to the center of the recession. It is obtained by changing the values of m in Col. 4 to cubic feet and then dividing by Q_o. Col. 7 gives the times of flow in each reach of pipe. These times between ponds are based on length and average velocity as it is believed that for these small lines the higher flow rate of wave velocity can be ignored, particularly inasmuch as the summation would not be affected by the changes that refinement would produce.

Direct addition of the ordinates of all the hydrographs gives the hydrograph of discharge at the outlet for the system. The exact shape of the rising side is necessarily approximate since the discharges are based on the critical hydraulic grades for maximum discharge; and, during the rising side, partial flows, lower hydraulic grades, and somewhat different discharges from the ponds can take place. The error in ignoring this condition is quite small and affects only the shape of the rising side, not the peak values.

For this seven-pond system with a 16-min pipe flow time, under the application of the two-year, 10-min rain, the maximum discharge will be reached in about 10 min and will persist for somewhat longer than 40 min from the beginning of the critical 10-min rain. Particularly noteworthy is the fact that for a period of more than 20 min all hydrographs are flat topped and their values can be added directly.

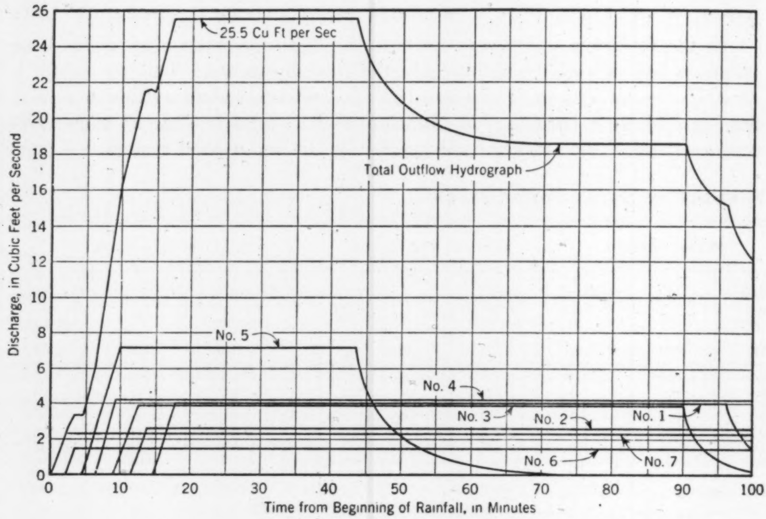


FIG. 13.—SUMMATION OF HYDROGRAPHS FOR SEVEN-POND SYSTEM

For the three-pond example, it was shown that there would be no appreciable difference in the discharge rate at the outlet under the occurrence of the

TABLE 8.—SUMMARY OF DESIGN DATA FOR SEVEN-POND SYSTEM

Pond or pipe No.	Initial pond elevation	DISCHARGE (CU FT PER SEC)		Length (ft)	Slope (ft per ft)	Pipe diameter (in.)	DISCHARGE (CU FT PER SEC)		Equilibrium pond elevation
		Initial	Total				Revised	Equilibrium	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	115.10	4.9	....	....	....	....	3.8	4.0	115.18
1-2......	....	....	4.9	280	0.0036	15	3.8	4.0	....
2	114.10	1.7	....	....	....	....	2.8	2.47	114.03
2-3......	....	....	6.6	590	0.0016	21	6.6	6.47	....
3	113.15	3.2	....	....	....	....	3.0	3.95	113.08
3-4......	....	....	9.8	440	0.0020	24	9.6	10.42	....
4	112.25	2.9	....	....	....	....	5.9	4.16	112.16
4-5......	....	....	12.7	550	0.0025	27	15.5	14.58	....
5	110.85	8.0	....	....	....	....	5.5	7.12	110.94
5-6......	....	....	20.7	370	0.0016[a]	33	21.0	21.70	....
6	111.20	1.5	....	....	....	....	....	1.50	111.20
6-7......	....	....	22.2	460	0.0016[a]	36	27.0	23.2	....
7	111.50	1.0[b]	....	....	....	....	....	2.3	....
7-outlet ..	....	....	23.2	500	0.0016[a]	36	27.0	25.5	111.37
Outlet....	108.7	....	....	....	....	....	....	....	....

[a] Average. [b] Actual value, 0.0.

two-year, 10-min rain, the two-year, 30-min rain, or the ten-year, 30-min rain. Plotting similar to that of Fig. 13 shows that both the first and last of these rains, which represent the most severe conditions, have long flat topped hydrographs extending for 20 min or more. The two-year, 30-min rain, because two of the inlets use very little pondage, comes almost to a peak, but still has a flat top of about 5-min duration.

TABLE 9.—TIME CHARACTERISTICS OF OUTFLOW HYDROGRAPHS FOR SEVEN-POND SYSTEM

Pond No. (1)	Inflow, Q_p (cu ft per sec) (2)	Outflow, Q_o (cu ft per sec) (3)	Mass run-off, m (in.) (4)	Inflow time, t_2 (min) (5)	Average hydrograph time, t_{avg} (min) (6)	Flow time (min) (7)
1...........	12.5	4.0	0.73	3.2	83	1.4
2...........	10.1	2.47	0.59	2.4	109	3.6
3...........	11.0	3.95	0.57	3.6	82	2.2
4...........	14.5	4.16	0.66	2.9	99	2.5
5...........	13.5	7.12	0.89	5.3	42	1.7
6...........	12.8	1.5	0.58	1.2	203	2.4
7...........	10.3	2.3	0.50	2.2	147	2.3
Outlet......	84.7	25.5	....	....	....	16.1

SUMMARY

Where surface drainage of airports permits the development and utilization of pondage in the flat turfed areas, accepted hydrologic knowledge concerning retention, infiltration, and overland flow is directly applicable. The development of supply rates requires an analysis of actual storms and their related infiltration values so selected as to result in surface supply rates produced, on the average, by the rains which give the critical average rates of the desired frequency. For facility of design, hydrographs of inflow to ponds can be considered triangular in shape.

The pondage resulting from the very flat grades required in airports can be allowed to "float" on the line and has the fortuitous characteristic of very small depth changes (and consequent minor differences in actual hydraulic gradient between ponds) for relatively large pondage volume changes. Once commercial pipe sizes are chosen for a definite duration and frequency, their capacities vary insignificantly for rains of longer duration or greater frequency. Pondage so flattens the hydrograph that direct addition of pond outflow rates can be used, obviating the laborious offsetting of each outflow hydrograph.

ACKNOWLEDGMENTS

This paper resulted from studies initiated by W. W. Horner, Past-President, ASCE, and had the benefit of his collaboration and constructive criticism. The studies were part of a "Report on Basis of Design for Field and Off-Field Drainage Systems for Lambert-St. Louis Municipal Airport" submitted to Milton M. Kinsey, president of the Board of Public Service of St. Louis, and reviewed by Guy Brown, M. ASCE, chief engineer, and E. J. A. Gain, Assoc. M. ASCE, engineer, Division of Sewers and Paving, Board of Public Service of St. Louis,

DISCUSSION

H. M. Williams,[11] **Assoc. M. ASCE.**—With the increasing tendency for airport planning, design, and construction to become major fields of endeavor for the private engineer, there is a definite need for such papers as Mr. Jens' excellent contribution on airport drainage, which crystallizes recent technical developments. Certain phases of the paper, however, do excite further discussion.

The assertion that airfield drainage areas have "accurately determinable characteristics" (see "General Discussion of Method") should be viewed with discretion. Such features may be accurately determined on the drafting table, but when translated to field construction and field maintenance conditions they frequently suffer major divergencies from the design plans. In recent field drainage investigations by the Corps of Engineers, United States Army, inability to define drainage boundaries permanently and accurately has been encountered—even where low dikes or artificial drainage divides were constructed with a degree of engineering control not usually possible in normal construction operations. Inability to maintain constant drainage areas usually resulted from vehicular or airplane traffic over the turfed areas, or from the gradual consolidation and deterioration of the small drainage divides. Rutting by wheeled traffic has also been observed to cause concentration or channeling of flow with a resultant change in the runoff characteristics of the drainage area. Erosion, in some cases, resulted in increasing or decreasing the design drainage area. In one unusual condition a turfed airfield test area about 75 acres in extent frequently indicated runoff in excess of 100%, despite careful and extensive investigation and examination of automatic measuring devices, drainage divides, topography, soils, and ground-water conditions. In connection with paved surfaces, several instances have been noted where an apparent increase in drainage area resulted from wind driving surface water over low drainage divides. The foregoing data are presented, not to discredit the desirable design refinements suggested in this excellent paper, but rather to remind the reader of the wide gap that frequently exists between the design and the actual construction and functioning of field installations. It is suggested that use of reasonable safety factors, based on sound engineering judgment and experience, must still be provided, despite the niceties of design justified by recent hydrologic research.

Reference is made to the method used in several recent airport drainage papers (including the author's) for determining the outflow peak discharge by subtracting the available storage from the peak of the inflow hydrograph. The method assumes a constant head-discharge relation on the drain inlet and results in an outflow hydrograph with an extended flat peak (see Fig. 8). This assumption is an acknowledged approximation that has hitherto been considered not to introduce serious error. However, current studies in the Airfields Branch Office of the Chief of Engineers on design methods for combination drains have indicated that, when a varying head-discharge relation more nearly reflecting field conditions is used, a considerable increase in the peak outflow may be expected over that obtained by the author's method.

---

[11] Senior Engr., U. S. Corps of Engrs., Washington, D. C.

In the case of the combination drain it was noted that the peak discharge obtained by routing with the variable head-discharge relation was approximately 1.5 times the average discharge obtained by the author's method for conditions where the ratio of volume of storage to supply was greater than 30%. Between storage supply ratios of 30% and zero, the ratio of the routed peak to the average peak apparently decreased from 1.5 to unity as a straight-line relation. In the case of a typical field inlet where the flow is collected at a point rather than along a length of combination drains, it is believed that the foregoing ratio of 1.5 reduces to approximately 1.3. This relation conforms, in general, with the results of other routing studies.[12]

The writer is not fully convinced of the necessity of using storm patterns in the design of airfield drainage systems. Inability to predetermine infiltration values accurately, the "leveling effect" on the hydrograph of a relatively small amount of storage, and the major divergencies between the storm pattern for a given occurrence and the average pattern, all contribute to this conviction.

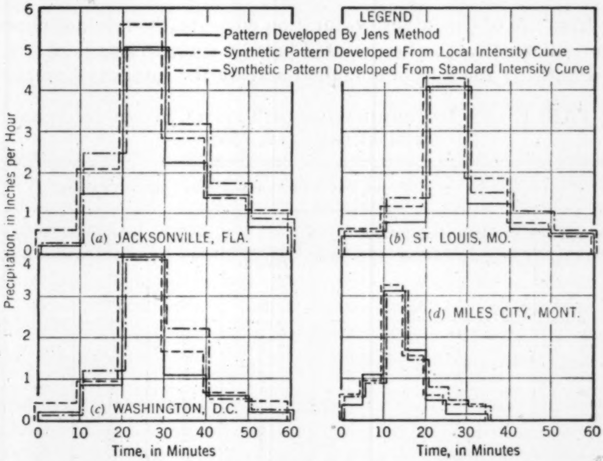


FIG. 14.—COMPARISON OF RAINFALL PATTERNS DEVELOPED BY VARIOUS METHODS FOR SELECTED LOCALITIES

However, when justification for using storm patterns is present, the amount of work required in developing the pattern may discourage its use, particularly in the case of small projects with limited engineering funds. In such cases it appears possible to approximate the storm pattern with a minimum amount of effort by an analysis of the rainfall-duration relation for the locality. This approach is suggested by the similarity between the procedure used by the author to secure intensities for storm patterns and the normal procedure for determining the intensity-duration curve. Disregarding pattern arrangement and considering only intensities, the author's method indicates the design frequency only for the selected critical period (other periods having intensities generally less than design frequency), whereas the use of the intensity-duration curve will give design frequency for all periods. However, the differences resulting from use of the intensity-duration curve method do not appear to be serious, as indicated by the four rainfall pattern comparisons shown in Fig. 14, developed both by the author's method and by the intensity-duration curve.

---

[12] "Hydraulics of Miami Flood Control Project," *Technical Report*, Miami Conservancy Dist., Dayton, Ohio, Pt. VII, p. 91.

TABLE 10.—Typical Computations for Storm Pattern Intensities, Based on Intensity-Duration Curve Developed from 10-Year Record at Miles City, Mont.

Time (min.) (1)	Intensity[a] (in. per hr) (2)	Total rainfall (in.) (3)	Rainfall increment (in.) (4)	Intensity for 5-min period (in. per hr) (5)
5	3.12	0.26	0.26	3.12
10	2.30	0.38	0.12	1.44
15	1.85	0.46	0.08	0.96
20	1.60	0.53	0.07	0.84
25	1.39	0.58	0.05	0.60
30	1.24	0.62	0.04	0.48
35	1.12	0.65	0.03	0.36
40	1.02	0.68	0.03	0.36

[a] From intensity-duration curve.

TABLE 11.—Determination of Typical Storm Pattern for Miles City, Mont.

Date of storm	Storm Pattern Arrangement (5-Min Periods)							
	Preceding Periods			Maximum period	Subsequent Periods			
	3	2	1		1	2	3	4
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
6/10/25	....	....	2	1	3	....	....	....
6/15/25	....	....	3	1	2	....	....	....
8/10/25	....	....	3	1	2	....	....	....
8/28/25	....	....	3	1	2	....	....	....
8/18/26	....	....	4	1	5	3	2	6
7/8/30	....	....	3	1	2	4	....	....
9/8/30	....	....	3	1	2	4	5	....
4/21/32	3	2	4	1	6	5	....	....
6/7/32	....	....	....	1	2	3	....	....
7/19/32	6	8	7	1	2	3	4	5
8/16/32	....	3	4	1	2	5	7	6
7/10/35	....	4	3	1	2	5	6	....
6/6/28	....	....	....	1	2	....	....	....
7/28/28	....	....	3	1	2	4	5	....
Total	9	17	42	14	36	36	29	17
No. of items	2	4	12	14	14	9	6	3
Total No. of items	4.5	4.2	3.5	1	2.6	4.0	4.8	5.7
Pattern arrangement	6	5	3	1	2	4	7	8

In Fig. 14, it should be noted that the local intensity curves are based on 10-year records except for St. Louis, Mo., where Fig. 2 was used. A typical computation for the synthesized pattern is indicated in Table 10.

The order in which the various intensities are to be assembled into a complete pattern can be obtained by a brief study of the relative magnitudes of the component parts of a few typical storm patterns. The procedure is illustrated in Table 11 for Miles City, Mont. It is interesting to note the similarity of the pattern arrangement obtained in Fig. 14, regardless of the widely separated geographical locations involved. Also of interest is the reasonable pattern obtained by use of the standard rainfall intensity-duration curves proposed by G. A. Hathaway,[4] M. ASCE, with the resultant elimination of the work necessary to establish the intensity-duration curve.

It should be noted that the rainfall data used for this study were taken from the annual weather bureau reports for the period prior to 1936. The method of reporting excessive intensities used since 1936 makes the use of these later records exceedingly difficult for developing storm patterns by the author's method. However, it should be noted that intensity-duration curves developed from records prior to 1936 are normally low by from 5% to 10%, as pointed out by the late D. L. Yarnell,[13] M. ASCE.

W. I. HICKS,[14] ESQ.—In this exposition of the fundamental principles used in drainage design for the Lambert-Municipal Airport at St. Louis (Mo.), it should be noted that the computations are based on data for a specific location, and that the data probably will be altered elsewhere. The method is based on utilization of a depressed turfed area, lying between paved runways, for the reduction of runoff volume by soil infiltration and for the reduction of runoff intensity by ponding. The interposition of open soil between the paved area and the drainage inlet allows the maximum use of these reductive factors. The paper shows conclusively that, when turfed impounding areas are available, a substantial saving in cost can be effected in drainage construction.

The method of developing the supply curve (Table 1) departs from the conventional analysis, which would involve computation of net rainfall for the significant rainstorms of record, development of a probability series for the different time durations, and determination of the predominant pattern of net rainfall intensities. The author has used a method of averaging gross and net rainfall of storms selected so as to make the average of the gross rainfall intensity identical with the two-year rainfall-duration curve in Fig. 2. The frequency of the selected values of gross rainfall intensity ranges from about ten years to less than one year (as determined by W. W. Horner, Past-President, ASCE, and F. L. Flynt,[15] Assoc. M. ASCE).

Comparison of the net rainfall patterns of the individual storms in Table 1 with the uniform rate pattern shows wide divergence with durations of individual storms varying from 5 min to 20 min within the significant 20-min duration. If Eq. 2 is altered in form for a supply duration of 20 min, a length of 185 ft, a slope of 1.75%, and a 0% imperviousness, and if it is set up for actual durations of 20 min, 15 min, 10 min, and 5 min, the results are, for the respective durations,

$$q = \sigma \tanh^2 (0.795\, \sigma^{0.5}) \qquad (7a)$$

$$q = 1.33\, \sigma \tanh^2 (0.689\, \sigma^{0.5}) \qquad (7b)$$

$$q = 2.00\, \sigma \tanh^2 (0.561\, \sigma^{0.5}) \qquad (7c)$$

---

[13] "Rainfall Intensity-Frequency Data," by D. L. Yarnell, *Miscellaneous Publication No. 204*, U.S.D.A., Washington, D. C., August, 1935.

[14] Civ. Engr., Storm Drain Div., Dept. of Public Works, Bureau of Eng., Los Angeles, Calif.

[15] "Relation Between Rainfall and Run-Off from Small Urban Areas," by W. W. Horner and F. L. Flynt, *Transactions*, ASCE, Vol. 101, 1936, p. 173, Fig. 23.

and

$$q = 4.00\,\sigma \tanh^2 (0.398\,\sigma^{0.5}) \qquad (7d)$$

In Eqs. 7, the value of σ used is that for the 20-min duration of supply.

For a 20-min duration $\sigma = 1.0$ in. per hr, the values of q for actual time durations of 20 min, 15 min, 10 min, and 5 min, respectively, are 0.435, 0.480, 0.524, and 0.576 in. per hr, with added amounts explained by the fact that the supply ceases before equilibrium flow has been established; and the excess detention above equilibrium detention for given values of q causes the intensity of runoff to increase for a short time after the supply ceases. This increase is greater for small durations of supply than for large ones. The author apparently has used the method of G. A. Hathaway,[4] M. ASCE, in deriving the values in Table 2 and Fig. 10, with possibly some alteration because of infiltration after cessation of supply. For a 20-min duration $\sigma = 2.0$ in. per hr; similar values of q (before the addition in intensity because of surplus detention at the end of supply) are 1.30, 1.50, 1.76, and 2.14 in. per hr.

These analyses show a fairly wide divergence in values of q derived from a given volume of net rainfall when its duration varies from 20 min to 5 min, and suggest the alternative of basing the conception of duration on net rainfall rather than on gross rainfall. The effect of the change in analysis would be most significant in cases of summing hydrographs where no deliberate ponding is possible.

The infiltration curves in Fig. 4 are reasonably safe for all soils in which artificial compaction does not occur; it might be assumed that an open sandy soil would dispose of all rainfall by infiltration, with some temporary ponding for heavy storms, if it were possible to prevent clogging of the sand by the washing in of fine detritus and oil. The shape of the curve (rate of infiltration decreasing with time), the small value of antecedent precipitation for the heavy thunderstorms prevalent in St. Louis, and the frequent occurrence of the "advanced" type of storm make the tendency toward uniform supply rates greater in St. Louis than in Los Angeles (Calif.). There the heavy antecedent precipitation and the "delayed" type of storm reduce the infiltration rate to practical uniformity, particularly for the high intensities used in design. However, Table 1 indicates no universal trend toward uniformity.

In studying the derivation of the triangular hydrograph in Figs. 7 and 8, the writer derived an expression for the time required to secure practical equilibrium (t_e) in terms of average equilibrium detention (δ_a) and supply intensity (σ). When the parenthetical expressions in Eqs. 2 and 3 reach a value of 2.45, the square of the hyperbolic tangent is approximately 0.97, and practical equilibrium has been attained

$$t_e = \frac{2.45 \times 60\, l^{0.5}}{\sqrt{3{,}520}\, K\, S^{0.25}\, \sigma^{0.5}} = \frac{2.475\, l^{0.5}}{K\, S^{0.25}\, \sigma^{0.5}} \qquad (8)$$

Using the relation, $\delta_a = \delta/K$, and a value of 2.45 for the parenthetical expression, Eqs. 3 and 8 are combined to produce

$$t_e = \frac{150\,\delta_a}{\sigma} \qquad (9)$$

The writer has conducted a series of tests of overland flow on a number of surfaces, including clipped sod.[16] An examination of the data discloses, for clipped sod, the relation,

$$t_e = \frac{96\,\delta_a}{\sigma} \qquad \text{.........................(10)}$$

A comparison of the triangular intensity hydrograph with the one derived by Eq. 2 shows that, when the time duration of supply is less than 0.6 of the equilibrium time (as defined by Eq. 9), the intensities on the ascending leg of the triangular hydrograph exceed those of the hydrograph in Eq. 2, and the triangular hydrograph is overly safe. When the duration of supply is about 0.6 of the equilibrium time, the straight leg of the triangular hydrograph averages with the curved line from Eq. 2; and when the duration of supply equals and exceeds the time of equilibrium, the triangular hydrograph becomes increasingly deficient on the ascending leg.

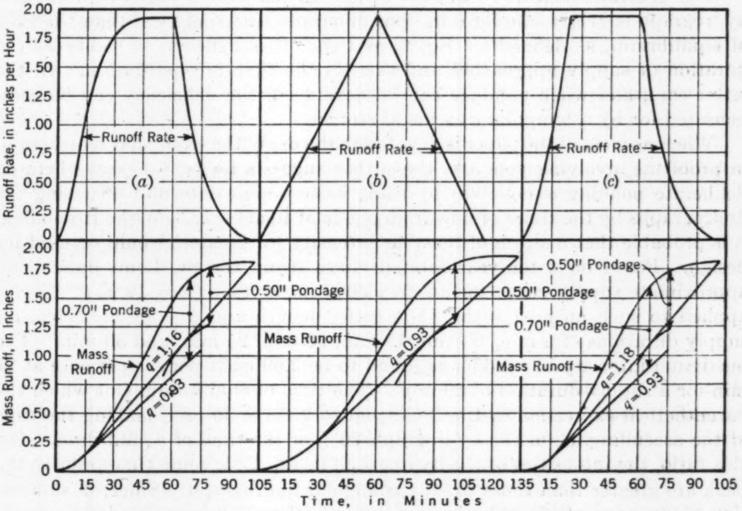


FIG. 15.—COMPARISON OF HYDROGRAPHS ON 50% IMPERVIOUS AREA WITH TURFED LENGTH OF 185 FT AND SLOPE OF 1.75%

For examining the effect of this divergence on the outflow curve under ponding, hydrographs were constructed for an area 50% impervious with a turfed length of 185 ft, a slope of 1.75%, and an approximate ten-year uniform supply curve for 60-min duration (3.86 in. per hr on the turfed area) using: (*a*) Eq. 2, (*b*) the triangular hydrograph derived from Eq. 2, and (*c*) Eq. 10 and the value of δ_a derived from Eq. 3.

These hydrographs are delineated in Fig. 15; in the top row are the intensity hydrographs and in the bottom row, the mass hydrographs. The intensity hydrograph in Fig. 15(*c*) is derived from the mass hydrograph, the reason being that the experimental runoff data for this case were measured volumetrically, not as intensities. The close agreement of the mass runoff curves in Figs. 15(*a*) and 15(*c*) furnishes experimental confirmation for Eq. 2, in so far as storage problems are concerned.

---

[16] Discussion by W. I. Hicks, of "Surface Runoff Determination from Rainfall Without Using Coefficients," by W. W. Horner and S. W. Jens, *Transactions*, ASCE, Vol. 107, 1942, pp. 1099, et seq.

An allowable pondage of 0.50 in. was assumed and measured downward from the mass hydrographs to form short arcs. The outflow intensity is measured as the slope of a straight line tangent to the inflow hydrograph on the accretion side and also to the short arc. The outflow intensity for the triangular hydrograph in Fig. 15(*b*) is 0.93 in. per hr, approximately 80% of the average intensity for the hydrographs in Figs. 15(*a*) and 15(*c*). The storage volume required to make the outflow intensities in Figs. 15(*a*) and 15(*c*) equal to the intensity of the hydrograph in Fig. 15(*b*) is approximately 0.70 in., or a 40% increase; the increase in depth of pondage is 12%.

The general conclusion from the foregoing analyses is that the triangular hydrograph is overly safe for durations of supply materially less than the time of equilibrium, as defined by Eq. 9, and that it is deficient in safety as the duration of supply approaches and exceeds the time of equilibrium. In the latter case, and when pondage can be developed, the deficiency can be compensated for by a temporary excess of storage.

Whereas most of the paper is devoted to the use of the triangular hydrograph in problems involving pondage, the author suggests its use (in cases where no deliberate ponding is possible) by using basic design data and offsetting the hydrographs by the times of flow from one inlet to another. Without ponding, it is probable that a storm of ten-year intensity (or greater) would be used for design. Based on a ten-year rainfall curve (from the St. Louis data[15]), an approximate supply curve for an area 50% impervious ($\sigma = 19.05\, t^{-0.38}$) was applied to such an area with 185-ft turfed length and a slope of 1.75% for supply durations of 5 min, 10 min, 15 min, 20 min, 25 min, and 30 min. The maximum intensity of runoff was found to be 2.69 cu ft per sec per acre at 21 min for a supply duration of 20 min. The time of equilibrium (for which the parenthetical expression in Eq. 2 is equal to 2.45) is 30 min, making the time of the ascending leg of the hydrograph 70% of the time of equilibrium. For this ratio, the intensities of the hydrograph in Eq. 2 for some time prior to the peak are greater than those for the triangular hydrograph. Since, in summation, the process of offsetting hydrographs for time of flow in the drain usually requires adding the peak of the local hydrograph to the ascending leg of the drain hydrograph, use of the triangular hydrograph can lead to an unsafe cumulative error. Using a sparse turf cover, which will diminish the time of equilibrium, would accentuate such an error.

In Figs. 9 and 12 the summations of entrance head (1.5 h_v) and friction head (h_f) in the tabulations are in excess of the difference between the surface elevations of the upper pond and of the pool into which the pipe system empties. This is caused by the measurement of the entrance head plus the friction head for a given length of pipe from the surface of the upstream pond to a point below the surface of the downstream pond. In Fig. 9 the entrance head is 0.34 ft and the friction head is 2.16 ft; these are measured downward from the upper pond at El. 128.34 to El. 125.84 in the downstream pond. If the full velocity head of 0.24 ft were recovered, the lower pond could not rise above

El. 126.08, whereas El. 126.30 is used for the next reach of pipe. The elevations on the two profiles have been computed precisely and it would be helpful if the author were to elaborate on his computation of the hydraulic grade lines.

To summarize, the writer suggests the use of an alternative method of determining the frequency and intensity pattern of the net rainfall, and the exclusion of the triangular hydrograph in cases where ponding cannot be used. The flexibility of the ponding method of design allows the use of the triangular hydrograph with its comparative ease of computation. The fundamental conception of the full use of infiltration and pondage in reducing the cost of drainage structures cannot be questioned.

E. J. A. Gain,[17] M. ASCE.—The methods given by the author were used by him to design the drainage system for the expansion of the Lambert-Municipal Airport at St. Louis, Mo., and were presented for approval in the unpublished "Report on Basis of Design for the Field and Off-Field Drainage Systems." Since the proposed methods were relatively new and had no precedent in the sewer design practice of the City of St. Louis, they were examined critically and reviewed with respect to their fundamental equations, assumptions, basic data, and their proposed uses.

In his recommendations to the chief engineer of sewers and paving (Guy Brown, M. ASCE) the writer presented the following summary to indicate the author's basis of design and its departure from current practice:

1. It is proposed to design drainage facilities for the runoff produced at the following rainfall rates—

Terrain	Frequency
Field areas	2 years
Terminal area	10 years
Off-field area	10 years

2. The rainfall available for storm runoff is termed the "supply" and is equal to rainfall minus losses due to soil infiltration, depression storage, and surface retention on plant cover. Variation of rainfall rates within the period of assumed uniform design rate and the effect of preceding rainfall on infiltration capacities are recognized factors affecting the supply.

3. Supply develops into surface runoff flowing to a collecting pond or inlet, or to the margin of a collecting stream or channel according to equations of surface flow developed by the late Robert E. Horton, M. ASCE. The effects of (*a*) length of overland flow, (*b*) slope, and (*c*) retention are recognized factors in the equation.

4. Variations in the supply rates are reflected in variable runoff rates which are reduced in intensity during the period of overland flow by the retarding effects of overland surfaces and by the storage capacity of the flowing sheet of runoff.

5. These variations in rates are further modified and their fluctuations reduced during the period of stream or channel flow by the storage capacity of the channel so that, by the time that runoff reaches the entrance of the inlet or pipe system, the large fluctuations in rainfall rates have been reduced and

---

[17] Engr. in Chg. of Sewer Design, Board of Public Service, Dept. of the President, St. Louis, Mo.

leveled to rates which are substantially uniform for considerable lengths of time.

6. The proposed basis of design as outlined in paragraphs from 1 to 5 is based on the work of Mr. Horton.[6,18,19] Experimental work by C. F. Izzard, Assoc. M. ASCE, of the United States Public Roads Administration, suggests that Mr. Horton's equations do not hold strictly true for all conditions of flow and that further research is necessary. Nevertheless, the Horton equations are the best that are available and the method is much better than any of the former methods in use. It differs essentially from older methods in current use in that the transformation of rainfall of a given rate to design flow of a determined rate follows the same course as that which occurs in nature, with recognition being given to all the factors modifying that course (within the bounds of existing knowledge, data, and economic expediency). The older methods usually simplified the transformation by using a given percentage of the rainfall rate to give the design unit flow. Slope, length of flow, shape of drainage area, and kind of surface cover were acknowledged only in a general way to cover only a few general cases.

7. The proposed design of field drainage embodies the use of storage ponds in interrunway areas, thereby enabling a reduction in size of required pipe lines for design flow and also providing considerable reserve capacity for the retention of excessive runoff without greatly exceeding design limits. By their use, lines designed for a runoff from a two-year storm can accommodate a ten-year storm with only a slight rise in pond elevation.

(Steps 8 to 10 are omitted from this discussion.)

11. The basis of design proposed in this paper follows more closely the conditions that actually occur in nature than any system used before because recognition is given to all the factors pertaining to them which are known or recognized. It is in keeping with the most recent advances in hydrology and their application to design, reducing the range of guesswork and allowances for the unknown or uncertain factors entering into a design; and it makes possible a design that is rationally expected to function more closely as designed. It thus makes possible a design that will be more economical for a given set of design assumptions than any other so far proposed.

Viewed as a whole, the author's methods are an excellent example of the application of two factors in drainage which, until about 1938, received little attention—namely:

a. Determination of realistic values of runoff from a storm of a given frequency by taking into account the infiltration, surface slope, kind of surface, depression storage, and surface storage of the flowing sheet; and

b. Allowance of storage whether in sheet flow, ponds, scuppers, channels, or sewers in determining the design capacities of pipe lines or storm channels. His work is a distinct contribution in furthering the use of more refined methods of drainage design.

---

[18] "Analysis of the Hydrograph," by Robert E. Horton, *Publication No. 101*, Soils Science Soc. of America, 1935, p. 73.

[19] "Analysis of Runoff Plat Experiments with Varying Infiltration Capacity," by R. E. Horton, *Transactions*, Am. Geophysical Union, Pt. IV, 1939, pp. 693–711.

In application, the methods require a high degree of completion of designed surface elevations. To know the size of subdrainage areas is not sufficient. This must be supplemented by additional information concerning the site, ground slopes, ponding areas, type of surface, and allowable distance and elevations. Changes in the latter can affect the design even if there is no change in the size of the subarea. The necessary computations are considerably increased in complexity, in detail, and in amount. This is a condition that usually characterizes refined methods in design, which are justified by the following results—(*a*) closer conformity of predicted to actual functioning; (*b*) greater economy in initial cost and in operation; and (*c*) the intangible satisfaction of the designer with the results of his work. Although, in many cases encountered in municipal engineering, the increased work involved may be a deterrent to its wider application (and so may help to retain the wide use of the more easily applied but less accurate so-called rational method), the increased accuracy of the newer method can be combined with the greater ease of application of the rational method by using the former to determine a convenient and full range of P-values in the well-known formula, $Q = A\ P\ I$, for the many typical conditions common in the larger municipalities. In this equation P is the ratio of runoff to rainfall for given conditions.

The use of designed ponds has special application to airport drainage and relatively little to municipal practice, except possibly for large park areas under limited conditions. The saving in cost of drainage by their use can be determined by noting in Table 7 the considerable difference between inflow rates and outflow rates. This indicates that, with ponding, pipe sizes can be reduced to accommodate storm flows that average from one fourth to one third of the values accumulated without ponding. Although individual values vary considerably from the average for the entire line, they indicate clearly the importance of using ponding whenever it is feasible.

There is a great need for necessary research to supplement the relatively small amount of available data required for a satisfactory application of the newer methods, and to confirm the data that have already been obtained. Available records, particularly of rainfall, need to be retabulated to put them in the detailed form that is needed for newer uses. Fortunately, the work that is necessary is not wasted since it is also applicable to other related problems. Data concerning local rainfall patterns, kinds of soil, and infiltration values for different soils and different cover are required; but, except for rainfall data, such information is not generally available, even for the larger centers of population where the need is probably greatest. The information that is available leaves much to be desired in necessary data.

E. F. Brater,[20] Assoc. M. ASCE.—This welcome addition to the literature of surface runoff gives in detail a practical technique for selecting a design rainfall pattern and infiltration capacity curves, as well as methods for determining the benefits of pondage in reducing peak discharge—all of which should prove useful to the designing engineer. Mr. Jens suggests that the form of the hydrograph may be determined by Eq. 2, which is a formula developed by the late R. E. Horton,[6,21] M. ASCE. The writer recognizes that the validity of this formula is not of major consequence in the over-all picture of Mr. Jens' paper. However, there are such definite limitations inherent in this expression that a discussion of them seems to be warranted. The writer wishes to outline briefly Mr. Horton's derivation of this formula and then mention what seem to be its more serious weaknesses. The notation of the original derivation will be changed wherever necessary to agree with that of the author.

---

[20] Associate Prof. of Civ. Eng., Univ. of Michigan, Ann Arbor, Mich.

The storage equation is written in differential form as

$$\sigma\, dt - q\, dt = d\delta_{av} \qquad (11)$$

It is then assumed that the relationship between discharge and depth of detention at the lower margin is

$$q = K_s\, \delta^m \qquad (12)$$

and that

$$\delta = K\, \delta_{avg} \qquad (13)$$

in which K_s, m, and K are constants for any given runoff plot. The value of q from Eq. 12 and the value of δ from Eq. 13 may now be substituted in Eq. 11, which gives

$$dt = \frac{1}{K} \frac{d\delta}{(\sigma - K\, \delta^m)} \qquad (14)$$

If σ is held constant and m is taken as 2, Eq. 14 may be integrated to obtain

$$t = \frac{1}{K\, K_s\, 2 \sqrt{\frac{\sigma}{K_s}}} \log \left[\frac{\sqrt{\frac{\sigma}{K_s}} + \delta}{\sqrt{\frac{\sigma}{K_s}} - \delta} \right] + C \qquad (15)$$

The value of C is found to be zero by assuming that $\delta = 0$ when $t = 0$. Eq. 15 may then be solved for δ, giving

$$\delta = \sqrt{\frac{\sigma}{K_s}} \frac{(e^B - 1)}{(e^B + 1)} \qquad (16)$$

in which, for simplification of typography,

$$B = 2\, K \sqrt{K_s\, \sigma} \qquad (17)$$

If Eq. 16 is introduced into Eq. 12,

$$q = \sigma \frac{(e^B - 1)^2}{(e^B + 1)^2} \qquad (18)$$

Eq. 18 may also be written in the form,

$$q = \sigma \tanh^2 2\, K \sqrt{\sigma\, K_s}\, t \qquad (19)$$

Mr. Horton then assumed that K_s could be evaluated from the Manning formula,

$$V = \frac{1.486}{n} r^{2/3} s^{1/2} \qquad (20)$$

---

[21] "Hydrologic Interrelations of Water and Soils," by R. E. Horton, *Proceedings*, Soils Science Soc. of America, Vol. 1, 1937, p. 401.

Assuming that r may be replaced by $\delta/12$, and multiplying both sides of Eq. 20 by δ, the following expression for discharge per foot width of channel is obtained:

$$q' = \frac{1.486}{n}\left(\frac{\delta}{12}\right)^{5/3} s^{1/2} \dots\dots\dots\dots (21)$$

Converting from cubic feet per second to area-inches per hour and regrouping terms, Eq. 21 becomes

$$q = \frac{1{,}020}{n\,l}\, s^{1/2}\, \delta^{5/3} \dots\dots\dots\dots (22)$$

in which l is the length of the plot in feet. By comparing Eqs. 12 and 22, it will be seen that

$$K_s = \frac{1{,}020\, s^{1/2}}{n\,l} \dots\dots\dots\dots (23)$$

If this value of K_s is substituted in Eq. 19,

$$q = \sigma \tanh^2\left(\frac{2\,K\,\sigma^{1/2}\sqrt{1{,}020}\, s^{1/4}\, t}{n^{1/2}\, l^{1/2}}\right) \dots\dots\dots\dots (24)$$

Eq. 24 is equivalent to Eq. 2 if $\frac{2\sqrt{1{,}020}}{n^{1/2}}$ is taken as $\frac{\sqrt{3{,}520}}{60}$.

Even the least of Mr. Horton's many contributions to hydrology often served the important function of stimulating study and discussion. The preceding derivation is surely one that causes the reader to probe more deeply. The runoff process provides a most complex hydraulic problem, and it would indeed be surprising if the answer were to be found in such a simplified mathematical treatment. In the first place, as in ordinary open channel problems, there are the slope, roughness, Reynolds' number, and outlet conditions to evaluate. In addition, the inflow to the plot is not only a variable with time but it varies spatially as well. Furthermore, the flow conditions deviate even farther from normal flow conditions because the raindrops, falling on the surface with nearly zero momentum in the direction of flow, cause a readjustment in the velocity distribution. A similar effect on the velocity distribution is caused on pervious areas by the draining off as infiltration of the low velocity water in contact with the ground surface. It would seem that a correct mathematical treatment would have to include most of the foregoing conditions. Because many of them were not considered in this derivation, the effect of their omission on the accuracy of the results to be expected from the use of Eq. 24 must be considered.

The first assumption made was that supply (σ) is a constant. Even if precipitation rate were assumed to be constant, σ varies with time somewhat in the manner shown by

$$\sigma = p - c_1 - c_2\, e^{-kt} \dots\dots\dots\dots (25)$$

in which p is the precipitation rate, and c_1, c_2, and k are constants. It will readily be seen that the presence of this expression for σ in Eq. 14 would have required a different integration process. Furthermore, if the integration were carried out with σ replaced by a function of t, there is no reason to believe that the result would be of the same form as Eq. 19. One very apparent consequence of the assumption that σ is a constant is that Eq. 24 shows q to become zero at the instant that σ becomes zero, thus making the recession side of the hydrograph a vertical line. Consequently, it is necessary to derive the recession curve by another method.

The second assumption was that *m*, the exponent of δ in Eq. 12, equals 2. This value was probably chosen by Mr. Horton because it is within the range of values of *m* and at the same time it made Eq. 14 easy to integrate. Eq. 14 may also be readily integrated for values of *m* equal to 1 and 3, but the resulting equations are quite different from Eq. 19. Although it is possible for two equations of different form to represent similar paths over limited regions, this cannot be assumed to be the case unless it has been demonstrated. It follows then that Eqs. 19 and 24 may only be used with assurance for plots having *m* equal to 2. The question thus arises as to whether 2 is a frequently occurring value of *m*. In the case of steady, uniform, laminar flow, both theory and experimental results show that *m* equals 3, whereas, for turbulent flow, test results culminating in the development of the Manning formula (Eq. 20) show that *m* is approximately 5/3. This is illustrated by the test results plotted in Fig. 16, for which the values of *q* and δ were obtained from studies[22] at the U. S. Waterways Experiment Station at Vicksburg, Miss. It should be noted that the points plotted in Fig. 16 follow the slope of 3 in the laminar flow region and the slope of 5/3 in the turbulent range. In the transition range, the plotted points show that *m* does not change gradually from 3 to 5/3, but changes rather quickly from 3 to 1 or less, and then gradually changes to 5/3. Thus, the region in which *m* equals 2 seems to be limited to a short range of Reynolds' numbers. Tests reported by W. O. Ree[23] and V. J. Palmer[24] for uniform steady turbulent flow through heavy grass cover show that *m* is approximately 5/3 for this case also.

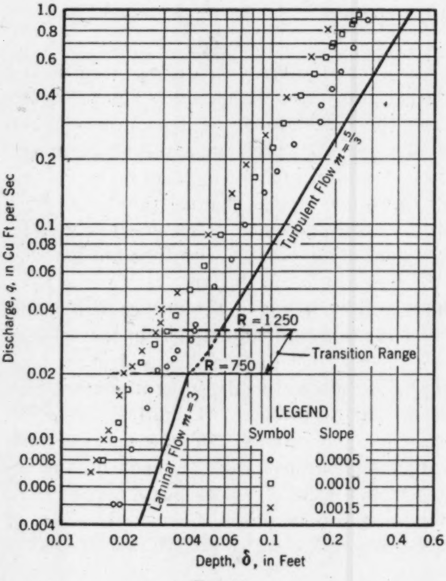


FIG. 16

One cannot predict with assurance that the relations between average depth and discharge for the unsteady conditions found on runoff plots will follow the same pattern as the uniform steady flow data plotted in Fig. 16. However, results obtained by C. F. Izzard,[10] Assoc. M. ASCE, for runoff resulting from the application of artificial rainfall on paved and turfed surfaces indicate that m equals 3 in the laminar range. Mr. Izzard's tests did not extend into the transition or turbulent ranges. The writer has encountered a number of plots[25] for which m equals 1 or less, thus indicating the presence of a transition range similar to that shown in Fig. 16.

---

[22] "Studies of River Bed Materials and Their Movement, With Special Reference to the Lower Mississippi River," *Paper No. 17*, U. S. Waterways Experiment Station, Vicksburg, Miss., January, 1935.

[23] "Some Experiments on Shallow Flows Over a Grassed Slope," by W. O. Ree, *Transactions*, Am. Geophysical Union, 1939, Pt. IV, p. 653.

[24] "Retardance Coefficients for Low Flow in Channels Lined with Vegetation," by Vernon J. Palmer, *ibid.*, April, 1946, p. 187.

Mr. Horton recognized that m would not usually equal 2 and suggested that Eq. 24 might still be made to give approximately correct results by a corresponding distortion of the values of K_s.[20] However, it should be noted, with reference to Fig. 16, that varying K_s simply changes the vertical position of any straight line. Therefore, a line having a slope of 2 can have but one point in common with a set of test results which falls along a slope other than 2. Consequently, a hydrograph synthesized on the basis of arbitrarily chosen values of K_s and m will arrive at the ultimate peak on a path which must be different from the correct one. If a particular rate of precipitation were to continue long enough for outflow to become equal to supply ($q = \sigma$), the error introduced is probably of little importance, but for storms of lesser duration there will be an error in the predicted peak discharge. When $q = \sigma$, no equation is needed to determine the peak discharge, because it is equal to the precipitation rate minus the infiltration capacity.

Although that part of the derivation of Eq. 24 previously discussed is based on the assumption that m equals 2, the value of K_s is determined from a formula for fully developed turbulent flow, applying only to the case where m equals 5/3. Furthermore, the expression for K_s (Eq. 24) so obtained does not apply to laminar flow, because in that case the exponent of s is 1 rather than $\frac{1}{2}$, and n does not vary with the roughness, but only with the Reynolds' number. The same limitations would apply in varying degrees to flow in the transition range.

Finally, before using Eq. 24 it is necessary to estimate K, which is the ratio of the depth at the margin to the average depth as shown by Eq. 13. Mr. Jens has used the value of 1.5 as suggested by Mr. Horton. However, K varies with the slope, outlet conditions, roughness of the surface, infiltration capacity, and rainfall intensity. For very flat surfaces the depth will increase in the upstream direction and K will be less than 1. For slopes less than critical, the profile is influenced by the presence of critical depth at the outlet, whereas for steeper slopes this is not the case. For any slope, the outlet may be drowned out, causing backwater, and in the case of slopes greater than critical, a hydraulic jump would occur. Sheet flow over irregular surfaces at high velocities is characterized by the presence of many small hydraulic jumps and standing waves. All these factors briefly mentioned heretofore combine to add to the difficulty of choosing a suitable value of K. The writer suggests that Eq. 12 could be written in terms of δ_{avg} rather than δ, thus avoiding the necessity of estimating K.

It is believed that the previous discussion shows that the derivation of Eq. 24 is based on a number of assumptions that are not entirely consistent with one another and that are likely to differ from the conditions prevailing on any particular runoff plot. Consequently, despite the fact that it was derived from the basically correct storage equation, Eq. 24 is an empirical formula. Two of the assumptions influenced the method of integration to such an extent that the form of Eq. 24 probably bears little resemblance to what might be called the correct form. The fact that K_s must be distorted to overcome arbitrary assumptions as to m introduces one of the weaknesses found in the "rational formula." Finally, it is necessary to employ some other method for deriving the recession side of the hydrograph—Eq. 24 being applicable only to the rising side. Perhaps none of these objections would be serious if the magnitude of the deviation of results obtained by the use of Eq. 24 from the correct results could be predicted for any set of conditions. Because this apparently cannot be done, the writer prefers a method in which the effect of the various assumptions is clearly evident.

---

[25] "Sprinkled Plat Runoff and Infiltration Experiments on Arizona Desert Soils," by E. L. Beutner, R. R. Gaebe, and R. E. Horton, *Technical Publication No. 38*, SCS, U.S.D.A., September, 1940.

A graphical procedure which permits the derivation of both the rising and falling sides of the hydrograph, similar to that described by W. W. Horner, Past-President, ASCE, and Mr. Jens[3] in connection with gutter flow and by C. R. Hursh and the writer[26] for determining outflow from channel storage on small watersheds, might be used. In this method, the storage equation, together with a relation between discharge and average depth of detention—

$$q = K_s \, \delta^m_{\text{avg}} \qquad \text{(26)}$$

—are solved for successive time intervals. The constants in Eq. 26 must be determined from tests on terrain comparable to that upon which they are to be used, and are usually determined from the recession side of a hydrograph. Although the values of K_s and m are usually somewhat different for the rising and falling sides of a hydrograph, the magnitude of the error introduced in this manner can be tested by using the relation derived from the recession curve to synthesize the rising side of a known hydrograph. Many such checks made by the writer indicate that this effect is small.

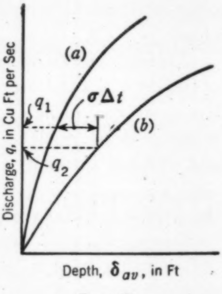


FIG. 17

The storage equation is written

$$\delta_1 + \sigma \, \Delta t - \frac{(q_1 + q_2)}{2} \Delta t = \delta_2 \qquad \text{(27)}$$

in which the subscripts 1 and 2 denote values occurring at the beginning and end, respectively, of a time interval Δt, and σ is the average supply during an interval. Eq. 27 may be rearranged as follows:

$$\left(\delta_1 - \frac{q_1 \, \Delta t}{2} \right) + \sigma \, \Delta t = \left(\delta_2 + \frac{q_2 \, \Delta t}{2} \right) \qquad \text{(28)}$$

Values of $\left(\delta - \frac{q \, \Delta t}{2} \right)$ and $\left(\delta + \frac{q \, \Delta t}{2} \right)$ may then be plotted against corresponding values of q, as illustrated by curves (*a*) and (*b*) of Fig. 17. Knowing the discharge at the beginning of any interval (q_1), the value of $\sigma\ \Delta t$ may be added to the corresponding abscissa of curve (a) and the value of q_2 read from curve (b) in the manner illustrated. The process is then repeated for successive intervals until the entire hydrograph is derived.

---

[26] "Separating Storm-Hydrographs from Small Drainage-Areas into Surface- and Subsurface-Flow," by C. R. Hursh and E. F. Brater, *Transactions*, Am. Geophysical Union, 1941, Pt. III, p. 863.

Carl F. Izzard,[27] Assoc. M. ASCE.—The main purpose of this discussion is to present pertinent data on the hydraulics of overland flow as related to airport drainage. As noted by the author, the United States Public Roads Administration conducted, with the cooperation of the United States Soil Conservation Service on the experimental phase, an investigation of overland flow on paved and turf surfaces. A report summarizing these studies[28] indicates that the rate of runoff under a uniform rate of supply can be determined for any surface by use of a single dimensionless hydrograph that takes the place of the Horton formula (Eq. 2). The latter was developed from very limited ex-

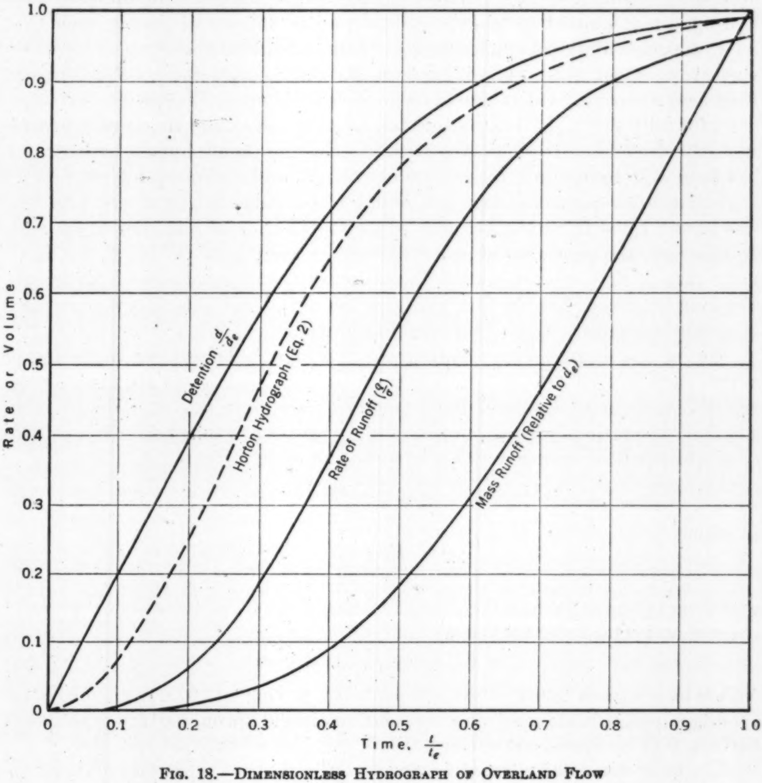


FIG. 18.—DIMENSIONLESS HYDROGRAPH OF OVERLAND FLOW

perimental data on short plots, whereas the dimensionless hydrograph integrates the results of several hundred tests on a variety of lengths, slopes, and surfaces with two different rainfall intensities. The principal difference between the two curves, as shown in Fig. 18, is that the Horton curve starts rising sooner after the beginning of supply. Aside from the fact that the dimensionless hydrograph affords a more accurate representation of overland runoff, it is also easier to use because a single curve takes the place of the families of curves needed to facilitate use of the Horton equation. Although the dimensionless hydrograph is limited to lengths of flow and intensities of supply such that $\sigma\, l < 500$, this limitation includes most of the conditions occurring on interrunway areas on airfields. When experiments being conducted by the Corps of Engineers are completed, it should be possible to determine detention characteristics for fully turbulent flow beyond the limits of the Public Roads Administration experiments. However, the writer, using the dimensionless hydrograph, has been able to obtain a reasonable reproduction of an actual hydrograph on a combined pavement and turf area where the average length of overland flow was about 1,000 ft with maximum supply intensities of 5 in. per hr on turf. In this case, the peak rate of runoff did not exceed 0.5 in. per hr, so the maximum rate of flow was within the limitation of the dimensionless hydrograph. For long lengths of flow on turf, the duration of the high intensity supply rates is always a small fraction of the equilibrium time.

---

[27] Senior Highway Engr., Public Roads Administration, Washington, D. C.

[28] "Hydraulics of Runoff from Developed Surfaces," by Carl F. Izzard, *Proceedings*, Highway Research Board, National Research Council, Vol. 26, 1946, pp. 129–150.

Use of the dimensionless hydrograph is simple. Time from beginning of supply is measured as a ratio to the time of equilibrium (t/t_e) required for the runoff rate to become substantially equal to the supply rate σ. The ordinate q/σ gives the ratio of rate of runoff to supply rate at the time t/t_e. To evaluate t_e in minutes it is necessary to know detention at equilibrium, d_e. Thus,

$$t_e = \frac{120\, d_e}{\sigma} \qquad (29)$$

in which

$$d_e = \frac{K_d\, L^{1/3}\, \sigma^{1/3}}{2.92} \qquad (30)$$

and is measured in inches; the detention coefficient K_d is expressed in terms of slope S and retardance coefficient c as

$$K_d = (0.0007\, \sigma + c)\, S^{-1/3} \qquad (31)$$

Values of c are as follows: Smooth asphaltic pavement, 0.007; crushed slate roofing paper, 0.0082; concrete pavement on airfield apron, 0.012; tar and sand surface, 0.017; clipped sod surface, 0.046; and dense bluegrass turf, 0.06.[28]

The dimensionless hydrograph can also be used to compute rates of runoff resulting from storms involving varying rates of supply. The method is described in detail elsewhere[28] and depends on the principle, developed from study of experimental data, that the rate of runoff is a function of the detention at any time. Thus, when the supply rate changes, the rate of runoff immediately adjusts to the rate which would have existed at the new supply rate with the given amount of detention. For runoff on turf, especially with pondage, which is almost always involved, the writer agrees with Mr. Jens that the average supply rate can be used to determine the peak rate, so that it is not ordinarily necessary to compute hydrographs with varying supply rates.

A dimensionless curve has also been developed to compute rates of runoff during recession.[28] This curve is in terms of a function which involves the detention and rate of runoff at the beginning of the recession so that the total area under the recession curve will equal the amount of detention. Inasmuch as the present discussion is concerned with triangular approximations of hydrographs, there is no necessity of using the dimensionless recession curve.

The Public Roads Administration experiments included a few tests on turf plots with various percentages of impervious area draining across the turf. The typical hydrograph shows a sharp increase in rate of runoff as soon as runoff from the impervious area reaches the lower margin of the turf.[28] A fair approximation of the rising hydrograph can be obtained by using the dimensionless hydrograph if the amount of detention in Eq. 29 is reduced in proportion to the reduction in detention resulting from having pavement instead of turf on the upper part of the area. A simple equation to determine this reduction is

$$\frac{d_c}{d_t} = 1 - \left(\frac{l_p}{l}\right)^{4/3} \left(1 - \frac{K_p}{K_t}\right) \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

in which d_c and d_t are equilibrium detention values for the composite area and a turf area of the same length l, respectively; l_p/l is the ratio of paved to total length; and K_p and K_t are detention coefficients for the paved and turf areas, respectively. The equation thus takes into consideration different slopes on paved and turf areas, since that factor is included in Eq. 31. The detention (for the composite area) used in Eq. 29 is the detention for a turf area of the same total length multiplied by the ratio d_c/d_t. The supply rate used is the supply rate for the turf as determined by Mr. Jens (with a proportionate increase in rainfall intensity to determine infiltration on turf area) converted to equivalent supply rate for the total length of overland flow, including the paved area.

To illustrate this procedure, the overland flow length of 185 ft shown in Fig. 7 is taken, and it is assumed that the upper 50 ft are paved, making the ratio of paved to turf area 0.27. The net supply rate on the composite area is 0.85 in. per hr for 30 min. From Eq. 31, values of K_p and K_t are 0.07 and 0.233, respectively, assuming slopes of 0.01 (on concrete pavement) and 0.0175 (on dense turf) and a rainfall intensity of about 3 in. per hr (on pavement). Detention on 185 ft of turf of uniform slope is 0.43 in. based on a supply rate of 0.85 in. per hr. This value is multiplied by 0.88 (computed by Eq. 32) to obtain 0.38 in. for the detention at equilibrium on the composite area. From Eq. 29 the equilibrium time is found to be 54 min so that the value of t/t_e at the time supply ended is $30/54 = 0.56$ and that of q/σ is 0.65 from the dimensionless hydrograph. The rate of runoff at 30 min is thus $0.65 \times 0.85 = 0.55$ in. per hr.

A slight error is involved in this method because the actual supply rate on the paved area is different from that on the turf area. The error is negligible because detention on the pavement is a very small proportion of the total detention on inter-runway areas.

Commonly, the width of paved area will be greater than the average width of turf area. To allow for the increased detention volume on pavement resulting from this situation, the ratio of areas may be substituted for the ratio of lengths used in Eq. 32.

Values of q_p computed by the dimensionless hydrograph method for the supply rates given in Table 2 give closely similar results except for short durations on 0% impervious areas, where the values were about 75% of those in Table 2. The reason for the difference is probably the fact that the Horton equation gives higher rates of runoff where the duration of supply is substantially less than the time of equilibrium. Using Eq. 32 in estimating detention on 30% impervious areas, the values of q_p are practically the same as those in Table 2 for all durations, but tend to be a little higher than those in Table 2 for the shorter durations.

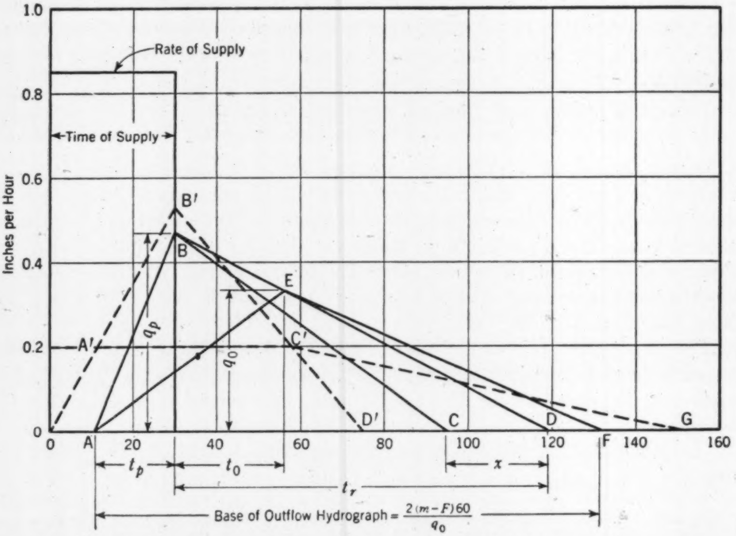


FIG. 19.—COMPARISON OF TRIANGULAR HYDROGRAPH WITH FLAT TOPPED HYDROGRAPH

With reference to the method of computing the rate of outflow from a pond as illustrated in Fig. 8, the writer, after examining actual hydrographs of runoff from ponded areas on airfields,[29] feels that the flat topped outflow hydrograph is not a good representation and that a triangular hydrograph as shown in Fig. 19 would be more nearly correct. The latter is more consistent with actual hydrographs because ponding begins at the beginning of the inflow hydrograph and reaches a maximum at a time after the ocurrence of the peak rate of inflow.

The example in Fig. 7 is used to illustrate this point. The average supply rate for 30 min is 0.85 in. on 185 ft of turf on a 1.75% slope. From Eqs. 31 and 30, it is determined that $K_d = 0.233$ and $d_e = 0.431$ in., from which $t_e = 61$ min by Eq. 29. At 30 min, $t/t_e = 30/61 = 0.49$. In the dimensionless hydrograph, Fig. 18, $q/\sigma = 0.55$ for this value of t/t_e, and therefore

---

[29] "Report on Drainage Verification at Military Establishments," Louisville Engr. Dist., Ohio River Div., Corps of Engineers, War Dept., Louisville, Ky., June, 1947.

$q_p = 0.55 \times 0.85 = 0.47$ in. per hr; also, from the detention and mass runoff curves (Fig. 18), detention $= 0.81 \times 0.431 = 0.349$ in. and mass runoff $= 0.17 \times 0.431 = 0.073$ in. The sum of detention and mass runoff is 0.422 in., which checks the amount of rainfall (0.85 in. per hr × ½ hr) within 0.003 in., the error being due to using only two significant figures in reading the values from the curve. Knowing q_p (0.47 in. per hr), the rising time t_p is computed as $\frac{2 \times 0.073 \times 60}{0.47} = 19$ min, which places the beginning of the runoff at 11 min after beginning of supply. The recession time t_r similarly becomes $\frac{2 \times 0.349 \times 60}{0.47} = 89$ min. The latter time can be adjusted by Mr. Jens' method for infiltration during recession. Thus, if $f = 0.26$ in. per hr, by Eq. 5, $X = \frac{89 \times 0.26}{(1.5 \times 0.47) + 0.26} = 24$ min.

The peak rate of outflow is estimated as

$$q_o = q_p \left(1 - \frac{p}{m}\right) \qquad (33)$$

in which p is represented by the area ABE (Fig. 19) and m is the runoff in inches before deducting infiltration during recession on the assumption that the total supply is effective in producing the peak outflow. The time after end of supply at which peak outflow occurs is

$$t_o = t_r \left(\frac{p}{m}\right) \qquad (34)$$

In the example, if $p = 0.125$ in., then $\frac{p}{m} = \frac{0.125}{0.425} = 0.29$ and $q_o = 0.47 \times 0.71 = 0.33$ in. per hr, occurring at $t_o = 89 \times 0.29 = 26$ min after the end of supply. The base of the outflow hydrograph must be such that the area equals supply minus infiltration during recession. The latter amounts to $\frac{0.47 \times 24}{2 \times 60} = 0.094$ in., making net runoff $= 0.425 - 0.094 = 0.331$ in. The base of the outflow hydrograph, therefore, is $\frac{2 \times 0.331 \times 60}{0.33} = 120$ min.

For comparison, the dotted lines show the inflow and outflow hydrographs which would result from using the same assumptions in Mr. Jens' method. Triangle AEF is believed to be a more true representation of outflow from a pond, because the peak occurs after supply has ended, which is necessarily the case, whereas the flat peak A′C′ begins before any substantial runoff has occurred. Summation of a series of hydrographs with flat tops would give an excessively high rate of runoff early in the storm. Summation of triangular outflow hydrographs would come closer to giving true peak outflow rates, if hydrographs are offset to allow for routing through conduits.

Mr. Jens does not indicate how the ponded water is discharged during recession. In Fig. 19, this has been shown as C′G with area C′D′G equal to area A′B′C′, the volume of pondage. In Fig. 13, the recession curves are not shown as straight lines, which raises a question as to what method Mr. Jens used for computing recession curves.

The writer was unable to determine how the use of the triangular outflow hydrograph would affect the final design of a system, since insufficient basic data were given in the example. It would be instructive if the author could recompute one set of data by this principle giving the results in his closing discussion.

The basic concept of the triangular outflow hydrograph was used by the writer in an unpublished report written in November, 1941. The method was not applied then because no good procedure for estimating the peak rate of inflow had been developed.

The writer doubts that the method of developing average supply patterns as illustrated in Table 1 is logically correct. As an analogy, the arithmetic mean of the number of spots on a die is 3.5 but the probability of rolling 3.5 is zero. In other words, the arithmetic mean has no possibility of occurring. On the other hand, the probability of rolling any given number from 1 to 6 is 1 in 6.

The probability of a given amount of rainfall in a given duration is fairly well established from rainfall intensity-frequency data such as those compiled by the late D. L. Yarnell,[13] M. ASCE. Table 1 shows that there is considerable variation in the pattern of rainfall intensities. Certain patterns result in higher rates of runoff than others. Obviously, the probability of encountering one of the critical patterns is not the same as the probability of equaling or exceeding a given total amount of rain in a given period. For example, if the probability of the amount of rain is 1 in 2, and the probability of the pattern being critical is 1 in 3, then the resultant probability is 1 in 6. The other five storms would not be critical.

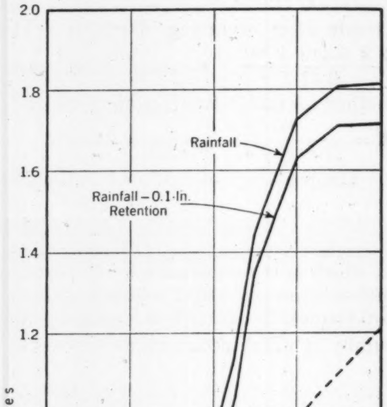


FIG. 20.—MASS CURVES FOR STORM OF SEPTEMBER 12, 1925, FROM TABLE 1

The writer is not prepared to suggest exactly how the statistical analysis should be made. That study should be undertaken by a mathematical statistician who is fully aware of the limitations of statistical methods.

The writer wishes to suggest a method of estimating the supply rates for a given rainfall intensity pattern and infiltration capacity curve which gives results closely comparable to those in Table 1 and is easier to follow. The method requires only a mass rainfall curve, a mass infiltration curve, and an estimate of the initial retention. For example, Fig. 20 shows mass rainfall for the storm of September 12, 1925, taken from Table 1. The curve is replotted 0.1 in. lower to allow for 0.1-in. retention. Then the infiltration mass curve is moved over until it touches at one point only—namely, at 50 min. Prior to that time, rainfall intensity was lower than infiltration rate so all the rainfall was absorbed. Beyond 50 min, rainfall intensity exceeds infiltration rate and in the next 10 min the excess is 0.10 in., as shown by the net mass rainfall curve. The net rainfall rate for that period is thus 0.6 in. per hr. At 90 min, the rainfall rate drops below the infiltration rate and no additional net rainfall occurs. Whether or not infiltration capacity recovers during periods when rainfall is less than infiltration is not fully established. It might be assumed, however, that the infiltration curve could be shifted to the right until the ordinate measured to the mass rainfall less retention curve is 0.65 in. at the time when excess rainfall resumes, 0.65 in. being the maximum net mass rainfall previously attained.

In conclusion, the writer believes that the principles for estimation of peak rates of runoff from ponded inter-runway areas on airfields as suggested by Mr. Jens are fundamentally sound. These changes relate to detailed procedures and are presented so that other engineers working on similar problems may consider the alternatives and develop still better procedures.

Stifel W. Jens,[30] M. ASCE.—Messrs. Williams, Hicks, and Izzard all question the validity of the technique involved in processing the rainfall data. Mr. Williams doubts the need for establishing a storm pattern, and recites some of the reasons why he believes the considerable effort involved in setting up a correct storm pattern is not warranted. Acknowledging the reasonableness of these criticisms, the method of the paper does approach the realities whereas the more usual "rational method" techniques, which utilize one average intensity only, clearly do not. There can be no denial that the amount of detailed work is likely to discourage its use, and Mr. Williams' constructive suggestion for a simpler approach is very interesting. The striking similarity of the storm patterns for the widely separated geographical locations is encouraging. Use of the method illustrated by Tables 10 and 11 should recognize, however, that a pattern made up entirely of intensities taken from the ordinary intensity-duration curve, rearranged in an order suggested by a brief analysis of the average pattern of a number of recorded storms, is an almost impossible combination of events. Certainly such a storm pattern departs more from the statistically valid pattern than do those developed as suggested in the paper.

The user of the Williams method should also keep in mind the fact that a pattern rainfall incorporating all the intensities for the various durations for one frequency, necessarily will result in a generally higher net rainfall pattern, because the infiltration capacity will be equaled or exceeded earlier and oftener.

---

[30] Partner, Horner and Shifrin, St. Louis, Mo.

A correct pattern of net supply would undoubtedly have more intervals of rainfall intensities lower than the related infiltration capacities.

The incidental illustration of the usefulness of the Hathaway standard rainfall intensity-duration curves points to a possible utilization of these for areas in which no satisfactory local record is available. To the writer, the Yarnell[13] curves are preferable, if used carefully with an understanding of their limitations. The supposition that the paper uses the method of G. A. Hathaway, M. ASCE, with some modifications is correct, but the background and initial application of the infiltration-overland flow technique were originally developed by W. W. Horner, Past-President, ASCE, in the studies made for the drainage of the Washington (D. C.) National Airport in 1939.

Mr. Hicks calls attention to the fact that only one third of the twelve storms of Table 1 have net rainfalls of 20-min duration; of the remaining eight, two have 15-min net rainfalls, four have 10-min supplies, and two sustain a supply rate for only 5 min. The wide variation in outflow rates for each of these different net supply rates leads Mr. Hicks to the sound conclusion that the intensity-duration curve should be based on the duration and intensity of net rainfall. The objective of the designer is to know, for any specific part of a drainage system, the frequency with which certain rates of flow will be equaled or exceeded. Obviously, if the true frequency of supply rates of different durations could be determined, the foregoing objective would be immeasurably closer. Lack of sufficient data, together with the need to recognize specifically, and in some detail, more of the many factors influencing net rainfall has made common the use of the frequency of gross rainfalls of various durations.

The probable gross rainfall pattern of a desired frequency is applicable to a sizable area. However, a different net rainfall pattern is required for each area with a different soil-cover-condition complex. Different soil, cover (and stage of growth), antecedent moisture, and season, to mention but a few of the many factors involved, each affect the net supply. The probabilities are that gross rainfall frequencies will continue for some time to come to be the guiding criteria in so far as frequency enters into storm sewer design. The important practical aspect of this fact is always to keep in mind the often considerable difference between the frequency of a given gross rainfall and the frequency of the runoff derived therefrom. Judgment will be better guided in the light of such recognition of the realities.

Mr. Izzard's illustration of the impossibility of an arithmetic mean actually occurring under certain circumstances does not seem to the writer to be altogether comparable to the determination of an average storm pattern. The desired objective was to evaluate the effects, on the critical period net rainfall, of immediately antecedent precipitation as it might occur on the average during the storm frequency under study. In the light of the almost infinite variety of patterns of rainfall intensities, it does not appear likely that a statistical study would reveal a storm that occurred on the average once every one, two, five, or ten years and actually repeated itself in every detail the number of times implicit in its indicated frequency. The method of developing average supply patterns, as illustrated in Table 1, is open to criticism of its statistical validity, but nonetheless it is believed to be a reasonable attempt to give quantitative expression to some of the actual happenings which are qualitatively known.

Professor Brater's review of the development of the Horton overland flow formula (Eqs. 2 and 24) and his analysis of its limitations are particularly convenient for comparison with Mr. Izzard's dimensionless hydrograph of overland flow. It is interesting to note that the latter's formula evaluates some of the factors Professor Brater lists as not reflected in the Horton equation. The additional retardance to sheet flow due to raindrop impact is recognized in Eq. 31 by the factor 0.0007 σ. The "effect on the velocity distribution * * * by * * * infiltration of the low velocity water in contact with the ground surface" is reflected indirectly in both Eqs. 2 and 7 in that the net supply rate is used and retention or depression storage and infiltration have been abstracted from gross rainfall.

Although Eq. 25 attempts to set up a mathematical expression for the variation of supply with time, practical application of either the Horton equation or the Izzard equation has involved the use of a stepped pattern net rainfall, each step considered to have a uniform rate over its short time. Mr. Izzard has indicated the principle on which both his and Mr. Horton's equations can be used to reflect changing rates of supply—namely, that the rate of runoff is directly related to the amount of detention at any time.

Both overland flow equations are admittedly only valid for the rising side of the hydrograph. In recognition of this, Mr. Izzard has developed from his experimental work another dimensionless equation[28] to compute rates of runoff during recession.

Professor Brater's illustration (Fig. 16) corroborates Mr. Izzard's experimental findings that for sheet flow, where $i L < 500$, the exponent $m = 3$, which indicates that the flow is steady, uniform, and laminar. Everyone interested in this matter awaits with eagerness the findings of the current experimental program of the Corps of Engineers to determine the interrelationships where the flow is fully turbulent. If the comparable curves computed by Eq. 2 with an exponent of 3 instead of 2 are added to Fig. 18, the close agreement of the two sets of curves is a tribute to the sound analytical approach of the late R. E. Horton, M. ASCE, and the careful experimental work of Mr. Izzard. The latter's dimensionless hydrograph for the rising side and his companion dimensionless recession hydrograph are based on analysis of a considerable volume of experimental data and from a practical viewpoint suggest that the theoretical limitations so well discussed by Professor Brater, even if individually serious, are apparently either canceled out in a compensatory fashion, or are properly reflected in the empirical formulas.

No one can quarrel with Professor Brater's preference for "* * * a method in which the effect of the various assumptions is clearly evident." One of the major objectives of the method discussed in this paper is an attempt to reflect more clearly the sequence of events as they occur. The writer prefers the approach involving the use of an overland flow equation such as the Izzard dimensionless hydrograph (simpler in form and easier to use than the cumbersome hyperbolic function of the Horton equation). Had the Izzard overland flow equations been available and understood, at the time of the studies leading to this paper, they would have been used. The very material amount of detailed computation required to change to the Izzard approach militated against a change-over.

The difficulties inherent in obtaining representative discharge storage relationships for both the rising and recession sides of the hydrograph of sheet flow are many and involve, in the writer's opinion, no less departure from the realities than the use of the overland flow equations. Professor Brater suggests that the constants of Eq. 26 "* * * are usually determined from the recession side of a hydrograph" of overland flow. That is exactly what Mr. Izzard has done in establishing the relationship between discharge and detention storage. It is believed that the effect of the sheet flow detention storage on the rates of net rainfall or supply is most practically expressed by an empirical formula. The many factors and their complex interrelationship, so well discussed by Professor Brater, will probably never be expressible in any practical rigorous mathematical form.

Mr. Gain has lent clarity to the paper in his succinct summary of the attempt by the method to follow logically and realistically the development of runoff from rainfall. Even if a workable combination of the more detailed and realistic methods, such as used in the paper, can be effected with the "rational" method, in the long run the more detailed method will be most desirable, because the designer can visualize the process much more clearly as it develops. Of course, it is highly probable that, in the evolution of better founded methods of predicting runoff from precipitation, some such intermediate step, as is suggested by Mr. Gain, may occur.

Although it is true that municipal practice ordinarily can make no use of the rate-reducing storage of deliberate pondage, in many very flat areas of intensive development (such as the wide, alluvial river bottoms at Kansas City, Mo., East St. Louis, Ill., and similar industrial developments along the great rivers of the United States), storm sewer design does not attempt to provide inlet and sewer capacities adequate for the very short times of concentration actually prevailing at upper ends of lines. The very small mass values of net rainfall involved in the very short-time high intensity rainfalls make it economic and practical actually to design upper end inlets and laterals for the lower intensities related to longer durations. In other words, in such circumstances, a minor amount of actual pondage around the inlets prevails for a relatively short period when the storm of design frequency occurs.

Everyone interested in the application of the newer hydrology seconds wholeheartedly Mr. Gain's plea for additional data and research. To cite one of the outstanding examples, the vast amount of experimental data on rainfall, infiltration, and runoff collected by the Soil Conservation Service at its experimental watersheds during the 1930's should be processed and be made available at the earliest possible moment.

The field investigations of airport drainage made by the Corps of Engineers and discussed by Mr. Williams constitute a valuable supplement to the hydrologic information used in design methods such as that suggested by this paper. Wheeled traffic over turfed airport areas should be practically nil on civil fields, but the other difficulties encountered in maintaining constant drainage areas indicate the need for conservative judgment in design. Because of the unavoidably flat ridges and drainage divides, the writer is particularly impressed by the possibility of "* * * wind driving surface water over low drainage divides." Since nothing practical can be done to prevent this, it can only be recognized through the employment of reasonable safety factors as suggested by Mr. Williams.

If Mr. Izzard's example to illustrate the use of Eq. 32 would use a value of net supply rate of 1.26 in. per hr for the composite area, it would employ the actual supply rate interpolated from Fig. 6 for a 27% impervious area and a 30-min duration of supply. The outflow rate at the lower edge of the composite area, as computed by Eqs. 7, 8, 9, and 10, is 1.06 in. per hr. Examination of Fig. 10 indicates a peak runoff rate of 1.12 in. per hr for 135 ft of turf length and 30% imperviousness—in keeping with Mr. Izzard's close check of the peak runoff rates in Table 2, except for short durations of supply on wholly pervious areas.

The limitations of the triangular hydrograph are brought out by Mr. Hicks' analysis as illustrated by Fig. 15 and by Mr. Williams' citation of the general conclusions arising out of studies of the Airfields Branch, Office of the Chief of Engineers, United States Army. Mr. Izzard indicates that brief examination of actual hydrographs suggests a peaked triangular outflow hydrograph as illustrated in Fig. 19. The criterion of the ratio of available pondage to mass runoff is an excellent guide to the designer. However, all the discussion points to the need for further study and research to develop a practical tool to take the place of the triangular hydrograph and at the same time to lend itself to reasonably accurate evaluation of pondage.

Mr. Hicks shows clearly the error of summating triangular hydrographs where ponding is not present. With the Izzard dimensionless hydrograph, actual inflow rate curves can be developed readily and then offset for addition. Mr. Hicks' discussion of Fig. 15 shows the small change in pondage depth necessary for an appreciable difference in required pondage volume. This fact is fortuitous. Examination of the dimensionless hydrograph supports Mr. Hicks' statement that, up to a duration of about 0.6 of the equilibrium time, the triangular hydrograph is overly safe; as it passes through that point and continues on to equilibrium, the straight ascending leg of the triangle falls below the curve values. The differences are not great until the duration involves a prolonged flow at equilibrium rate.

The writer cannot agree with Mr. Izzard's assumption that "* * * ponding begins at the beginning of the inflow hydrograph." Until the overland flow rate into the collecting inlet exceeds the capacity of the outlet, no pondage could result. One other assumption involved in the method illustrated by Fig. 19 assumes that the peak outflow rate occurs at the moment when the pondage has all been drained. This does not appear likely. If Mr. Izzard's triangular hydrograph of outflow from ponds is correct, it bears out his conclusion that the flat topped hydrograph, when summated, would give a considerably higher rate of runoff early in the storm.

In adding outflow rates for ponded areas, it was felt unnecessary to develop the recession part of the ponded hydrograph, inasmuch as the peak value was all that was significant for design purposes. If it had been necessary, it would have been developed on the assumption that Mr. Izzard makes in Fig. 19. The recession curves in Fig. 13 were sketched in with the area beneath them equal to the volume of pondage less infiltration during the recession side of the curve.

Fig. 20 indicates a variation in technique for the development of a net rainfall curve. This deserves consideration in evolving a less laborious methodology for handling the problem.

Mr. Hicks inquires concerning the hydraulic grade lines, as shown in Figs. 9 and 12. These were developed originally on the presumption that the pond would not have a free hydraulic relationship with the collecting pipe line—that is, an intervening short connection would prevent the pond from having a direct influence on the hydraulic gradient of the upstream pipe. Admittedly, the actual hydraulics involved in the pond is complex and suggests the need for some careful research to indicate the actual hydraulic conditions that would prevail under circumstances comparable to airport ponded drainage.

The writer is very appreciative of the considerable interest evidenced by the provocative discussions.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2349

LEAST SQUARES ADJUSTMENT OF TRIANGULATION NET BETWEEN GEODETIC STATIONS

By E. F. Coddington [1] and O. C. J. Marshall,[2] Members, ASCE

With Discussion by Messrs. Julius L. Speert, C. H. Ney, H. W. Hemple, Lansing G. Simmons, and E. F. Coddington and O. C. J. Marshall

Synopsis

Local triangulation nets, both urban and rural, can be precisely adjusted to large-scale geodetic triangulation control with comparatively little effort if all the angle, distance, and azimuth relations involved are reduced to simultaneous equations expressed in terms of direction corrections and then adjusted by least squares methods. The derivation of the correction equations is illustrated by a sample net containing three quadrilaterals. Application of the method is also illustrated by adjustment of a local net for Logan County, Ohio.

Introduction

The United States Coast and Geodetic Survey (U. S. C. and G. S.) triangulation system, which has 25-mile spacing of stations as its goal, has been adjusted to the local triangulation systems of at least two cities. As a result, Pittsburgh, Pa., and Columbus, Ohio, each have a triangulation system for local horizontal control to which the U. S. C. and G. S. has adjusted the overall net. The triangulation system for each of these cities was completed before geodetic control was available, but the U. S. C. and G. S. program has progressed to such an extent that it is available or can be made so for any future local requirement.

On several occasions, local cadastral surveys have been conducted without making a complete connection to available geodetic control. The objective of this paper, therefore, is to describe a method that will enable engineers in charge of supplementary triangulation projects to make full use of the control which the U. S. C. and G. S. has made available.

Note.—Published in September, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Prof. Emeritus of Geodetic Eng., Ohio State Univ., Columbus, Ohio.

[2] Prof. of Surveying and Geodesy, Ohio State Univ., Columbus, Ohio.

SAMPLE PROBLEMS

To illustrate the problems involved, assume a system of three complete quadrilaterals connecting two fixed triangulation stations (1 and n in Fig. 1) of the type established by the U. S. C. and G. S. It is assumed that for the sample net all eight angles in each quadrilateral have been measured with the same relative accuracy consistent with the requirements of the project and that the azimuth and length of a starting base line (1–2 in Fig. 1) have been determined with adequate precision. In this example, the azimuth and length of a terminal base line, 7–n, are also assumed, for the purpose of illustrating the formation and use of the azimuth and length equations. In practice, a terminal base line would seldom be needed so close to the starting base line.

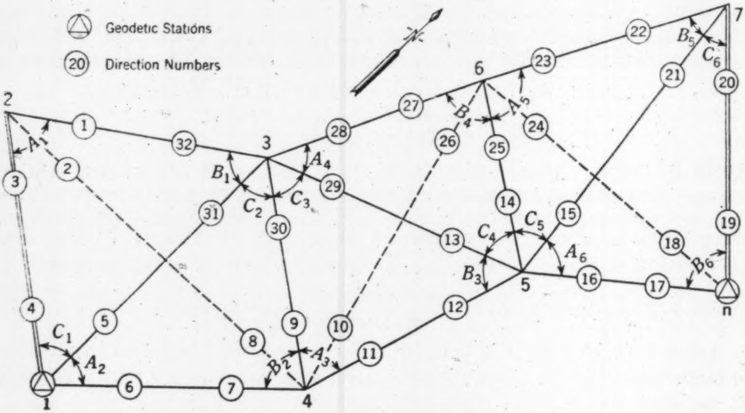


FIG. 1.—TRIANGULATION NET OF THREE COMPLETE QUADRILATERALS BETWEEN GEODETIC STATIONS

Several procedures are available for the preliminary adjustment, any of which will determine preliminary angles to be used in computing triangles from which the accumulated errors in latitude and departure between fixed stations can be found. In the first of these procedures the angle equations of each quadrilateral may be satisfied by the approximate method.[3,4] This is the equivalent of a rigid solution which disregards the relation of the sides, and is accomplished very rapidly where the figures are simple quadrilaterals or center point triangles with no overlapping figures involved. The preliminary solution also may be set up as a least squares solution involving the usual three angle equations and one side equation for each quadrilateral, leaving all tie equations for the final adjustment. Where there are no overlapping figures or other complications, these quadrilaterals may be adjusted separately. As a third possibility, the preliminary adjustment may also be made as a least squares solution including all condition equations involved in the net (angle, side,

---

[3] "The Principles and Practice of Surveying," by C. B. Breed and G. L. Hosmer, Vol. 2, "Higher Surveying," John Wiley & Sons, Inc., New York, N. Y., 5th Ed., 1938.

[4] "Least Squares in Engineering," by E. F. Coddington and O. J. Marshall, Ohio State Univ., Columbus, Ohio, 1941.

azimuth, and length equations) except the latitude and departure equations. When this solution is completed, the resulting latitude and departure closure is likely to be smaller than when any less inclusive preliminary adjustment is used.

In the example illustrated in this paper, the first of the foregoing methods is used. The first step is the preliminary computation of a selected set of triangles of the net, starting with the base line 1–2 and proceeding through the strongest pair of triangles of each quadrilateral to the final station, n. The triangles used in this example are those adjacent to the diagonal shown by a full line in each quadrilateral. The other diagonal in each case (not used in computation of triangles) is shown by a dashed line. The stronger set of triangles should be selected by the "strength of figure" theory, but use of triangles adjacent to the shorter diagonal in each quadrilateral will generally result in the selection of the stronger figure.

A selected traverse from station 1 to station n may then be computed along any of these calculated triangle sides by latitudes and departures. The distance by which this computed traverse fails to fit the gap between the original and the final fixed stations represents the closing error of the survey. Because the preliminary adjustment makes the triangles used geometrically consistent, the closing error will be independent of the traverse route selected between 1 and n. In practice it is advisable to close each quadrilateral by latitudes and departures, since this procedure provides a needed check on the computation of the triangles. For purposes of obtaining the relative closure, however, the computed straight-line distance from station 1 to station n should be used as the length of the survey.

The closing error of the connecting traverse is then used to make a combined readjustment of the entire triangulation net so as to obtain an exact fit between fixed stations. To make this readjustment, two equations—one for latitude and one for departure—are used in combination with the equations resulting from other geometrical conditions of the net. The constant terms for all equations in this final adjustment are based on the corrected angles resulting from any preliminary solution. Hence, those condition equations which are used in both preliminary and final adjustments will show zero for the constant term in the final adjustment. All equations are handled simultaneously in the final least squares adjustment of the net.

DERIVATION OF TIE EQUATIONS

In deriving the previously mentioned equations, the measured base is considered as fixed in length and direction and as originating at one of the fixed coordinated points. The effect on the location of n, the end point of the connecting traverse line, caused by a change in each angle of the net is also considered.

Each triangle of the net is examined separately in deriving the relation between the angle changes and the change of the end point. Considering the triangle 123 (Fig. 1), the combined effect of its angle changes will be an expansion or contraction of the side 1–3, with point 1 remaining fixed, and a rotation of side 1–3 about point 1. The change of side 3–2 does not require consideration, because the rotation of side 1–3 will be due solely to the change in angle *C*, and the expansion of side 1–3 will be due to the combined effects of the changes in angles A and B.

EXPANSION AND CONTRACTION OF THE NET

In triangle 123,

$$\frac{1\text{–}3}{2\text{–}1} = \frac{\sin A}{\sin B} \quad \text{(1a)}$$

therefore,

$$1\text{–}3 = 2\text{–}1\left(\frac{\sin A}{\sin B}\right) \quad \text{(1b)}$$

and

$$\log(1\text{–}3) = \log(2\text{–}1) + \log\sin A - \log\sin B \quad \text{(1c)}$$

If the changes in the angles of this triangle are assumed to be v_A, v_B, and v_C, the tabular differences for log sin A and log sin B are assumed as δ_A and δ_B, respectively, and the length of the variable side 1–3 is designated by L, then

$$d\log_{10}\sin A = \delta_A v_A \quad \text{(2a)}$$

$$d\log_{10}\sin B = \delta_B v_B \quad \text{(2b)}$$

$$d\log_{10}(2\text{–}1) = 0 \quad \text{(2c)}$$

and

$$d\log_{10} L = \delta_A v_A - \delta_B v_B \quad \text{(2d)}$$

If it is also assumed that

$$\log_{10} L = N \quad \text{(3a)}$$

then

$$\frac{dL}{L} = \log_e 10\, dN \quad \text{(3b)}$$

However,

$$\log_e 10 = \frac{1}{\log_{10} e} \quad \text{(3c)}$$

therefore,

$$\frac{dL}{L} = \frac{dN}{\log_{10} e} = \frac{dN}{0.434294} = \frac{d\log_{10} L}{0.434294} \quad \text{(3d)}$$

Substitution of Eq. 2*d* in Eq. 3*d* gives

$$\frac{dL}{L} = \frac{\delta_A v_A - \delta_B v_B}{0.434294} \quad \text{(4a)}$$

from which

$$dL = L\left(\frac{\delta_A v_A - \delta_B v_B}{0.434294}\right) \quad \text{(4b)}$$

Numerical Illustration.—The validity of this formula may be demonstrated by assigning definite values to the angles of triangle 123 and a definite length to the base 2–1 in Fig. 1. From these original values the length 3–1 is first computed. Then, with changes to the three angles assumed as noted previously, the length 3–1 is recomputed. The difference, in these computed lengths, should correspond to the value of dL obtained by Eq. 4*b*. Thus, if it is assumed that angles A, B, and C measure 68° 24′ 17.81″, 59° 47′ 36.45″, and 51° 48′ 05.74″, respectively (making a total of 180° 00′ 00.00″); that v_A, v_B, and v_C are $+5''$, $-3''$, and $-2''$, respectively; and that 2–1 = 13,672.64: Then $\delta_A = 0.83$, $\delta_B = 1.22$, and log (3–1) = log 13,672.64 + log sin 68° 24′ 17.81″ − log sin 59° 47′ 36.45″.. Solution of this equation for 3–1 by seven-place log tables gives:

	Original angles	Corrected angles
log 13,672.64 =	4.1358523	4.1358523
log sin A =	+9.9683934	+9.9683976
	4.1042457	4.1042499
log sin B =	−9.9366229	−9.9366193
log (3–1) =	4.1676228	4.1676306
3–1 =	14,710.34	14,710.61
Difference =	....	0.27

Solving by Eq. 4b, $dL = L\left(\frac{\delta_A v_A - \delta_B v_B}{0.434294}\right) = \frac{14{,}710}{0.434294}(0.00000083 \times 5 + 0.00000122 \times 3) = 0.27$, which agrees with the difference previously found.

The values of δ_A and δ_B in this problem appear as eight places of decimals taken from a seven-place log table. This result is due to the fact that intervals in the table are for 10 sec of arc, whereas δ_A and δ_B are changes of log sin A and log sin B for a 1-sec change in the angle. Use of the sixth decimal place as the unit for the tabular differences δ_A and δ_B is also in accordance with the usual practice in setting up the side equation in adjustment of a quadrilateral.

General Application.—Returning to Eq. 4a, $\frac{dL}{L}$ is the relative change of the length of the side 3–1, due to the angle changes v_A and v_B. This ratio might be expressed as a percentage of change or rate of change. If no angle changes are made in the remainder of the net, all the remaining triangle sides will receive this same relative increase. Furthermore, all linear dimensions of the remainder of the net will receive this relative increase. The same may be stated concerning the line n–1 or any component of it, in which n is the designation of the end or final station.

Referring to Fig. 2, the x-component of n–1 is $x_n - x_1$. Let Δx be the change in the length of this component. Then $\frac{\Delta x}{x_n - x_1}$, the relative change in the length of this component, is

$$\frac{\Delta x}{x_n - x_1} = \frac{dL}{L} = \frac{\delta_A v_A - \delta_B v_B}{0.434294} \quad \text{(5a)}$$

from which

$$\Delta x = (x_n - x_1)\frac{\delta_A v_A - \delta_B v_B}{0.434294} \quad \text{(5b)}$$

which gives the amount the end point of the net is moved in an east-west direction as a result of a definite change in angles A and B of the beginning triangle.

Similarly,

$$\Delta y = (y_n - y_1)\,\frac{\delta_A\, v_A - \delta_B\, v_B}{0.434294} \dots\dots\dots\dots\dots\dots (6)$$

Both Δx and Δy are obtained as a product of two factors, either of which may be plus or minus. When the latter factor $(\delta_A\, v_A - \delta_B\, v_B)$ is positive, the net is expanded by the changes v_A and v_B; when it is negative, the net is contracted.

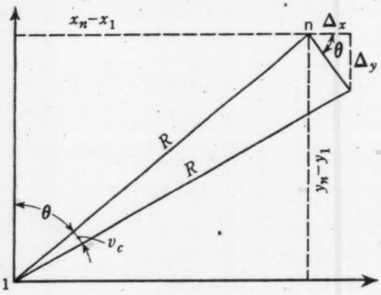


FIG. 2.—DISPLACEMENT FACTORS INVOLVED IN ANGULAR ROTATION ABOUT POINT 1

If an expanding net and positive $(x_n - x_1)$ are assumed, Δx is positive. If the end point falls west of the starting point, $x_n - x_1$ will be negative, which results in a negative Δx with an expanding net. A similar analysis also would apply to Δy. To derive Δx and Δy with proper signs, care must be taken to subtract the coordinate of the starting point from that of the end point. The computations indicated by these equations are accomplished in tabular form as illustrated by the numerical example to follow.

ROTATION DUE TO CHANGE OF ANGLE C

The effect of adding an increment to the C angle of the triangle under consideration is to produce a right-hand rotation of the line 3–1 about the fixed point 1, and such an angle increment is assumed as positive.

Any change in angle C produces a rotation of the whole net about the fixed point 1, so that the line n–1 rotates through the same angle increment as the line 3–1. Therefore, in Fig. 2, let v_C represent the increment to the angle C; let R be the distance n–1; and let θ be the bearing of line n–1. The length of the arc described by point n is $\frac{R\, v_C}{206{,}265}$, in which v_C is expressed in seconds. The east-west component of this arc is

$$\Delta x = \frac{R\, v_C}{206{,}265}\cos\theta = \frac{y_n - y_1}{206{,}265}\, v_C \dots\dots\dots\dots\dots (7a)$$

Also, the north-south component is

$$\Delta y = -\frac{R\, v_C}{206{,}265}\sin\theta = -\frac{x_n - x_1}{206{,}265}\, v_C \dots\dots\dots\dots (7b)$$

Eq. 7a with Eq. 5b, and Eq. 7b with Eq. 6, give

$$\Delta x = \frac{x_n - x_1}{434{,}294}\,(\delta_A\, v_A - \delta_B\, v_B) + \frac{y_n - y_1}{206{,}265}\, v_C \dots\dots\dots (8a)$$

and

$$\Delta y = \frac{y_n - y_1}{434{,}294}(\delta_A v_A - \delta_B v_B) - \frac{x_n - x_1}{206{,}265} v_C \qquad (8b)$$

In the triangle just completed A and B may be designated as expansion angles; and C, as the rotation angle. Line 3–1 may now be regarded as fixed and increments may be applied to the angles of triangle 134 to produce an expansion of line 4–3 and a rotation of this line about point 3, now also held fixed. Angle C is always at the intersection of the new fixed base and the next base to be established—in this case at point 3. Angle A must be opposite the next base or at point 1.

Expressions similar to Eqs. 8*a* and 8*b* can be derived to determine the components of motion Δx and Δy of the end point n caused by the application of increments to the angles of this second triangle, still holding the remaining angles of the net as fixed. With point 3 regarded as the initial point for this triangle, these equations become

$$\Delta x = \frac{x_n - x_3}{434{,}294}(\delta_A v_A - \delta_B v_B) - \frac{y_n - y_3}{206{,}265} v_C \qquad (9a)$$

$$\Delta y = \frac{y_n - y_3}{434{,}294}(\delta_A v_A - \delta_B v_B) + \frac{x_n - x_3}{206{,}265} v_C \qquad (9b)$$

It should be noted that the signs of the v_C-terms in Eqs. 9 are in opposition to those written for the first triangle—because the rotation point is at that end of the fixed line which gives a left-hand rotation when the angle at C is given a positive increment v_C.

In Fig. 1, triangle 345 has 3–4 as the fixed base with the point of rotation still at point 3. Thus, the signs in the equations for this triangle will be the same as in those for triangle 134. In this manner, all the triangles are used until the end point of the net is reached. The combined effect of the increments of the angles of each triangle applied in order is given by the expressions:

$$\Delta y = \sum \left[\frac{y_n - y_c}{434{,}294}(\delta_A v_A - \delta_B v_B) \mp \frac{x_n - x_c}{206{,}265} v_C \right] \qquad (10a)$$

$$\Delta x = \sum \left[\frac{x_n - x_c}{434{,}294}(\delta_A v_A - \delta_B v_B) \pm \frac{y_n - y_c}{206{,}265} v_C \right] \qquad (10b)$$

A sample numerical solution, based on the triangulation net of Fig. 1, is given in Fig. 3. Section (*a*) of these computations lists the fixed values of stations and bases; section (*b*), the angle adjustments for the entire net. Section (*c*) shows the preliminary angle adjustment for one quadrangle, on which the side equation, triangle computations, azimuths, and traverse computations of sections (*d*), (*e*), (*f*), and (*g*), respectively, are based. Together with similar computations for the remaining quadrangles of the net sections (*c*), (*d*), (*e*), (*f*), and (*g*) yield the preliminary coordinates shown in section (*h*) for all the stations in the net, as well as the *y*-component and *x*-component of the closure, as shown in section (*i*).

The side equation for one quadrangle, determined from section (*d*) of Fig. 3, is

$$\begin{aligned} &-3.30\,(1) + 5.77\,(2) - 2.47\,(3) - 1.57\,(4) + 3.66\,(5) - 2.09\,(6) \\ &\quad - 2.43\,(7) + 5.06\,(8) - 2.63\,(9) - 1.47\,(30) + 3.02\,(31) \\ &\qquad - 1.55\,(32) - 7.4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots \quad (11a) \end{aligned}$$

(*a*) Fixed Values

Station	Coordinates N	Coordinates E	Base	Length	Azimuth
1	20,000	20,000	1-2	14,352.09	124° 27′ 16.53″
n	50,615.79	41,078.07	n-7	15,009.58	132 08 04.58

(*b*) Angle Adjustment

Angle	Measured value	Preliminary adjustment	Correction	Final value
		Quadrilateral 1234		
− 1+ 2	32° 32′ 56.2″	56.46″	+1.19″	57.65″
− 2+ 3	40 28 47.8	47.59	+2.06	49.65
− 4+ 5	53 21 02.6	02.39	−0.85	01 54
− 5+ 6	45 12 16.7	16.41	−1 05	15 36
− 7+ 8	40 57 53.9	53.61	−0.16	53.45
− 8+ 9	38 44 11.8	11.99	−0.65	11.34
−30+31	55 05 37.8	37.98	+1.86	39.84
−31+32	53 37 13.3	13.57	−2.40	11.17
		Quadrilateral 3456		
− 9+10	39° 55′ 29.4″	28.42″	+1.41″	29.83″
−10+11	31 37 48.2	48.13	−2.47	45.66
−12+13	51 49 15.9	15.84	−0.24	15.60
−13+14	54 52 46.5	47.66	+0.89	48.55
−25+26	41 40 07.2	08.36	+1.82	10.18
−26+27	40 02 45.2	45.44	−0.86	44.58
−28+29	43 24 18.3	18.54	−1.85	16.69
−29+30	56 37 28.6	27.61	+1.30	28.91
		Quadrilateral 567n		
−14+15	49° 29′ 07.7″	09.56″	+0.15″	09.71″
−15+16	56 44 32.6	33.24	−1.83	31.41
−17+18	35 13 57.2	57.84	+1.41	59.25
−18+19	49 42 26.7	25.46	+0.90	26.36
−20+21	38 19 04.7	03.47	−0.48	02.99
−21+22	35 00 26.8	26.78	+1.75	28.53
−23+24	56 58 04.3	04.29	−2.17	02.12
−24+25	38 32 17.5	19.36	+0.27	19.63

(*c*) Preliminary Angle Adjustment—Quadrilateral 1234

	Angle	Measured value						Preliminary adjustment
	− 1+ 2	32° 32′ 56.2″	+ 0.22″	56.42″	+ 0.04″	....		56.46″
B_1	−31+32	53 37 13.3	+ 0.23	13.53	+ 0.04	....		13.57
C_2	−30+31	55 05 37.8	+ 0.22	....		38.02″	− 0.04″	37.98
	− 8+ 9	38 44 11.8	+ 0.23	....		12.03	− 0.04	11.99
		59.1						00.00
		+0.9						
	− 7+ 8	40° 57′ 53.9″	− 0.25″	....		53.65″	− 0.04″	53.61″
A_2	− 5+ 6	45 12 16.7	− 0.25	....		16.45	− 0.04	16.41
C_1	− 4+ 5	53 21 02.6	− 0.25	02.35″	+ 0.04″	....		02.39
	− 2+ 3	40 28 47.8	− 0.25	47.55	+ 0.04	....		47.59
		01.0		59.85		00.15		00.00
		−1.0		+0.15		−0.15		

Adjusted Angles

	Angle	
A_1	− 1+ 3	73° 01′ 44.05″
	− 4+ 6	98 83 18.80
B_2	− 7+ 9	79 42 05.60
	−30+32	108 42 51.55
		360 00 00.00

Fig. 3.—Sample Numerical Solution of Triangulation Net Shown in Fig. 1 (See Text)

(*d*) SIDE EQUATION

Angle	Log sin	Angle	Log sin
− 1 + 2	9.7307092 + 3.30 (− 1 + 2)	− 2 + 3	9.8123659 + 2.47 (− 2 + 3)
−30 + 31	9.9138620 + 1.47 (−30 + 31)	−31 + 32	9.9058528 + 1.55 (−31 + 32)
− 7 + 8	9.8166366 + 2.43 (− 7 + 8)	− 8 + 9	9.7963953 + 2.63 (− 8 + 9)
− 4 + 5	9.9043388 + 1.57 (− 4 + 5)	− 5 + 6	9.8510300 + 2.09 (− 5 + 6)
	9.3656366		9.3656440
	6440		
	−7.4		

(*e*) TRIANGLE COMPUTATIONS, QUADRILATERAL 1234

1-2	14,352.09 ft	log = 4.1569145
B_1	53° 37′ 13.57″	colog sin = 0.0911472
A_1	73 01 44.05	log sin = 9.9806632
C_1	53 21 02.39	log sin = 9.9043388
1-3	17,050.03 ft	log = 4.2317252
2-3	14,302.12 ft	log = 4.1554005
1-3	17,050.03 ft	log = 4.2317252
B_2	79° 42′ 05.60″	colog sin = 0.0070535
A_2	45 12 16.41	log sin = 9.8510300
C_2	55 05 37.98	log sin = 9.9138620
3-4	12,297.27 ft	log = 4.0898087
1-4	14,211.52 ft	log = 4.1526407

(*f*) AZIMUTHS

1-2	124° 27′ 16.53″
A_1	73 01 44.05
2-3	231 25 32.48
$B_1 + C_2$	108 42 51.55
3-4	302 42 40.93
B_2	79 42 05.60
4-1	43 00 35.33
$C_1 + A_2$	98 33 18.80
1-2	124 27 16.53 (check)

(*g*) TRAVERSE COMPUTATIONS

Course	Bearing	Distance	Cosine	Sine	Latitude	Departure
1-2	N 55° 32′ 43.47″ W	14,352.09	0.5657529	0.8245755	8,119.73	−11,834.38
2-3	N 51 25 32.48 E	14,302.12	0.6235291	0.7818001	8,917.79	11,181.40
3-4	S 57 17 19.07 E	12,297.27	0.5404073	0.8414034	− 6,645.53	10,346.96
4-1	S 43 00 35.33 W	14,211.52	0.7312369	0.6821236	−10,391.99	− 9,694.01
					0.00	− 0.03

(*h*) PRELIMINARY COORDINATES

Station	N	E
1	20,000.00	20,000.00
2	28,119.73	8,165.63
3	31,037.52	19,347.03
4	30,391.99	29,694.00
5	43,052.89	32,913.22
6	48,355.23	24,077.12
7	60,685.70	29,917.68
n	50,615.18	41,078.47
Fixed n	50,615.79	41,078.07

(*i*) $\Delta y = -0.61;\ \Delta x = +0.40$

Let L = length of line (7-n)

Preliminary $\log L = 4.1763772$

Fixed $\log L = 4.1763685$

$\Delta \log L = +8.7$

FIG. 3.—(*Continued*)

or, dividing by 2.918, the average coefficient is

$$-1.13\,(1) + 1.98\,(2) - 0.85\,(3) - 0.54\,(4) + 1.26\,(5) - 0.72\,(6) - 0.83\,(7) + 1.73\,(8) - 0.90\,(9) - 0.50\,(30) + 1.03\,(31) - 0.53\,(32) - 2.536 = 0 \qquad (11b)$$

EXPLANATION OF TABLE 1

Table 1, which contains the data necessary to compute the coefficients of the unknown angle corrections, provides a column for each triangle used. The headings of these columns are the number designations of the stations at which the C angles are located. Stations 3 and 5 are repeated because two C angles are located at each of them and these stations are featured in determining the algebraic signs of the coefficients. Reference to Eqs. 10*a* and 10*b* will show the significance of the numbers in the body of the table.

Extreme care should be taken in respect to the algebraic signs of the coefficients, with due regard for the order of differences and also for the signs of the products. These signs may be checked directly from Fig. 1. Considering the latitude coefficients, assume that all the *A* angles of the several triangles receive plus corrections, which will cause an expansion of each of the opposite triangle sides. This series of expansions indirectly will cause, for each triangle, an expansion of the line connecting the corresponding *C* angle station and the end station, n. When this expansion causes n to have a north component of motion, the corresponding sign of the coefficient will be plus. If the component of motion is south, the sign will be minus. Thus, if the *C* angle is south of n, the sign of the corresponding coefficient will be plus; if a *C* angle station lies north of n, the corresponding sign will be minus.

TABLE 1.—DATA FOR LATITUDE AND DEPARTURE EQUATIONS OF SAMPLE NET

Item	STATION AT *C* ANGLE (POINT OF ROTATION)					
	1	3	3	5	5	7
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Coordinates[a] N	20,000.00	37,037.52	37,037.52	43,052.89	43,052.89	60,685.70
Coordinates[a] E	20,000.00	19,347.03	19,347.03	32,913.22	32,913.22	29,947.68
Angles *A*......	73° 01′ 44.05″	45° 12′ 16.41″	71° 33′ 16.55″	43° 24′ 18.54″	95° 30′ 23.65″	56° 44′ 33.24″
Angles *B*......	53° 37′ 13.57″	79° 42′ 05.60″	51° 49′ 15.84″	81° 42′ 53.80″	35° 00′ 26.78″	84° 56′ 23.30″
Angles *C*......	53° 21′ 02.39″	55° 05′ 37.98″	56° 37′ 27.61″	54° 52′ 47.66″	49° 29′ 09.56″	38° 19′ 03.47″
δ_A............	0.65	2.09	0.71	2.23	−0.20	1.38
δ_B............	1.55	0.38	1.66	0.30	3.01	0.18
$y_n - y_c$........	30,615	13,578	13,578	7,562	7,562	−10,071
$\frac{y_n - y_c}{434,294}$........	0.0705	0.0313	0.0313	0.0174	0.0174	−0.0232
$x_n - x_c$........	21,078	21,731	21,731	8,165	8,165	11,131
$\frac{x_n - x_c}{434,294}$........	0.0485	0.0500	0.0500	0.0188	0.0188	0.0256
$\frac{y_n - y_c}{434,294}\delta_A$.....	0.0458	0.0654	0.0222	0.0388	−0.0035	−0.0320
$-\frac{y_n - y_c}{434,294}\delta_B$...	−0.1093	−0.0119	−0.0520	−0.0052	−0.0524	+0.0042
$\mp\frac{x_n - x_c}{206,265}$.....	−0.1022	0.1054	0.1054	−0.0396	−0.0396	0.0540
$\frac{x_n - x_c}{434,294}\delta_A$.....	0.0315	0.1045	0.0355	0.0419	−0.0038	+0.0353
$-\frac{x_n - x_c}{434,294}\delta_B$...	−0.0752	−0.0190	−0.0830	−0.0056	−0.0566	−0.0046
$\pm\frac{y_n - y_c}{206,265}$.....	0.1484	−0.0658	−0.0658	+0.0367	+0.0367	0.0488

[a] Coordinates of end station: N = 50,615.18; E = 41,078.47.

There is an occasional exception to this rule; for example, when the *A* angle in question is greater than 90°, an increase causes a contraction of the opposite side and the sign in question is reversed.

Since an increase in the magnitude of any *B* angle causes a contraction of the side opposite *A*, the sign of the corresponding coefficient will be the reverse of that derived from the *A* angle. As in the case of the *A* angle, if the *B* angle is more than 90°, the resulting sign is again reversed.

The signs of the latitude coefficients due to an assumed plus correction to the C angles may now be considered. In this case, these displacements of n are due to a rotation of each of the lines connecting the C angle stations with station n. It must first be determined if the rotation is clockwise or counterclockwise about the C angle station in question, and then whether this rotation causes a north or south movement of n. The signs of the latitude coefficients in the problem under consideration may then be checked by Fig. 1.

Angle Change Coefficients.—Inspection of the A angles will show that the one in the second triangle at station 5 is greater than 90°, thus the sign of the v_A-coefficient will be reversed. Also, C angle stations 1, 3, and 5 are south of n, but station 7 lies north of n. Therefore, the first four are regular and plus, the fifth is minus because the angle is greater than 90°, and the last is minus because 7 lies north of n.

Since there are no angles greater than 90° among the B angles, all the v_B-coefficients will be minus except the last, which is plus because 7 lies north of n.

Determination of the v_c-coefficients for the various triangles gives the following: In triangle 1, the rotation is clockwise, and the displacement has a south component. The sign is therefore minus; in triangles 2 and 3, C angle station 3, the rotation is counterclockwise, which produces a north displacement of n, so that these two signs are plus; in triangles 4 and 5, C angle station 5,

TABLE 2.—COMPUTATION OF COEFFICIENTS IN LATITUDE EQUATION OF SAMPLE NET (SEE TABLE 1)

Direction No.	TRIANGLE DESIGNATION						Final coefficients
	123	134	345	356	567	57n	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(1)	−0.0458	....	....	....	....	....	−0.0458
(2)	....	....	....	....	....	....	....
(3)	+0.0458	....	....	....	....	....	+0.0458
(4)	+0.1022	....	....	....	....	....	+0.1022
(5)	−0.1022	−0.0654	....	....	....	....	−0.1676
(6)	....	+0.0654	....	....	....	....	+0.0654
(7)	....	+0.0119	....	....	....	....	+0.0119
(8)	....	....	....	....	....	....	....
(9)	....	−0.0119	−0.0222	....	....	....	−0.0341
(10)	....	....	....	....	....	....	....
(11)	....	....	+0.0222	....	....	....	+0.0222
(12)	....	....	+0.0520	....	....	....	+0.0520
(13)	....	....	−0.0520	+0.0396	....	....	−0.0124
(14)	....	....	....	−0.0396	+0.0396	....	....
(15)	....	....	....	....	−0.0396	+0.0320	−0.0076
(16)	....	....	....	....	....	−0.0320	−0.0320
(17)	....	....	....	....	....	−0.0042	−0.0042
(18)	....	....	....	....	....	....	....
(19)	....	....	....	....	....	+0.0042	+0.0042
(20)	....	....	....	....	....	−0.0540	−0.0540
(21)	....	....	....	....	+0.0524	+0.0540	+0.1064
(22)	....	....	....	....	−0.0524	....	−0.0524
(23)	....	....	....	....	+0.0035	....	+0.0035
(24)	....	....	....	....	....	....	....
(25)	....	....	....	+0.0052	−0.0035	....	+0.0017
(26)	....	....	....	....	....	....	....
(27)	....	....	....	−0.0052	....	....	−0.0052
(28)	....	....	....	−0.0388	....	....	−0.0388
(29)	....	....	−0.1054	+0.0388	....	....	−0.0666
(30)	....	−0.1054	+0.1054	....	....	....	....
(31)	+0.1093	+0.1054	....	....	....	....	+0.2147
(32)	−0.1093	....	....	....	....	....	−0.1093

the rotation is clockwise and the displacements of n are south, producing minus coefficients; and in triangle 6, *C* angle station 7, the rotation is counterclockwise, producing a north displacement and plus coefficient.

The departure equation coefficients may be checked by a similar analysis, by considering the *A* angle changes as expansions, the *B* angle changes as contractions, and the *C* angle changes as rotations, clockwise or counterclockwise, and then determining whether the resulting displacements are east (+) or west (−).

TRANSFORMATION TO DIRECTIONS

Corrections to the angles *A*, *B*, and *C* are designated by the symbols v_A, v_B, and v_C. However, the direction method is generally used in reduction of a triangulation net; and, since each angle is the difference of two directions, each angle correction will be the difference between corresponding direction corrections. As is customary, direction corrections are designated by numbers within parentheses.

It will be found convenient to use the form of Tables 2 and 3 to complete the final latitude and departure equations, with a column allowed for each triangle. Since there are three angles in each triangle, three pairs of coefficients will appear in each column. For example, in triangle 123, Fig. 1, the

TABLE 3.—COEFFICIENTS FOR DEPARTURE EQUATION OF SAMPLE NET (SEE TABLE 1)

Direction No.	Triangle Designation						Final coefficients
	123	134	345	356	567	57n	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(1).........	−0.0315	....	....	....	....	....	−0.0315
(2).........	....	....	....	....	....	....	....
(3).........	+0.0315	....	....	....	....	....	+0.0315
(4).........	−0.1484	....	....	....	....	....	−0.1484
(5).........	+0.1484	−0.1045	....	....	....	....	+0.0439
(6).........	....	+0.1045	....	....	....	....	+0.1045
(7).........	....	+0.0190	....	....	....	....	+0.0190
(8).........	....	....	....	....	....	....	....
(9).........	....	−0.0190	−0.0355	....	....	....	−0.0545
(10).........	....	....	....	....	....	....	....
(11).........	....	....	+0.0355	....	....	....	+0.0355
(12).........	....	....	+0.0830	....	....	....	+0.0830
(13).........	....	....	−0.0830	−0.0367	....	....	−0.1197
(14).........	....	....	....	+0.0367	−0.0367	....	....
(15).........	....	....	....	....	+0.0367	−0.0353	+0.0014
(16).........	....	....	....	....	....	+0.0353	+0.0353
(17).........	....	....	....	....	....	+0.0046	+0.0046
(18).........	....	....	....	....	....	....	....
(19).........	....	....	....	....	....	−0.0046	−0.0046
(20).........	....	....	....	....	....	−0.0488	−0.0488
(21).........	....	....	....	....	+0.0566	+0.0488	+0.1054
(22).........	....	....	....	....	−0.0566	....	−0.0566
(23).........	....	....	....	....	+0.0038	....	+0.0038
(24).........	....	....	....	....	....	....	....
(25).........	....	....	....	+0.0056	−0.0038	....	+0.0018
(26).........	....	....	....	....	....	....	....
(27).........	....	....	....	−0.0056	....	....	−0.0056
(28).........	....	....	....	−0.0419	....	....	−0.0119
(29).........	....	....	+0.0658	+0.0419	....	....	+0.1077
(30).........	....	+0.0658	−0.0658	....	....	....	....
(31).........	+0.0752	−0.0658	....	....	....	....	+0.0094
(32).........	−0.0752	....	....	....	....	....	−0.0752

following relations for the latitude equation are found:

$$0.0458\, v_A = 0.0458\, [(3) - (1)] = -\,0.0458\, (1) + 0.0458\, (3) \ldots\ldots (12a)$$

$$-\,0.1093\, v_B = -\,0.1093\, [(32) - (31)] = 0.1093\, (31) - 0.1093\, (32) \ldots\ldots (12b)$$

$$-\,0.1022\, v_C = -\,0.1022\, [(6) - (5)] = 0.1022\, (5) - 0.1022\, (6) \ldots\ldots\ldots (12c)$$

The final coefficient of each direction correction is found by adding the individual coefficients opposite that direction horizontally in each table.

AZIMUTH EQUATION

If the measured (field) angles are used in the derivation of equations for the final adjustment, the *C* angles, or corresponding directions, must be used in deriving the azimuth equation. However, if a preliminary adjustment of the angles in each quadrilateral is made, as in the sample net, the angles are consistent throughout the net and any chain of directions connecting the two fixed bases may be used. Reference to Fig. 1 shows that the simplest form of the azimuth equation is given by the following combination of directions:

$$-\,(4) + (6) - (7) + (11) - (12) + (16) - (17) + (19) + 2.50 = 0 \ldots (13)$$

In Eq. 13, the constant term is found by the calculations shown in Table 4.

TABLE 4.—DETERMINATION OF CONSTANT TERM IN EQ. 13

Line or angle	AZIMUTH		
	°	′	″
1−2..	124	27	16.53
− 4+ 6..	98	33	18.80
− 7+11..	151	15	22.15
−12+16..	212	55	46.30
−17+19..	84	56	23.30
	672	08	07.08
3×180°....	−540	00	00.00
n−7.......	132	08	07.08
n−7 (fixed).	132	08	04.58
$\Delta\alpha$........			+02.50

The *A* and *B* angles are used to write an equation based on the length conditions connecting the two measured bases. Both of these distance measurements should be reduced to the standard elevation governing the coordinates of the two fixed (U. S. C. and G. S.) stations. If the net is to be computed on one of the state-wide systems of coordinates, the field measurement of each base must be reduced to sea level distance, and further corrected to the scale of the standard system.

A system of local coordinates is sometimes preferable for a small area such as a county or metropolitan district. For rural counties, it is customary to reduce all measured distances to sea level datum. In a large city, however, it may be desirable to make the published results of the control survey agree as nearly as possible with values which the local surveyor would find by accurate measurement on the ground. This can be accomplished by a system of local coordinates, reducing all distances (including distances between fixed stations) to a mean elevation selected for the district.

The distance equation for the sample net under consideration is derived from the computations shown in Table 5, and has the form:

$$\begin{aligned} &-\,0.65\,(1) + 0.65\,(3) - 2.09\,(5) + 2.09\,(6) + 0.38\,(7) - 1.09\,(9) \\ &\quad + 0.71\,(11) + 1.66\,(12) - 1.66\,(13) - 2.23\,(28) + 2.23\,(29) \\ &\qquad + 1.55\,(31) - 1.55\,(32) + 8.7 = 0 \ldots\ldots\ldots\ldots\ldots (14) \end{aligned}$$

TABLE 5.—Determination of Coefficients in Eq. 14

Side or angle (1)	Value ° ′ ″ (2)	Log or log sin (3)	Side or angle (4)	Value ° ′ ″ (5)	Log or log sin (6)
1-2....	....	4.1569145	7-n...	....	4.1763685
A_1.....	73 01 44.07	9.9806632 +0.65 (− 1+ 3)	B_1....	53 37 13.57	9.9058528 +1.55 (−31+32)
A_2.....	45 12 16.41	9.8510300 +2.09 (− 5+ 6)	B_2....	79 42 05.60	9.9929465 +0.38 (− 7+ 9)
A_3.....	71 33 16.55	9.9770948 +0.71 (− 9+11)	B_3....	51 49 15.84	9.8954690 +1.66 (−12+13)
A_4.....	43 24 18.54	9.8370533 +2.23 (−28+29)	B_4....	81 42 53.80	9.9954436 +0.30 (−25+27)
A_5.....	95 30 23.65	9.9979912 −0.20 (−23+25)	B_5....	35 00 26.78	9.7586718 +3.01 (−21+22)
A_6.....	56 44 33.24	9.9223180 +1.38 (−15+16)	B_6....	84 56 23.30	9.9983041 +0.18 (−17+19)
		3.7230650			3.7230563
		3.7230563			
log ΔL		+8.7			

A Practical Application

The principles and methods described in the preceding section were used to advantage on a triangulation project in Logan County, Ohio, where it was decided to develop a sample area between four U. S. C. and G. S. triangulation stations to show some of the problems involved in extending geodetic control in a rural area. In addition, the completed project was intended to illustrate

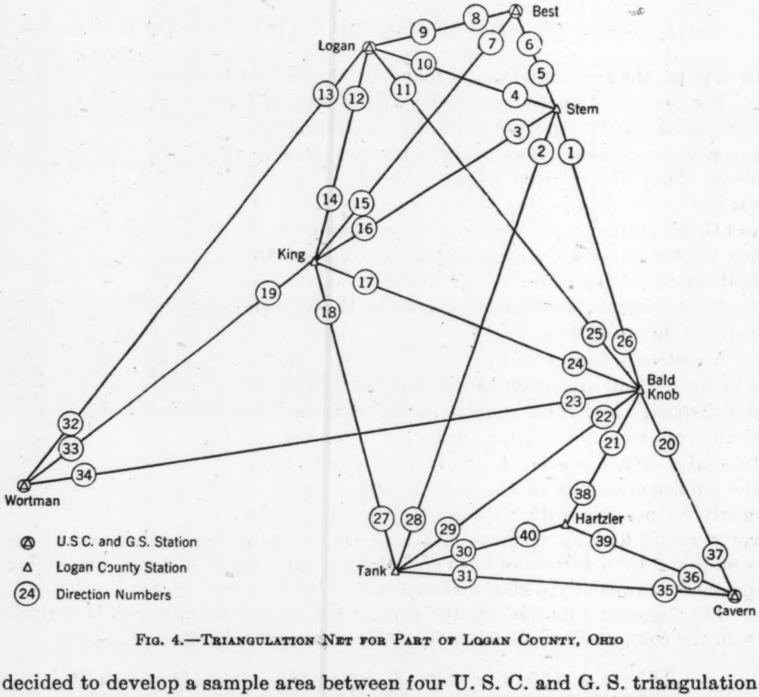

Best
Logan
Stem
King
Bald Knob
Wortman
Hartzler
Tank
Cavern
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
U.S C. and G.S. Station
Logan County Station
24 Direction Numbers

Fig. 4.—Triangulation Net for Part of Logan County, Ohio

some of the values of horizontal control for section, township, and other survey corners in such an area. Rolling terrain called for triangulation rather than traverse for the basic control, even considering the fact that triangulation alone does not tie in any section corners.

A reconnaissance party set out to establish a chain of quadrilaterals connecting U. S. C. and G. S. stations "Logan" and "Best" on the north with "Cavern" and "Wortman" on the south. The net shown in Fig. 4 was the result. It has several overlapping figures, a rather undesirable characteristic in a demonstration project, which should be kept as simple as possible. However, triangulation reconnaissance for daylight observation without the aid of towers is not always easy.

TABLE 6.—COORDINATES OF UNITED STATES COAST AND GEODETIC SURVEY STATIONS IN LOGAN COUNTY NET

Station	U. S. C. AND G. S. COORDINATES		LOCAL COORDINATES	
	Latitude ° ′ ″	Longitude ° ′ ″	North	East
(1)	(2)	(3)	(4)	(5)
Logan....	40 21 38.509	83 45 33.675	79,610.12	97,393.08
Best......	40 21 54.618	83 43 18.054	81,241.39	107,891.53
Cavern...	40 14 59.354	83 40 21.441	39,227.22	121,599.65
Wortman.	40 16 38.714	83 51 06.039	49,228.66	71,628.59
Origin....	40 25 00.000	83 45 00.000	100,000.00	100,000.00

The net shown is well controlled by the U. S. C. and G. S. stations at each of the four extreme corners. It was therefore considered complete in itself and adjusted for final coordinates. Since the project was intended to establish a local coordinate system for the county, the local coordinates of the four known stations were computed as shown in Table 6.

The angles were observed in daylight with a direction theodolite reading to single seconds. Special signals were constructed for the purpose, including two heliotropes for use on the longer lines. Triangle closures on the seventeen triangles averaged 2.23 sec, with a maximum of 5.80 sec.

LEAST SQUARES ADJUSTMENT

The first adjustment involving nineteen equations (thirteen angle equations and six side equations) was completed by least squares methods commonly used for this type of problem. A set of preliminary adjusted angles resulted which made all triangles in the net entirely consistent, but did not require that the total net fit between fixed triangulation stations in either latitude or departure.

Starting with the fixed line Logan-Wortman and using corrected angles from the first adjustment, the lengths and azimuths of lines Logan-Bald Knob and Bald Knob-Cavern were computed with the triangles shown in Fig. 5. By use of these two lines and the known coordinates of Logan, the local coordinates of Cavern were found to be N 39,228.14 and E 121,599.38, which, when compared with the known coordinates of Cavern given in Table 6, show a closing error of +0.92 ft in latitude and −0.27 ft in departure. This discrepancy cannot be adjusted satisfactorily by any simple arbitrary method such as is commonly used for traverses, but should be eliminated by small changes in all the angles shown in Fig. 5. Corrections to other angles of the net also will be involved, since the triangle closures and side equation relations set up in the original least squares solution must not be disturbed. To accomplish this result, a complete new solution is needed using the former nineteen condition equations based on corrected angles plus two additional equations to take out the errors in latitude and departure, respectively. The coefficients for the first nineteen equations will be the same as before, but the constant term will be zero in each case since the adjusted equations are based on corrected angles. The new solution is accomplished efficiently by adding the necessary columns to the tabular form of the preliminary solution.

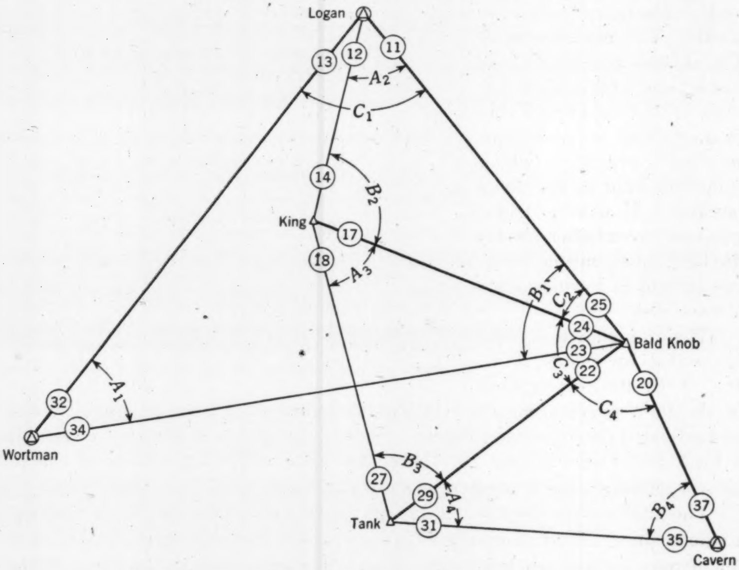


FIG. 5.—TRIANGLES FOR DERIVATION OF LATITUDE AND DEPARTURE EQUATIONS OF LOGAN COUNTY NET

FORMATION OF LATITUDE AND DEPARTURE EQUATIONS

The latitude and departure equations take the form of Eqs. 10*a* and 10*b*, in which x_n and y_n are the fixed coordinates of the end point, and x_c and y_c are the coordinates of the station at the *C* angle of each triangle used. Also, the constant terms are

$$\Delta y = y_n - y'_n \quad \text{.....................} (15a)$$

$$\Delta x = x_n - x'_n \quad \text{.....................} (15b)$$

in which x'_n and y'_n are preliminary coordinates of the end point as found from the first adjustment.

Derivation of the coefficients of the latitude and departure equations is greatly facilitated by the tabular arrangement shown in Tables 7 and 8. These tables are similar to Tables 1, 2, and 3 for the sample net of Fig. 1.

TABLE 7.—DATA FOR LATITUDE AND DEPARTURE EQUATIONS OF LOGAN COUNTY NET

Item	STATION AT *C* ANGLE (POINT OF ROTATION)			
	Logan	Bald Knob	Bald Knob	Bald Knob
(1)	(2)	(3)	(4)	(5)
Triangle	1	2	3	4
Coordinates[a] N	79,610.12	54,054.92	54,054.92	54,054.92
Coordinates[a] E	97,393.08	115,295.23	115,295.23	115,295.23
Angles *A*	43° 24′ 57.0″	51° 23′ 40.2″	53° 31′ 29.4″	40° 45′ 38.0″
Angles *B*	61° 13′ 00.8″	98° 33′ 08.5″	66° 47′ 38.8″	60° 57′ 05.0″
Angles *C*	75° 22′ 02.2″	30° 03′ 11.3″	59° 40′ 51.9″	78° 17′ 17.0″
δ_A	2.22	1.69	1.56	2.44
δ_B	1.16	− 0.31	0.90	1.17
$y_n - y_c$	−40,383	−14,828	−14,828	−14.828
$\frac{y_n - y_c}{434{,}294}$	−0.0930	−0.0341	−0.0341	−0.0341
$x_n - x_c$	24,207	6,304	6,304	6,304
$\frac{x_n - x_c}{434{,}294}$	0.0557	0.0145	0.0145	0.0145
$\frac{y_n - y_c}{434{,}294}\delta_A$	− 0.2065	− 0.0576	− 0.0532	− 0.0832
$-\frac{y_n - y_c}{434{,}294}\delta_B$	0.1079	0.0106	0.0307	0.0399
$\mp\frac{x_n - x_c}{206{,}265}$	0.1174	0.0306	0.0306	0.0306
$\frac{x_n - x_c}{434{,}294}\delta_A$	0.1237	0.0245	0.0226	0.0354
$-\frac{x_n - x_c}{434{,}294}\delta_B$	− 0.0646	− 0.0045	− 0.0131	− 0.0170
$\pm\frac{y_n - y_c}{206{,}265}$	0.1958	0.0719	0.0719	0.0719

[a] Coordinates of end station: N = 39,227.22; E = 121,599.65.

The latitude equation is written by inserting the coefficients in Col. 6, Table 8, using the symbols (11), (12), $\cdots$, (37) to denote the unknown corrections. The constant term represents the fixed N-coordinate of the end point (Cavern) minus the computed coordinate of that point, in this case −0.92. Multiplication of the equation by 10 for more convenient use in the solution gives

$$\begin{aligned}&-0.598\,(11) - 0.576\,(12) + 1.174\,(13) - 0.106\,(14) + 0.638\,(17)\\&-0.532\,(18) - 0.306\,(20) - 1.079\,(23) + 1.385\,(25) - 0.307\,(27)\\&+1.139\,(29) - 0.832\,(31) + 2.065\,(32) - 2.065\,(34) - 0.399\,(35)\\&\qquad + 0.399\,(37) + 9.2 = 0 \qquad \text{(16a)}\end{aligned}$$

By a corresponding procedure, the departure equation is written from coefficients in Col. 11, Table 8, as follows:

$$\begin{aligned}&-2.203\,(11)+0.245\,(12)+1.958\,(13)+0.045\,(14)-0.271\,(17)\\&+0.226\,(18)-0.719\,(20)+0.646\,(23)+0.073\,(25)+0.131\,(27)\\&-0.485\,(29)+0.354\,(31)-1.237\,(32)+1.237\,(34)+0.170\,(35)\\&\qquad-0.170\,(37)-2.7=0\;\ldots\ldots\ldots\ldots\ldots\ldots\;(16b)\end{aligned}$$

TABLE 8.—COMPUTATION OF COEFFICIENTS IN LATITUDE AND DEPARTURE EQUATIONS FOR LOGAN COUNTY NET

Direction No.	(a) LATITUDE EQUATION					(b) DEPARTURE EQUATION				
	Triangle				Final coefficient	Triangle				Final coefficient
	1	2	3	4		1	2	3	4	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
(11)	−0.1174	+0.0576	....	....	−0.0598	−0.1958	−0.0245	....	....	−0.2203
(12)	....	−0.0576	....	....	−0.0576	....	+0.0245	....	....	+0.0245
(13)	+0.1174	....	....	....	+0.1174	+0.1958	....	....	....	+0.1958
(14)	....	−0.0106	....	....	−0.0106	....	+0.0045	....	....	+0.0045
(15)	....	....	....	....	....	....	....	....	....	....
(16)	....	....	....	....	....	....	....	....	....	....
(17)	....	+0.0106	+0.0532	....	+0.0638	....	−0.0045	−0.0226	....	−0.0271
(18)	....	....	−0.0532	....	−0.0532	....	....	+0.0226	....	+0.0226
(19)	....	....	....	....	....	....	....	....	....	....
(20)	....	....	....	−0.0306	−0.0306	....	....	....	−0.0719	−0.0719
(21)	....	....	....	....	....	....	....	....	....	....
(22)	....	....	−0.0306	+0.0306	....	....	....	−0.0719	+0.0719	....
(23)	−0.1079	....	....	....	−0.1079	+0.0646	....	....	....	+0.0646
(24)	....	−0.0306	+0.0306	....	....	....	−0.0719	+0.0719	....	....
(25)	+0.1079	+0.0306	....	....	+0.1385	−0.0646	+0.0719	....	....	+0.0073
(26)	....	....	....	....	....	....	....	....	....	....
(27)	....	....	−0.0307	....	−0.0307	....	....	+0.0131	....	+0.0131
(28)	....	....	....	....	....	....	....	....	....	....
(29)	....	....	+0.0307	+0.0832	+0.1139	....	....	−0.0131	−0.0354	−0.0485
(30)	....	....	....	....	....	....	....	....	....	....
(31)	....	....	....	−0.0832	−0.0832	....	....	....	+0.0354	+0.0354
(32)	+0.2065	....	....	....	+0.2065	−0.1237	....	....	....	−0.1237
(33)	....	....	....	....	....	....	....	....	....	....
(34)	−0.2065	....	....	....	−0.2065	+0.1237	....	....	....	+0.1237
(35)	....	....	....	−0.0399	−0.0399	....	....	....	+0.0170	+0.0170
(36)	....	....	....	....	....	....	....	....	....	....
(37)	....	....	....	+0.0399	+0.0399	....	....	....	−0.0170	−0.0170

Eqs. 16, when added to the preceding nineteen equations, gave final corrections to the angles, with the result that the position closure on the final station, Cavern, was 0.02 ft in latitude and 0.01 ft in departure. In this manner, the entire supplementary net was made consistent with U. S. C. and G. S. stations in the county. If it is assumed that U. S. C. and G. S. results are acceptable without local verification—certainly a logical assumption for a rural area—this method eliminates the need of measuring a base line or of determining a geodetic azimuth.

DISCUSSION

Julius L. Speert,[5] M. ASCE.—A strong case is presented in this paper for the adjustment of local triangulation nets to the basic geodetic datum when ties are available at two or more points. The subject is particularly timely because of two parallel but independent developments in recent years. The first of these is the continued expansion and the breakdown of the basic triangulation net by the establishment of a large number of closely spaced geodetic stations. This has brought geodetic control into many regions that previously had been without it. The second important development was the introduction and acceptance of the state plane coordinate systems.

These innovations make it possible to express geodetic data in terms of simple x-coordinates and y-coordinates that are convenient for use by all local surveyors. The old excuse that geodetic data are too complicated for purely local use is no longer applicable. To conform with the trend of bringing the strength of geodetic control within the scope of the local surveyor's use, the methods of geodetic computation must be simplified in their application to small local survey nets computed on a plane coordinate grid. The paper presents a worthy contribution in that direction.

It is appropriate to stress that the method of adjustment proposed by the authors is essentially the same as that recommended and used for many years by the United States Coast and Geodetic Survey and described in detail in its publications.[6,7] The principal improvement made by the authors is the adaptation of the method to computation in plane coordinates and their demonstration of its true simplicity.

The problem of adjusting a local net to the basic geodetic datum will fall naturally into one of two cases—according to whether the local net was established before or after the ties to the basic datum became available. Each case is subject to its own particular type of treatment.

In the first case, where the local net was established first, and properly adjusted internally before the geodetic datum was extended into the area, the problem becomes one of readjustment of datum. Perhaps the simplest method of datum readjustment is by the use of "isodiffs." This method has been described in its application to geodetic triangulation,[8] but the appropriate simplifications should be readily apparent to any engineer who is interested in using it on a plane coordinate system. The method is especially advantageous when either funds or time for complete recomputation and adjustment are not readily available. If the original adjustment was made by the least squares method of condition equations, and if the original computations are available to the computers, the addition of equations for the new conditions imposed by the ties (as recommended by the authors) offers an excellent opportunity to complete the adjustment without repeating the bulk of the original computations. Both methods take full advantage of the previous computations and adjustment.

---

[5] Topographic Engr., U. S. Geological Survey, Washington, D. C.

[6] "Application of the Theory of Least Squares to the Adjustment of Triangulation," *Special Publication No. 28*, U. S. Coast and Geodetic Survey, 1915.

[7] "Manual of Triangulation Computation and Adjustment," *Special Publication No. 138*, U. S. Coast and Geodetic Survey, 1934.

[8] "Readjustment of Triangulation Datum," by Julius L. Speert, *Transactions*, ASCE, Vol. 103, 1938, p. 1002.

In the second case, when ties to the basic geodetic datum are available at the time the local net is established, the original adjustment should, of course, be based on that datum. In this case the computer has greater latitude in his choice of method. The two basic methods of geodetic adjustment—condition equations and variation of coordinates—are both available. Both lend themselves readily to considerable simplification in their application to a small local net on a plane coordinate system.

The authors have presented an excellent description of the condition equation method. However, for this particular type of survey net the coordinate method is usually so far superior to the condition method as to economy of labor that it would appear to justify far more consideration than it has received. A detailed description of its application to a plane coordinate system was published as far back as 1935.[9] In this method there will always be just two normal equations for each new station established, regardless of the complexity of the net, or of the amount of overlapping of its component parts. Thus, the adjustment in Fig. 1 would require the solution of only eight simultaneous equations as compared to the sixteen equations of the condition method; and the adjustment in Fig. 4 would require the solution of ten equations instead of twenty one. The addition of further ties, check observations, or other complexities to the problem merely increases the advantages of the coordinate method over the condition method.

C. H. NEY,[10] ESQ.—An interesting and practicable method of adjusting a local triangulation net between two fixed points, that have been established with a higher degree of accuracy, has been outlined by the authors. As the positions of the survey stations are defined in terms of rectangular coordinates, surveyors and city engineers who have occasion to make use of this type of adjustment will be spared the necessity of familiarizing themselves with the computation of geodetic latitudes and longitudes.

With one of the fixed stations taken as the origin, the latitudes and departures of the other triangulation stations are determined. A traverse, composed of selected triangle sides joining the fixed points, is used in the formation of the condition equations of closure. For the determination of the absolute terms of these equations, the latitude and departure of the second fixed point must be computed through the traverse. The accuracy of the triangulation adjustment hinges upon the correct determination of these values, which, in reality, define the azimuth and length of the polar ray joining the two fixed points.

The authors compute the required latitude and departure of the second fixed point by plane trigonometry on the premise that the survey is of such limited extent that no error will result from neglecting spherical excess and the curvature of the earth. However, it should be kept in mind that there is a definite limit to the physical extent of the local survey beyond which the authors' methods of adjustment will not be applicable. In this discussion an attempt will be made to formulate a criterion for the determination of this limit.

---

[9] "Triangulation Adjustments in Plane Coordinate Systems," by R. M. Wilson, *Engineering News-Record*, October 3, 1935, p. 459.

[10] Chf., Triangulation Adjustments Div., Geodetic Survey of Canada, Dept. of Mines and Resources, Ottawa, Ont., Canada.

In 1928, W. M. Tobey, then Assistant Director of the Geodetic Survey of Canada, issued a bulletin[11] outlining a method whereby a traverse having courses of almost unlimited length may be treated in terms of latitudes and departures by an application of Legendre's theorem to the polar triangles formed by lines joining each traverse station with the origin.

Fig. 6(a) shows a traverse on the spheroid, and Fig. 6(b) shows a corresponding "Legendrian" traverse on a plane where the measured deflection angles, $\beta_2, \beta_3, \beta_4, \cdots, \beta_n$, are increased by $\frac{\epsilon_2}{3}, \frac{\epsilon_2 + \epsilon_3}{3}, \frac{\epsilon_3 + \epsilon_4}{3}, \cdots, \frac{\epsilon_{n-1} + \epsilon_n}{n - 1}$, and where the distances, $t_1, t_2, t_3, \cdots, t_n$, of the Legendrian traverse are identical with the corresponding values determined from actual measurement on the earth's surface. It is seen that the deflection angle at any point of the corresponding Legendrian traverse is greater than the corresponding deflection angle of the actual traverse by one third of the sum of the spherical excesses of the polar triangles adjacent to the polar ray.

The bearings, $b_1, b_2, b_3, \cdots, b_n$, of the courses of the Legendrian traverse are

$$b_1 = b_{1a} \dots\dots\dots\dots (17a)$$

$$b_2 = b_{2a} + \frac{\epsilon_2}{3} \dots\dots\dots\dots (17b)$$

$$b_3 = b_{3a} + \frac{2}{3}(\epsilon_2 + \epsilon_3) - \frac{\epsilon_3}{3} \dots\dots\dots\dots (17c)$$

and

$$b_n = b_{na} + \frac{2}{3}(\epsilon_2\epsilon + {}_3 + \cdots + \epsilon_n) - \frac{\epsilon_n}{3} \dots\dots\dots\dots (17d)$$

in which $b_{1a}, b_{2a}, \cdots, b_{na}$ are approximate bearings of the measured courses referred to the meridian through the origin, and computed from the north, as follows:

$$b_{1a} = \beta_1 \dots\dots\dots\dots (18a)$$

$$b_{2a} = \beta_1 + \beta_2 \dots\dots\dots\dots (18b)$$

and

$$b_{na} = \beta_1 + \beta_2 + \beta_3 + \cdots + \beta_n \dots\dots\dots\dots (18c)$$

The Legendrian latitudes and departures for the nth-station for the traverse are

$$L_n = L_{na} - [\tfrac{2}{3}(\epsilon_2 + \epsilon_3 + \epsilon_n) - \tfrac{1}{3}\epsilon_n] D_{na} \sin 1'' \dots\dots\dots\dots (19a)$$

and

$$D_n = D_{na} + [\tfrac{2}{3}(\epsilon_2 + \epsilon_3 + \epsilon_n) - \tfrac{1}{3}\epsilon_n] L_{na} \sin 1'' \dots\dots\dots\dots (19b)$$

in which L_n and D_n are the Legendrian values of the latitudes and departures, respectively, and

---

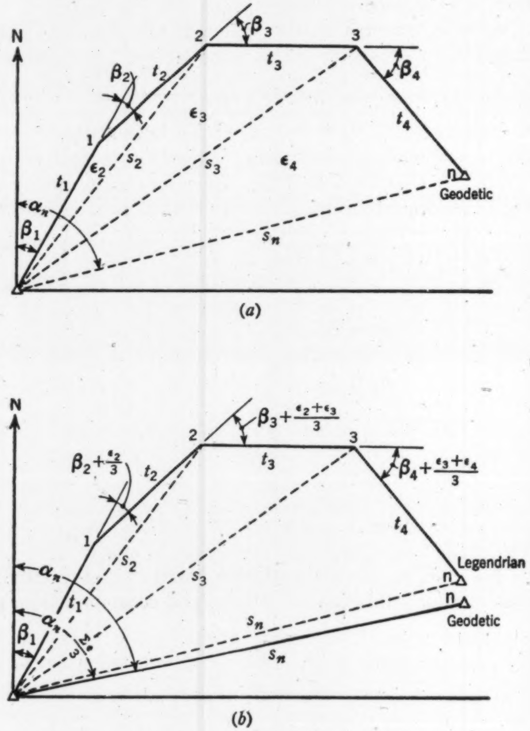
[11] "The Conversion of Latitudes and Departures of a Traverse to Geodetic Differences of Latitude and Longitude," by W. M. Tobey, *Publication No. 25*, Geodetic Survey of Canada, Ottawa, Ont., 1928.

$$L_{na} = t \cos b_{na} \ldots\ldots (20a)$$

and

$$D_{na} = t \sin b_{na} \ldots\ldots (20b)$$

are the approximate values as computed in ordinary traverse reductions. The addition of the corrective terms to the approximate latitudes and departures allows for reduction of traverses up to 120 miles in length with geodetic precision. Before the length and azimuth of a polar ray that extends from the origin to any one of the traverse stations may be determined, it is necessary to calculate the spherical excesses of the polar triangles.


N
2
3
1
n
β_3
β_2
β_4
β_1
α_n
t_1
t_2
t_3
t_4
ϵ_2
ϵ_3
ϵ_4
s_2
s_3
s_n
Geodetic
N
2
3
1
n
n
$\beta_3+\frac{\epsilon_2+\epsilon_3}{3}$
$\beta_2+\frac{\epsilon_2}{3}$
$\beta_4+\frac{\epsilon_3+\epsilon_4}{3}$
β_1
α_n
α_n
$\frac{\Sigma\epsilon}{3}$
t_1
t_2
t_3
t_4
s_2
s_3
s_n
s_n
Legendrian
Geodetic

(a)

(b)

FIG. 6

The spherical excesses of the polar triangles are determined from

$$\epsilon''_n = \frac{D_{na} L_{(n-1)a} - D_{(n-1)a} L_{na}}{2 R N \sin 1''} + K \ldots\ldots (21)$$

in which K is a second order term of negligible magnitude. It is seen that, in calculating the excesses, the approximate values of the latitudes and departures may be used. Attention must be given to the correct signs of the latitudes and departures—points north of the origin have a plus latitude and points east of the origin have a plus departure.

Having determined the corrected latitude and departure of the terminal station of the traverse, the azimuth and length of the polar ray from the origin to this station may be found. If ψ represents the angle on the plane between the initial meridian and the polar ray,

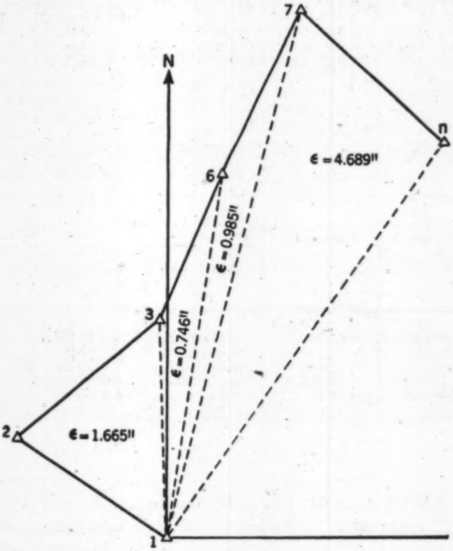


FIG. 7

$$\tan \psi = \frac{D_n}{L_n} = \frac{D_a + dD}{L_a + dL} \qquad (22)$$

The true azimuth corresponding to ψ is given by

$$\alpha_n = \psi_n + \left(\frac{\epsilon_2}{3} + \frac{\epsilon_3}{3} + \cdots + \frac{\epsilon_n}{3} \right) \qquad (23)$$

and the length of the polar ray is

$$S_n = \frac{D_n}{\sin \psi_n} = \frac{L_n}{\cos \psi_n} \qquad (24)$$

In adjusting local triangulation between widely separated fixed points by the methods suggested by the authors, the accuracy of the adjustment may be increased by taking into consideration the Legendrian corrections to the latitudes and departures.

Example 1.—Taking the length of the line 1–2 and the adjusted angles (figural adjustment only) pertaining to the authors' first numerical example,

TABLE 9.—APPROXIMATE LATITUDES AND DEPARTURES OF COURSES

Course or station	Bearing[a]	LOGARITHM OF:					Latitude[a,b] L_a	Departure[a,b] D_a
		$t^{(b)}$	cos b_a	sin b_a	Latitude[a]	Departure[a]		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1–2	304° 27′ 16.53″	4.4190809	9.7526268	9.9162301	4.1717077	4.3353110	14,849.36	−21,642.68
2–3	51° 25′ 32.48″	4.4175670	9.7948570	9.8930957	4.2124240	4.3106627	16,308.86	20,448.56
3–6	22° 40′ 54.78″	4.3508778	9.9650417	9.5861532	4.3159195	3.9370310	20,697.58	8,650.29
6–7	25° 27′ 37.33″	4.3974938	9.9556314	9.6333540	4.3531252	4.0308478	22,548.89	10,736.13
7–n	132° 08′ 07.08″	4.4385440	9.8266472	9.8701478	4.2651912	4.3086918	−18,415.83	20,355.97
n	....	....	....	....	....	....	55,988.86	38,548.27

[a] Approximate. [b] Measured in meters.

TABLE 10.—COMPUTATION

Station	Latitude (m)	Departure (m)	LOGARITHM		$L_m D_{m+1}$
			Latitude	Departure	
(1)	(2)	(3)	(4)	(5)	(6)
1	0.00	0.00			
2	14,849.36	−21,642.68	4.1717072	4.3353106	
3	31,158.22	− 1,194.12	4.4935729	3.0770480	− 17,731,893
6	51,855.80	+ 7,456.17	4.7147972	3.8725176	+ 232,322,069
7	74,404.69	+18,192.30	4.8716002	4.2598876	+ 943,375,906
n	55,988.86	+38,548.27	4.7481018	4.5860055	+2,868,175,414

[a] m =

TABLE 11.—EVALUATION OF ϵ-CORRECTIONS TO APPROXIMATE BEARINGS

Polar triangle	ϵ (″)	$\epsilon/3$ (″)	$2\epsilon/3$ (″)	$\Sigma\frac{2\epsilon}{3}$ (″)	CORRECTIONS TO APPROXIMATE BEARINGS		Course
					db = Col. 5 − Col. 3 (″)	Log db	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1–2–3	1.6646	0.5549	1.1097	1.1097	+0.5548	9.74414	2–3
1–3–6	0.7459	0.2486	0.4973	1.6070	+1.3584	0.13303	3–6
1–6–7	0.9851	0.3284	0.6567	2.2637	+1.9353	0.28675	6–7
1–7–n	4.6890	1.5630	3.1260	5.3897	+3.8267	0.58282	7–n
		2.6949					

TABLE 12.—LEGENDRIAN

Course	b_a	Log t (m)	db (″)	Log db	Sin 1″	$L_a = t \cos b_a$
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1–2	304° 27′ 16.53″	4.4190809				14,849.36
2–3	51° 25′ 32.48″	4.4175670	0.5548	9.74414	4.6855748	16,308.86
3–6	22° 40′ 54.78″	4.3508778	1.3584	0.13303	4.6855748	20,697.58
6–7	25° 27′ 37.33″	4.3974938	1.9353	0.28675	4.6855748	22,548.89
7–n	132° 08′ 07.08″	4.4385440	+3.8267	0.58282	4.6855748	−18,415.83

the lines 1–2, 2–3, 3–6, 6–7, and 7–n, were worked out and treated as a traverse (Fig. 7). Computed in the usual manner, the azimuth and length of the polar ray 1–n are found to be $\alpha_{1-n} = 34° 32' 50.65''$ and $\log S_{1-n} = 4.0542039$ m. The Legendrian reduction gives, for the same line, $\alpha_{1-n} = 34° 32' 50.72''$ and $\log S_{1-n} = 4.0542039$ m. For a survey of limited extent, as typified by this example, it is evident that no increased accuracy of practical importance is obtained by the application of the Legendrian principle.

Example 2.—To show the application of the Tobey method[11] for finding the azimuth and length of a long polar ray, the traverse of the preceding example (Fig. 7) has been recomputed using the original deflection angles. The length of each course has been converted to meters, however, and multiplied by 6. As this theoretical traverse extends over an area 35 miles from north to south and 38 miles from east to west, it is sufficiently large to benefit by the Legendrian refinement in making the computations.

OF SPHERICAL EXCESS

$L_{m+1} D_m$ (7)	Col. 6 − Col. 7		Log $m^{(a)}$ (10)	Spherical excess (11)
	Value (8)	Log (9)		
− 674,347,194	+ 656,615,300	8.81731	1.40400	1.6646
− 61,922,024	+ 294,244,093	8.46870	1.40400	0.7459
+ 554,766,175	+ 388,599,731	8.58950	1.40400	0.9851
+1,018,566,501	+1,849,608,912	9,26708	1.40400	4.6890

$\frac{1}{2 R N \sin 1''}$.

Fundamental data of the expanded traverse are as follows:

(*a*) Geographical position of the origin, Station 1, $\phi = 45° 00' 00.000''$ (assumed) and $\lambda = 75° 00' 00.000''$ (assumed);

(*b*) $\alpha_{1-2} = 304° 27' 16.53''$ (assumed) clockwise from the north;

LATITUDES AND DEPARTURES

Log sin b_a (8)	LATITUDES		$D_a = t \sin b_a$ (11)	Log cos b_a (12)	DEPARTURES	
	dL (m) (9)	$L_a + dL$ (m) (10)			dD (13)	$D_a + dD$ (14)
9.9162301	0.000	14,849.36	−21,642.68	9.7526268		−21,642.68
9.8930957	0.0550	16,308.80	+20,448.56	9.7948567	+0.0439	20,448.60
9.5861531	0.0570	20,697.52	8,650.29	9.9650417	+0.1363	8,650.43
9.6333539	0.1007	22,548.79	10,736.13	9.9556314	+0.2116	10,736.34
9.8701478	0.3776	−18,416.21	20,355.97	9.8266472	−0.3417	20,355.63
		55,988.27				38,548.32

(*c*) Measured lengths of courses—

Course	Log of length, in meters
1–2	4.4190809
2–3	4.4175670
3–6	4.3508778
6–7	4.3974938
7–n	4.4385440

(*d*) Measured deflection angles—

$$\begin{aligned}
\beta_1 &= +\ 304^\circ\ 27'\ 16.53''\\
\beta_2 &= +\ 106^\circ\ 58'\ 15.95''\\
\beta_3 &= -\ \ 28^\circ\ 44'\ 37.70''\\
\beta_6 &= +\ \ \ 2^\circ\ 46'\ 42.55''\\
\beta_7 &= +\ 106^\circ\ 40'\ 29.75''
\end{aligned}$$

The various steps of the reduction are given in detail hereinafter.—

(*a*) The determination of the approximate bearings of the courses (reckoned clockwise from the north and referred to the meridian through Station 1) is as follows:

$$\begin{array}{lcr}
\text{Azimuth 1–2} & = & 304^\circ\ 27'\ 16.53''\\
\beta_2 & = & \underline{106^\circ\ 58'\ 15.95''}\\
b_{2a}\ \text{(approximate bearing 2–3)} & = & 51^\circ\ 25'\ 32.48''\\
\beta_3 & = & \underline{-\ 28^\circ\ 44'\ 37.70''}\\
b_{3a}\ \text{(approximate bearing 3–6)} & = & 22^\circ\ 40'\ 54.78''\\
\beta_6 & = & 2^\circ\ 46'\ 42.55''\\
b_{4a}\ \text{(approximate bearing 6–7)} & = & \underline{25^\circ\ 27'\ 37.33''}\\
\beta_7 & = & \underline{106^\circ\ 40'\ 29.75''}\\
b_{5a}\ \text{(approximate bearing 7–n)} & = & 132^\circ\ 08'\ 07.08''
\end{array}$$

Steps (*b*), (*c*), (*d*), and (*e*), are shown in Tables 9 to 12, respectively, in which all measurements are given in meters.

(*f*) Azimuth and length of polar ray 1–n—

$$\operatorname{Tan}\psi = \frac{D}{L} = \frac{38{,}548.32}{55{,}988.26}$$

$$\begin{aligned}
\operatorname{Log}\tan\psi &= 4.58600546 - 4.74809697 = 9.83790849\\
\psi &= 34^\circ\ 32'\ 51.68''\\
\Sigma(\epsilon/3) &= \underline{+\qquad\ 2.69''}\\
\alpha_{1-n}\qquad &= 34^\circ\ 32'\ 54.37''
\end{aligned}$$

$$\begin{aligned}
\operatorname{Log} L_n &= 4.74809697\\
\operatorname{Cos}\psi &= \underline{9.91574505}\\
\operatorname{Log} S &= 4.83235192\\
S &= 67{,}975.32\ \text{m}
\end{aligned}$$

Thus, for the polar ray 1–n,

$$\alpha_{1-n} = 34° \; 32' \; 54.37''$$
$$\text{Log } S_{1-n} = 4.8323519$$
$$S = 67{,}975.42 \text{ m}$$

Had the values been computed without applying the Legendrian corrections, the result would have been

$$\alpha_{1-n} = 34° \; 32' \; 50.53''$$
$$\text{Log } S_{1-n} = 4.8323549$$
$$S = 67{,}975.89 \text{ m}$$

In general, in adjusting local triangulation nets between fixed primary stations by the methods suggested by the authors, it is desirable to have a criterion indicating the extent of the survey which may be accurately reduced. From the foregoing, it is evident that the sum of the spherical excesses of the polar triangles involved in the survey provides this criterion. From the application of the Legendrian principle it can be shown that

$$\alpha_{1-n} = \beta_1 + \beta_2 + \beta_3 + \cdots + \beta_0 - 180° + \Sigma(\epsilon'') \dots\dots\dots (25)$$

in which $\Sigma(\epsilon'')$ is the sum of the spherical excesses of the polar triangles. If $\Sigma(\epsilon'') = 0$,

$$\alpha_{1-n} = \beta_1 + \beta_2 + \cdots + \beta_0 - 180° \dots\dots\dots\dots\dots (26)$$

and the azimuth and the length of the polar ray joining the origin with the terminal station may be deduced with geodetic accuracy from the approximate latitudes and departures. The approximate value of ϵ for each polar triangle may be quickly deduced by scaling the area of each triangle, converting the area to square miles, and dividing by 76.5. Remembering that the Legendrian corrections to the latitudes and departures of the traverse courses are

$$dL_{\text{Leg}} = t \sin b_a \, db \sin 1'' \dots\dots\dots\dots\dots\dots (27a)$$

and

$$dD_{\text{Leg}} = t \cos b_a \, db \sin 1'' \dots\dots\dots\dots\dots\dots (27b)$$

respectively, and that

$$\tan \psi = \frac{\Sigma(D_a + dD_{\text{Leg}})}{\Sigma(L_a + dL_{\text{Leg}})} \dots\dots\dots\dots\dots\dots (28)$$

the effect of the spherical excesses on the azimuth and length of the computed polar ray may be readily determined. A cursory examination in this manner will show whether the desired accuracy may be attained without the application of Legendre's theorem.

H. W. Hemple,[12] M. ASCE.—One of several methods which may be used in adjusting local triangulation surveys to the federal geodetic triangulation network is illustrated in this paper. As the density of coverage of the federal net of control established by the United States Coast and Geodetic Survey (U.S.C. and G.S.) increases, more general utilization of the stations established thereon will be made to control supplemental surveys. Connections of local surveys to the federal triangulation net will make certain that these will be in their correct relative position with respect to other surveys so connected; and also will give assurance that the points established on the local survey can be re-established at any time, should the original markings be lost. The method of adjustment discussed by Professors Coddington and Marshall brings to the attention of engineers and surveyors one system by which their own surveys may be adjusted to agree with the over-all federal net.

---

[12] Chf., Div. of Geodesy, U. S. Coast and Geodetic Survey, Washington, D. C.

The U.S.C. and G.S. has completed in excess of 100,000 miles of first order and second order triangulation, forming a network of control spanning the entire United States. In addition to the main scheme stations, there are thousands of supplemental points located on these arcs. The positions of all points are expressed on the North American Datum of 1927. The national survey nets on Mexico and Canada are also related to this same datum, as is the triangulation extending into Alaska, to the tip of the Aleutian Islands, and to the Bering Strait. Since all triangulation stations on the national nets of North America are on a common datum, their locations are definitely established; and all such survey data are expressed in their correct relationship to one another. For any one station, the geographic position will satisfy only that particular location.

At certain intervals a measured base line is inserted into the scheme of triangulation. These base lines are measured with invar tapes, which have been standardized at the United States Bureau of Standards. Measurements are made with a definite tension, and temperature readings are taken at each tape length. At least three tapes are used, and measurements are made backward and forward. Checks are such that the distance is determined with an accuracy so that the actual error must not exceed 1/300,000 for first order triangulation.

The check on distances between base lines as carried through the triangulation must be 1/25,000 or better. In actual practice the average check is of the order of 1/80,000 for the 100,000 miles of triangulation in the United States. Azimuth control is provided by Laplace azimuths spaced at regular intervals. Observed astronomic azimuths are corrected for the effect of the deflection of the vertical through the use of precise longitude observations and are thereby reduced to true geodetic azimuths. Surveyors and engineers should have no hesitancy concerning the accuracy in connecting to the federal net.

To promote the more general use of federal control stations, the U.S.C. and G.S. has transformed the geographic positions thereof to plane coordinates (X and Y). For each state a definite system of plane coordinates has been adopted, and all federal horizontal control therein is expressed on this plane. The purpose of transforming geographic positions to plane coordinates was to facilitate the use of federal surveys by making provision for a simpler method of treating the computations involved for the local surveys when they are connected to federal triangulation schemes. These plane coordinate systems were adopted principally as a procedure by which the surveyor could connect his traverse surveys and adjust his work to fit into the federal net, utilizing only those simple computational methods which are in general use. Professors

Coddington and Marshall show how plane coordinates may also be used to adjust a local triangulation scheme.

After the arcs of triangulation are about 25 miles apart, the U.S.C. and G.S. has found it advisable to cover the entire remaining intermediate area with control, adjusting it simulataneously. In extending the area triangulation, the main scheme stations are about 8 miles apart. Supplemental stations are established at about 4-mile intervals along the main highways. Corrections necessary to fit properly the new work into the major networks surrounding it are then distributed over a relatively large number of observations and can be assimilated without causing too great a distortion in any one measurement or observation.

The authors mention urban control in Pittsburgh, Pa., and Columbus, Ohio. Other cities where geodetic control connected to the federal net has been accomplished are Rochester, N. Y.; Atlanta, Ga.; Boston, Mass.; Cleveland, Ohio; Hartford, Conn.; Minneapolis, Minn.; and Houston, Tex.; as well as many others. In June, 1947, the East Bay cities of the San Francisco (Calif.) area completed a comprehensive scheme of urban control rigidly connected to the federal net; and in December, 1947, the urban control for Cincinnati, Ohio, also rigidly connected to the federal net, was completed.

In the extension of the federal triangulation control, it is the policy of the U.S.C. and G.S. to establish base lines for length control in proximity to urban centers. These control the distance determinations in the federal net and will likewise aid in the control of the lengths for the urban triangulation whenever the cities establish a scheme of urban control. If such local control is available when the federal surveys are established in the locality, connections are made between the two schemes; and, if the bases and lengths of the urban control are of sufficient accuracy, the federal net is adjusted thereto.

The decade from 1937 to 1947 has marked a greatly augmented use of federal triangulation by engineers of state, municipal, and private organizations. It is confidently expected, that, as the ease and facility with which local survey connections may be made to the federal net become more generally known, this precise control will be used even more. Such papers as this are indicative that general interest is becoming increasingly manifest.

Lansing G. Simmons,[13] M. ASCE.—This subject has become of increasing interest in recent years, particularly to engineers and surveyors taking advantage of the rapidly expanding precise triangulation net of the United States. With the establishment of the state plane coordinate systems, the problem of adjusting triangulation directly on these plane grids, without reference to the geographic elements of position, length, and azimuth, is of concern to all involved in the establishment of high-grade horizontal control surveys. The authors have presented a solution which, although departing from correct theory in some instances, will produce excellent results for control schemes of local extent.

The writer cannot agree with the statement in the "Synopsis" to the effect that "Local triangulation nets, both urban and rural, can be precisely adjusted to large-scale geodetic triangulation with comparatively little effort * * *." Most readers will agree, particularly when it is realized that the actual solution of the nineteen simultaneous equations of the Logan County example does not appear in the paper. The importance and precision of the work should be considered before such a plan is adopted.

---

[13] Chf. Mathematician, Div. of Geodesy, U. S. Coast and Geodetic Survey, Washington, D. C.

The new features brought out are the development and the use of condition equations, in terms of rectangular coordinates, necessary for the elimination of the position closure between stations previously established. This has always been a troublesome point in the adjustment of local work, and has usually been handled by simply neglecting the condition and by permitting inconsistencies to exist between new and old work.

Any triangulation net worthy of an adjustment, including the rigid conditions of position closures, is usually worthy of a completely rigid and simultaneous adjustment. The authors state that one of three possibilities for a preliminary adjustment would be to include all condition equations except those for latitude and departure closures. These would be introduced later in a final adjustment. It is believed that all conditions, including those of position, should be contained in one adjustment. This system not only gives better results in general, but can be accomplished more rapidly than any combination of successive adjustments that would yield comparable results.

Although the paper is principally confined to local triangulation nets, readers should be warned of the fact that, in strictness, the observed angles of a a triangulation scheme are not plane angles and should not be so used in working on a plane grid. The difference is infinitesimal for short lines on a local grid, but it becomes quite appreciable in carrying a precise scheme across a state with longer lines. These differences between observed (geodetic) angles and grid angles result from the fact that a straight line on the earth is not exactly a straight line on the projection, and vice versa. Their magnitude can easily approach 1 sec or 2 sec and, at times, several seconds of arc.

As an example, in carrying the computation over a considerable distance, if the necessary corrections are not applied to the observed angles to obtain grid angles, no account is made of the slowly changing scale of the projection. This difference between geodetic and grid angles can also be illustrated by the fact that a large geodetic triangle must necessarily close to 180° plus the spherical excess, whereas a triangle on the grid is closed to exactly 180° in an adjustment. Therefore, it follows that that the algebraic sum of the corrections to the three angles of a triangle must be equal to the spherical excess and negative in sign.

In conclusion, the authors of the paper have dealt with an interesting subject and have effected a nice solution. The foregoing cautions are aimed at those who propose to adapt the method to extended triangulation of high precision. It is believed that the most satisfactory solution in these cases is to compute the work geodetically, converting the position of each station to the grid individually.

E. F. CODDINGTON[14] AND O. C. J. MARSHALL,[15] Members, ASCE.—The discussions express three general opinions concerning this paper: First, the need and timeliness of a simple method to fill in the 25-mile gaps of the first order triangulation nets provided by the United States Coast and Geodetic Survey (U. S. C. and G. S.); second, an excellent description of this control net by Mr. Hemple; and, third, a questioning by some of the discussers of the accuracy of the method.

Messrs. Ney and Simmons raise the question of the accuracy of the method, especially when extending basic control nets. Mr. Ney demonstrates that the computation errors were insignificant in the case of the synthetic net used by the writers. The Logan County (Ohio) net is smaller than the synthetic net. As to the question of rigid or correct theory raised by Mr. Simmons, it may be stated that practically all geodetic theory is applied by a succession of approximations. The number of significant terms in the series of approximations increases with the length of lines or with the size of area under consideration. The writers have been successful in applying mechanical integration to find accurate lengths of elliptical arcs on the earth's surface.

The writers are convinced that local rectangular coordinates are sufficiently accurate for areas under 25 miles square, or to fill in control in any 25-mile gap in the completed first order triangulation net executed by the U. S. C. and G. S. The local rectangular coordinates of all geodetic stations of the net adjusted to the first order control may be easily transferred back to geographic coordinates (latitude and longitude) in case state-wide coordinates may be desired for mapping purposes. This method was derived to enable city engineers to use local coordinates in making cadastral surveys tied to first order geodetic control.

---

[14] Prof. Emeritus of Geodetic Eng., Ohio State Univ., Columbus, Ohio.

[15] Prof. of Surveying and Geodesy, Univ. of Toronto, Toronto, Ont., Canada.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2350

THE GENERAL PRINCIPLES OF HIGHWAY TRANSITION CURVE DESIGN

By John Joseph Leeming,[1] Esq.

With Discussion by Messrs. Carl F. Meyer, Ralph L. Fisher, William R. Welty, Elmer R. Haile, Jr., Charles M. Noble, and John Joseph Leeming

Synopsis

The theoretical and experimental background of the published theory of highway transition curve design is explored in this paper. The writer shows (1) that the theory is fallacious in its disregard of the superelevation, and (2) that the experimental evidence produced for it is questionable.

A modified theory, called "the lateral force theory," is outlined. Experimental evidence is produced to show that the length of the transition curve is not a matter of great importance, and the proposal is made that it should be made a fixed proportion of the total length of the curve. It is also shown that the transition need not necessarily be spiral in form.

Simplified equations for the solution of the compound curve are given, based on the radius of the arc as parameter, together with a table of spiral functions. There is also a brief description of the proposed method of curve design, illustrated by a worked example.

1. Introduction

In designing transition curves the fundamental principle of the method described by John Oran Eichler, Assoc. M. ASCE, and Howard W. Eves[2,3] is, in the words of the authors,[4]

> "* * * that the rational way to achieve this transition [from tangent to curve] is to make the rate of gain of the 'centrifugal ratio,' P/W, constant throughout the curve. This is equivalent to the demand that the driver sharpen his curve by turning his steering wheel at a constant rate."

The symbol P in this context denotes the centrifugal force. In this theory it is also postulated that the rate of change of centrifugal force, for which

---

Note.—Published in October, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] County Surveyor, Dorsetshire, England.

[2] "Mathematical Examination of the Highway Transition Spiral," by John Oran Eichler and Howard W. Eves, *Transactions*, ASCE, Vol. 111, 1946, p. 959.

[3] *Ibid.*, p. 972.

[4] *Ibid.*, p. 960.

W. H. Shortt's[5] symbol C will be used, should be the same for all curves. Mr. Shortt, in England, suggested that this rate for C should be 1 ft per sec^3, claiming that this is the maximum rate that will pass unnoticed. The value has been accepted by others, although in the United States values of 2 ft per sec^3 and 3 ft per sec^3 have been proposed.

This theory gives the same length for a transition curve whether superelevation is present or not, although it is obvious that this factor must have some effect on a car.

Consideration of the foregoing from the viewpoint of first principles at once raises the question as to whether the rate of change of centrifugal force is the important effect for transition curve design. To find an answer it is necessary to consider the forces acting on a car traveling on a canted road.

2. THE LATERAL FORCE

Fig. 1 represents a car traveling on a curve of radius R, at a speed v, on a road canted at an angle α. The angle, and all forces, are considered to be positive as drawn. The centrifugal force is F_c. The reactions of the road on the car, parallel and normal to the surface, are designated by $j\,W$ and $n\,W$, respectively, and are called the lateral and normal forces. Then, resolving forces and eliminating W, the weight of the car,

$$j = \frac{v^2}{g\,R}\cos\alpha - \sin\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and

$$n = \frac{v^2}{g\,R}\sin\alpha + \cos\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which g is the acceleration due to gravity. The two quantities, j and n, are called the lateral and normal ratios, respectively. They are pure numbers, provided the units used are mutually consistent.

The normal force, $n\,W$, is very little greater than W under road conditions. Its increase due to curvature is a very small increase in an already existing force, acting in a direction that cannot affect the stability of the car or the comfort of the passengers. Therefore, it may be neglected.

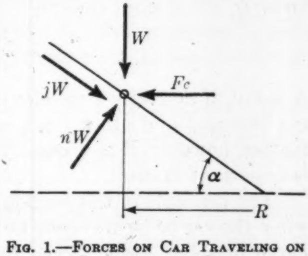


FIG. 1.—FORCES ON CAR TRAVELING ON A SUPERELEVATED CURVE

The lateral force, $j\,W$, is the key to the situation. It is the sum of the components of the centrifugal force, and of the weight of the car, along the surface of the road and is the force which tends to cause the car to skid or to overturn, or which causes the passengers discomfort by urging them sideways in their seats. It is also the force that must be applied by the driver through the steering wheel, as it is the force he must overcome in turning his wheel to make the car travel around a curve.

The full implications of the lateral force as a factor in road design were dealt with at some length in a previous paper,[6] but the most important must be mentioned in passing—namely, that a change in cant is equivalent to a change in curvature and that one change should never occur without the other.

---

[5] "A Practical Method for the Improvement of Existing Railway-Curves," by W. H. Shortt, *Proceedings*, Inst. C. E., Vol. 176, 1909, p. 97.

At a certain speed, called the "hands-off speed," the lateral ratio j is equal to zero. At this speed the car steers itself around the curve, provided the steering is properly adjusted, and the driver need not touch the wheel at all. The centripetal force is acting still, but it is applied to the car by the superelevation. If the curvature and the cant are changing together in due proportion (on a well-designed transition curve, for example), the wheel can be seen to turn itself if the proper speed is held. Alternatively, if the driver reaches the hands-off speed on a superelevated circular arc, he may then steer the car in a zigzag course by varying the speed without touching the wheel, which again will turn itself.

At any speed different from the hands-off, the driver must exert some force to follow the curve, but the direction as well as the magnitude of this force will be decided by whether the speed is more or less than the hands-off. If it is less he must steer away from, if more he must steer toward, the center of curvature. Thus, the presence of the superelevation does make a very vital difference to the driver, and also to the passengers, who are affected similarly. Therefore, it is the rate of change of lateral force that is the important result and should be kept constant on a transition curve.

A significant effect follows from this consideration. In Eq. 1, cos α approximates to unity. Then, at any constant speed, the requirement for maintaining a constant hands-off speed is that

$$\frac{k}{R} = \sin\alpha \text{........................} (3)$$

in which k equals v^2/g. For this condition a circular arc is not essential. Any curve for which it is possible to keep the curvature and superelevation proportional will do, including the conventional transition spiral.

Similarly, the condition for maintaining a constant rate of change of lateral force, Q, at constant speed is

$$Q = \frac{dj}{dt} = \frac{d}{dt}\left(\frac{k}{R} - \sin\alpha\right) \text{..................} (4)$$

A spiral, if correctly superelevated, will provide this condition, but its use is not essential and other types of curves can be used. Before discussing this matter, however, it is necessary to decide whether the rate of change of force is important in itself.

The principle of this section may be summarized in another manner. Assume the car to be traveling at a constant speed on a well-designed transition curve of spiral form. Then the rate of change of direction in plan will be proportional to C, the rate of change of centrifugal force, but the change will be made by two agencies. The first is by the rate of change of the superelevation. This is solely a function of road design, the driver has nothing whatsoever to do with it, and in most cases will be unaware of it. The second agency is the application of force by the driver at the steering wheel, and this is proportional to Q, the rate of change of lateral force. In his turn, the road engineer has nothing to do with this.

---

[6] "Road Curvature and Superelevation," by J. J. Leeming, *Road Paper No.* 7, Inst. C. E., October, 1942.

The whole essence of the theory put forward in this paper is that this latter effect must be studied and allowed for in design, because it may cause variation between theory and practice. However well designed a curve may be according to spiral principles, if the driver does not follow a constant Q proportional to the value of C on which the spiral has been designed, he is not following the design curve. It is surely logical that the curve should be laid out to conform to the driver's practice; and, if he prefers, and habitually uses, some method of turning his wheel other than the constant rate expressed by constant Q, then the type of transition should be chosen to conform with his habits—even if this curve is not of spiral form.

3. The Rate of Turning the Steering Wheel

If the rate of application of force by the driver at the steering wheel is defined as "the rate of turning the wheel," the foregoing principle raises two further questions of fact, namely:

1. Does the driver turn his wheel at a constant rate on any given transition curve?
2. Does he turn it at the same rate on all transition curves?

These two questions are not the same, although they have not hitherto been separated. The first decides the shape of the transition curve, but does not involve any condition as to its length. The assumption that this question can be answered affirmatively has formed the case for the spiral curve, although it is seen that other curves can also satisfy the condition. The second part, if answered similarly, means that all transition curves must be designed to provide the same rate of change of force, at some chosen speed. The assumption that both questions can be answered affirmatively forms the basis of the previous theory.

Examination of the literature setting forth the case for this affirmative answer to the two questions shows that it rests entirely on Mr. Shortt. All other writers have merely repeated his conclusions; but his experiments were made on uncanted railway tracks, and some doubts may be felt at once on examining the original paper. Mr. Shortt stated that he had found that a rate of change of centrifugal force of 1 ft per sec^3 was the maximum which would pass unnoticed. This statement must involve the idea of comfort, and it has been so read in the past by most people. However, it is impossible for an observer to analyze his feelings so delicately and accurately as to be able to decide beyond doubt that the rate of change of force affected his comfort, and not the force itself. Nevertheless, there must be some value of the force that will cause discomfort, and the possibility that some lower value might cause some lesser degree of discomfort obviously is present. Mr. Shortt did not even discuss this possibility, nor did he state how he arrived at his conclusions so that the reader could judge for himself.

F. G. Royal-Dawson[7] applied Mr. Shortt's principle to roads, without experimental proof of the validity of the transfer, and called the rate of $C = 1$ a "natural rate of turning." In this case the difficulty of the limit to the value of the force itself, previously referred to, was avoided by the expedient of stipulating a maximum value for the centrifugal ratio, usually 0.25. The superelevation was specified as some fraction of the centrifugal ratio, and it was then stated that the natural rate of turning should be adhered to, provided that the centrifugal ratio did not exceed this amount at some "design speed."

---

[7] "Curve Design," by F. G. Royal-Dawson, E. & F. N. Spon, Ltd., London, 1932.

This argument is not very convincing, quite apart from the fact that it contains the fundamental fallacy of using C and the centrifugal ratio, ignoring the superelevation. The theory gives the same length for a transition curve, no matter what the value of the superelevation. However, Mr. Shortt himself stated, in his original paper,[5] that "* * * a curve which cannot be given superelevation is more in need of easing at the ends than one which can be superelevated," thus implying that superelevation does affect the problem, although, mathematically, he and his followers ignored this principle.

There has also been a large school among British engineers, both railway and highway, that has not accepted the principles of Messrs. Shortt and Royal-Dawson on practical grounds. Finally, H. A. Warren[8] carried out some experiments which threw doubt on the validity of the rate of change of force as the criterion for transition curve design. These experiments were on a small scale, and there was some question as to the accuracy of the work, by which C was derived from the plotted path of the car.

Thus, it was thought that there was need for experimental work on the rate of turning the wheel. The writer, therefore, undertook this work, in collaboration with A. N. Black. A preliminary report on the results has been published,[9] but a summary is given herein, which includes later work.

4. Reasons for the Experimental Procedure

The experimental technique had to be based on the previous theory, because, despite its shortcomings, this held the field, and because it had found some measure of acceptance. Therefore, the experiments were designed to answer the two questions:

(1) How do drivers actually take curves?
(2) What is the criterion of comfort on curves?

The two questions of Section 3 are included in the first of these latter two.

Since the force the driver applies at the wheel is proportional to j, the best method of investigating these questions is to measure j as directly as possible. That part of the centripetal force applied through the superelevation (that is, $W \sin \alpha$), being independent of the driver and imposed on him by the road designer, is irrelevant to the purpose. To make the conclusions general, it is also essential that the tests be carried out under the most natural conditions possible, to avoid distractions to the driver that would tend to influence his method of driving, and also that the tests should include as many cars and drivers as possible.

---

[8] "Experimental Transition Curves," by H. A. Warren and E. R. Hazeldine, *Journal*, Inst. Municipal & County Engrs., London, March, 1939, p. 1021.

[9] "Road Curvature and Superelevation: Experiments on Comfort and Driving Practice," by J. J. Leeming and A. N. Black, *ibid.*, December, 1944, p. 137.

The method used by Mr. Warren was to trace out and survey the path of the car. This procedure, however, is open to a number of objections. In the first place it is inaccurate, because it involves some process of double differentiation of the path, to determine the centrifugal ratio, from which the superelevation must be deducted to find j. It is laborious in every way, and involves considerable apparatus as well as a technique which must infallibly distract the driver. Finally, the method can only be useful on a limited number of curves because of the labor involved—resulting from the fact that a detailed survey of the curve is necessary, since the superelevation must be known. Thus, the answer obtained would be limited to some extent, and would therefore be of less general application.

5. Experimental Procedure

From the considerations mentioned, it was decided that the ideal method would be to use a recording accelerometer. This instrument measures j directly, and Q can be determined by simply measuring the slope of the line on the record. Accordingly, such a device was designed and built, which, although simple to make with the materials available in wartime England, could fit into most cars.

The accelerometer consists of a 10-lb weight hung on springs from the baseboard of a strong wooden case. When the instrument is placed athwart the car, under the influence of the lateral force the weight shifts over against the springs, moving a stylus on the end of a long magnifying lever. This records the movement of the weight on a strip of metallic paper traveling over a drum driven by a phonograph motor. A fixed stylus provides a datum line.

This instrument was carried in cars traveling on ordinary journeys. Before starting, it was emphasized that the driver should not vary his usual driving methods in any way. When the car approached any curve thought suitable, the operator started the motor; and, for each curve, recorded its direction of curvature, the average speed of the car, and the comfort sensations of himself and the driver.

In the subsequent analysis, the individual curves were identified, and smoothed off to average straight lines if this could reasonably be done. For more than 90% of the curves it was found that the rate of change of force could be represented by sloping straight lines, and the curve as a whole by two such lines, with or without a length of straight line parallel to the datum—that is, of constant j—between them. Thus, the assumption that the driver turns his wheel at a constant rate on any given transition seems justified.

It soon appeared that there was considerable variability in a driver's practice. Records made on the same curve with the same car and driver at different times showed very marked differences even at the same speed, and Q varied between such wide limits as to make it quite obvious that there can be no natural rate of turning. In other words, the answer to question 2, Section 3, is that drivers do not always turn their wheels at the same rate. The variability also made it essential to use statistical methods, so that an extensive statistical analysis was started and a large number of records were taken in a variety of vehicles.

No conclusive evidence could be found of wide differences in practice between drivers, considering the general variability and the inevitable differences in speeds. Most of the records were taken, perforce, in the British type of light car, but a fair range of commercial vehicles was also included. Among these latter was one of the largest and heaviest rigid wheel base vehicles made—an eight wheeler—loaded with 15 tons of bricks. In England its speed is limited by law to 20 miles per hr, but allowing for this it did not seem to show any marked difference from the smaller cars, nor did the one American car it was possible to include. It was not possible to take records in an articulated vehicle, nor in a fast, heavy, passenger vehicle.

A marked tendency was found for Q to vary with the value of j reached on any curve. This tendency can be expressed by the regression equation:

$$\bar{Q} = 1.3\, e^{3.5j} \quad \text{.........................}(5)$$

in which e is the base of natural logarithms, and $\bar{Q}$ is the mean value of Q observed, in feet per second^3. This expression refers to entrance transitions, and is independent of speed. The scatter was moderately wide, and the equation represents only a trend. Another way of expressing this principle is that the driver turns his wheel faster on entering a sharp curve than on entering a flat one. Values of Q as low as 1 ft per sec^3 were comparatively rarely observed, and then only on curves in the lower ranges of j.

Another fairly marked tendency was for the driver to use a definite proportion of the curve as transition, with an over-all mean of two thirds. The use of the all-transitional curve was comparatively rare. There was also a marked tendency for the exit transition to be slightly shorter than the entrance one, although the difference is not great enough to be of practical importance. This latter effect is probably related to the caster action of the steering, which helps the driver on the exit, and opposes him on the entrance. The proportion of transition tended to vary with the value of j reached, and with the speed of the car and the length of the curve, according to the regression equation:

$$p = 0.34 + 6.3\,j - 25.5\,j^2 + 33.0\,j^3 - 0.002\,v + 0.007\,L/v \quad \text{......}(6)$$

In this expression, p is the mean proportion of transition observed (that is, the length of the two transitions combined, divided by the total length of the curve); v is the speed, in miles per hour; and L is the total length of the curve, in feet. For $j \geqq 0.1$, Eq. 6 gives values of p ranging from two thirds to one half. It should, however, be viewed with caution. Because of wartime restrictions in England, the data from which Eq. 6 was calculated do not include very large samples of curves taken at speeds higher than 50 miles per hr, although, since the completion of the regression analysis, more records at higher speeds have been made, and it appears unlikely that the conclusions will be materially altered.

The curves were also analyzed by class of road, including right-angle junctions and rotary intersections. This procedure gave a rough classification by the deviation angle of the curve, which appears to have little effect. This analysis cannot be made with any confidence by measuring the deviation angle of the curves themselves, because it is not possible to decide on the driver's angle of approach to the curve, and the records gave reason to believe that this varies fairly widely.

In recording comfort, no attempt was made to separate the sensations due to the force or to its rate of change. The observer was merely asked if a sensation of curvature was felt or not, according to the groups: (*a*) Easy, when no sensation of curvature was felt at all; (*b*) comfortable, when a sensation of curvature was felt, but not thought uncomfortable; (*c*) slightly uncomfortable; and (*d*) uncomfortable. The effects of the force and of its rate of change were separated in the statistical analysis.

As was to be expected, there was considerable variation in the assessment of comfort, but the application of statistical methods showed that comfort depends mainly on j, although high rates of Q tended to reduce the tolerable values of j slightly. Speed does not appear to affect the assessment of comfort. No uncomfortable or noticeable value of Q could be detected at low values of j.

6. Conclusions from the Experiments

The main conclusions from the experiments are:

(1) The driver does appear to turn his wheel at a constant rate on any given transition.

(2) There is no such thing as a constant rate of turning applicable to all transitions. The driver tends to turn his wheel faster when entering a sharp curve than when entering a flat one. This conclusion is in accordance with general observation, but is in contradiction to the previous theory.

(3) There is some marked tendency for drivers to use about two thirds to one half of the curve as the transition.

(4) As far as comfort is concerned, j is the important effect; and Q is unimportant.

From these conclusions it appears that there is no sound reason for using the rate of change of force as the criterion of transition curve design, and it is suggested that this theory should be abandoned. On the other hand, it may be urged that some value of C or Q in the range of 1, 2, or 3 ft per sec^3, according to opinion, is a value which is safe or comfortable, and that its use is convenient. To counter this argument it is necessary to consider the mathematics of the compound curve.

7. The Compound Curve

Mathematics.—Although the condition for constant Q can be satisfied by curves of other types than the usual spiral, this curve is not unsatisfactory provided the superelevation is properly designed to comply with the conditions of Eq. 3. It is so simple and easy to use, and so familiar, that there is little to be gained by abandoning it. At the same time, the lemniscate has its advocates, and, if preferred, there is no reason whatever against its use. Nevertheless, the spiral will be used herein. Its mathematics have been dealt with elsewhere[3] at some length and need not be repeated. However, since it has been shown that the rate of change of force is not important, it also follows that the length of the curve is relatively unimportant, and may be eliminated from the equations for the main dimensions of the curve, basing these on the

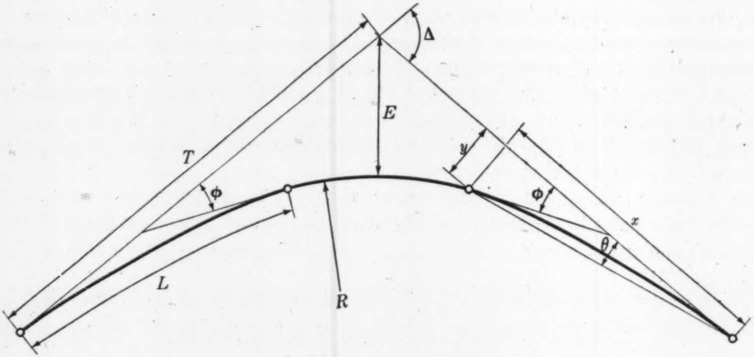


FIG. 2.—BASIC DIMENSIONS OF SYMMETRICAL CIRCULAR CURVE WITH TRANSITION SPIRAL AT BOTH ENDS

radius. These modifications were first published by I. Kursbatt[10] in a slightly different form.

Referring to Fig. 2,

$$x = N\,R \qquad (7a)$$

$$y = P\,R \qquad (7b)$$

$$T = R\left(M \tan\frac{\Delta}{2} + S\right) \qquad (8a)$$

$$E = R\left(M \sec\frac{\Delta}{2} - 1\right) = R\left[(M-1) + \left(\sec\frac{\Delta}{2} - 1\right) + (M-1)\left(\sec\frac{\Delta}{2} - 1\right)\right] \qquad (8b)$$

$$L = 2\,\phi\,R \qquad (8c)$$

$$\theta = \phi/3\text{—small correction} \qquad (8d)$$

and

$$\phi = \frac{p}{2\,(2-p)}\,\Delta \qquad (8e)$$

in which T is the total tangent distance from the point of intersection to the starting point of transition; E is the external distance from the intersection point to the nearest point of the curve; R is the radius at the point on the transition defined by the angle ϕ; x and y are the coordinates of the same point from the starting point of transition as origin:

---

[10] "Calculations for a Circular Curve with Transitions," by I. Kursbatt, *Highways, Bridges and Aerodromes*, London, February, 14, 1945, p. 226.

$$N = 2\phi\left(1 - \frac{\phi^2}{5\,(2!)} + \frac{\phi^4}{9\,(4!)} - \frac{\phi^6}{13\,(6!)} \cdots\right) \quad \text{.........} (9a)$$

$$P = 2\phi\left(\frac{\phi}{3} - \frac{\phi^3}{7\,(3!)} + \frac{\phi^5}{11\,(5!)} - \frac{\phi^7}{15\,(7!)} \cdots\right) \quad \text{........} (9b)$$

$$M = P + \cos\phi = 1 + \frac{\phi^2}{6} - \frac{\phi^4}{168} + \frac{\phi^6}{7{,}920} \cdots \quad \text{..........} (9c)$$

and

$$S = N - \sin\phi = \phi - \frac{\phi^3}{30} + \frac{\phi^5}{1{,}080} - \frac{\phi^7}{65{,}520} \cdots \quad \text{.........} (9d)$$

In these expressions, which apply whether or not a circular central arc is present, ϕ is in radians.

TABLE 1.—VALUES OF SPIRAL FUNCTIONS IN EQS. 8*d*, 8*e*, 9*a*, 9*b*, 9*c*, AND 9*d*

ϕ °	′	N	P	M	S	$\left(\frac{\phi}{3}-\theta\right)$ (sec)
0	30...	0.017453	0.000051	1.000013	0.008727	....
1	0...	0.034905	0.000203	1.000051	0.017453	....
1	30...	0.052356	0.000457	1.000114	0.026178	....
2	0...	0.069805	0.000812	1.000203	0.034905	....
2	30...	0.087250	0.001269	1.000317	0.043630	....
3	0...	0.104691	0.001827	1.000457	0.052355	....
3	30...	0.122127	0.002487	1.000622	0.061079	....
4	0...	0.139558	0.003248	1.000812	0.069802	....
4	30...	0.156983	0.004110	1.001028	0.078524	....
5	0...	0.174400	0.005074	1.001269	0.087244	....
5	30...	0.191809	0.006139	1.001535	0.095964	....
6	0...	0.209210	0.007305	1.001827	0.104682	....
6	30...	0.226601	0.008572	1.002144	0.113398	1
7	0...	0.243982	0.009940	1.002486	0.122112	1
7	30...	0.261351	0.011409	1.002854	0.130825	1
8	0...	0.278709	0.012979	1.003247	0.139537	2
8	30...	0.296054	0.014649	1.003665	0.148244	2
9	0...	0.313385	0.016420	1.004109	0.156950	2
9	30...	0.330702	0.018292	1.004577	0.165654	3
10	0...	0.348004	0.020264	1.005071	0.174356	3
10	30...	0.365290	0.022336	1.005591	0.183054	4
11	0...	0.382560	0.024508	1.006135	0.191751	4
11	30...	0.399812	0.026780	1.006705	0.200444	5
12	0...	0.417045	0.029152	1.007299	0.209134	5
12	30...	0.434260	0.031623	1.007919	0.217821	6
13	0...	0.451455	0.034194	1.008564	0.226504	7
13	30...	0.468629	0.036865	1.009234	0.235184	8
14	0...	0.485783	0.039634	1.009930	0.243861	8
14	30...	0.502913	0.042502	1.010650	0.252533	9
15	0...	0.520022	0.045469	1.011395	0.261203	10
15	30...	0.537106	0.048535	1.012166	0.269867	11
16	0...	0.554166	0.051699	1.012961	0.278528	13
16	30...	0.571200	0.054961	1.013781	0.287185	14
17	0...	0.588209	0.058322	1.014626	0.295837	15
17	30...	0.605191	0.061780	1.015497	0.304485	17
18	0...	0.622146	0.065335	1.016392	0.313129	18
18	30...	0.639072	0.068988	1.017311	0.321767	20
19	0...	0.655969	0.072738	1.018256	0.330401	21
19	30...	0.672836	0.076584	1.019226	0.339029	23
20	0...	0.689673	0.080527	1.020220	0.347653	25
20	30...	0.706479	0.084566	1.021240	0.356271	27
21	0...	0.723252	0.088702	1.022282	0.364884	29
21	30...	0.739993	0.092933	1.023351	0.373492	31
22	0...	0.756700	0.097260	1.024444	0.382093	33
22	30...	0.773373	0.101682	1.025561	0.390689	35
23	0...	0.790010	0.106198	1.026703	0.399279	38
23	30...	0.806612	0.110810	1.027870	0.407863	40
24	0...	0.823178	0.115515	1.029061	0.416441	43
24	30...	0.839706	0.120315	1.030276	0.425013	46
25	0...	0.856196	0.125208	1.031516	0.433578	48

Table 1 gives these spiral functions to six decimal places, for angles to 25° at half degree intervals. Corrections for the polar deflection angle in Eq. 8*d* are also given, although this is usually negligible for small values of ϕ.

The second form of Eq. 8*b* will be found rather simpler for calculation purposes, in spite of its more complicated appearance.

In Eq. 8*e*, if $p = \frac{2}{3}$, $\phi = \Delta/4$, and if $p = \frac{1}{2}$, $\phi = \Delta/6$.

The Practical Problem.—Under practical road conditions the engineer will have two tangents, meeting at an angle Δ, which will have been measured on the ground. If there is unlimited room available, no complications are involved; the curve can be made as flat as desired. Frequently, however, some obstacle or other restricts either the exterior distance E (Fig. 2), or the tangent distance T. The problem is to make the best use of the room available.

Whichever of these distances is limited can be substituted in Eq. 8*a* or Eq. 8*b* to find R, provided the value of ϕ is known so that the values of the appropriate functions may be determined from Table 1. When $\phi/\Delta = \frac{1}{2}$, the curve will be all-transitional; and, as ϕ/Δ decreases, R increases, with a corresponding reduction in j, but the length of the transition decreases, thus increasing Q. Finally, when $\phi/\Delta = 0$, the curve is a circular arc without transitions and R is a minimum for the conditions, but Q will be infinite.

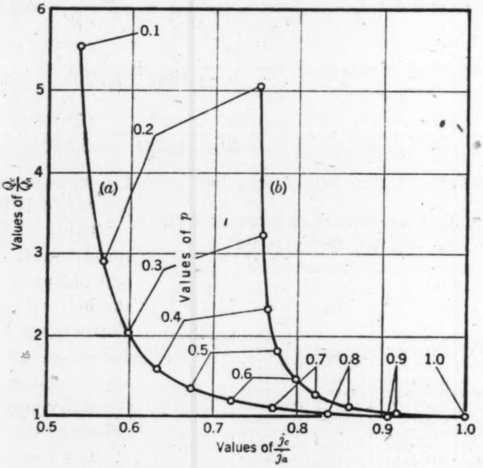


FIG. 3.—PLOT OF $\frac{Q_c}{Q_a}$ VERSUS $\frac{j_c}{j_a}$ FOR $\Delta = 60°$ AND
(*a*) Fixed T
(*b*) Fixed E

If the superelevation is assumed to be proportional to the centrifugal ratio, a quite usual practice, and if the value of j on any compound curve is denoted as j_c and that for the corresponding all-transitional curve as j_a, with the corresponding values of Q denoted as Q_c and Q_a, the following expressions can be written for the ratios between these quantities: For fixed E,

$$\frac{j_c}{j_a} = \frac{M_\phi \sec \Delta/2 - 1}{M_{\Delta/2} \sec \Delta/2 - 1} \qquad (10a)$$

and, for fixed T,

$$\frac{j_c}{j_a} = \frac{M_\phi \tan \Delta/2 + S_\phi}{M_{\Delta/2} \tan + S_{\Delta/2}} \qquad (10b)$$

For either condition,

$$\frac{Q_c}{Q_a} = \left(\frac{j_c}{j_a}\right)^2 \frac{\Delta}{2\,\phi} \qquad (11)$$

The symbols M_ϕ and S_ϕ represent the values of M and S for the value of ϕ used, and $M_{\Delta/2}$ and $S_{\Delta/2}$, those for $\Delta/2$. These expressions depend only on the relative values of the angles, and express the ratio of the value of j or Q on any compound curve on which E or T is fixed to that on the all-transitional curve for the same conditions. Fig. 3 gives curves for Q versus j for $\Delta = 60°$, although the effect of Δ is so small that they may be considered approximately general. In effect, these curves show the effects on j and Q caused by the various values of ϕ/Δ chosen by the engineer, although the latter has been expressed as a proportion of transition.

TABLE 2.—SUGGESTED RATES OF SUPERELEVATION

Radius, in feet	SUPERELEVATION	
	1 on:	Feet per foot
1,200[a]...	14½	0.069
1,400....	17	0.059
1,600....	19	0.053
1,800....	22	0.045
2,000....	24	0.042
2,500....	30	0.033
3,000....	36	0.028
3,500....	37	0.027
4,000....	38	0.026
4,500....	39	0.026
5,000....	40	0.025
7,500[b]...	48[b]	0.021[b]
>7,500....	....[c]	....[c]

[a] 1,200 ft or less. [b] 5,000 to 7,500 ft, superelevation for 1 on 40 to 1 on 48, or 0.025 ft per ft to 0.021 ft per ft, according to the nature of the material. [c] Normal camber or crossfall retained.

From Fig. 3 it is evident that the higher proportions of transition greatly increase j without reducing Q to any appreciable extent. The difference in Q between the all-transitional curve and one with 0.9 of transition is negligible, in the ratio of 1.00 to 1.03, although the difference in j is more important, in the ratio of 1.00 to 0.92. Thus, there is no justification whatever for the all-transitional curve. It merely increases j unnecessarily.

Thus, the case against a fixed rate of C or Q is clinched. No matter what rate is chosen, for a fixed E or T, and fixed speed, long transitions and eventually all-transitional curves are demanded as the room available decreases. As it decreases further, the ridiculous demand may be made that the transition be longer than the whole curve, even at low values of j. This argument is usually countered by ruling that in such cases the "design speed" must be reduced—making nonsense of the whole matter, because this theory completely ignores the driver, who cannot be expected to observe the "design speed," as he is, of course, completely unaware of its existence. It appears, in fact, that the driver slows down to reduce j and not to reduce Q and that, if a design speed is specified, it should be on the basis of j and not of Q. It is therefore suggested that the rate of change of force should be abandoned as a method of transition curve design.

8. ALTERNATIVE METHOD OF TRANSITION CURVE DESIGN

Although this paper only purports to deal with general principles, some new procedure must be suggested, since the effect of the argument is that the previous theory should be abandoned.

For the moment, the "proportion rule" first proposed elsewhere[9] is favored. This rule states that curves should be designed to have about two thirds of their length transitional, although the writer now inclines to the belief that one half is a better proportion. The exact length of the transition is not, however, a matter of extreme importance, so that ϕ may be taken to the nearest half degree.

The proposed procedure is illustrated by the following example, which is, however, only intended to demonstrate the method. The values used are not put forward as recommendations, nor is the example intended to show that the rule is sound. The required information includes:

a. The maximum speed for which the road is intended, say (for purpose of example only), 80 miles per hr;

b. The maximum value of j considered safe at that speed, say, 0.15;

c. The rates of superelevation[11] to be used, say, as shown in Table 2; and

d. The proportion of transition, say, two thirds, so that $\phi = \Delta/4$, taken to the nearest half degree.

By substituting the foregoing known values in Eq. 1, the minimum permissible radius of curve will be found, in this example, to be 2,300 ft.

Next, the deviation angle, Δ, and the greatest lengths of T and E allowed by the topography are measured on the site. For the example, let $\Delta = 22° 38'$, and let the maximum possible $E = 55$ ft. Then $\phi = 5° 30'$; and, from Eq. 8*b*, using Table 1 to obtain the values of the spiral functions, $R = \dfrac{55.00}{0.001535 + 0.019828 + 0.000030} = 2{,}570.93$ ft. This value is permissible, resulting in $j = 0.13$, and a superelevation rate of 0.04 ft per ft. If not permissible, some reconsideration of the alinement is necessary.

From Eq. 8*a*, $T = 2{,}570.93\ (1.0015\ 5 \times 0.200122 + 0.095964) = 762.01$ ft, and $L = 2{,}570.93 \times 0.191986 = 493.58$ ft. The spirals may be set out in any convenient manner, the ten-chord method being especially suitable. The method has the merit of extreme simplicity.

9. CONCLUSION

The whole argument of this paper runs counter to established practice. It is submitted, nevertheless, that the rejection of the current theory is based on sound reasoning supported by experimental analysis, and that the new concept is far more in accord with common sense and with everyday observation. The objection will no doubt be raised that it takes the whole problem back to rule-of-thumb days, and that it is vague and unscientific. However, it does appear that vagueness is inherent to the subject, and that past theoretical and mathematical treatment is not justified by facts.

Far from the length of a transition curve being "a theoretically determinable quantity," as one of the enthusiasts for the previous theory has claimed, it varies widely at the driver's whim. Therefore, there seems to be little object in going to great lengths to find it. The principal aim is to arrive at some reasonable compromise. This, it is claimed, the proportion rule accomplishes.

10. ACKNOWLEDGMENTS

The author's most grateful thanks are due to Mr. Black and to Mr. Kursbatt.

---

[11] "Memorandum on the Layout and Construction of Roads," *Memorandum No. 575*, Ministry of Transport, London, 1943, p. 7.

DISCUSSION

Carl F. Meyer,[12] **M. ASCE.**—Because they lead to a suggestion for a new method of transition curve design, Mr. Leeming's experiments appear to throw some light on a perplexing problem. The author's statement (under the heading, "9. Conclusion,") that "The whole argument of this paper runs counter to established practice" should, in itself, be enough to stimulate discussion. However, the phrase "established practice" may be misleading; considering all the interrelated factors, there is hardly any established practice, at least in the United States. This fact may be noted by examining the recommendations of such bodies as the United States Public Roads Administration and the various state highway departments in the United States, wherein marked variations will be found in the length of transition and in the superelevation rate for identical conditions of radius and design speed.

It is true, however, that most attempts at rationalizing this problem have been based on the use of some constant rate of turning, C; but no conclusive evidence exists as to whether C should be 1, 2, or some other value. Therein lies an important reason for the great variation in transition lengths that are found.

The principal objection which the author has to the previous theory (see "Synopsis"), is its "disregard of the superelevation." Mention is made (under the heading, "3. The Rate of Turning the Steering Wheel"), for example, of F. G. Royal-Dawson's method which

> "* * * contains the fundamental fallacy of using C and the centrifugal ratio, ignoring the superelevation. The theory gives the same length for a transition curve, no matter what the value of the superelevation."

Although there may be a fallacy, it should be emphasized that in the previous theory superelevation is not completely ignored or disregarded. Present practice bases superelevation on R, according to some variable allowance for side friction and up to some practicable maximum. The length of transition is then based on R, design speed, and some value of C. Joseph Barnett,[13] M. ASCE, for example, suggests $C = 2$, giving $L = 1.6 \frac{v^3}{R}$. Thus, L is dependent on the superelevation, although possibly not in accordance with correct dynamic theory.

There is little doubt that the crux of the problem lies in conclusive answers to the two questions posed by the author in Section 3. He is to be commended for his ingenious efforts to answer them. After studying Mr. Leeming's paper, describing his experiments and the resultant conclusions, the writer suggests that answers to the following might aid in forming a better understanding of the experimental procedure and in judging the validity of the conclusions:

---

[12] Prof. of Civ. Eng., Worcester Polytechnic Inst., Worcester, Mass.

[13] "Transition Curves for Highways," by Joseph Barnett, U. S. Bureau of Public Roads, U. S. Govt. Printing Office, Washington, D. C., 1938.

(1) Can the author supply a sketch of the accelerometer setup and a copy of a typical (Q–j)-graph?

(2) What were the ranges of values of Q and j corresponding to the four groups of comfort experienced by the observer? It would be interesting to compare values of j with the side friction factors found by R. A. Moyer,[14] Assoc. M. ASCE, and Mr. Barnett.[15]

(3) Were the comfort sensations those of the same driver and observer, or were different pairs of individuals used?

(4) Did the test curves traversed include those with transitions as well as those without; and, if so, did this difference affect the results?

(5) If, as stated (under the heading, "5. Experimental Procedure"), "* * * no attempt was made to separate the sensations due to the force or to its rate of change," how was it possible to arrive at the conclusion (under the heading, "6. Conclusions from the Experiments") that "(4) As far as comfort is concerned, j is the important effect; and Q is unimportant?"

The author was frank in mentioning the limited scope of his experiments imposed by wartime restrictions on speeds. The availability of only one American car was unfortunate, notwithstanding the author's finding that the type and the weight of vehicle seemed to have little effect on the results.

The finding (under the heading, "5. Experimental Procedure"), stating, "Another fairly marked tendency was for the driver to use a definite proportion of the curve as transition, * * *" seems quite significant provided that many different drivers were involved. This tendency, taken in connection with the mathematical analysis represented by Fig. 3, is substantial argument against large values of p, and is especially devastating to the all-transitional curve favored by some engineers. Assuming that these and later extensive experiments justify the author's conclusions, the practical application still remains to be developed. In doing this the author seems to favor the "proportion rule" (Section 8), but apparently only "for the moment." Is his reticence due to the feeling that something better will be found, or is he doubtful as to the proper value of p to use?

Although the proportion rule is easy to apply, a moderate variation in the proportion of transition results in a rather large change in transition length. For instance, in the example worked by the author in Section 8, using $p = \frac{2}{3}$, the resulting values of R and L were about 2,570 ft and 494 ft, respectively. A similar computation with all data unchanged except $p = \frac{1}{2}$ will give approximate values for R and L of 2,670 ft and 350 ft. The degree of curve remains about the same, but the transition length is reduced about 30%. As the author noted (under the heading, "8. Alternative Method of Transition Curve Design"), "The exact length of the transition is not, however, a matter of extreme im-

---

[14] "Skidding Characteristics of Automobile Tires on Roadway Surfaces and Their Relation to Highway Safety," by R. A. Moyer, *Bulletin No. 120*, Iowa Eng. Experiment Station, Ames, Iowa, 1934.

[15] "Safe Side Friction Factors and Superelevation Design," by Joseph Barnett, ***Proceedings***, Highway Research Board, National Research Council, Washington, D. C., Vol. 16, 1936.

UNIVERSITY OF MICHIGAN LIBRARIES

portance * * *." Apparently flexibility in the matter of transition length is obtained in the proportion rule method by varying the value of p.

In the practical application of the proportion rule, the values of R and L found by using a particular value of p would be slightly altered for convenience in staking (at least by most engineers in the United States). In the example given in Section 8, for instance, the computed values were $R = 2{,}570.93$ ft (or $D = 2°13' +$) and $L = 493.58$ ft. Probably 500-ft spirals joining a 2°10′ or 2°15′ curve would be used. Of course, E would not then be exactly 55 ft, but a rigid requirement permitting no deviation from a fixed E is rarely met.

The tables of spiral functions (Table 1), for which the equations use the radius as parameter, are very useful for the special cases involving a maximum value of E or T. Since these situations happen frequently, it is to be hoped that engineers will not reject the author's proportion rule solely because of the absence of such special tables in route surveying textbooks, or among the tables issued by the American Railway Engineering Association (AREA). This omission is not a stumbling block, for the usual AREA ten-chord tables and formulas may be used with only slight added effort. A quick slide rule solution of the example in Section 8 demonstrates this fact.

First, solve Eq. 8*e* for $\phi = 5.7°$ and round off to 5°30′ as in the given example. (Incidentally, it should be realized that Eqs. 8*c* and 8*e* apply also to the AREA spiral, and may be derived by using the basic relation $\phi = \frac{L\,D}{200}$.)

Then, in the AREA formula—

$$E = (R + o) \operatorname{exsec} \frac{\Delta}{2} + o \dots\dots\dots\dots\dots\dots (12a)$$

—there are two unknowns (R and o); but since the length o is always small by comparison with R it may be neglected temporarily and R solved for (approximately $\frac{E}{\operatorname{exsec}\frac{\Delta}{2}} = \frac{55}{0.01983} = 2{,}770$ ft). Thus (from AREA tables), $D =$ approximately 2°04′ and $o = 4.25$ ft. Next, recompute a closer value of R from

$$R + o = \frac{E - o}{\operatorname{exsec}\frac{\Delta}{2}} \dots\dots\dots\dots\dots\dots (12b)$$

giving $R = 2{,}555$ ft. Then $D = 2°14'$ (or 2°15′). Finally, L (from Eq. 8*c*) $= 492$ ft. In practice this would probably be rounded to 500 ft, using ten 50-ft chords.

Despite the author's objections to the current theory, certain of the methods that contain the several objections he has raised give results surprisingly close to the method he now proposes, as may be illustrated by simple computations using the Barnett tables.[13] In the Barnett method full superelevation is computed for 75% of the design speed up to a maximum rate of 0.10 ft per ft. The length of transition approximates that obtained from—

$$L = 1.6\,\frac{v^3}{R} \dots\dots\dots\dots\dots\dots\dots\dots (13)$$

—which is derived by using $C = 2$. The Barnett tables are built for values of degree of curve varying by 30′ or 1° increments, and transition lengths varying by 50-ft or 100-ft increments. In Table 3 several combinations of D and L are given which fit closely the conditions of the author's example, namely, design speed = 80 miles per hr, $\Delta = 22°38'$, and $E = 55$ ft (approximately). Values of D, L, T, and E were obtained quickly by simple interpolation from the Barnett tables; the values of ϕ, p, e (superelevation rate), and j have been computed for comparison with the author's values in Section 8. It will be observed that the combinations using 350-ft and 500-ft transitions compare closely with the values obtained by the "proportion rule" using $p = \frac{1}{2}$ and $p = \frac{2}{3}$, respectively. The largest difference seems to be in the matter of superelevation rate, the values of which could be scaled downward toward the author's recommendations (with consequent increase in j) without changing the values of the other quantities in Table 3.

TABLE 3.—SUITABLE TRANSITIONS FOR EXAMPLE OF SECTION 8

D	L	T	E	ϕ	p	e	j
2°00′	300	723.6	48.2	3.0	0.42	0.084	0.065
2°00′	350	748.7	58.6	3.5	0.47	0.084	0.065
2°30′	400	659.2	48.4	5.0	0.61	0.10	0.086
2°30′	500	709.5	51.1	6.25	0.71	0.10	0.086
2°30′	600	759.8	52.1	7.5	0.80	0.10	0.086

It was not the writer's intention to show the preceding comparison for the purpose of discounting the author's alternative method of transition design. On the contrary, it is felt that the author has made a genuine contribution to this interesting problem. As for the proposed method (under the heading, "9. Conclusion"), taking "* * * the whole problem back to rule-of-thumb days * * *," there appears to be no more, if as much, rule of thumb than exists in the present theory. Even though the author feels that his method may meet resistance because it runs counter to present theory, the writer is encouraged to feel that engineers are willing to reject anything found unsound (as stated in the "Conclusion") in favor of "* * * sound reasoning supported by experimental analysis * * *." The modern engineer does not prefer old-established error to novel truth.

RALPH L. FISHER,[16] ASSOC. M. ASCE.—A realistic approach to the problem of the transition curve has been presented by the author, who states that "vagueness is inherent to the subject" and cumbersome "mathematical treatment is not justified by the facts."

The basic function of a spiral curve is to enable a driver to develop the necessary steering angles (and friction values) with "safety and comfort" within the limits of his lane. On a tangent or on a circular curve a driver is subject to constant, known conditions and it is only on the spiral that he must make adjustments. It is awkward to drive on curves that are transitional throughout. Even when traveling on a tangent, a driver is constantly moving the steering wheel in order to keep on his path. On a curve of constant radius these adjustments in steering are also necessary. If the radius of the curve is constantly changing, the driver's task is made more difficult.

---

[16] Engr. of Design, State Highway Dept., Trenton, N. J.

UNIVERSITY OF MICHIGAN LIBRARIES

Mr. Leeming suggests that curves have from one half to two thirds of their length transitional. Except for the rare case of equal length curves, this would make every spiral different, even those having the same radius of circular curve. If spirals replace a large part of the circular curve, they must perform a function for which they are not suited, that is, change in direction. For this latter function, the circular curve, with its constant conditions, is best fitted. It would be better design practice to confine the spirals to each end and thereby be able to use a larger radius for the circular curve, thus creating more uniformity of operating conditions.

The alinement, grades, superelevation, and spirals should be designed as a unit. Too often the spirals are added more or less as an afterthought, with the result that the spirals or the superelevation runouts are at bridges, vertical curves, minimum grades, or other unfortunate locations—thus creating problems that need never have occurred. Any transition curve should provide a path conforming to driver behavior. Observation indicates that most drivers tend to "cut corners" and to use more throw than is provided by the cubic spiral. The designers of the Pennsylvania Turnpike developed a 2.5 power spiral, which gives more throw than the cubic spiral. It was not used because of lack of time to make the necessary field tests and because of lack of tables for the field forces.

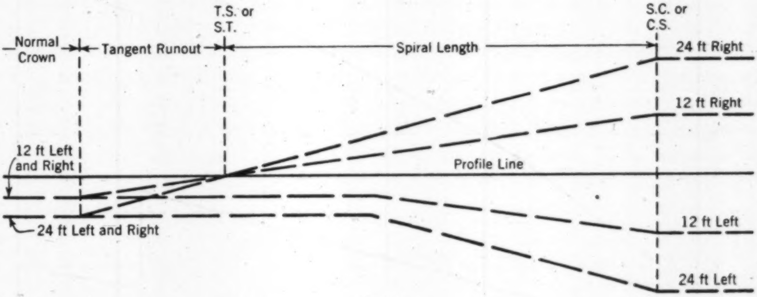


FIG. 4

The writer agrees with Mr. Leeming that (within limits) the length of the transition is not a matter of great importance. As he states, there is considerable variation in a driver's practice, even on the same curve and at the same speed. For the same conditions, a driver does not develop the steering angles at the same rate. It is believed that the amount of throw is of more importance than the rate of curvature of the transition spiral.

Before decision as to the lengths of the spirals can be made, the method of obtaining the superelevation must be determined. This requires careful study for multilane roads or divided roadways with narrow islands. In the colder regions consideration must be given as to where snow can be ploughed on the superelevated sections. The cross section should be designed so that during periods of thawing and freezing, water from the melting snow will not run on the pavement and freeze, thus becoming a hazard. At sags, care should be taken that the low side does not go below certain minimum elevations for surface drainage or water table conditions. Detailed studies may have to be made at bridges where vertical clearance is limited.

For the foregoing reasons, the length of spiral is rarely governed by the distance necessary for change in centrifugal acceleration, as distances given by this method are generally too short. With the flat grades used on most highways today, the length of spiral is generally governed by the minimum gutter grades. If the lengths of the spirals and tangent runouts are selected to fit these flat grades, they can be used throughout, and there will not be the necessity of using one length for the steeper grades and having "special cases" for the flatter grades.

Fig. 4 shows the usual method of obtaining superelevation with spirals. In cases of 5% superelevation, a spiral 400 ft long is required if the relative slope of the outer edges of the pavement with respect to the center line is held to a change of 0.3%. With a center line grade of 0.5% this means that one edge of the pavement would have a grade of 0.2%. However, the shoulder can be warped to give minimum gutter grades of 0.3%.

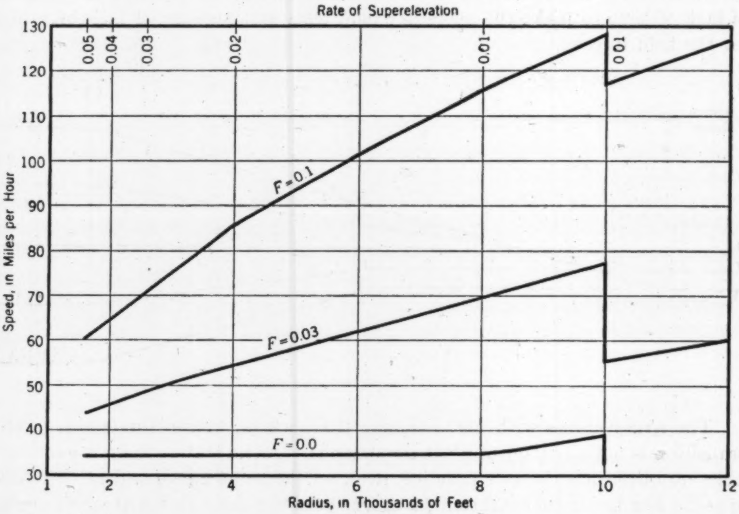


FIG. 5

Uniformity of operating conditions is the keynote of modern highway design. In practice, the design speed is usually selected first and then the other elements, including transition curves, are designed to be consistent with this design speed. As an example, with an assumed design speed of 60 miles per hr and a minimum radius of 1,600 ft, for a friction factor of 0.1, E would equal 0.05 from

$$V^2 = \frac{R\,E'}{0.067} \qquad (14)$$

UNIVERSITY OF MICHIGAN LIBRARIES

For other radii, the superelevation varies inversely as the radius.

Fig. 5 shows the loci for $F = 0.0$ (balanced forces), $F = 0.03$ (assumed icy surfaces), and $F = 0.1$. The locus for balanced forces indicates uniform and consistent operating conditions and the speed is low enough for most trucking units. The break in the curves at the 10,000-ft radius point is due to the change from a superelevated section to a normal crowned section.

TABLE 4.—RATE OF SUPERELEVATION FOR 400-FT CUBIC SPIRAL TO CURVE OF 1,600-FT RADIUS HAVING 5% SUPERELEVATION

Distance from T.S. (ft) (1)	Rate of superelevation (%) (2)	Radius (ft) (3)	Distance from T.S. (ft) (1)	Rate of superelevation (%) (2)	Radius (ft) (3)
0	0	∞	250	3.12	2,560
50	0.62	12,900	300	3.75	2,130
100	1.25	6,400	350	4.37	1.830
150	1.87	4,280	400	5.0	1,600
200	2.50	3,200			

For the cubic spiral, the degree of curve and the superelevation varies directly with the length from the point of tangency (T.S.). Table 4 shows this relationship for a 400-ft spiral to a curve of 1,600-ft radius with 5% superelevation. The curvature of this spiral varies from infinity to 1,600 ft, and this curvature is fixed. It seems apparent that, if this rate of curvature is suitable for a transition to the minimum radius, it would also be suitable for curves of larger radius if the proper length of the sharper end were deleted. In other words, by selecting a proper segment of this spiral to meet a given radius, it would be suitable for all curves of 1,600-ft, or greater, radius. This seems to be more logical than having a spiral with a different rate of curvature for each radius.

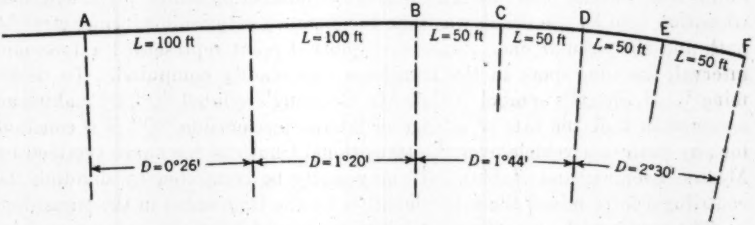


FIG. 6

Fig. 6 was prepared from Table 4. This series of curves gives a throw of 6.40 ft for the 1,600-ft radius, compared with a throw of 4.17 ft for the cubic spiral. This additional throw is believed to conform more nearly to driver behavior. However, if desired, by changing the radii slightly, these curves could approximate the cubic spiral very closely. The transitions from point A to point B would be used for curves of 3,500-ft radius to 3,200-ft radius; from A to C, for 3,200 ft to 2,500 ft; from A to D, for 2,500 ft to 2,100 ft; from A to

E, for 2,100 ft to 1,800 ft; and, from A to F, for 1,800 ft to 1,600 ft. In actual practice, it would probably be necessary to develop two series of transition curves, one for general conditions and another of minimum length for cases where the middle ordinate or tangent lengths are restricted. All elements of these curves as required by local surveying practice could be easily computed, thus doing away with the necessity for any tables or computations by the field parties. This is a factor to be considered, particularly during periods when most highway departments are experiencing a shortage of trained personnel.

William R. Welty,[17] Assoc. M. ASCE.—It is gratifying that the research of the English engineering profession in this hereto virtually unexplored subject is made available to the American engineer in this fine paper. The fact that transition curves in the past have been designed on fallacious assumptions does not require that the same procedure continue in the future.

The writer had the opportunity to examine the spiral experimentally, using the photographic method developed by Bruce D. Greenshields,[18,19] Assoc. M. ASCE. This study [20] of the transition curve was instituted by the writer due to recurring doubts as to the validity of the conventional spiral transition theory. The work of Kenneth A. Stonex and Charles M. Noble,[21] M. ASCE, H. A. Warren,[8] and Mr. Leeming and A. N. Black,[9] certainly casts doubt on the preconceived theories, if it does not entirely shatter them. In the study using the photographic method, a specially arranged motion picture camera and timer were set up at a high vantage point (for example, on a skycraper or in a blimp) above and close to the curve to be studied. The camera was arranged so as to take one picture every eighty eighth of a minute. Thus, the motion of the vehicle in feet between successive frames gives the vehicle's velocity in miles per hour. For examination of the vehicle's path, its position was plotted to scale for each 1/88-min interval. The points at the approximate center of the curved path were fitted graphically with a circular arc of the smallest radius which would pass through the greatest number of points. The length of transition used by the driver was then the distance between his former straight path and the circular one. Since each plotted point represented a 1/88-min interval, the time spent in the transition was readily computed. To determine W. H. Shortt's symbol "*C*" or Mr. Leeming's symbol "*Q*", by making an assumption that the rate of change of lateral acceleration "*Q*" is a constant for any path of a vehicle over the transitional length of the curve (verified by Messrs. Leeming and Noble), "*Q*" may easily be computed by dividing the centrifugal force minus the superelevation by the time spent in the transition.

The study, although of very limited scope and by no means comprehensive, did seem to bear out Mr. Leeming's conclusion that the value "*Q*" is by

---

[17] Traffic Designing Engr., State Highway Dept., Austin, Tex.

[18] "Photographic Method of Studying Traffic Behavior," by Bruce D. Greenshields, thesis presented to the University of Michigan at Ann Arbor, in 1934, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

[19] "Traffic Performance at Urban Street Intersections," by Bruce D. Greenshields, Donald Schapiro, and E. L. Erickson, *Technical Report No. 1*, Yale Bureau of Highway Traffic, New Haven, Conn., 1947.

[20] "A Method for Studying the Paths of Motor Vehicles on Curves," by William R. Welty, Bureau of Highway Traffic, Yale Univ., New Haven, Conn., May, 1946.

[21] "Curve Design and Tests on the Pennsylvania Turnpike," by Kenneth A. Stonex and Charles M. Noble, *Proceedings*, Highway Research Board, National Research Council, Washington, D. C., Vol. 20, 1940, p. 429.

UNIVERSITY OF MICHIGAN LIBRARIES

no means a constant. A direct relationship seems to exist between "j" and Q; Q seemed to have a value between $31.4\,j$ and $47.2\,j$. Measurements were not made of high speed operation, but some values measured with an accelerometer on the Pennsylvania Turnpike by Messrs. Stonex and Noble approximately fitted the graphical relationship.

According to the Shortt theory of a constant C or Q, the length of transition will vary with the cube of the velocity. It was found in the photographic study that for the same curve the faster drivers used the shortest transitions, which is in opposition to the former theory. The writer believes that the reason the faster drivers use a shorter transition than do the slower drivers may be explained in the following manner:

1. Fast drivers expect on entering a curve to develop considerable side thrust.
2. Since the side thrust is uncomfortable to the driver (and he expects that it will be), he develops the maximum side thrust quickly, continues it only as long as is necessary, and gets out of the curve as soon as possible.
3. On the other hand, the slow driver, since motorists are humanly lazy, turns the steering wheel at a minimum rate to save work, thus using a long transition. A long transition does not, in this case, prolong the discomfort of a high value of side thrust, since at slow speeds a high value of side thrust is not reached.

The accelerometer method of measurement possesses certain advantages: First, the rate of change of lateral acceleration may readily be determined; and, second, the transitional and circular lengths of the path are easily secured. On the other hand, it is believed, as recognized by Mr. Leeming, to be quite possible that the fact the driver knows his actions are being recorded, in the case of the accelerometer method, may cause significant changes from his normal behavior. By use of the photographic method, measurements may be made without the driver's knowledge.

ELMER R. HAILE, JR.,[22] Assoc. M. ASCE.—Although his argument may, as the author states, run counter to established practice, it is in general agreement with observations made by the writer several years ago. At that time the writer measured the rate of turning of the steering wheel on entrance to a curve, and concluded that no definite trend could be discerned because of wide variations in the rate observed during several test runs. Mr. Leeming is to be commended for carrying his experimentation and analysis through to logical conclusions.

The writer agrees with the author's four conclusions (under the heading, "6. Conclusions from the Experiments"). He does not agree, however, that the proportion rule is applicable in all cases, and this discussion is intended to point out certain limitations in its use. With reference to the third conclusion, the writer believes that the tendency for drivers to use from two thirds to one half of the curve as the transition is due to an inherent characteristic of motoring that is not generally recognized. This might be called the characteristic of "smoothing" the path of the vehicle.

---

[22] Highway Engr., U. S. Public Roads Administration, Arlington, Va.

Accepted design theory is based on the assumption that the vehicle behaves like a locomotive fixed on rails, whereas, in reality, each driver selects the path for his vehicle, and constantly tends to smooth out any "kinks" in the road. This action is usually subconscious and, in fact, unavoidable, stemming from the fact that the driver guides his vehicle by the appearance of the road many car lengths ahead. In contrast, the railroad locomotive is guided solely by the rails through the pony wheels. The inability of the driver to see the road at the front wheels is occasionally embarrassing—for example, when attempting to drive up a ramp or onto a hydraulic lift at a service station. However, in high speed driving on the highway, this inability to see the front wheels is comfortable rather than uncomfortable.

Some examples of this tendency of drivers to smooth the courses of their vehicles are as follows:

(1) In making a right turn at a street intersection, drivers tend to swerve left before turning right, and also to swing wide into the cross street in order to obtain a longer radius of curvature. This behavior is often observed, and persists in spite of many collisions and near collisions that result from it.

(2) A circular curve, when viewed from a distance, appears to break sharply from the tangent. There is a tendency to smooth out this break in the same manner as described in item (1). This smoothing is better known as the transition.

(3) An all-transitional curve, say, with a length of 300 ft and a central radius of 250 ft, appears to bend sharply in the middle. Drivers invariably traverse the middle of such a curve with a radius longer than the design radius, and, if traffic permits, they fail to confine their vehicles to the proper traffic lane.

(4) A short curve with a small deviation angle appears to be a bad break when it is located between two long tangents. Drivers invariably "cut the corner," particularly at high speeds.

In the foregoing examples, the underlying motivating force is the aim of the driver to select a smooth path between the location of the vehicle at any instant and some point ahead that appears in the field of vision. In slow city driving, the driver may gage his path to a point less than 100 ft ahead, but in fast rural driving he may keep his eyes on the road several hundred feet ahead.

The properties of two curves computed by conventional methods are compared with curves computed by the proportion rule, as follows:

Conventional Method.—

Curve 1.—Assume that $v = 40$ miles per hr; $\Delta = 35°$; and that the tangents are limited. Then the minimum allowable $R = 409$ ft and $L = 250$ ft, using a maximum superelevation of 0.10 ft per ft.[23] Then $\phi = 17° 30'$; $T = 256$ ft; and $E = 27$ ft.

Curve 2.—Assume that $v = 60$ miles per hr; $\Delta = 21°$; and the tangents are limited. Then the minimum allowable R = 955 ft and $L = 350$ ft, using a maximum superelevation of 0.10 ft per ft.[23] Then $\phi = 10° 30'$; $T = 353$ ft; and $E = 22$ ft.

---

[23] "Transition Curves for Highways," by Joseph Barnett, U. S. Bureau of Public Roads, U. S. Govt. Printing Office, Washington, D. C., 1938, p. 55.

UNIVERSITY OF MICHIGAN LIBRARIES

Each of the curves given is transitional throughout because $\phi = \Delta/2$.

Proportion Method.—Applying the proportion rule, with p equal to, say, one half ($\phi = \Delta/6$):

Curve 1.—$\Delta = 35°$; $\phi = \Delta/6 = 5°50'$; and $L = 250/2 = 125$ ft. Then $R = 614$ ft; $T = 256$ ft; and $E = 31$ ft. Using[24] $j = 0.16$ and superelevation $= 0.10$, allowable $v = 49$ miles per hr, an increase of 9 miles per hr over the conventional curve.

Curve 2.—$\Delta = 21°$; $\phi = \Delta/6 = 3°30'$; and $L = 350/2 = 175$ ft. Then $R = 1{,}432$ ft; $T = 353$ ft; and $E = 25$ ft. Using[25] $j = 0.14$ and superelevation $= 0.10$, allowable $v = 72$ miles per hr, an increase of 12 miles per hr over the conventional curve.

To substantiate the fact that it is possible to permit speeds higher than those permitted by conventional theory, the following paragraph is quoted from an unpublished report by E. W. Allfather, dated July 20, 1940, describing the results of using an aeroplane bank indicator to determine the safe speeds to be painted on curve signs on the Blue Ridge Parkway in Virginia and North Carolina:

> "It was demonstrated that, for short double spiral curves, signs may safely show speeds fifteen miles per hour in excess of the theoretical speed obtained at the point of maximum curvature. However, it is doubtful if this rule could be applied to long double spirals as too much distance would be covered while traveling through degrees of curvature more nearly approaching the maximum degree obtained at the S.C.S. [spiral-curve-spiral]."

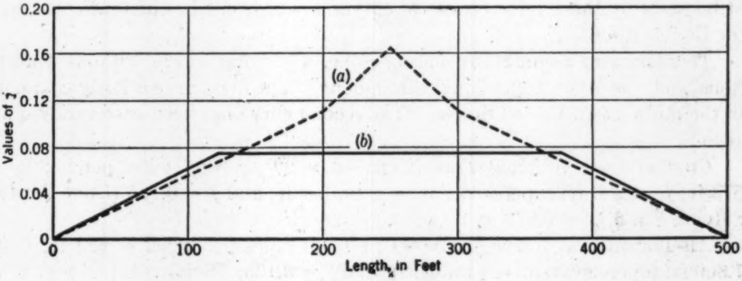


Fig. 7.—Comparison of Values of j at $v = 40$ Miles per Hr and 0.10 Superelevation for

(a) Double Spiral; $p = 1$, $R = 409$ ft

(b) Circular Curve with Spirals; $p = \frac{1}{2}$, $R = 614$ ft

Fig. 3 proves the case against all-transitional curves so well that further proof seems unnecessary. However, for the record, Fig. 7 was plotted to show the theoretical values of j that would be obtained if a vehicle travels at 40 miles per hr over the conventional double spiral, compared to those for the circular curve with spirals for which p equals one half. On the conventional double spiral, maximum superelevation was attained before the midpoint on the curve, as recommended.[26]

---

[24] "Transition Curves for Highways," by Joseph Barnett, U. S. Bureau of Public Roads, U. S. Govt. Printing Office, Washington, D. C., 1938, p. 5.

[25] *Ibid.*, p. 6.

[26] *Ibid.*, p. 9.

Although it is perhaps irrelevant to the present paper, the writer would like to know the values of j and Q which the observers described as "uncomfortable" and "slightly uncomfortable." It is presumed that correction was made for body roll in measuring j with the accelerometer.

In Section 1, the author noted that the formulas generally used for determining the length of required transition do not take superelevation into account. In 1936 the writer proposed a formula which allowed for the effect of superelevation[27] but recommended that it not be used because the lengths of transition computed with the formula were much too short for values of j near zero. This advice is in line with Mr. Leeming's observation that Q is unimportant. However, one should not overlook the fact that the minimum length of transition may be fixed by the distance required for changing from a crown section to a fully superelevated section, especially on multiple-lane roads.

In making a statistical analysis it is of paramount importance that cognizance be taken of all variables and special conditions. The author noted that his experiments were made principally with the British type of light car, at speeds seldom exceeding 50 miles per hr. Some of his tests were made on right-angle junctions and rotary intersections. It would be helpful if Mr. Leeming could describe the curves most frequently used in his tests. Very possibly, few, if any, of the curves he tested were comparable with the curves on the Pennsylvania Turnpike which have radii varying from 38,197 ft to 955 ft and an average length of about 1,400 ft.[21] It would be interesting to know what proportions of his tests were run on (*a*) curves without transitions, (*b*) curves with transitions, and (*c*) all-transitional curves. It seems likely that analyses of his tests by these three classes of curves would result in different constants for Eqs. 5 and 6.

To illustrate, assume two curves—one, a circular curve without transitions, and, the other, transitional throughout. The first curve is fairly common in the midwestern United States. The second curve has been used rarely if at all.

Circular Curve Without Transitions.—$\Delta = 90°$; $v = 40$ miles per hr; $R = 573$ ft; $L = 900$ ft; superelevation = $\frac{3}{4}$ in. per ft; and $j = 0.125$ (from Eq. 1). Solving Eq. 6, $p = 0.87$.

All-Transitional Curve.—$\Delta = 90°$; $v = 40$ miles per hr; $R = 573$ ft; $L = 1{,}800$ ft; superelevation = $\frac{3}{4}$ in. per ft; and $j = 0.125$. Solving Eq. 6, $p = 1.03$ (an impossible solution, as p cannot exceed 1).

In neither of the foregoing illustrations does p approximate values between two thirds and one half which the author states is the case for $j \overset{=}{>} 0.1$. Perhaps the inference to be drawn is that the curves tested did not include many curves similar to those illustrated herein.

If driving tests to measure p should be made on the foregoing circular curve, the length of transition employed by the driver of the car could hardly exceed 300 ft because the offset required for inserting a spiral transition of that length is 6.5 ft. This means that, in order to stay on the road, the driver would probably make his transitions somewhat shorter than 300 ft. If he used 200 ft, the length of circular curve remaining would be 700 ft and the value of p would be $\frac{400}{400 + 700} = 0.36$. It is apparent, therefore, that the measured value of p would range from around 0.3 to 0.4 for a circular curve similar to the curve considered.

---

[27] *Transactions*, ASCE, Vol. 102, 1937, p. 1096.

If driving tests should be made on the foregoing all-transitional curve, using the same line of reasoning, it appears that the value of p would range from about 0.8 to 1.0. This discussion should emphasize the desirability of qualifying statistical analyses such as those represented by Eqs. 5 and 6 by setting forth the limits of all variables.

In conclusion, it is believed that the author's proportion rule is applicable to all short curves, particularly those for which customary design methods result in values of p greater than two thirds. At other curves, especially those of great length, the value of p has little significance. Indeed, when the radius is greater than about 4,000 ft, there are no noticeable consequences if p is allowed to become zero, which is to say that the curve need not have transitions.

There is need for a comprehensive study to determine the length of transition required to satisfy the behavior of the average driver, but, until it can be determined for various radii and lengths of curves, and for a logical range of speeds, there seems to be insufficient justification for changing current practice.

Charles M. Noble,[28] M. ASCE.—In this refreshing paper the author questions the rather complacent attitude toward the subject of highway transition curve design.

Originally the cubic equation, generally utilized in the United States, was developed for railway practice and has been perfected by the railroads to suit actual operating equipment and speeds in the light of many years of practical experience. Safety and passenger comfort are the criteria. Fixed rails guarantee an accurate adherence to the designed path of the spiral so that the human equation in steering the vehicle is entirely absent. No real and searching investigation has been made to develop a transition curve suited entirely to highway vehicles and drivers.

As pointed out by the author and the writer,[29] the use of superelevation cancels out part of or all the centrifugal acceleration (depending on the speed of the vehicle), and this changes the fundamental assumptions in present spiral theory. Primarily, the transition curve enables the vehicle to attain superelevation safely within the lane path by avoiding the introduction of premature lateral forces, which are present if the superelevation is "run out" on the tangent ahead of the curve. If the spiral is long enough to attain this objective, there is some evidence to indicate that it will also be long enough to absorb safely the application of residual (after deducting the effect of superelevation) centrifugal forces.

These matters are clearly perceived by the author, and in the "Synopsis" he succinctly states:

---

[28] State Highway Engr., New Jersey State Highway Dept., Trenton, N. J.

[29] "Thoughts on Highway Design Research as Related to Safety of Vehicle Operation," by Charles M. Noble, *Proceedings*, Highway Research Board, National Research Council, Washington, D. C., Vol. 17, 1937, p. 247.

> "* * * (1) that the theory is fallacious in its disregard of the superelevation, and (2) that the experimental evidence produced for it is questionable."

The writer wishes to support these statements and urges the implementation of an adequate, scientifically instrumented research program so that the real and practical facts relating to highway transition curve design may become known.

In 1937 E. R. Haile, Jr.,[30] Assoc. M. ASCE, and T. T. Wiley,[31] Assoc. M. ASCE, proposed that spiral lengths be proportioned on the basis of rotational change in attaining the curve superelevation. The latter stated the railroads had determined that superelevation should not be introduced in track (4 ft 8½-in. gage) at a rate greater than 1¼ in. per sec of time at the design speed. This is equivalent to slightly more than 2% of cross slope, from which he concluded that:

> "Superelevation on a highway should not be introduced at a rate exceeding 0.02 ft per ft per sec, and the length of spiral on a curve with a superelevation of 1 in. per ft should approximate the distance traveled in 4 sec at the design speed."

It is worthy of note that this principle and the precise maximum value recommended by Mr. Wiley—namely, 2% cross gradient per sec of time—were utilized in designing the spiral lengths on the Pennsylvania Turnpike.

Apparently the author is not familiar with the high speed curve tests conducted on the Pennsylvania Turnpike, which were fully reported.[21] These tests were extended to the top speeds (105 miles per hr) of the stock car models used. A fifth wheel and electrical recording on flags spaced at measured intervals were utilized to obtain accurate speed measurements. Actual path measurements for both the front and rear of the car were taken and recorded. It is believed that a careful study of these test results by the author will be interesting and may suggest additional methods of approach to this problem.

The writer believes that in so far as possible all roadway features should be designed to conform with the requirements of both vehicle and driver,[32] and therefore wishes to support and emphasize the following quotation from the paper (under the heading, "3. The Rate of Turning the Steering Wheel"):

> "It is surely logical that the curve should be laid out to conform to the driver's practice; and, if he prefers, and habitually uses, some method of turning his wheel other than the constant rate expressed by constant *Q*, then the type of transition should be chosen to conform with his habits—even if this curve is not of spiral form."

This policy of matching road design with the characteristics of both driver and vehicle has received impetus in recent years and is now regarded as accepted procedure by American engineers.

As suggested by the author, it is entirely conceivable that driver-vehicle requirements may dictate another form of curve than that represented by the cubic equation. Experience in driving on the Pennsylvania Turnpike indicates that cubic equation spirals are faster than the simple curve they connect and may have a tendency to lure the driver into the curve too fast. This is not serious, but it indicates that the cubic equation may not be the ideal sought.

---

[30] *Transactions*, ASCE, Vol. 102, 1937, p. 1097.

[31] *Ibid.*, p. 1105.

[32] "Engineering Design of Superhighways," by Charles M. Noble, *Proceedings*, Am. Road Builders' Assn., 1941, p. 183.

There is some indication that it may be desirable to treat sharp curves and flat curves differently. It is believed that there has been confusion in dealing with the subject of transition curves as between sharp and flat curves. Some proponents advocate making the curve transitional throughout, and even Mr. Leeming proposes that the transitions should be a fixed proportion of the entire curve. To most practical American highway engineers it appears preposterous to transform a simple curve 3,000 ft or 4,000 ft long into transitional curves throughout, or to have a fixed proportion of their length transitional. The tendency of drivers to "cut corners" on entering a curve should also be considered as one of the factors in the problem.

To develop further the sharp-flat curve idea, it may be illustrative to divide curves arbitrarily into two groups: Those sharper than 1,000-ft radius and those flatter than 1,000 ft. There is a possibility that the cubic equation may be suited to curves sharper than 1,000-ft radius, whereas another form of transition may be required for easier curves. There is also some evidence that curves flatter than a 3,500-ft radius do not require spirals at all for design speeds up to 70 miles per hr.

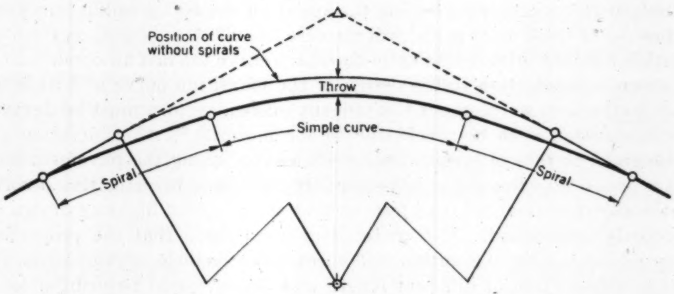


FIG. 8

For example, assume a curve with 3,000-ft radius, a superelevation of 0.03 ft per ft, and a design speed of 70 miles per hr. (On modern American main trunk highways a curve of this radius may vary from 500 ft to 5,000 ft in length.) If the rotational theory for determining spiral length is applied, a 154-ft spiral is required. (For simplicity, it is assumed that the roadway surface is level at the beginning of the transition.) This will result in a "throw" of 0.34± ft (Fig. 8). This offset or throw is so small that it does not appear to meet the requirements of drivers who cut corners on entering curves. If Mr. Leeming's proportional rule is applied, and assuming that the 3,000-ft radius curve is 1,260 ft long, then a 420-ft spiral is required, which will result in a throw of 2.5± ft. This throw or offset may more nearly meet the corner cutting tendency of the driver, but if a cubic equation is utilized much of its length will be of such slight curvature that it is meaningless in practical driving. To be specific, a driver will traverse 211 ft of the spiral before the curve sharpens to a 6,000-ft radius, and 360 ft, before reaching a 3,500-ft radius. Thus, it appears that the throw may meet the requirements of the driver, but that the curve is too flat on the end and should be shortened 210 ft, indicating a transition curve length of 210 ft. However, such a curve must have different characteristics from a cubic equation in order to obtain the desired throw within a length of 210 ft.

On the other extreme, assume a curve with 500-ft radius, a superelevation of 0.10 ft per ft, and a design speed of 43 miles per hr. If the rotational theory is applied, a 315-ft spiral is required, which will yield a throw of 8.24± ft. This offset may be entirely sufficient to meet the corner cutting tendency of the driver, whereas the offset of 0.34 ft for the cubic equation spiral on a 3,000-ft radius curve appears insufficient. Furthermore, it is necessary for the driver to traverse only 45 ft of the 315-ft spiral before reaching a radius of 3,500 ft. If the author's proportional rule is applied, assuming the curve is 700 ft long, a 230-ft spiral is required, which will result in a throw of 4.4± ft. It is believed that this transition is too short for a 500-ft radius curve.

In considering the operating possibilities on the 3,000-ft and 500-ft curves, it is interesting to note that increasing the speed on the 3,000-ft curve from 70 miles per hr to 80 miles per hr will increase the unbalanced centrifugal ratio, f, to only 0.113, whereas increasing the speed on the 500-ft radius curve from 43 miles per hr to 53 miles per hr will increase f to 0.275. Therefore, the longer transition will give drivers who enter the sharp curve too fast an opportunity to slow down. In addition to the design of the transition curve and its length, proper methods of warping out the tangent roadway crown must be devised if the entire design is to be acceptable to the motorist.[33] Consideration must also be given to rate of longitudinal grade change along the pavement edges.

It is hoped that this paper will stimulate additional investigation involving scientifically instrumented road tests so that this phase of highway design may be properly established. The writer does not believe that the proportional theory advocated by the author will meet driver-vehicle requirements, inasmuch as a transition of different length and throw would be utilized for the same radius curve whenever it varied in length. Until new forms are proved superior in meeting driver-vehicle requirements, use of the cubic equation will undoubtedly continue in the United States. The author is to be congratulated for the presentation of material on an important subject, and it is hoped that he will continue his study and investigation.

John Joseph Leeming,[34] Esq.—It is gratifying to find so much agreement with the principles proposed in the paper, in spite of the fact that the proportion rule does not find general acceptance. Indeed, the writer must admit that he is not fully satisfied with the rule in its present form. It is probably too rigid, and tends to give excessively long transitions on the longer curves. Since the paper was written, the work of the experiments has been concluded and the statistical analysis is in progress, although it will be some time before the latter will be complete. It seems possible that some sliding scale of reduced proportion of transition for longer curves may result, although this is only conjecture. For the moment, the writer is inclined to agree with Mr. Haile that the rule should only apply to short curves. In spite of its shortcomings, however, the rule has some experimental background, and does represent some attempt to relate transition curve design to facts. The writer cannot agree with Mr. Haile that there is "insufficient justification for changing current practice," since the latter has no theoretical or experimental justification at all.

---

[33] "Developments in Curve Design, Speed and Sight Distance," by Charles M. Noble, *Roads and Streets*, January, 1942, p. 25.

[34] County Surveyor, Dorsetshire, England.

Messrs. Noble and Fisher both object to the proportion rule on the ground that it involves spirals with different rates of curvature for different radii of curves. Nevertheless, it appears from the experiments that drivers do vary the scale of the transitions they use. This is the meaning of the observed variable Q, since Q is a measure of the scale of the spiral. To demand a constant shape of spiral, as Mr. Fisher does, is to return to the principle of the constant C, which the writer has shown to be unsound.

Professor Meyer and others desire further particulars of the experimental technique. Present space hardly permits this; but the original paper has been deposited in the Engineering Societies' Library. In reply to Mr. Noble, however, the writer did read the report on the high speed tests on the Pennsylvania Turnpike,[21] and his study of the report influenced his own experiments somewhat. The latter, however, were intended to deal with ordinary touring conditions and with the ordinary driver, whereas the Pennsylvania tests were designedly carried out at extreme speeds with highly skilled drivers, and were mainly directed toward finding the maximum safe value of j at high speeds.

It is not possible to state what proportion of the curves analyzed had transitions. In the first stage of the work the mean proportions of transition, for various classes of roads worked out as follows:

Type of road	Average value of P	Number of curves
First class	0.64	507
Second class	0.67	379
Unclassified	0.67	370
Rotary intersections	0.64	43
Right-angle junctions	0.67	38
Total		1,337

Values are not available for the total number of curves analyzed. The first class roads included a fair proportion of curves with transitions (although some of these were of compound circular arcs) and superelevation; the second class, a few; and the remainder, probably none at all. The experimental results give no reason to believe that the presence of a transition either makes for any greater consistency on the part of the driver, or prevents him from cutting corners. As a result of his observations, the writer is convinced that the latter process is due to the principle stated by Mr. Haile (and with which the writer is in complete agreement) that "* * * the driver guides his vehicle by the appearance of the road many car lengths ahead," combined with the geometrical properties of the compound curve as shown in Fig. 3. Ordinarily, the driver does not begin to turn before he reaches the start of the curve. Whenever the records could be compared with the road conditions, the length of the curve on the record was not appreciably greater than the corresponding length of the curve on the ground, and often, it was shorter. If, then, the driver begins to turn at about the start of the spiral, the quicker he turns his wheel, the flatter will be the curve he follows and the less will be the value of j (that is, the force he has to apply at the wheel), although the throw or cut will be greater. Since he has no objection to a high value of Q, and since he is, in Mr. Welty's words, "humanly lazy," the driver takes the line of least resistance and cuts the curve if its general appearance leads him to believe that it would be feasible. This will usually be the case if a sharp curve has unduly long transitions. On the other hand, short transitions anticipate the cut, as it were, and lead to as flat a curve as the conditions allow. A high value of Q does not increase the driver's work.

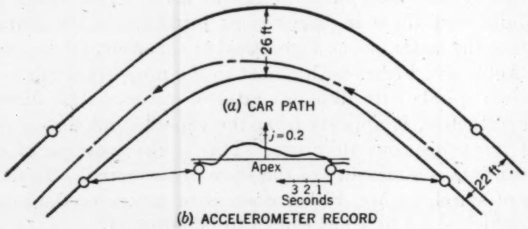


FIG. 9.—TYPICAL RECORD OF CAR ACTION ON ALL-TRANSITIONAL CURVE AT 39 MILES PER HR

The foregoing, however, is not a complete explanation, as the following example shows. A careful analysis was made of an all-transitional curve, laid out and constructed with great care early in World War II. A diagram of the curve is shown in Fig. 9 and the design data for the lemniscate curves of the inner and outer edges are given in Table 5. The deviation angle, I, of the entire curve is 39° 26′; the hands-off speed, 22.5 miles per hr; and the superelevation at the apex, 0.07. The curve is admittedly sharp, especially so, as it was on an important road which carried much traffic. It was located at the start of a diversion of an originally straight road, built for the construction

TABLE 5.—DESIGN DATA FOR LEMNISCATE CURVES IN FIG. 9

Description	Outer curve (ft)	Inner curve (ft)
External distance, E	32.2	35.0
Tangent distance, T	297.6	323.5
Half length of curve, $\frac{L}{2}$	290.2	315.7
Minimum radius, R	456.5	496.0

of an aerodrome, but owing to wartime conditions there was only a minimum of construction. Recordings were made with the accelerometer, tied in with the tangent points of the curve, and checked by timing. Other traffic was observed and timed. A typical record, taken at 39 miles per hr, is shown in Fig. 9, in which it can be seen that the record is markedly unsymmetrical. The car's probable path is shown, and the observations are confirmed that most of the faster traffic followed a similar line. (The English left-hand rule of the road must be remembered in interpreting Fig. 9.) In this case, the most probable explanation is that the driver follows the first half of the curve fairly closely, although some slight tendency to cut the apex is possible. When he reaches the apex, the sharpening of the curve passes unnoticed until the swerve at the three-quarters point becomes obvious, and a sharp exit is made. It was also noticed that traffic in the opposite direction cut the curve very noticeably, yet in this case the recordings were of a more normal type.

In reply to the questions raised by Mr. Haile, the analysis of comfort is not yet complete, as it involves a statistical analysis of great complexity. Provisionally, it may be stated that about half the observers found the value of $j = 0.13$ combined with the value of $Q = 4$ ft per sec^3 to be imperceptible. A similar number of observers thought $j = 0.22$ in combination with $Q = 5$ ft per sec^3 to be uncomfortable. In the later work, no distinction was made between the two classes of discomfort. So far, it has been impossible to state whether speed has any effect on the estimation of comfort. Body roll is included in these values of j, and, since it is one of the sensations of the driver, it should be included in analyzing comfort. Mr. Noble[33] found that this effect would amount to about 10% of the real value of j. British cars tend to be more stiffly sprung than American cars, and their body roll might be less than this.

Mr. Welty's experiments were unknown to the writer, and are of interest in that they agree so well with his own conclusions, even though the value of Q found was higher than that of the writer and Mr. Black. Derivation of Q by the analysis of the car path by any graphical method is probably less accurate than by the direct accelerometer method. The writer does not think that the accelerometer unduly influenced the drivers. He himself was one of them, and he found that he soon became accustomed to the presence of the instrument and paid no further attention to it. Some deliberate swerves were made for study, and recorded, but the results were not used in the analysis.

With reference to Mr. Fisher's contribution, the use of the term "friction factor" was deliberately avoided although the writer was aware that it had been used in the United States. The term "lateral ratio" was adopted after careful thought, because he found a frequent tendency for confusion, both in the literature on the subject and in discussions with his colleagues, between the friction factor and the coefficient of friction as understood in dynamics.

If the writer understands Mr. Noble correctly, the method of transition design, mentioned by him as proposed by Mr. Haile,[30] and by T. T. Wiley,[31] Assoc. M. ASCE, which determines the length of the curve "* * * on the basis of rotational change in attaining the curve superelevation," is open to the same theoretical objection as the method of W. H. Shortt. Referring to Eq. 1, Mr. Shortt based the length of transition on C, which is proportional to $\frac{d}{dt}\left(\frac{v^2}{R}\right)$, disregarding the superelevation. Those of Messrs. Haile and Wiley, on the other hand, are apparently based on $\frac{d}{dt}$ ($\sin \alpha$), neglecting C. Both should be considered, if the rate of change of force is a criterion. In his early work the writer derived a formula for the length of transition based on Q, and was faced by the difficulty, mentioned by Mr. Haile, that, when j is zero (that is, at the hands-off speed), the length of transition becomes zero. This result is nonsense, yet it is logically demanded by the theory. The fallacy was suspected to lay in the use of the rate of change of force, and the experiments confirmed the suspicion. It is true that very short transitions may produce a sharp rise in the superelevation, but this rise does not effect the rate of change of force. Any unpleasant effect from the rise is very possibly due to the psychological effect of a sudden rotation of the horizon. There is also the question of the unsightliness of a sharp rise in the curb or in the edge of the roadway.

The writer must dispute Professor Meyer's claim that the Shortt formula $\left(L = 1.6 \frac{v^3}{R} \text{ if } C = 2\right)$ makes L dependent on superelevation. Mr. Shortt defined C as the rate of change of centrifugal force. The only other terms are v and R, and there is no term which could be said to include, in any way, superelevation. The fact that the latter may be based in some way on R does not affect the formula, which gives the same length of curve whatever the relation between superelevation and R.

In conclusion, the writer strongly supports Mr. Noble's plea for adequate research, and hopes that this paper will help in achieving that end.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

TRANSACTIONS

Paper No. 2351

CLASSIFICATION AND IDENTIFICATION OF SOILS

By ARTHUR CASAGRANDE,[1] M. ASCE

WITH DISCUSSION BY MESSRS. RALPH E. FADUM; JAMES H. STRATTON; DONALD J. BELCHER; J. A. HAINE AND J. W. HILF; JACOB FELD; KENNETH S. LANE; GEORGE F. SOWERS; RENÉ S. PULIDO Y MORALES; RAYMOND F. DAWSON; D. F. GLYNN; L. F. COOLING; A. W. SKEMPTON, AND R. GLOSSOP; MILTON VARGAS; DONALD M. BURMISTER; M. G. SPANGLER; E. W. LANE; B. K. HOUGH, JR.; AND ARTHUR CASAGRANDE.

SYNOPSIS

The purpose of this paper is threefold: (1) To review and compare existing soil classifications used in civil engineering, with emphasis on their limitations; (2) to present a new soil grouping tentatively adopted by the United States Engineer Department in 1942 for use on airfield projects; and (3) to outline field identification procedures, requiring no apparatus. In abbreviated form, this paper was presented at the Annual Meeting of the Society in New York, N. Y., in January, 1943.

1. INTRODUCTION

The two most controversial chapters in the study of soils for civil engineering purposes are: (*a*) Soil classification, the most confused chapter; and (*b*) the shear strength of soils, the most difficult chapter. If the art of soil classification is far from satisfactory, the confusion is often made worse in that users are unaware of its limitations and apply it for purposes other than that originally intended.

Thorough knowledge of not only one but all classification systems important in civil engineering is the best protection against pitfalls and is the best assurance of an intelligent application of an individual classification. In addition, it provides tools with which the engineer can fashion, if necessary, a new classification to fit his needs in applying soil mechanics to a particular problem. It also frees him from the danger of becoming narrow-minded in this field. Those who really understand soils can, and often do, apply soil mechanics without any formally accepted soil classification.

NOTE.—Published in June, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Prof. of Soil Mechanics and Foundation Eng., Graduate School of Eng., Harvard Univ., Cambridge, Mass.

Limitations of Soil Classification Systems.—It is not possible to classify all soils into a relatively small number of groups such that the relation of each soil to the many divergent problems of applied soil mechanics will be adequately presented. Some systems require detailed tests to permit proper grouping of soils; others are specifically designed so that classification is possible on the basis of field identification without quantitative test results. Some classifications were created principally to facilitate making soil surveys, whereas others are chiefly to aid in digesting and correlating empirical knowledge in a specific engineering problem.

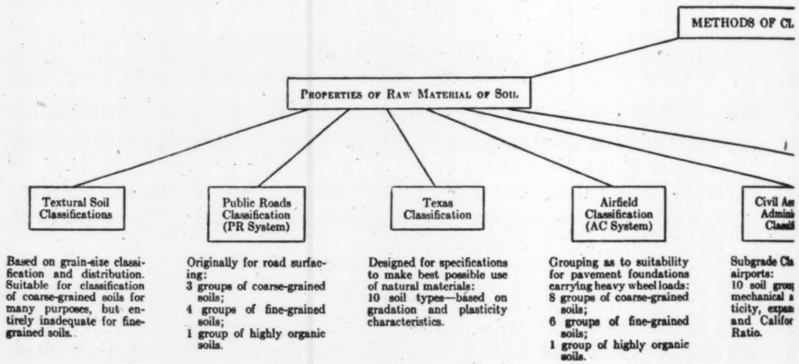

METHODS OF CL
PROPERTIES OF RAW MATERIAL OF SOIL
Textural Soil Classifications
Public Roads Classification (PR System)
Texas Classification
Airfield Classification (AC System)
Civil A
Admini
Classi
Based on grain-size classification and distribution. Suitable for classification of coarse-grained soils for many purposes, but entirely inadequate for fine-grained soils.
Originally for road surfacing: 3 groups of coarse-grained soils; 4 groups of fine-grained soils; 1 group of highly organic soils.
Designed for specifications to make best possible use of natural materials: 10 soil types—based on gradation and plasticity characteristics.
Grouping as to suitability for pavement foundations carrying heavy wheel loads: 8 groups of coarse-grained soils; 6 groups of fine-grained soils; 1 group of highly organic soils.
Subgrade Cl
airports:
10 soil grou
mechanical
ticity, expa
and Califor
Ratio.

FIG. 1.—METHODS OF

Development of Systems.—To some extent, soil classifications used in civil engineering reflect in their chronological order the development of soil mechanics. In its beginnings this leaned heavily on the soil testing methods developed by agricultural soil science. Since this science originally stressed classification by grain size, the first soil classifications used by civil engineers were textural classifications. Most of these were based on the relative contents of three grain-size fractions called sand, silt, and clay. Later classifications used the entire grain-size curve.

Probably the first departure from the grain-size analysis as the exclusive basis for classification was proposed in 1908 by A. Atterberg in Sweden for agricultural purposes. Later, the Swedish Geotechnical Commission devised and used a similar classification in its extensive research. Still later, the United States Bureau of Public Roads classification was created, with identification and classification of fine-grained soils based almost exclusively on tests other than the grain-size analysis. Finally, new soil classifications have been developed to fill the needs in design and construction of airfields for heavy airplanes.

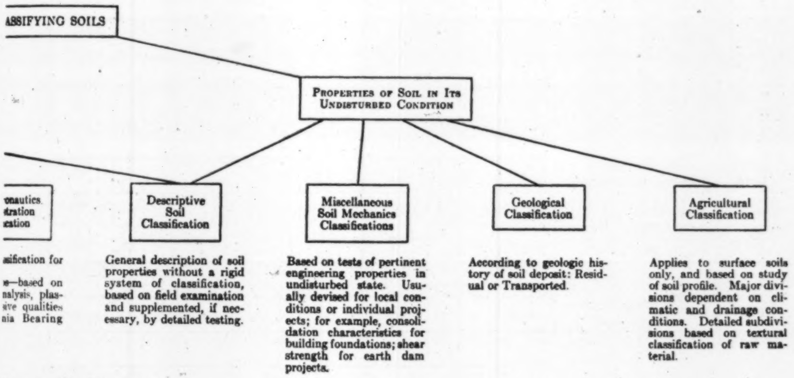
The essential elements of all classifications described in this paper are summarized in Fig. 1.

2. GRAIN-SIZE SCALES

The division of grain sizes into fractions with arbitrarily standardized boundaries is usually called grain-size classification. It is not really a soil classification but rather a grain-size scale, and will be so termed in this paper, to prevent confusion. Its purpose is to form the basis of grain-size classifications of soils, usually termed textural soil classifications.

The most important grain-size scales, summarized in Fig. 2, will be discussed.

United States Bureau of Soils Grain-Size Scale.—This scale is the one most commonly used in the United States. For practical purposes, and particularly for the textural soil classification as developed by the Bureau, the fraction between 1 mm and 2 mm has always been grouped together with the sand fraction (1).[2]


ASSIFYING SOILS
Properties of Soil in Its Undisturbed Condition
nautics
tration
cation
sification for
e—based on
nalysis, plas-
sive qualities
nia Bearing
Descriptive Soil Classification
General description of soil properties without a rigid system of classification, based on field examination and supplemented, if necessary, by detailed testing.
Miscellaneous Soil Mechanics Classifications
Based on tests of pertinent engineering properties in undisturbed state. Usually devised for local conditions or individual projects; for example, consolidation characteristics for building foundations; shear strength for earth dam projects.
Geological Classification
According to geologic history of soil deposit: Residual or Transported.
Agricultural Classification
Applies to surface soils only, and based on study of soil profile. Major divisions dependent on climatic and drainage conditions. Detailed subdivisions based on textural classification of raw material.

CLASSIFYING SOILS

International Grain-Size Scale.—A. Atterberg's (2) scale was proposed as international standard in 1913 when the International Society of Soil Science (3) stated:

> "It was decided to accept the group division of soil grains recommended by Dr. Atterberg. For the time being the following nomenclature shall be used in the German language [English terms added by the author]:

Grains larger than 20 mm	Stein und Geröll	[stones]
Grains from 20 mm to 2 mm	Kies	[gravel]
Grains from 2 mm to 0.2 mm	Grobsand	[coarse sand]
Grains from 0.2 mm to 0.02 mm	Feinsand	[fine sand]
Grains from 0.02 mm to 0.002 mm	Schluff	[silt]
Grains finer than 0.002 mm	Kolloidale Teilchen oder Rohton	[colloidal particles or raw clay]"

A noteworthy exception to this proposal was taken in 1914 by J. Kopecky (4), who recommended a modification based on the following divisions: 0.2, 0.06, 0.02, 0.006, and 0.002 mm. It is of interest that the scale adopted by the Massachusetts Institute of Technology (M.I.T.), Cambridge, Mass. (Fig. 2), has essentially the same numerical values.

M.I.T. Grain-Size Scale.—The advantages of the M.I.T. scale are (*a*) that the 0.002-mm limit for the clay fraction has been adopted to correspond to the international grain-size scale; and (*b*) that it, like that of the Bureau of Soils, possesses the simplicity of division into three major fractions, and permits the plotting of results on a triangular chart, as discussed in Section 3, textural soil classifications.

---

[2] **Numerals in parentheses, thus: (1), refer to corresponding items in the Bibliography (see Appendix).**

United States Department of Agriculture (U.S.D.A.) Grain-Size Scale.—The U.S.D.A. modified the original Bureau of Soils scale by (1) lowering the upper boundary for the clay fraction from 0.005 mm to 0.002 mm, to bring this scale in line with the international standard and (2) designating the entire range from 2.0 mm to 0.05 mm as sand.

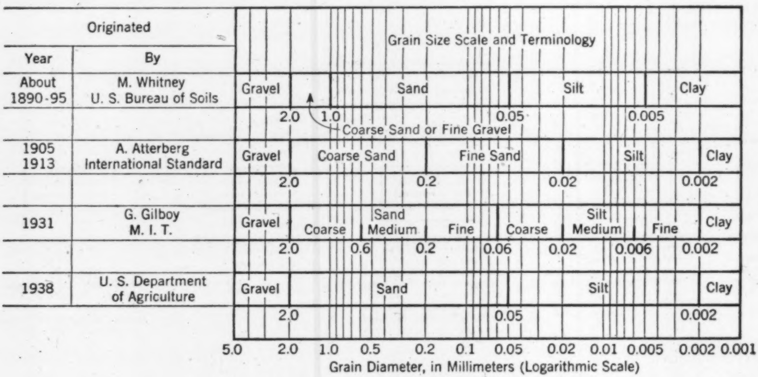


FIG. 2.—COMPARISON OF PRINCIPAL GRAIN-SIZE SCALES

Comparison of the M.I.T. and the new U.S.D.A. scales (Fig. 2) shows that their major divisions are practically identical. The various subdivisions are of no consequence since, in fact, they are never used in practical applications.

Limitations of Grain-Size Scales.—Below the range of the sieve analysis, all preceding grain-size scales represent merely two points of the grain-size curve. This is a very incomplete presentation as compared with the complete curve, which can be obtained readily by modern methods of wet-mechanical analysis.

The use of the terms "silt" and "clay" for arbitrary grain-size fractions is misleading. Such terms should be reserved exclusively for the description of soils having the corresponding physical characteristics. To designate the size fractions, one should either use the terms "sand sizes," "silt sizes," "clay sizes," together with the name of the classification used, or state the size without a name. For example, "30% clay sizes according to the Bureau of Soils Classification" would mean "30% < 0.005 mm." Obviously, designation of the grain size itself is much more brief and leaves less doubt as to its meaning. If a publication merely refers to 12% clay, the reader must usually guess what particular grain-size fraction is meant, or whether perhaps 12% of the total material consists of a plastic soil.

Because of the limited usefulness of the grain-size scales, the lack of a uniform standard is less serious than would be expected. Since the new U.S.D.A. scale is almost identical with the M.I.T. scale, either would probably cause less confusion if adopted as a United States standard. In the absence of such a uniform standard, it is desirable to adhere to the designation of the numerical limits of the grain-size fraction, omitting any names for the individual fractions.

In 1945, R. Glossop and A. W. Skempton (5) proposed the M.I.T. grain-size scale as a standard in their valuable contribution on this subject.

3. Textural Soil Classifications

Basis and Types.—The principle of classifying soils purely on the basis of their grain-size distribution dates back to the time when it was not yet realized that the physical properties of fine-grained soils, having the same grain-size distribution, can be widely different. The simplicity of textural soil classifications, and the fact that they can be applied with little experience, are the main reasons why they are still widely used.

In the United States the original textural classifications are based on the relative percentages of the sand, silt, and clay size fractions. They are facilitated by plotting these three fractions on a triangular chart. Therefore, they will be referred to as triangle textural classifications.

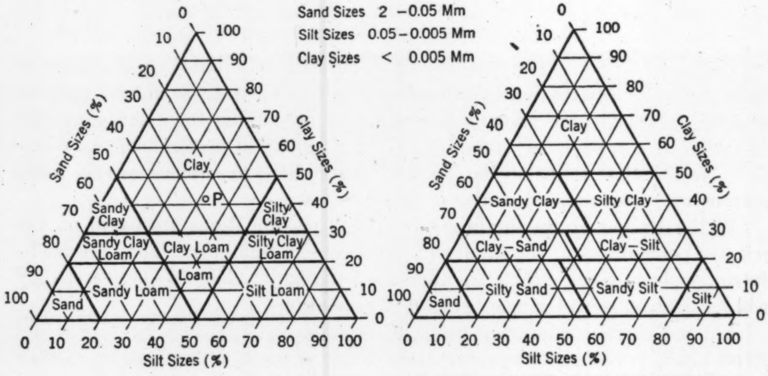
For individual engineering projects, textural classifications have been devised which are based on a comparison of the entire grain-size curves of soils with a set of master curves. These classifications will be called master curve textural classifications.

Triangle Textural Classifications.—The most widely used triangle textural classification is the U. S. Bureau of Soils classification (6) shown in Fig. 3. Soils are divided into the ten groups whose names appear inside the triangle. This classification is applied only to that part of a soil smaller than 2 mm in size, with the sand fraction extending from 2.0 mm to 0.05 mm.

The principle of the triangular plot is that any point represents a number on each of the three scales, and these three numbers always add up to 100. Therefore, to classify a soil, it is only necessary to determine the sand and silt fractions in per cent, and to find the corresponding point in the triangle. For example, if the fraction of a soil smaller than 2 mm in size contains 26% of sand sizes, and 32% of silt sizes (automatically making the contents of clay sizes 42%), then the position of this soil on the chart is found by the intersection of the 26% line on the sand scale with the 32% line on the silt scale. It can be seen that this point P (Fig. 3(*a*)) is to be classified as a "clay." If particles larger than 2 mm in size were present in quantities sufficient to be especially noticeable, this would be indicated by adding the term "gravelly" to the name of the soil.

Engineers who have used this classification have often noticed serious discrepancies between the names as derived from the triangle and the actual physical characteristics of the soils. Therefore, they have introduced other subdivisions and, in some cases, other names. For example, Fig. 3(*b*), used by the Lower Mississippi Valley Division of the U. S. Engineer Department, represents an attempt to adjust this method of classification to the soils of the lower Mississippi Valley. It should be noted that "loam," which is really an agricultural term, is not used in Fig. 3(*b*).

The difficulty of all triangle classifications lies in the fact that those physical characteristics of fine-grained soils which are of interest to the civil engineer often are not reflected by the three grain-size fractions. Point P in Fig. 3(*a*) may represent a highly plastic soil in one part of the United States and a slightly plastic silt in another locality.


Sand Sizes 2 –0.05 Mm
Silt Sizes 0.05 – 0.005 Mm
Clay Sizes < 0.005 Mm
Sand Sizes (%)
Clay Sizes (%)
Silt Sizes (%)
0 10 20 30 40 50 60 70 80 90 100
Clay
P
Sandy Clay
Silty Clay
Sandy Clay Loam
Clay Loam
Silty Clay Loam
Loam
Sandy Loam
Silt Loam
Sand
Clay
Sandy Clay
Silty Clay
Clay – Sand
Clay – Silt
Silty Sand
Sandy Silt
Sand
Silt

(*a*) U. S. BUREAU OF SOILS

(*b*) LOWER MISSISSIPPI VALLEY DIVISION, U. S. ENGINEER DEPT.

FIG. 3.—COMPARISON OF SOILS CLASSIFICATION TRIANGLES (SIZES IN MILLIMETERS: SAND FRACTION, 2 TO 0.05; SILT FRACTION, 0.05 TO 0.005; AND CLAY FRACTION, LESS THAN 0.005)

The more successful use of the classification in Fig. 3(*a*) in agricultural soil studies may be because in well-developed surface soils certain fine-grained materials such as uniform, nonplastic, rock flour do not exist. It would seem as if the weathering due to the mechanical, chemical, and biological forces, which are responsible for the development of the surface soils, tends to reduce the range of physical properties which a soil with a given grain-size distribution may possess; or, in other words, the grain-size distribution is more likely to reflect the physical properties of a weathered surface soil than of the underlying unweathered materials.

Master Curve Textural Classifications.—The development of modern methods of wet-mechanical analysis which yield the entire grain-size curve as readily as the three fractions used in the triangle classification has led to utilizing the entire grain-size curve for soil classification. This can be accomplished by plotting a set of typical grain-size curves, called master curves, covering the desired range of soils.

Examples of master curve textural classifications are the classifications developed by the Boston Metropolitan District Water Supply Commission (7), by the U. S. Engineer Office in Providence, R. I. (8), and by Donald M. Burmister, Assoc. M. ASCE (8*a*). In the Providence classification, the necessity for further differentiation of fine-grained soils into nonplastic and plastic soils is also recognized.

Textural soil classifications by means of master curves have been reasonably successful for noncohesive soils of the same general geologic origin. However, for most fine-grained soils the same objections apply as for the triangle classification.

TABLE 1.—COMPARISON OF GRAIN-SIZE DISTRIBUTION AND PHYSICAL CHARACTERISTICS OF SOME FINE-GRAINED SOILS

No.	Location (1)	Material (2)
1	Winnipeg, Canada	Very tough, highly plastic clay
2	Massena, N. Y.	Soft, medium plastic, Laurentian clay
3	Massena, N. Y.	Soft, somewhat organic, silty Laurentian clay of low plasticity
4	Vera, Wash.	Kaolin-type clay, with typical lack of toughness
5	California	Diatomaceous earth with characteristics of clayey rock flour
6	Wayland, Mass.	Diatomaceous earth with characteristics of nonplastic rock flour
7	Missouri	Lean, silty clay
8	Hartwick, N. Y.	Typical rock flour

TABLE 1.—(*Continued*)

No.	PERCENTAGE BY WEIGHT SMALLER THAN: (DIAMETERS IN MILLIMETERS) 0.05 (3)	0.005 (4)	0.002 (5)	0.001 (6)	Liquid limit (7)	Plastic limit (8)	Plasticity index (9)	Dry strength (10)	AC group symbols[a] (11)
1	90	75	54	35	96	29	67	Very high	CH
2	96	77	62	50	55	26	29	Medium	CH
3	92	70	50	37	37	24	13	Medium	CL–OL
4	96	76	50	34	55	31	24	Low	MH–CH[b]
5	96	70	47	34	122	80	42	Very low	MH
6	92	52	27	15		Nonplastic		Almost none	MH
7	90	26	16	10	35	22	13	Medium	CL
8	88	20	10	6	24	21	3	Very low	ML

[a] Airfield Classification System symbols; see Table 4. [b] The new KH-group.

Limitations of Textural Soil Classifications.—Table 1 shows the large range of soils covered by almost identical grain-size curves. Soils Nos. 1 to 5 range from a highly plastic, very tough clay having high dry strength, through kaolin-type clays lacking toughness and having only low dry strength, to diatomaceous soils with very low plasticity and only slight dry strength.

It is also instructive to compare the grain-size distribution of soils Nos. 6 to 8 (Table 1) with their other characteristics. The highest percentage of grains smaller than 0.005 mm is shown by the diatomaceous earth which is practically noncohesive and nonplastic. The clay (No. 7) with only 26% material smaller than 0.005 mm actually shows more plasticity and greater dry strength than soil No. 5 containing three times as much material smaller than 0.005 mm.

Many similar discrepancies between grain-size distribution and physical characteristics of fine-grained soils could be cited. Nevertheless, the grain-size curves of fine-grained soils are still extensively used as the exclusive basis for soil classification. In contrast to the misleading impressions which these curves may convey, the pertinent physical properties of such soils can easily be distinguished on the basis of simple field identification procedures.

4. Agricultural Soil Classification

A classification of surface soils for agricultural purposes, called pedological classification, is a complicated system in which the classification of the physical properties of an individual soil layer represents merely one element. Since surface soils have "grown" under the influence of climate, vegetation, type of underlying parent material, topographic features, and age, all these factors, and particularly their influence on the chemistry of the surface soils, play a major role in a pedological soil classification.

Of main interest to the civil engineer is that part of such a classification which describes the physical properties and composition of the surface soils and of the underlying parent materials. In the United States, such description is confined to a textural classification.

Because in a textural classification soils with different physical characteristics would be classified the same, A. Atterberg in Sweden suggested the classification (3) of the physical properties of soils for agricultural purposes shown in Table 2. Further subdivisions are made to distinguish between soils rich in humus or lime, and soils containing gravel, stones, and diatoms. To assist in identification, Atterberg proposed the use of his plasticity tests as well as a dry strength number based on the dry strength of the soil. Thus, his classification emphasizes pertinent physical characteristics whose differentiation is not possible by using grain-size analysis alone.

TABLE 2.—Atterberg Soil Classification

Major divisions	Secondary divisions	Description
A.—Clays (Plastic Soils)		
	I.—Sticky Clays (Highly Plastic)	This group contains only the heaviest clays
	II.—Loamy Clays (Not Sticky)	Subdivided into medium heavy and fairly heavy clays
B.—Loams (Nonplastic, More or Less Cohesive Soils)		
	I.—Fairly Heavy Loams	Clayey loams
	II.—Light Loams	Sandy loams and loess soils
C.—Sand, *Mo*, and Silt Soils (Noncohesive Soils)		
	I.—Capillarity Greater Than 34 Cm	Fine-grained sandy soils; dust loess; subdivided by mechanical analysis
	II.—Capillarity Less Than 34 Cm	Coarse, dry sandy soils, useful only for forestry

Use of Agricultural Soil Maps in Civil Engineering.—The surface soils of large areas of the United States and other countries have been surveyed and mapped from an agricultural standpoint. Often it is possible to correlate, by means of field observations and tests, the classifications used on such maps with the pertinent engineering properties of these soils so as to save much time in soil surveys for roads and airfields, and in finding sources of satisfactory material for fill construction. Also, with sufficient experience, or advice from agricultural soil experts, the engineer can derive information on the subdrainage characteristics of a given area.

A systematic correlation between agricultural soil classifications, as found on the soil maps, and the engineering properties, has been published for the State of Indiana (9).

5. Geologic Soil Classifications

Every comprehensive soil investigation should also include a geologic classification. In most cases differentiation between the two major subdivisions, that is, residual and transported soils, will be possible on the basis of a general field examination. Further subdivisions of the transported soils, such as glacial, fluvial, lacustrine, marine, aeolian, and volcanic, are a useful and sometimes necessary part of a comprehensive soil classification which, however, may require the assistance of a geologist. When geologic maps are available, they should be utilized in connection with foundation investigations for important civil engineering projects.

6. United States Bureau of Public Roads Classification System

Development and Limitations.—The Public Roads (PR) Classification (10) was developed in the years 1927 to 1929 from extensive research that the U. S. Bureau of Public Roads had directed particularly toward the use of soils in secondary road construction. The original definition of its groups is based on the stability characteristics of soils when directly acted upon by wheel loads. This classification, therefore, is essentially a grouping applicable to the use of soil in a road surface or as a base beneath a thin bituminous wearing surface. Unfortunately, it has been termed "subgrade classification." Attempts to use it for judging the characteristics of subgrades beneath a substantial thickness of pavement and base have frequently led to disqualification of excellent subgrade and subbase materials. The susceptibility of soils to frost action is also not properly recognized in this classification because at the time it was developed, knowledge of frost action was still limited.

A simplified presentation of the PR classification system is shown in Table 3. It differs from the usual presentation (10) to place more emphasis on those characteristics which aid in field identification. To understand the use of the letter A in the group designations, it is noted that the original PR classification also provided for three additional groups designated B–1, B–2, and B–3. These soon fell into disuse and need not be considered here.

Classification Procedure.—The original publication (10) contains recommendations for classifying soils based on grain-size distribution, plasticity, and shrinkage characteristics, as well as on other tests which are not so commonly used. The numerical limits of the test results, which determine the group classification of a particular soil, have undergone many revisions, the latest (11) being dated February, 1942.

Modifications of PR System.—Some of the shortcomings of the PR system were recognized by soils engineers in various highway departments, who tried either to adapt it to their local soil conditions and problems, or to develop new classifications. In New Hampshire, where frost action is the most destructive agent, engineers have eliminated the A–1 group entirely, since soils in this group (such as some well-graded glacial tills) were found to be just as subject to frost heaving as soils in the A–2 group. Therefore, all sandy and gravelly soils of the A–1 and A–2 groups were classified as A–2 materials, and the new A–2 group was subdivided into three subgroups, depending on their frost-heaving characteristics. The definition of the A–3 group was not changed; all soils in this group are considered better subgrades than the soils in the combined A–1 and A–2 groups (12).

TABLE 3.—SIMPLIFIED CHART OF PUBLIC ROADS CLASSIFICATION SYSTEM

Major divisions		Group	General description	Examples
Coarse-Grained Soils		A–1	Coarse with fines—well graded; considerable dry strength and practically no swelling and shrinkage.	Sand-clays in Southern States; some types of glacial till.
		A–2	Coarse with fines—compared with A–1 either deficient in dry strength or subject to volume change.	Many types of glacial deposits and of residual soil.
		A–3	Coarse-grained soils containing less than 10% passing No. 200 mesh and having no dry strength.	Clean, well graded glacial gravel; beach and dune sands.
Fine-Grained Soils	Silts	A–4	Silts and clays of low plasticity	Rock flour; many sandy and silty clays; loess.
		A–5	Elastic silts	Highly micaceous silt; diatomaceous earth.
	Clays	A–6	Highly colloidal types of inorganic clays; medium to high plasticity.	Some inorganic clays from southern and tropical regions.
		A–7	The great majority of inorganic clays and some types of organic clays; medium to high plasticity.	Glacial clays; some types of gumbo; kaolin-type clays; some organic clays.
Peat and Muck		A–8	Materials having very high compressibility.	Peat and organic clays.

Realizing that the PR system is seriously deficient for classifying subgrades, highway engineers have given consideration to a radical revision of this system without changing the designations A–1, A–2, etc. Under a tentative proposal, all gravelly and coarse sandy soils of the A–3 group, which are excellent subgrades, would be classified as A–1 materials, together with those originally defined as A–1. Equally radical changes are proposed for the fine-grained soils. If this is adopted, it would not be possible to gain a clear idea of the type of soil from the PR designation, except by noting the "edition" of the classification. Such confusion could easily be avoided by giving this radical modification an entirely new designation not mistakable for the original, as by using a prefix letter other than A.

7. TEXAS HIGHWAY DEPARTMENT SOIL CLASSIFICATION SYSTEM

About 1939, the Texas Highway Department developed a classification of Texas soils for writing specifications such that natural materials would be used to best advantage in road construction (13). The ten soil types are based essentially on gradation and plasticity characteristics. Type 1, which is typical, is defined as follows:

> Type 1 soil is a material which will retain high stability under the direct impact of traffic loads. Soils of this type are the only materials permitted in the upper six inches of the finished roadway. This type of material can also be used for the roadbed below six inches, if economically available. Type 1 soil includes caliche, rock, shell, iron ore, or gravel with good grading, and excellent soil binder. The detailed requirements vary depending upon whether the aggregate is angular or rounded by nature.

The other soil types are classified in the order of decreasing value in road construction. Types 7 to 9 represent clays of increasing plasticity and type 10 represents a highly compressible material, usually of organic character, having a liquid limit greater than 90.

8. CIVIL AERONAUTICS ADMINISTRATION (CAA) SOIL CLASSIFICATION SYSTEM

The CAA system, established in 1944 for classifying soils in the order of their value as subgrade of airport pavements, is based on the mechanical analysis, plasticity characteristics, expansive qualities, and California Bearing Ratio (CBR).

Soils are divided into groups designated by the symbols E–1 to E–10. Their characteristics are briefly described by A. H. Hadfield (14):

> "* * * there are two groups: the E–1 to E–4 'granular soils,' containing 50% or more of sand and the E–5 to E–10 'non-granular soils,' containing less than 55% sand (the E–5 has 65% or less sand).
>
> "The granular soils are further divided into non-frostheave soils, groups E–1 and E–2, and soils subject to frost heave, groups E–3 and E–4. The E–1 soil is a free draining, non-plastic sand corresponding to the PRA classification A–3. The E–2 soil is a sand containing slightly more silt and clay than the E–1. The E–3 corresponds to a non-plastic and moderately plastic PRA A–2 type, and the E–4 is equivalent to the PRA A–2 plastic type.
>
> "In the non-granular group the E–5 soil corresponds to the non-plastic or moderately plastic A–4 silt, and the E–6 to the more plastic A–4 silts, the A–4 and A–6 silty clays, and the A–6 or A–7 clays of low plasticity. The E–7, E–8 and E–9 groups include clay soils of average plasticity, high plasticity and very high plasticity, respectively, covered by PRA's A–6 and A–7 soils groups. The E–10 is the highly elastic soil classed as A–5 by PR."

9. AIRFIELD CLASSIFICATION SYSTEM FOR U. S. ENGINEER DEPARTMENT

Development.—As a result of difficulties in the use of other classifications for the design and construction of military airfields, the Office of the Chief of Engineers tentatively adopted in 1942 a new soil classification which will be referred to as the Airfield Classification (AC) System. It was originally developed by the writer for instruction in Army courses on "Control of Soils in Military Construction" offered at the Graduate School of Engineering, Harvard University, Cambridge, Mass., from 1942 to 1944.

TABLE 4.—AIRFIELD CLASSIFICATION SYSTEM—PHYSICAL CHARACTERISTICS OF SOIL GROUPS AS AFFECTING CONSTRUCTION OF FOUNDATIONS FOR AIRFIELD PAVEMENTS[a]

1		2	3	4		5	6
Major divisions		Soil groups and typical names	Group symbol	GENERAL IDENTIFICATION (ON DISTURBED SAMPLES)		Observations and tests relating to material in place	Principal classification tests (on disturbed samples)
				Dry strength[b]	Other pertinent examinations		
Coarse-grained soils	Gravel and gravelly soils	Well-graded gravel and gravel-sand mixtures, little or no fines	GW	None	Gradation Grain shape Examination of binder wet and dry Durability of grains	Dry unit weight or void ratio Degree of compaction Cementation Stratification and drainage characteristics Ground water conditions Traffic tests Large-scale load tests California bearing ratio tests	Mechanical analysis
		Well-graded gravel-sand mixtures with excellent clay binder	GC	Medium			Mechanical analysis, liquid and plastic limits on binder
		Poorly graded gravel and gravel-sand mixtures, little or no fines	GP	None			Mechanical analysis
		Gravel with fines, silty gravel, clayey gravel, poorly graded gravel-sand-clay mixtures	GF	Very slight to high			Mechanical analysis, liquid and plastic limits on binder if applicable
	Sands and sandy soils	Well-graded sands and gravelly sands, little or no fines	SW	None			Mechanical analysis
		Well-graded sand with excellent clay binder	SC	Medium to high			Mechanical analysis, liquid and plastic limits on binder
		Poorly graded sands, little or no fines	SP	None			Mechanical analysis
		Sand with fines, silty sands, clayey sands, poorly graded sand-clay mixtures	SF	Very slight to high			Mechanical analysis, liquid and plastic limits on binder if applicable

TABLE 4.—(*Continued*)

1		2	3	4		5	6
Major divisions		Soil groups and typical names	Group symbol	GENERAL IDENTIFICATION (ON DISTURBED SAMPLES)		Observations and tests relating to material in place	Principal classification tests (on disturbed samples)
				Dry strength[b]	Other pertinent examinations		
Fine-grained soils containing little or no coarse grained material	Fine-grained soils having low to medium compressibility; liquid limit < 50	Silts (inorganic) and very fine sands, *mo*, rock flour, silty or clayey fine sands with slight plasticity	ML	Very slight to medium	Shaking test and plasticity	Dry unit weight, water content and void ratio	Mechanical analysis, liquid and plastic limits if applicable
		Clays (inorganic) of low to medium plasticity sandy clays, silty clays, lean clays	CL	Medium to high	Examination in plastic range	Consistency, undisturbed and remolded	Liquid and plastic limits
		Organic silts and organic silt-clays of low plasticity	OL	Slight to medium	Examination in plastic range, odor, color	Stratification, root holes and fissures Drainage and groundwater conditions	Liquid and plastic limits from natural condition and after oven drying
	Fine-grained soils having high compressibility; liquid limit > 50	Micaceous or diatomaceous fine sandy and silty soils, elastic silts	MH	Very slight to medium	Shaking test and plasticity	Traffic tests Large-scale load tests	Mechanical analysis, liquid and plastic limits if applicable
		Clays (inorganic) of high plasticity, fat clays	CH	High to very high	Examination in plastic range	California bearing ratio tests	Liquid and plastic limits
		Organic clays of medium to high plasticity	OH	Medium to high	Examination in plastic range, odor, color	Compression tests	Liquid and plastic limits from natural condition and after oven drying
Fibrous organic soils with very high compressibility		Peat and other highly organic swamp soils	Pt	Readily identified		Consistency, texture and natural water content	

TABLE 4.—(*Continued*)

3	7	8		9	10	11	12	13[d]	14	15
Group symbols	Value as foundation when not subject to frost action	VALUE AS WEARING SURFACE FOR STAGE OR EMERGENCY CONSTRUCTION		Potential frost action	Compressibility and expansion	Drainage characteristics[c]	Field compaction characteristics and equipment	Solids at optimum compaction (lb per cu ft) and void ratio, e	California bearing ratio for compacted and soaked specimen	Comparable groups in PR classification
		With dust palliative	With bituminous surface treatment							
GW	Excellent	Fair to poor	Excellent	None to very slight	Almost none	Excellent	Excellent; crawler tractor, rubber tired equipment	>125 e <0.35	>50	A-3
GC	Excellent	Excellent	Excellent	Medium	Very slight	Practically impervious	Excellent; tamping roller, rubber tired equipment	>130 e <0.30	>40	A-1
GP	Excellent	Poor	Poor to fair	None to very slight	Almost none	Excellent	Good to excellent; crawler tractor, rubber tired equipment	>115 e <0.45	25–60	A-3
GF	Good to excellent	Poor to good	Fair to good	Slight to medium	Almost none to slight	Fair to practically impervious	Good to excellent; crawler tractor, rubber tired equipment, tamping roller	>120 e <0.40	>20	A-2
SW	Excellent	Poor	Good	None to very slight	Almost none	Excellent	Excellent; crawler tractor, rubber tired equipment	>120 e <0.40	20–60	A-3
SC	Excellent	Excellent	Excellent	Medium	Very slight	Practically impervious	Excellent; tamping roller, rubber tired equipment	>125 e <0.35	20–60	A-1
SP	Good	Poor	Poor	None to very slight	Almost none	Excellent	Good to excellent; crawler tractor, rubber tired equipment	>100 e <0.70	10–30	A-3
SF	Fair to good	Poor to good	Poor to good	Slight to high	Almost none to medium	Fair to practically impervious	Good to excellent; crawler tractor, rubber tired equipment, tamping roller	>105 e <0.60	8–30	A-2

TABLE 4.—(*Continued*)

3	7	8		9	10	11	12	13[d]	14	15
Group symbols	Value as foundation when not subject to frost action	VALUE AS WEARING SURFACE FOR STAGE OR EMERGENCY CONSTRUCTION		Potential frost action	Compressibility and expansion	Drainage characteristics[c]	Field compaction characteristics and equipment	Solids at optimum compaction (lb per cu ft) and void ratio, e	California bearing ratio for compacted and soaked specimen	Comparable groups in PR classification
		With dust palliative	With bituminous surface treatment							
ML	Fair to poor	Poor		Medium to very high	Slight to medium	Fair to poor	Good to poor; close control essential; rubber tired roller	>100 e <0.70	6–25	A–4
CL	Fair to poor	Poor		Medium to high	Medium	Practically impervious	Fair to good; tamping roller	>100 e <0.70	4–15	A–4 A–6 A–7
OL	Poor	Very poor		Medium to high	Medium to high	Poor	Fair to poor; tamping roller	>90 e <0.90	3–8	A–4 A–7
MH	Poor to very poor	Very poor		Medium to very high	High	Fair to poor	Poor to very poor	>100 e <0.70	<7	A–5
CH	Poor to very poor	Very poor		Medium	High	Practically impervious	Fair to poor; tamping roller	>90 e <0.90	<6	A–6 A–7
OH	Very poor	Useless		Medium	High	Practically impervious	Poor to very poor	<100 e >0.70	<4	A–7 A–8
Pt	Extremely poor	Useless		Slight	Very high	Fair to poor	Compaction not practical			A–8

[a] Table 4 represents the third revision since it was first prepared by the author in 1942. [b] For binder = fraction passing No. 40 U. S. standard mesh. [c] These characteristics do not apply to undisturbed materials having fissures and root holes, such as most surface soils. [d] These weights apply only to soils having specific gravities from 2.65 to 2.75.

In the AC system, it is recognized that a clean gravelly soil, which would be classified A–3 in the PR system, is a better foundation material for pavements than a gravelly soil of the A–1 or A–2 groups, because it is not subject to frost action and is free draining. The AC system also recognizes that a gravelly soil is generally a better foundation material for heavy wheel loads than a sand.

The fifteen groups of the AC system represent a considerably greater number than are used in the other systems. However, the grouping is specifically so devised that soils can readily be classified by an experienced soils engineer on the basis of visual and manual examination.

10. Soil Groups and Group Symbols of AC System

Table 4, representing the third revision by the writer, summarizes the essential information on the AC system including the group symbols. Each symbol consists of two letters which may be considered as initials of the name of the most typical soil of that group. Their meaning will be understood from the following outline:

Coarse-grained soils are subdivided into:

(1) Gravel and gravelly soils; symbol G.
(2) Sands and sandy soils; symbol S.

The gravels and the sands are each divided into four groups:

(*a*) Well-graded, fairly clean material; symbol W, in combinations GW and SW.
(*b*) Well-graded material with excellent clay binder (corresponding to PR A–1 group); symbol C, in combinations GC and SC.
(*c*) Poorly graded, fairly clean material; symbol P, in combinations GP and SP.
(*d*) Coarse materials containing fines, not covered by preceding groups; symbol F, in combinations GF and SF.

Fine-grained soils are subdivided into three types:

(1) The inorganic silty and very fine sandy soils; symbol M, which stands for the Swedish terms *mo* and *mjala* (flour), used for fine-grained nonplastic, or slightly plastic, soils and, in the international grain-size scale, designating the fractions from 0.2 mm to 0.02 mm and from 0.02 mm to 0.002 mm, respectively.
(2) The inorganic clays; symbol C.
(3) The organic silts and clays; symbol O.

Each of these three types of fine-grained soils is grouped according to its liquid limit into:

(*a*) Fine-grained soils having liquid limits less than 50; that is, of low to medium compressibility; symbol L, in combinations ML, CL, and OL.
(*b*) Fine-grained soils having liquid limits greater than 50; that is, of high compressibility; symbol H, in combinations MH, CH, and OH.

Important Note.—In classifying fine-grained soils the term compressibility refers to the slope of the virgin pressure-void ratio curve and not to the actual condition of the undisturbed soil which may be partly dried or otherwise preconsolidated. Experience has proved that the liquid limit is a good measure of relative compressibility.

Highly organic soils, usually fibrous, such as peat and swamp soils having very high compressibility, are not subdivided and are placed in one group; symbol Pt.

Use of Group Symbols.—The sequence of the group symbols need not be memorized, but the meaning of the symbols and the sequence of the major divisions, that is, G–S–L–H should be learned.

When a material does not clearly fall into one group, boundary classifications such as GW–SW or ML–CL should be used.

A study of the soil names in Col. 2, Table 4, will show that these represent no new soil terminology, but are those that experienced soils engineers use when describing a soil on the basis of visual and manual examination.

Emphatically the group symbols are not intended to be substituted for detailed descriptions. However, the use of these symbols is proper when comparing physical characteristics of different soil groups, as in Table 4, references (15)(16).

11. Coarse-Grained Soil Groups of AC System

GW–SW Groups.—These soils are well graded. They comprise gravelly and sandy soils having little or no fines. The presence of the fines must not noticeably change the strength characteristics of the coarse-grained fraction and must not interfere with its free-draining characteristics. In areas subject to frost action, the material should not contain more than about 3% of soil grains smaller than 0.02 mm in diameter. Where frost action is not a factor, contents of the order of 10% passing No. 200 mesh may not be objectionable. The GW and SW soils would be classified in the PR system as A–3 materials.

GC–SC Groups.—These soils also are well graded. They comprise gravel-sand mixtures and sands with an excellent clay binder, or other natural cementing agent, so proportioned that the mixture shows negligible swelling and shrinkage. Thus, dry strength of GC and SC materials is usually provided by a small amount of highly colloidal clay, and in some cases by the cementation of calcareous material (as in coral), or iron oxides (as in iron ore gravels). The letter C in these group symbols conveys the meaning of both "clay" and "cementation."

Materials in the GC and SC groups correspond to the A–1 group in the PR system. They may be used as road surfacing material because they are stable under wheel loads both in wet and dry condition. However, they are usually subject to frost action.

GP–SP Groups.—These soils are poorly graded. They comprise gravels and sands, containing little or no fines, as stated for the GW and SW groups. Materials in the GP and SP groups can be subdivided into:

(*a*) Uniform gravels, such as the clean, round-grained gravel deposited in gravel banks along the middle courses of rivers;

(*b*) Uniform sands, such as beach sand and dune sand; and

(c) Nonuniform mixtures of very coarse material with fine sand, with intermediate sizes lacking, as often results during excavation from mixing of gravel and sand layers.

These groups would be classified A-3 in the PR system.

GF-SF Groups.—Any coarse-grained material which contains detrimental fines (either silty or clayey) and which does not qualify for the GC-SC groups is classified in the GF-SF groups. Silty gravels, silty sands, poorly graded gravel-sand-clay mixtures, and sand-clay mixtures are typical examples. In the PR system they would be classified as A-2 materials.

Identification of Coarse-Grained Soils.—In field identification of coarse-grained materials, the gradation, grain size, shape, and the mineral composition are examined by spreading a dry sample on a flat surface.

The division between gravelly sands and sandy gravels has not been standardized. Many highway engineers classify gravelly sands containing only 20% to 30% of gravel sizes (retained on No. 10 mesh) as gravelly soils. An exact point of division is unimportant, except for specifications, and boundary cases may be classified as GW-SW, and so forth.

Considerable experience is required to differentiate, on the basis of a visual examination, between some well-graded soils and poorly graded soils. When in doubt, a mechanical analysis should be made and the grain-size curve plotted, from which the gradation characteristics can be judged.

Examination of fines is confined to the fraction passing No. 40 mesh (United States Standard), which fraction is usually called the "binder." In the absence of a sieve, this separation can be performed tolerably well by hand. The binder is mixed with water, and its plasticity characteristics and dry strength are examined. The dry strengths are listed in Col. 4, Table 4. For more accurate identification, quantitative determination of the liquid and plastic limit of the binder, and the grain-size curve of the entire material are required.

Identification of active cementing agents other than clay is usually not possible by visual and manual examination, since such agents require a curing period of days, and even weeks. It is on the basis of the actual use of such soils that they are classified into the GC-SC groups. In the absence of such experience, they may be classified tentatively into other groups, neglecting any possible development of strength due to cementation.

Depending on the use to which a coarse-grained soil is put, the durability of its grains may require careful examination. Pebbles and sand grains consisting of sound, igneous rock are easily identified. Weathered material is recognized from its discolorations and the relative ease with which the grains can be crushed. Pebbles consisting of weathered granitic rocks and quartzite are not necessarily objectionable when used for the foundations of pavements. In the process of compaction, some round-grained quartzite gravels, found in southeastern United States, will break up into much smaller, but angular and quite durable, quartz fragments which will form a well-graded stable base. Disintegrated granite in California is extensively used in highway construction. On the other hand, coarse-grained soils containing fragments of shaley rock may be dangerous because alternate wetting and drying may result in partial or complete disintegration. This tendency can be identified by a slaking test. The particles are first thoroughly oven-dried, or sun-dried, then submerged in water for at least 24 hours; finally, their strength is examined and compared with their original strength. Some types of shales will completely disintegrate when subjected to such a slaking test.

The properties of coarse-grained soils in the undisturbed state (natural deposite or man-made fill) require examination if they are to be used in that condition. Depending on the problem, several of the observations and tests listed in Col. 5, Table 4, will be required.

12. Fine-Grained Soil Groups of AC System

In the transition between coarse-grained and fine-grained soils, usually those soils containing more than 50% of material smaller than 0.1 mm in size (passing No. 150 mesh) are classified as fine-grained soils (ML or CL); and soils containing less than 50% smaller than 0.1 mm in size are classified as coarse grained (SF or SC). A non-plastic, silty, fine sand with 40% to 60% passing a No. 150 mesh would be classified as an SF–ML soil.

Of basic importance in the classification of fine-grained soils is the plasticity as measured by the Atterberg liquid and plastic limit tests, or estimated by manual examination (17)(18)(19)(20)(21).

The Plasticity Chart.—The writer's method of using the liquid and plastic limit values for soil identification is illustrated in the plasticity chart, Fig. 4.

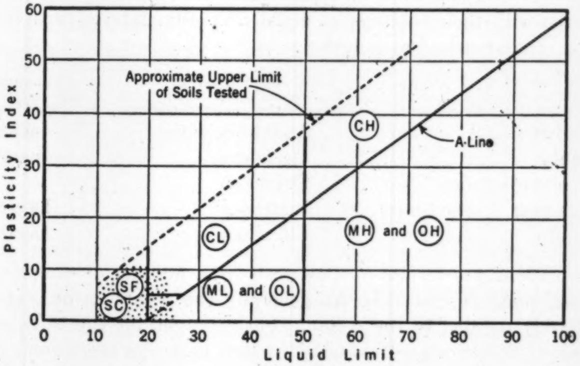


Fig. 4.—Plasticity Chart for AC System

The inclined A-line represents an important empirical boundary. It goes through the point on the base line having a liquid limit of 20, and through the point with a liquid limit of 50 and a plasticity index of 22.0. Therefore, its equation is: Plasticity index = 0.73 × (liquid limit − 20).

The A-line represents the empirical boundary between typical inorganic clays (CL–CH groups) which are generally above the A-line, and plastic soils containing organic colloids (OL–OH groups) which are below it. Also located below the A-line are typical inorganic silts and silty clays (ML and MH groups), except for liquid limits less than 30, for which values inorganic silts may range slightly above the A-line.

For soils having liquid limits less than 25, there is considerable overlapping of several soil groups as shown by the dotted area in Fig. 4. For example, a soil with a liquid limit of 20 and a plasticity index of 8 could be classified according to its grain-size composition and physical characteristics either as a sandy clay (CL), or a very clayey sand (SF), or it may be such a well-graded mixture of sand and clay as to fall into the SC group. Frequently, such soils will be classified as boundary cases CL–SC or CL–SF; or SF–SC. A soil with a liquid limit of 20, and a plasticity index of 0 to 2, may be either a typical rock flour (ML), or a very silty sand with a trace of plasticity (SF), or a boundary case designated SF–ML. In spite of this overlapping, an experienced soils engineer will be able to differentiate between these soil groups purely on the basis of field identification.

Relation Between Position on Plasticity Chart and the Physical Properties.—Experience has shown that the compressibility, when compared at equal preconsolidation load, is approximately proportional to the liquid limit, and that materials having the same liquid limit possess approximately equal compressibility. In comparing the physical characteristics of soils having the same liquid limit, it will be found that, with increasing plasticity index, the cohesive characteristics (toughness and dry strength) increase, whereas permeability decreases.

The behavior of soils at equal liquid limit, with plasticity index increasing, as compared with their behavior at equal plasticity indexes, with liquid limit increasing, is summarized as follows:

Characteristic	Comparing soils at equal liquid limit with plasticity index increasing	Comparing soils at equal plasticity index with liquid limit increasing
Compressibility	About the same	Increases
Permeability	Decreases	Increases
Rate of volume change	Decreases	....
Toughness near plastic limit	Increases	Decreases
Dry strength	Increases	Decreases

After plotting the results of limit tests on a number of samples from the same fine-grained deposit (Fig. 5), generally the points lie on a straight line approximately parallel to the A-line. For example, for the thick stratum of the so-called "Soft Boston Blue Clay," which underlies large areas of Greater Boston, the liquid limit ranges from 30 to more than 70 along a line corresponding approximately to the lower boundary of the area marked "Glacial Clays." Of hundreds of limit determinations on this clay, the average liquid limit is 47 and the average plastic limit is 23.

Position of Typical Inorganic Clays.—The belt of glacial clays in Fig. 5 seems to cover most inorganic clays from Boston, Detroit, Chicago, and northeastern Canada. Much tougher clays are represented by the line showing test results on clays from one locality in Venezuela and by the belt which covers gumbo clays from several southern states.

Kaolin-Type Clays.—Interesting types of fine-grained soils are the kaolin-type clays, usually derived from feldspar in granitic rocks. On the plasticity chart, they plot below the A-line. Although they are generally referred to as clays, they possess some of the characteristics of inorganic silts. They have relatively low dry strength; and in the wet state show some reaction to the shaking test, which is characteristic of the ML–MH groups. However, they are extremely fine grained and smooth, and often contain as great a clay-size fraction as many tough clays located well above the A-line, as shown by soil No. 4 in Table 1. In Fig. 5 the kaolin-type clays in a belt slightly below the A-line are from widely separate deposits. They may be classified as boundary cases, ML–CL, and MH–CH. Test results on a different type of kaolin (from Mica, Wash.) are located along a line which lies in the MH area well below the A-line.

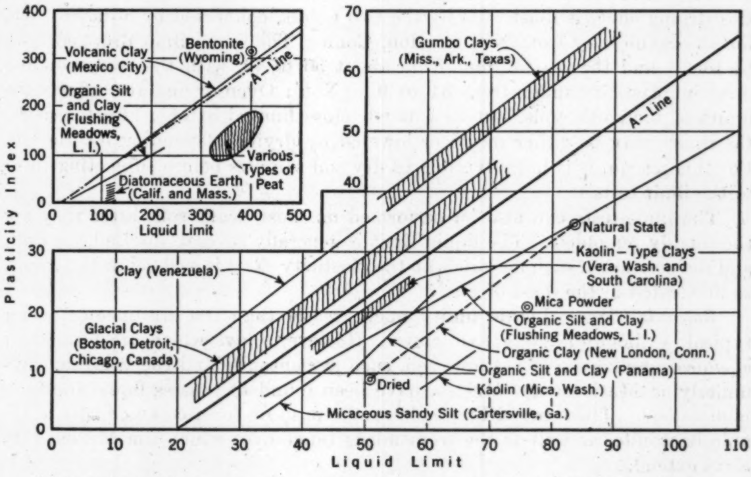


Fig. 5.—Relation Between Liquid Limit and Plasticity Index for Typical Soils

Inorganic Silts, Rock Flour, Micaceous, and Diatomaceous Soils.—The common types of inorganic silts and rock flour have liquid limits less than 30 and are located in the ML area, Fig. 4. A typical rock flour, such as is found in glaciated areas, is usually nonplastic.

Eolian deposits of the loess-type have liquid limits ranging usually between 25 and 35 and vary from nonplastic (that is, the base line) to the A-line, and in exceptional cases also into the CL group slightly above the A-line.

Micaceous soils have a substantially greater liquid limit than a similar soil without the admixture of mica, as illustrated in Fig. 5 by the micaceous, sandy silt from Cartersville, Ga.

Fairly pure diatomaceous earth is generally nonplastic, with liquid limits above 100. With other fine-grained admixtures, diatomaceous soils were found to range throughout the ML and MH groups. The locations on the plasticity chart of two types of fairly pure diatomaceous earth are shown by the small shaded area in the inset of Fig. 5.

Organic Soils.—Even a very small admixture of organic colloidal matter will cause an inorganic clay to increase its liquid limit, without an appreciable change in the plasticity index, thus causing a relative shift to a position below the A-line. In Fig. 5 are shown the positions of several organic silts and clays. Most of the organic silts, silt clays, and clays deposited by the rivers along their lower reaches near the Atlantic seaboard (for example, Hudson River silt, and Potomac River silt) range in liquid limit between 40 and more than 100 and are located on the plasticity chart from 5 units to 15 units below the A-line.

An important property of all fine-grained, organic soils is the radical drop in plasticity due to oven-drying. This is due to the fact that organic colloids are very sensitive and undergo irreversible changes upon drying. In Fig. 5 this drying effect is illustrated by the two points, connected by a dashed line, for an organic clay from New London, Conn. The liquid limit dropped from 84 to 51, and the plastic limit from about 50 to 42, with the corresponding drop in plasticity index from 34 to 9. (Note: Oven-drying also affects the limits of inorganic soils, but to a much more limited extent. Furthermore, the limits may be either raised or lowered by drying, depending on the soil. For this reason, it is not advisable to dry soil samples before subjecting them to the limit tests.)

The limit tests can also be performed on most peaty soils after they are thoroughly remolded. The liquid limit is generally several hundred per cent, and the position of such materials on the plasticity chart is well below the A-line, as illustrated in the inset of Fig. 5.

Volcanic Clays.—Liquid limits much greater than 100 are uncommon for typical, sedimentary, inorganic clays. However, clays which are formed by chemical breakdown of volcanic ash, such as bentonite and the volcanic clays underlying Mexico City, Mexico, have been found to possess liquid limits as high as 500. The inset in Fig. 5 shows the position for a commercially available bentonite, as well as the tremendous range over which the Mexico City clays extend.

13. Field Identification of Fine-Grained Soils

Basis of Field Identification.—It has been shown how the limit tests, supplemented if necessary by other tests, are used to classify fine-grained soils. However, the AC system is so devised that it is possible, with experience, to classify most soils correctly on the basis of field identification procedures alone. The easiest way of learning field identification is under the guidance of an experienced man. To learn without such assistance, one should compare systematically the numerical test results for typical soils in each group with the "feel" of the material while performing the field identification procedures.

The distinguishing characteristics between coarse-grained soils and fine-grained soils have already been described. The principal procedures for field identification of fine-grained soils are (*a*) the shaking test, (*b*) the examination of plasticity characteristics, and (*c*) the examination of dry strength. In addition, observation of color and odor is of value, particularly for organic soils.

Shaking Test.—In this method a wet pat of soil is alternately shaken in the palm of the hand and then squeezed between the fingers. A nonplastic, fine-grained soil will become "livery" and show free water on the surface while being shaken. Squeezing will cause the water to disappear from the surface and the sample to stiffen and finally crumble under increasing finger pressure, like a brittle material. If the water content is just right, shaking the broken pieces will cause them to liquefy again and flow together. The speed with which the pat changes its consistency and the water on the surface appears or disappears, is distinguished as (1) rapid, (2) sluggish or slow, and (3) no reaction. Rapid reaction to the shaking test is typical for nonplastic, uniform, fine sand (SP, SF) and inorganic silts (ML), particularly of the rock flour type; also for diatomaceous earths (MH). With decreasing uniformity the reaction becomes more sluggish. Even a slight content of colloidal clay will impart some plasticity and slow up materially the reaction to the shaking test; this is true, for example, of slightly plastic, inorganic, and organic silts (ML, OL); very silty clays (CL–ML); and many kaolin-type clays (ML, ML–CL, MH, and MH–CH). Extremely slow or no reaction to the shaking test is typical for all clays above the A-line (CL, CH), as well as for highly plastic organic clays (OH).

Plasticity Characteristics.—Examining of the plasticity characteristics should start with a small sample having a soft, but not too sticky, consistency. This sample is alternately rolled into a thread and folded into a lump, and the increase in stiffness of the thread is observed as the plastic limit is approached. After the water content has dropped below the plastic limit, the pieces of the threads should be lumped together, and a slight kneading action on the lump continued until it crumbles.

The higher the position of a soil above the A-line on the plasticity chart (CL, CH), the stiffer are the threads near the plastic limit, and the tougher are the lumps below the plastic limit. Examples for such tough clays are the gumbo clays (CH) shown in Fig. 5. For soils slightly above the A-line (as the glacial clays (CL, CH)), the threads at the plastic limit are medium stiff and the lumps soon start crumbling when the water content drops. Soils below the A-line (ML, MH, OL, and OH) show, with very few exceptions, weak threads near the plastic limit. For organic soils and micaceous inorganic soils, located considerably below the A-line, the threads are very weak and spongy. All soils below the A-line, except OH soils near the A-line, lack cohesiveness when the water content drops below the plastic limit, and the lumps will readily crumble.

When working in a locality where the air humidity is fairly constant, the length of time to reach the plastic limit furnishes a rough indication of the magnitude of the plasticity index. For example, for a gumbo clay with a liquid limit of 70 and a plasticity index of 50 (CH), or a highly plastic organic clay with a liquid limit of 100 and a plasticity index of 50 (OH), a very much longer time will elapse before the plastic limit is reached, as compared with a glacial clay of the CL group. For a slightly plastic silt (ML) the plastic limit will be reached very quickly. To prevent a misleading conclusion it is important that the examination of the plasticity characteristics be started for all soils at about the same consistency.

Dry Strength.—The resistance of a piece of dried soil to crushing by finger pressure is an indication of the character of the colloidal fraction of a soil.

Almost no dry strength characterizes nonplastic ML and MH soils. The dried lumps will fall apart under the slightest finger pressure. Typical rock flour and fairly pure diatomaceous earth are examples.

Low dry strength is representative of all soils of low plasticity, located below the A-line, and of very silty, inorganic clays in the CL group, slightly above the A-line.

Medium dry strength is typical of most clays of the CL group, and of those clays of the CH, MH (kaolin-type clays), and OH groups which are located near the A-line.

High dry strength is indicative of most CH clays, also of those CL clays located high above the A-line, as well as some organic clays of the OH group having very high liquid limits and located near the A-line.

Very high dry strength is developed only by those clays of the CH group located high above the A-line, as the gumbo clays shown in Fig. 5.

Color.—In field soil surveys color is often helpful in distinguishing between various soil strata, and, with sufficient preliminary experience with the local soils, also in identifying the individual soils. To the experienced eye certain dark or drab shades of gray and brown, including almost black colors, are indicative of fine-grained soils containing organic-colloidal matter (OL, OH). In contrast, clean and bright-looking colors, including medium and light gray, olive green, brown, red, yellow, and white, are generally associated with inorganic soils.

Odor.—Organic soils of the OL and OH groups usually have a distinctive odor which, with experience, can be used as an aid in identification. This odor is especially apparent in fresh samples. It is gradually reduced by exposure to air, but can again be brought out by heating a wet sample.

Observations and Tests Relating to Material in Place.—For foundations, the properties of soils in the undisturbed state require special examination. Most important are the denseness of coarse-grained soils and the consistency of fine-grained soils. Also, the consistency of the same soil remolded with its natural water content should be observed. For example, a clay may be described as "stiff and brittle in undisturbed state and very soft and sticky in remolded state." Where the drainage characteristics are of importance, one should observe stratification, root holes, and fissures. Depending on the problem, other tests, such as those listed in Col. 5, Table 4, may be required.

14. Possible Expansion of AC System

General.—Numerous suggestions have been received for the expansion of the airfield classification system. Some have recommended the use of subscripts or letters and numbers added in parentheses, to supply more detailed information. Such additional symbols are useful abbreviations but have the disadvantage that they can be read only by those who have taken the trouble to acquire a working knowledge of such a "shorthand" system. In general, readers of reports and publications should not be burdened with more than a minimum number of symbols; full description in words is preferred.

For soil specialists it may be desirable to arrive at an agreement on the use of possible symbols for describing soils more in detail or on the addition of soil groups. For this purpose the most practical of the suggestions received are summarized below, including the writer's ideas, based on experience accumulated since the new system was formulated.

Coarse-Grained Soils.—Simple field identification methods permit differentiation within the GP and SP groups between uniform soils (such as beach sand) and those materials covering a wide range of sizes but of poor gradation (as a mixture of gravel and fine sand which is deficient in intermediate sizes). By adding the new groups GU and SU for uniform gravels and sands, respectively, and restricting the use of the GP and SP symbols to those materials which have a wide but poor gradation, it is easy to provide for this differentiation.

To distinguish between those soils in the GF and SF groups which possess a silt binder and those having a plastic (clay) binder, the subscripts m and c, respectively, can be used, for example, GF_c, SF_m. The gradation of the coarse fraction could also be described by subscripts, such as, S_uF_m, S_pF_c, and S_wF_m.

Fine-Grained Soils.—The entire range of plasticity is at present divided into the two main groups designated with the letters L and H. Experience has indicated that the L group covers too large a range and that with practice one can distinguish by field identification between soils having a liquid limit ranging well below 35 and those from 35 to 50. The soils in such an intermediate group could be identified with the letter I. Thus, CL would designate a very lean clay having a liquid limit below 35, and located in the plasticity chart above the A-line; CI would be a clay with a liquid limit between 35 and 50, above the A-line; and CH would be a fat clay with a liquid limit larger than 50, above the A-line.

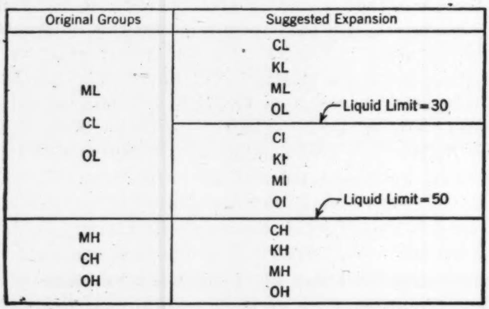
In localities where kaolin-type clays require attention, it would be desirable to add such clays as separate groups for which the symbols KL, KI, and KH suggest themselves. This illustrates the flexibility of this classification, which permits adding new groups without affecting the main framework. It should be kept in mind, however, that any expansion should not be beyond the possibilities of field identification. From that viewpoint there is no objection to introducing the kaolin-type clays as a separate group, because anyone who has handled such clays and learned their "feel" will be able to recognize them readily.

A comparison of the original groups of fine-grained soils with this possible expansion is illustrated in Fig. 6. It will be noted that the order has also been changed to conform better to the order of usefulness of soils in foundation and earthwork engineering.

15. DESCRIPTIVE SOIL CLASSIFICATION

Describing a soil in detail on the basis of a careful visual and manual inspection, utilizing more or less generally accepted nomenclature, classifies this soil, although no rigid system is used. This approach will be designated as a "descriptive soil classification." As such, it may be considered the oldest and most widely used classification. Not only field engineers and boring foremen, but also soil mechanics experts use this extensively, often in preference to a more rigid system.

The disadvantage of the descriptive classification lies in a lack of uniformity of soil terminology. The abuse of classifications by giving only the group number of a soil without any further information, has perhaps been the greatest obstacle to the development of a more uniform soil nomenclature. Therefore, the application of any system of classification as an excuse to reduce the description to a minimum or to eliminate it entirely must be strongly condemned. Instead, the main emphasis should always be on a complete verbal description of every pertinent characteristic of a soil, in its undisturbed state as well as after it is reworked.


Liquid Limit = 30
Liquid Limit = 50

Original Groups	Suggested Expansion
ML CL OL	CL KL ML OL
	CI KI MI OI
MH CH OH	CH KH MH OH

FIG. 6.—SUGGESTED EXPANSION OF AC GROUPS FOR FINE-GRAINED SOILS

For example, a soil might be described as "olive green, homogeneous, medium-plastic, silty clay, medium stiff and brittle in the undisturbed state, very soft when remolded." This descriptive classification will convey to the reader, familiar with this terminology, a fairly clear picture of this soil. In contrast, if it is designated as an A–4 or A–7 soil, the information is entirely inadequate for any engineering use.

If a soil surveyor is forced to describe each soil in detail, he will develop the habit of examining it with greater thoroughness than if he is asked merely to give the group number or symbol. He can, for convenience, use his own abbreviations for frequently recurring characteristics thereby creating a more detailed classification than is afforded by any existing rigid classification. Such abbreviations may be in the form of numerous subdivisions of certain classes within an existing system.

For some purposes the descriptive classification should emphasize specific soil characteristics. Regarding compaction, it was often found useful to group soils into (*a*) noncohesive, coarse-grained soils (that is, soils having no dry strength); (*b*) typical cohesive soils (soils having considerable dry strength); and (*c*) those fine-grained soils having little or almost no dry strength.

Descriptive Adjectives.—The most common adjectives used in describing soils are:

Grain Size

(1) Very coarse
(2) Coarse
(3) Medium coarse
(4) Medium fine
(5) Fine
(6) Very fine

Grain Shape

(1) Angular
(2) Subangular
(3) Subrounded
(4) Rounded
(5) Well rounded

Gradation

(1) Very uniform
(2) Uniform
(3) Poorly graded
(4) Fairly well graded
(5) Well graded
(6) Very well graded

Compactness

(1) Very loose
(2) Loose
(3) Medium dense
(4) Dense, firm, compact
(5) Very dense, well compacted

Consistency

a. Consistency Grading—
  (1) Very soft
  (2) Soft
  (3) Medium stiff
  (4) Stiff
  (5) Hard

b. Other Consistency Characteristics—
  (1) Brittle
  (2) Friable
  (3) Elastic
  (4) Spongy
  (5) Sticky

Plasticity

a. Plasticity Grading—
  (1) Nonplastic
  (2) Trace of plasticity
  (3) Medium plastic
  (4) Highly plastic

b. Cohesiveness Near Plastic Limit—
  (1) Very weak
  (2) Weak
  (3) Firm
  (4) Medium tough
  (5) Tough
  (6) Very tough

Dry Strength

(1) Very slight
(2) Slight
(3) Medium
(4) High
(5) Very high

Examples of Descriptive Soil Classification.—Examples of descriptive soil classification will indicate how the adjectives can be combined into a detailed description:

(1) Uniform, fine, clean sand with rounded grains. Equivalent to A–3; SP (or SU in expanded AC system).

(2) Well-graded, gravelly silty sand; angular gravel particles, ½-in. maximum size; rounded and subangular sand grains; a very silty binder of low dry strength with no plasticity; well compacted and moist in undisturbed state. Equivalent to A–2–3; GW–GF.

(3) Organic silt; medium stiff and brittle in undisturbed state and very soft and sticky in remolded state; medium dry strength; weak and spongy at plastic limit; crumbles readily below plastic limit. Equivalent to A–7, OL–OH.

(4) Highly plastic (fat) clay; stiff at the plastic limit; forms a tough lump when drying is continued considerably below the plastic limit; medium stiff in undisturbed state, soft and sticky when remolded. Equivalent to A–6–7; CH.

Acknowledgment.—The writer wishes to express his appreciation to S. D. Wilson for stimulating assistance.

APPENDIX. BIBLIOGRAPHY

(1) **"Mechanical Analysis of Soils,"** *Bulletin No. 4*, U.S.D.A., 1896.

(2) **"Die Rationelle Klassifikation der Sande und Kiese,"** by A. Atterberg, *Chemiker-Zeitung, No. 15*, 1905.

(3) *Internationale Mitteilungen für Bodenkunde* (rept., meeting of the International Comm. for Mech. and Physical Soil Investigations, October 31, 1913), Vol. IV, No. 1, 1914.

(4) **"Ein Beitrag zur Frage der neuen Einteilung der Körnungsgruppen,"** by J. Kopecky, *Internationale Mitteilungen für Bodenkunde*, Vol. IV, No. 2/3, 1914.

(5) **"Particle-Size in Silts and Sands,"** by R. Glossop and A. W. Skempton, *Journal*, Inst. C. E. (Great Britain), December, 1945, p. 81. (*a*) p. 102, Table V.

(6) **"Grouping of Soils on the Basis of Mechanical Analysis,"** by R. O. E. Davis and H. H. Bennett, *Circular No. 419*, U.S.D.A., July, 1927.

(7) **"Permeability Determination, Quabbin Dam,"** by Stanley M. Dore, *Transactions*, ASCE, Vol. 102, 1937, p. 682.

(8) Discussion of the paper, **"Graphical Representation of the Mechanical Analyses of Soils,"** by Frank B. Campbell, *ibid.*, Vol. 104, 1939, p. 169. (*a*) p. 156.

(9) **"The Formation, Distribution and Engineering Characteristics of Soils,"** by D. J. Belcher, L. E. Gregg, and K. B. Woods, *Research Series No. 87*, Purdue Univ., Lafayette, Ind., January, 1943.

(10) **"Interrelationship of Load, Road and Subgrade,"** by C. A. Hogentogler and Charles Terzaghi, *Public Roads*, May, 1929, p. 1.

(11) **"Classification of Soils and Control Procedures Used in Construction of Embankments,"** by H. Allen, *ibid.*, February, 1942, p. 263.

(12) **"The Application of Soil Mechanics to Highway Foundation Engineering,"** by J. O. Morton, *Proceedings*, International Conference on Soil Mechanics and Foundation Eng., Cambridge, Mass., Vol. III, 1936, p. 244.

(13) **"Materials and Tests,"** *Manual of Instructions*, Texas Highway Dept., Vol. 3, Pt. 4, 1939.

(14) **"Airport Pavement Design,"** by A. H. Hadfield, *Journal*, Boston Soc. of Civ. Engrs., July, 1944, p. 157.

(15) *Engineering Manual*, Chapter XX, Office of the Chf. of Engrs., War Dept., March, 1943.

(16) **"Soil Tests for Design of Runway Pavements,"** by T. A. Middlebrooks and G. E. Bertram, *Proceedings*, 22d Annual Meeting, Highway Research Board, National Research Council, December, 1942, p. 144.

(17) **"Über die physikalische Bodenuntersuchurg und über die Plastizität der Tone,"** by A. Atterberg, *Internationale Mitteilungen für Bodenkunde*, Vol. 1, 1911.

(18) Statens Järnvägars Geotekniska Kommission 1914–1922, Statens Järnvägar, Geotekniska Meddelanden, Stockholm, 1922.

(19) **"Erdbaumechanik auf Bodenphysikalischer Grundlage,"** by K. Terzaghi, Vienna, 1925.

(20) **"Adaptation of Atterberg Plasticity Tests for Subgrade Soils,"** by A. M. Wintermyer, *Public Roads*, Vol. 7, August, 1926, p. 119.

(21) **"Research on the Atterberg Limits of Soils,"** by A. Casagrande, *ibid.*, Vol. 13, October, 1932, p. 121.

(22) **"Preparation of Soil Samples for Mechanical Analysis and Determination of Subgrade Constants,"** Designation D 421–39, *A.S.T.M. Standards*, 1944, pp. 614–615.

(23) **"Exploration of Soil Conditions and Sampling Operations,"** by H. A. Mohr, *Publication No. 376*, Graduate School of Eng., Harvard Univ., Cambridge, Mass., 3d Ed., November, 1943, pp. 9–12.

(24) **"Definitions of Soil Components,"** Rept. of Committee VII on Foundations and Soil Mechanics, *Civil Engineering Bulletin*, Civ. Eng. Div., Am. Soc. for Eng. Education, March, 1947.

(25) **"The Preparation of Subgrade,"** by O. J. Porter, *Proceedings*, Highway Research Board, National Research Council, Washington, D. C., Vol. 18, 1938.

(26) **"Soil Survey Practice in the United States,"** by Levi Muir and William F. Hughes, *ibid.*, Vol. 19, 1939, p. 467.

(27) **"Engineering Significance of National Bureau of Standards Soil Corrosion Data,"** by K. H. Logan, *Paper No. RP 1171*, National Bureau of Standards, Washington, D. C., January, 1939.

(28) **"Soil-Cement Mixtures, Laboratory Handbook,"** Portland Cement Assn., Chicago, Ill., 1939–1942.

(29) **"State-Wide Highway Planning Survey Soil Study,"** *Bulletin No. 6*, Nebraska Dept. of Roads and Irrig., Lincoln, Nebr., 1939.

(30) **"Special Papers on the Pumping Action of Concrete Pavements,"** *Research Reports No. 1–D*, Highway Research Board, National Research Council, Washington, D. C., Vol. 25, 1945.

(31) **"Characteristics and Uses of Loess in Highway Construction,"** by R. E. Bollen, *American Journal of Science*, May, 1945, p. 283.

(32) **"Special Papers on the Pumping Action of Concrete Pavements,"** *ibid.*, 1946.

(33) **"Flexible Pavement Test Section For 300,000 lb. Airplanes, Stockton, California,"** by R. A. Freeman and O. J. Porter, *Proceedings*, Highway Research Board, National Research Council, Washington, D. C., Vol. 25, 1945, p. 23.

(34) **"Longitudinal Cracking of Concrete Pavements on State Highway 13 in Clark and Taylor Counties, Wisconsin,"** by H. F. Janda, *ibid.*, Vol. 15, 1935, p. 157.

(35) **"Experimental Soil-Cement Stabilization at Cheboygan, Michigan,"** by W. S. Housel, *ibid.*, Pt. II, Vol. 17, 1937, p. 46.

(36) **"Soil-Cement Stabilization in Missouri,"** by F. V. Reagel, *ibid.*, p. 66.

(37) **"Sampling, Soil Classification and Cement Requirement—North Carolina,"** by L. D. Hicks, *ibid.*, Vol. 19, 1939, p. 521.

(38) **"Experimental Sand-Clay Base Course in North Carolina,"** by James A. Kelley, *Public Roads*, April–May–June, 1947, p. 299.

(39) *Transactions*, ASCE, Vol. 110, 1945, p. 324.

(40) **"Application of Geologic and Soils Principles to Highway Research,"** by F. R. Olmstead, paper presented at the seminar on engineering geology, Section on Eng. Geology, U. S. G. S., Denver, Colo., November, 1946.

(41) **"Aviation Engineers,"** U. S. War Dept., *Technical Manual TM 5-255*, U. S. Govt. Printing Office, Washington, D. C., April 15, 1944.

(42) **"Terra, a Philosophical Essay of Earth,"** by John Evelyn, London, 1678.

(43) **"Traite Experimental et Analytique de la Poussee des Terres,"** by M. Mayniel, D. Colas, Paris, 1808, p. 102.

(44) *Ibid.*, Book IV, p. 166.

(45) **"Treatise on Soils and Manures,"** by a Practical Agriculturist, London, 1818.

(46) **"Classification System for Composite Soils,"** by D. M. Burmister, *Engineering News-Record*, July 31, 1941, p. 61.

(47) **"Practical Methods for the Classification of Soils,"** by D. M. Burmister, *Proceedings*, Purdue Conference on Soil Mechanics, Purdue Univ., Lafayette, Ind., July, 1940, p. 129.

(48) **"Soil Investigations,"** Eng. Manual for Civil Works, Exhibit 4, Pt. VI, Office of the Chf. of Engrs., Dept. of the Army, Washington, D. C., September, 1945.

(49) **"Engineering For Dams,"** by W. P. Creager, J. D. Justin, and J. Hinds, John Wiley & Sons, Inc., New York, N. Y., Vol. III, 1945, p. 625.

(50) **"L'Analyse Méchanique,"** by H. Geesner, Dunod, Paris, 1936, p. 329.

(51) **"Classification of Highway Subgrade Materials,"** *Proceedings*, Highway Research Board, National Research Council, Washington, D. C., Vol. 25, 1945, pp. 376–392.

(52) **"Mechanical Composition of Soil in Relation to Field Descriptions of Texture,"** by T. J. Marshall, *Bulletin No. 224*, Council for Scientific and Industrial Research, Commonwealth of Australia, Melbourne, 1947.

(53) **"Micropedology,"** by Walter L. Kubiena, Iowa State College Press, Inc. (formerly Collegiate Press, Inc.), Ames, Iowa, 1938.

(54) **"Soil-Cement Mixtures, Laboratory Handbook,"** Portland Cement Assn., Chicago, Ill., 1946.

(55) **"Report of Subcommittee on Sediment Terminology,"** *Transactions*, Am. Geophysical Union, December, 1947, pp. 936–938.

(56) **"A Scale of Grade and Class Terms for Clastic Sediments,"** by C. K. Wentworth, *Journal of Geology*, Vol. 30, 1922, pp. 377–392.

(57) **"Mechanical Composition of Clastic Sediments,"** by J. A. Udden, *Bulletin*, Geological Soc. of America, Vol. 25, 1914, pp. 655–744.

(58) **"Sealing the Lagoon Lining at Treasure Island with Silt,"** by Charles H. Lee, *Transactions*, ASCE, Vol. 106, 1941, p. 577.

(59) **"Earth Materials Investigation Manual,"** Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo., November, 1947.

DISCUSSION

Ralph E. Fadum,[3] Assoc. M. ASCE.—The critical review of the soil classification systems used in civil engineering, as presented by the author, is a worthy contribution to the literature of soil mechanics. The author is to be commended for directing attention to the inadequacies of the much used textural classification systems according to which all soils are distinguished on the basis of grain-size composition. He has submitted ample evidence to support his thesis that the grain-size composition does not, by itself, reflect the significant physical properties of fine-grained soils.

In the proposed Airfield Classification (AC) System the author distinguishes between two primary soil groups—the so-called coarse-grained group and the fine-grained group. Sands and gravels comprise the first group and silts and clays the latter group. Soils of the coarse-grained group are distinguished primarily on the basis of grain-size composition. In general, a visual examination will suffice to differentiate the eight soil types included in this group. The six soil types included in the fine-grained group, on the other hand, are distinguished primarily by the degree of plasticity that they exhibit as measured by the simple, manual tests described by the author. It is noteworthy that in accordance with this system fourteen soil types, which have significantly different physical properties, can, with experience, be recognized by simple visual and manual tests that can be made without the aid of laboratory facilities.

The proposed system is flexible and, accordingly, suitable for classifying soils with respect to performance characteristics under a wide variety of conditions. The author has referred to the AC system as the Airfield Classification System. This name connotes a restrictive meaning that does not do justice to its general applicability.

The author recommends (see heading, "12. Fine-Grained Soil Groups of AC System") that 0.1 mm be used as the boundary size between the coarse-grained and fine-grained groups—

> "* * * usually those soils containing more than 50% of material smaller than 0.1 mm in size (passing No. 150 mesh) are classified as fine-grained soils (ML or CL); and soils containing less than 50% smaller than 0.1 mm in size are classified as coarse grained (SF or SC)."

This arbitrary limit provides a convenient method of separating a soil into its component fractions for detailed laboratory analysis. If a soil is dispersed in water, the particles larger than approximately 0.1 mm will settle out very quickly; those smaller than this size will remain in suspension. Thus, the coarse-grained and fine-grained fractions can be separated conveniently by a process of successive washing and decanting. After drying, the coarse fraction s suitably prepared for a dry sieve analysis. The wash water containing the fines can be used for hydrometer analysis (if the gradation of this fraction is considered significant), or the fines can be recovered from the wash water and subjected to other tests that may be prescribed for the so-called binder fraction.

---

[3] Prof. of Soil Mechanics, School of Civ. Eng. and Eng. Mechanics, Purdue Univ., Lafayette, Ind.

The binder fraction is usually defined (22) as that fraction of a soil the constituents of which pass a standard No. 40 sieve. The writer is of the opinion that the results of the physical tests such as the Atterberg limit tests as determined from the binder fraction, consisting of the fraction of a soil smaller in size than 0.1 mm, would be more significant than the results obtained from the same tests performed on a binder fraction consisting of all material passing the No. 40 sieve.

In his discussion of the various methods of classifying soils the author includes one that he refers to as descriptive soil classification (see heading, "15. Descriptive Soil Classification"). This method, which is used extensively, consists in "Describing a soil in detail on the basis of a careful visual and manual inspection, utilizing more or less generally accepted nomenclature * * *." The author states (see heading, "15. Descripture Soil Classification") that: "The disadvantage of the descriptive classification lies in a lack of uniformity of soil terminology." The following remarks are directed particularly toward the soil terminology used in a descriptive soil classification system.

Confusion in terminology has arisen from the practice of differentiating soil types quantitatively on the basis of grain size composition, from the use of provincial names such as Boston Blue Clay, Chicago Blue Clay, etc., and from the use of terms such as marl and loam that do not convey sufficiently specific meanings. To be acceptable and useful, terminology used in a descriptive soil classification system should:

1. Consist of terms with which a boring foreman, an earthwork engineer, and a soil mechanician are equally conversant;
2. Describe the significant characteristics that can be determined quickly without the aid of laboratory facilities;
3. Be discriminating; and
4. Convey a well-defined meaning.

For purposes of soil identification and descriptive classification it is suggested that the following three size fractions be recognized as significant in describing the grain-size composition of a soil—the gravel fraction, the sand fraction, and the fine-grained fraction (composed of particles smaller than 0.1 mm in size). The composition in terms of these fractions can be expressed conveniently by a number system, as follows: 50–10–40 designates a soil composed 50% by weight of gravel, 10% by weight of sand, and 40% by weight of fines; and 0–20–80 designates 20% by weight of sand and 80% by weight of fines. If graphical representation is preferred, a triangular chart similar to those shown in Fig. 3 could be used. Reasonable estimates of composition according to these fractions suitable for identification purposes can be made without the aid of laboratory facilities.

As stated, sands and gravels are distinguished solely on the basis of the size of their constituent particles. Factors other than grain size, however, must be considered in distinguishing between silts and clays. The author has directed attention to the fact that the grain-size composition of two fine-grained soils may be identical; yet, the physical properties can be significantly different. The effect of mineralogical composition, which is reflected by the shape of the soil particles, dominates the effect of grain size on the properties of a fine-grained soil. Soils composed essentially of "clay-size" minerals, as defined by the various classification scales in current use, may exhibit either silt-like or clay-like properties. According to engineering concepts a fine-grained soil or fraction is identified on the basis of the degree of plasticity and dry strength that it exhibits. A clay-like soil can be made to exhibit plastic properties by an adjustment of its water content and it has considerable strength when air-dried. (The term plastic is used, herein, to describe that property of a material which enables it to be deformed quickly without exhibiting a noticeable elastic rebound or volume change and without cracking or crumbling.) A silt-like soil cannot be made plastic and it exhibits little or no strength when air-dried. There are no simple, direct methods suitable for routine identification and classification purposes by which the clay-like constituents of the fine-grained fraction can be separated quantitatively from the silt-like constituents. It is suggested, therefore, that the terms silt and clay be used in a qualitative rather than in a quantitative sense.

In light of the foregoing remarks and in the interest of promoting agreement on soil terminology to be used in a descriptive soil classification system, the following summarized recommendations are proposed:

1. The terms "gravel," "sand," "silt," and "clay," used singly or in combination, shall form the basic elements of a soil name. The name shall be qualified by the use of carefully selected adjectives so as to make the description of a soil complete and discriminating.
2. Distinction between silts and clays shall not be based upon the size of the constituent particles.
3. The terms "silt" and "clay" shall be used in a qualitative rather than in a quantitative sense.
4. The gravel fraction, the sand fraction, and the fine-grained fraction shall be recognized as the significant fractions in describing the grain-size composition of a soil.
5. The fine-grained fraction (or binder fraction) shall be defined as that fraction of a soil the constituents of which are smaller than 0.1 mm in size.
6. The fine-grained fraction shall be designated "silt" if it cannot be made plastic and if it exhibits little or no strength when air-dried. (Careful distinction must be made between inorganic and organic silts (23).)
7. The fine-grained fraction shall be designated "clay" if it can be made plastic by an adjustment of its water content and if it exhibits considerable strength when air-dried.

The foregoing recommendations were submitted by the writer to the American Society for Engineering Education for use in formulating a set of standard definitions (24).

JAMES H. STRATTON,[4] M. ASCE.—The very nature of military operations in active theaters of war precludes other than "rule of thumb" methods in the design of airfields and roads, and the fluidity of the military situation generally requires that their construction and improvement be undertaken by stages. Improvements in bearing capacities must generally be accomplished in the face of enemy activity and increasing severity of use by the engineer's own forces.

---

[4] Col., U. S. Army, Superv. Engr., The Panama Canal, Diablo Heights, Canal Zone.

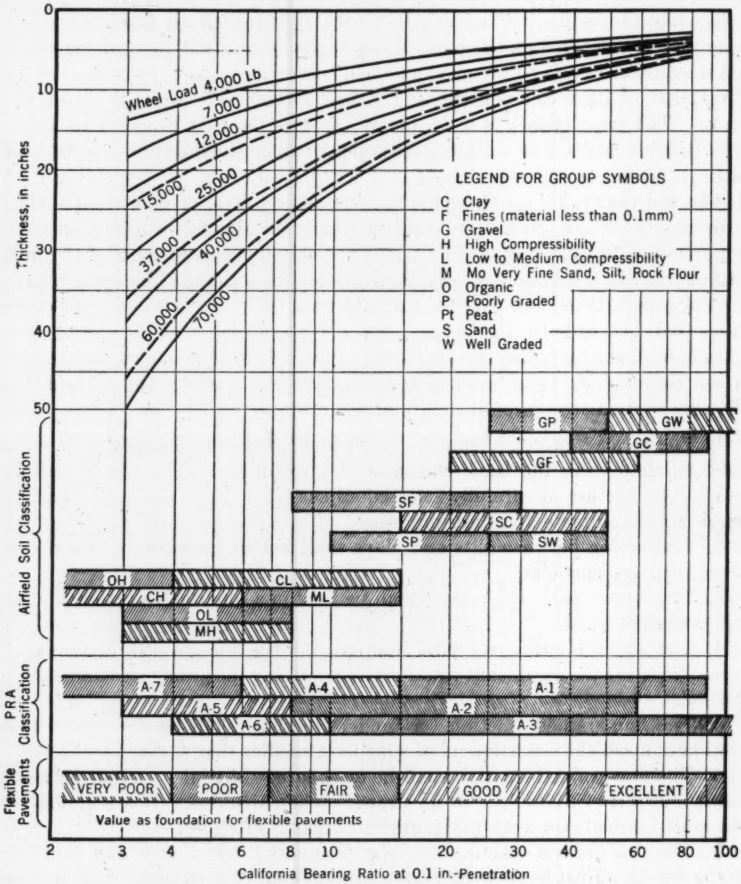


Fig. 7.—Tentative Recommendations for Combined Thickness of Pavement and Base Having Satisfactory Bearing Ratio

The military engineer must know, at all times, the capacities of his roads and airfields in terms of both traffic volume and loading intensities. This calls for rapid interpretations or analyses, and therefore some standardization of the "rule of thumb" design method is required to insure against faulty designs or interpretations of the strengths of roads and airfields which would adversely affect the plans of the military commander.

In the spring of 1942, the Corps of Engineers, United States Army, adopted the California highway method of evaluating the bearing values of road subgrades and extended the design curves developed by O. J. Porter, M. ASCE (25),[4a] to the higher loadings later required for airfields (see Fig. 7). An attempt was made to correlate the Public Roads Administration soils classification with the bearing-value curves in order to provide a design method for military road and airfield construction in the zone of operations through simple soil classification tests. As a result of the inability of the Corps of Engineers satisfactorily to correlate the Public Roads Administration soils classification with design procedures, Professor Casagrande was requested to study methods of soils classification. The material for the paper was taken from the resulting study.

Prior to adopting the Casagrande classification, a canvass was made of all district and division engineers with the interesting result that, of the forty-four offices canvassed, thirty nine were in favor of the adoption and only five raised objections. The objections were largely concerned with the confusion that would result in the adoption of revised soils classification when construction was at a peak. However, in the light of the favoring majority opinion, it appeared that a change at that time would actually facilitate the transition by giving a wide opportunity for correlation with the Public Roads Administration system of classification that would aid in later revisions and improvements of the Casagrande classification. Accordingly, the Casagrande classification was adopted as standard for the Corps of Engineers.

Objections to the soil classification method were interposed by some of the most experienced soil technicians, who condemn all soils classification methods on the ground that their use encourages casual design conclusions whereas exhaustive investigations and study are required. In weighing this objection it must be noted that "symbols" identified with soils classification are a form of "engineering shorthand" which will always be popular in the profession despite the concern of those who recognize the complexity of soils and the difficulties in predicting their behavior. The Corps of Engineers does not propose to substitute a soil classification method for needed investigations and tests and orderly design procedures except where circumstances preclude other than rough or rudimentary methods of design.

It was the conclusion of the Corps of Engineers that the correlation of the soil classification methods proposed by Professor Casagrande with the bearing values of subgrades and base courses would be helpful in developing a "rule of thumb" method of design that could be employed where extensive investigations, tests, and deliberate design methods were infeasible.

The virtues ascribed by districts and divisions of the United States Engineer Department (U.S.E.D.) to the Casagrande method of classification may be summarized by stating that the improved classification is in consonance with general progress in soils engineering. One division of the U.S.E.D. commented favorably on the method because of its mnemonic characteristics.

The Lower Mississippi Valley Division, U.S.E.D., noted that a soil derivative of loess, frequently encountered in the Mississippi Valley, consisted of from 40% to 60% of silt and partakes of the nature of both an A-4 and an A-5 soil, and thus is not classifiable by the Public Roads Administration method. This division found that this soil also was not classifiable by the Casagrande method and that further study with a view to cataloging this soil is required. The same division also pointed out that soils classification correlation with the California Bearing Ratio (CBR) values must be predicated on standard methods of soil compaction and testing for the bearing values. The United States Waterways Experiment Station at Vicksburg, Miss., investigated a wide variety of soils and reported that the bearing values correlated very well with its interpretation of the Casagrande classification with one exception, which fell 3% outside the estimated range of CBR values as specified by Professor Casagrande for the particular soil type.

One district suggested that the percentage of soil binder and its plasticity index be indicated by numerical values in the classification of gravels and sands; for example, GW-3,1 would indicate a well-graded gravel with approximately 30% soil binder and a plasticity index of approximately 10%; likewise, SP-0 would indicate a poorly graded sand with no fines. If this plan were adopted, the GC, GF, SC, and SF groups would be eliminated. For the fine-graded soils, this district proposed correlating the plasticity index and moisture condition with the classification letters. For example, C5-3 would represent a "fat" clay with a plasticity index of 50% and a moisture content of 30%. There are certain virtues in this suggestion which may be realized indirectly by adopting the suggestion of the Upper Mississippi Valley Division, U.S.E.D., that the CBR number be added after the letter symbols; thus, for example, SF-25 would indicate a soil to be a sand with an excess of fines and would give a picture of its internal structure. This suggestion is of value since the CBR is indicative of behavior under load. It is probable that this form of identification will be used when bearing tests have been made.

The North Atlantic Division, U.S.E.D., stated that the suggested soil groups are satisfactory with the exception of the gravels and sands in the fine groups—that is, GF and SF, respectively. These could be subdivided advantageously into additional groups reducing the wide latitude now covered by them. It was suggested by this division that an additional gravel and sand group, designated GL and SL, respectively, be added to the Casagrande classification, to include gravels and sands with excess of fines.

The Casagrande classification is not expected to supplant local names entirely for special soils, such as loess, buckshot, gumbo, etc. These names have special meaning in the localities of occurrence and it is expected that they will be used to supplement the Casagrande classification.

The foregoing discussion was prepared for delivery at the Annual Meeting of the Society in January, 1943. Subsequently, the discusser's duties deprived him of opportunity to participate in engineering activities where he might have noted the results of use of the Casagrande soils classification system. It would be of interest to learn if the method of classification found favor and if it assisted military engineers in the construction of roads and airfields in the theater of operations.

Donald J. Belcher,[5] Assoc., M. ASCE.—Professor Casagrande's review of the present status of soil classification and identification provides a well-organized framework that is necessary to a perspective of the current situation in this field. The trial and error process, plus native ingenuity, has produced a "system" for each large organization dealing with soil in the engineering sense. This review organizes soil classifications and identification and places them in juxtaposition so that they may be evaluated and their relationships noted. All too often various methods are treated with an intolerance that indicates a lack of scientific maturity or perspective. Much of this intolerance appears when something "fails"—that is, the system is wrong. The same people throw golf clubs and tennis rackets because their game is poor.

Perhaps there have been many instances where each system has been applied too broadly—used beyond the limitations originally perscribed. Professor Casagrande points this out with respect to the Public Roads (PR) system. It is true of other systems as well and cannot be considered as a fundamental fault. In this respect a highly desirable development of Fig. 1 would be to introduce a vertical distribution based on grain size. The author indicates this particularly under the heading, "3. Textural Soil Classifications," where it is to be noted that such a method is inadequate for fine-grained soils. Similar limitations can be placed on several of the others, particularly those in Fig. 1 under the division dealing with "Properties of Soil in its Undisturbed Condition."

In the detailed discussions of the various classifications, especially those under the heading, "4. Agricultural Soil Classification," and "5. Geologic Soil Classification," some additional attention will add to the perspective that can be gained from the over-all treatment of the subject. Primarily, it is erroneous to describe the first method as an "Agricultural Soil Classification." Engineers are in fact unintentionally perpetuating a careless description of a science; this is the proper place to correct this concept in the minds of civil engineers. To tell an engineer that this is an agricultural method (as has been done for so long) is to send him elsewhere looking for an engineering method. Unfortunately, to describe it as a pedological soil classification is a dubious improvement. It is, more simply, the science of soil formation; the classification is based on all the properties of the soils. Therein lies an advantage not found in other methods.

First, and for the record as it stands in the original paper, it would be more complete to state that it is a classification for the purpose of describing a soil and the conditions under which that soil exists. On that basis there is no fundamental leaning toward any one field of application, but any field of interest may interpret the data for its own particular use. If there is a technical predilection toward agriculture on the part of some pedologists, it is attributable to the fact that agriculture supports their programs by the allotment of funds.

It is to be regretted that an influential paper such as this should label the pedological system as a complicated one. On the contrary, its principles are simple and logical, particularly when the method is interpreted for engineering use. It also possesses simplicity of another order: With this system a soil is classified in three dimensions, length, width, and depth. With one well chosen auger boring a soil area can be classified, whereas in other methods the area shrinks to the size of the auger hole and a sample has little significance beyond the hole from which it was obtained.

---

[5] Associate Prof., Civ. Eng., Cornell Univ., Ithaca, N. Y.

It has been commonplace to find the pedological classification described in the literature as a classification of surface soils for agricultural purposes. Although Professor Casagrande described it in this way, he also states that the underlying parent materials are included. The term "surface soils" is a relative one. When it is used in conjunction with "agricultural purposes," "surface soil" is immediately interpreted as meaning 6, 8, or at most 10 in. of depth below the surface. For the sake of clarification this point should be examined in some detail and emphasis added to Professor Casagrande's inclusion of the parent material, which may extend to considerable depths. Actually, there is no arbitrary boundary or basement beyond which lies the realm of the geologist, and there is no particular need for such a limitation.

No one will advocate this method for material existing at depths common to deep foundations. Therefore, at depths of from 80 ft to 100 ft it will never be of value. Depths of from 60 ft to 80 ft are still much too great for any general use of this method, but sand dunes of that depth are found in some areas that are or can be classified on the pedological basis. Also, there are extensive areas of deep wind-blown silt that reach these and even greater depths (from 100 ft to 200 ft). These may be called Aeolian (geologic classification) or loess deposits but they also coincide with pedologic units. Along the Lower Mississippi Valley these are classified as the Memphis soil; to the north, especially in Iowa, Illinois, and Missouri, they are described as Clinton soils; and in the State of Washington as Palouse.

As the surface is approached, in the 40-ft to 60-ft depth range, the application of pedology becomes somewhat greater although only geologic deposits that form the parent materials are as yet under consideration. In this range are included much rock and drift in the glaciated areas. In those soils weathered from rock there are valid relationships in many areas between the soil mantle and the character of the rock at such depths. In the drift areas the character of the mass of of material below the surface is sufficiently well described, so that general compaction and several other requirements (26) can be anticipated on the basis of the series name of a soil (a unit of pedological classification).

Each 10-ft increment, as the surface is approached, emphasizes two trends: First, that the application of the pedological principles increases to its greatest value in the 10-ft zone immediately below the surface; and, secondly, that the dollar volume of highway and airport construction increases to its greatest volume within the same general zone.

This somewhat brief illustration is intended to show that, on a national basis, a sliding scale of applicability is necessary. In smaller units, such as within a state or district, the scale becomes more fixed. It will be found that the significance of the pedological method, as it applies to soils in the depth, is a function of the type of deposit or formation. In areas where bare rock is exposed, or, where very shallow soils exist on rough terrain, the value of the method vanishes.

Climate is also important. Within the United States climatic differences are sufficiently great that they strongly influence soil development. In the humid regions much water percolates downward through the soil, constantly leaching the soil and rock, and in general building up accumulations of clay at depths of several feet below the surface. In this process soluble materials are removed. In the west, however, the situation is often reversed. There, rainfall and melting snows in the mountains are a contrast to the desert climate prevailing in the intervening valleys a few thousand feet below. Much of the water proceeds as ground water to the valley area, where it is largely dissipated by evaporation from the surface. This reversed process precipitates salts near the surface to form the alkali and caliche soils. These are also areas where the clays may easily become saturated with exchangeable ions and thus take on characteristics of abnormally high volume change and plasticity. Such soils occur as far east as western Minnesota, where small areas of alkaline soil are associated with the Fargo (clay) series.

At present relatively little correlation has been accomplished in this way, but it is known that pavement performance, base thickness requirements, and many construction details are related to the individual soil series. The alkali attack on concrete slabs, head walls, culverts, and steel pipe is a severe action in many of these areas (27).

If this method is adaptable to engineering use, then who is using it, where is it being used, and how? An examination of this field reveals the following:

The Portland Cement Association in its work on soil-cement stabilization uses and recommends the use of the pedological method as a basis for obtaining soil data (28).

> "Missouri [(26)] uses the pedological system as used by the U. S. Bureau of Chemistry and Soils, in making soil surveys for highway purposes. Missouri takes an attitude similar to that of Michigan, in that, of the three best known methods of classifying soils, the pedological system seems best to suit their needs. The other two methods seem to be limited in their scope because the classification by texture, as in the triaxial chart takes into account no other soil properties and the classification as developed by the Public Roads Administration is more an index of the behavior of disturbed soils and so does not take into account field conditions. The pedological system is essentially a reflection of drainage, topography, parent material, weathering agencies, as well as textural class. Missouri, however, does require that samples of each horizon of each soil type be taken and submitted to the testing laboratory for analysis. Field density tests are also made of each horizon in order that the theoretical soil shrinkage can be calculated for each horizon.
>
> "The Michigan State Highway Department [(26)] has found that the pedological scheme of soil classification provides an excellent means of studying soils for general highway purposes, and that it serves as a foundation for all other soil studies in the department. The soil type, in addition to serving as a unit in mapping, also serves as a unit to which laboratory test results, road behavior observations, construction problems, and maintenance experiences are related. In other words, the soil type serves as a means for classifying and coordinating the accumulated soil experience of the entire highway department personnel."

The Nebraska Department of Roads and Irrigation (29) states:

> "The pedological system of soil classification which has international recognition, was selected.
>
> "By adopting this method of classifying and indentifying soils, it was possible to make use of a wealth of available scientific information on soils,

including the published county soil maps and reports, and to supplement such knowledge with the test data and information obtained by observation on this survey as well as that which may be obtained through future investigations."

Tennessee and Nebraska have established (30)(31) a correlation between the soil series name and the texture of the material, as well as a further correlation between these and pavement performance.

The writer has observed a distinct positive correlation between the pumping of rigid pavements and the occurrence of the Superior clay series near Superior, Wis.; the Sharkey clay series near New Orleans, La.; and the Susquehanna clay series in Texas (partially reported in (32)).

There is a fair but incomplete correlation between the California Bearing Ratio (CBR) value and the soil series. This is indicated in the results of United States Engineer Department tests on the Clermont and Brookston soil series in Indiana; and by inference in the reported CBR experiments at Stockton, Calif. (33).

The North Carolina Department of Highways reports (32) "Pumping was not encountered on soils of the Norfolk or Ruston series. Pumping occurred on the Portsmouth and Plummer series * * *." It occurred also on the Lufkin, Elkton, and Coxville series.

In 1935, H. F. Janda (34) reported that excessive longitudinal cracks in a particular concrete pavement were confined to an area of Colby silt loam. W. S. Housel, M. ASCE (35), and F. V. Reagel (36) have related soil-cement construction to the soil series and L. D. Hicks (37) has established the relationship between the principal horizons of several soil series and the cement required for proper stabilization. Public Roads Administration (PRA) studies (38) show results of typical soils of the coastal plain (Norfolk, Portsmouth, and Ruston) and their relationship to base course performance in North Carolina.

Professor Casagrande's description of "5. Geologic Soil Classifications" is significant. It would be of considerable value if the method could be evaluated in greater detail, or perhaps less subtly; for truly "it is damned by faint praise."

The weakness of the method seems to be lodged in the broad basis of the classification and in the recognition of age and general origin as the most important factors. For those reasons it can seldom be used in engineering without considerable modification. An evaluation of this situation was concisely expressed by Karl Terzaghi, M. ASCE (39), in summing up the results of an investigation in this field:

> "There are no definite relations [between geologic classifications and engineering properties], because for a given geologic origin and petrographic character of a stratum the strength of the material, rock, or soil depends on conditions that are not, and cannot be, recognized by a purely geological investigation. For almost every geologically well-defined stratum the possible scattering of its physical properties from the statistical average is so important, that one cannot even make an acceptable guess based on precedent."

Essentially the pedological classification includes a refined geological classification and the parent material of a soil is directly related to materials that are mapped and described by geologists. F. R. Olmstead (40) and others have shown this in various sections of the United States.

J. A. HAINE,[6] ESQ., AND J. W. HILF,[7] ESQ.—A development of soils classification for engineering purposes is presented to the profession by the author's excellent Airfield Classification (AC) System, which one of the writers studied under him at Harvard University in Cambridge, Mass., in 1943. From personal observation in the Southwest Pacific theater of operations, the AC system proved definitely valuable to United States Army engineer officers who were required to design airdromes rapidly in the field. The AC system was found to have two advantages over other current classification systems:

(1) The symbols used by Professor Casagrande are descriptive and easy to associate with actual soils, and experience in teaching this system to military personnel, who had no previous soils training, showed that it could be learned more quickly than systems depending solely on memory.

(2) The ability to assign definite engineering properties and even numerical values of design criteria, within reasonable limits, to the soil groups (such as the California Bearing Ratio) greatly facilitated the design of foundations for flexible pavements.

Although the AC system was devised for military roads and airdromes, it can be used profitably in foundation explorations and in prospecting for materials for rolled earth dams. In adapting this system to earth dams, it becomes necessary to expand two of the soil groups, and to specify certain limits of particle size and percentages that the author has not definitely stated in the paper.

Two of the fifteen soil groups, GF and SF, should be subdivided in order to distinguish the character of the fines in these soils which may be critical in high embankments because of the phenomenon of pore fluid pressures. This modification is easily accomplished without affecting the definitions of the original soil groups, by substituting GF (silty), GF (clayey), SF (silty), and SF (clayey), for GF and SF, respectively—mentioned by the author as a possible expansion of the AC system.

Specifications for rolled earth dams usually limit the maximum size of gravel in the rolled fill to 5 in. for a 6-in. compacted layer. In the investigation of sources of borrow materials for such structures, it follows that sizes larger than 5 in. should be excluded from the soil classification, although the importance of reporting oversize rock in exploration logs is recognized. With a range of coarse grains, from 0.1 mm to 5 in., the lower limit of gravel size (defined as 1 mm or 2 mm by the various grain-size classifications) should be raised to $\frac{1}{4}$ in. in order to simplify field identification, and to make the sand and gravel ranges more nearly equal on the grain-size curve. This definition more nearly corresponds with the coarse aggregate of concrete (No. 4 sieve), and was used by the Army (41).

A question often asked by those studying the AC system is, "How much gravel must be present in order to classify a coarse-grained soil as gravel, rather than as sand?" Professor Casagrande does not list quantitative limits for the soil groups in Table 4. Undoubtedly, such limits could be provided, but

---

[6] Civ. Engr., Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

[7] Civ. Engr., Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

TABLE 5.—SOIL CLASSIFICATION FOR ROLLED EARTH DAMS (ADAPTED FROM AIRFIELD CLASSIFICATION SYSTEM)

1		2					3	4	5	6			
Major divisions		Field identification (Excluding material more than 5 in. in diameter)					Group symbol	Soil groups and typical names	Permeability	RELATIVE DESIRABILITY FOR EMBANKMENT USE			
										Homogeneous	Zone 1[c]	Zone 2[c]	Zone 3[c]
Coarse-grained soils	Gravel and gravelly soils	More than 50% of sample is coarse grains (>0.1 mm[a])	More than 50% of coarse grains >¼ in.	Good gradation	Substantial amounts of all coarse sizes present (up to maximum size in sample)	Just enough clayey fines to bind coarse grains together	GC	Well-graded gravel-sand mixtures with excellent clay binder	Impervious	1	1	—	—
						"Clean" material (absence of appreciable amount of fines)	GW	Well-graded gravels and gravel-sand mixtures; little or no fines	Pervious	—	—	1	1
				Poor gradation	Predominantly more of one or more coarse sizes		GP	Poorly graded gravels and gravel-sand mixtures; little or no fines	Very pervious	—	—	—	2
					"Dirty" material (excess of fines)	Fines are silty (nonplastic)	GF[b] Silty	Very silty gravel, poorly graded gravel-sand-silt mixtures	Semipervious to impervious	2	2	2	—
						Fines are clayey (plastic)	GF[b] Clayey	Clayey gravel, poorly graded gravel-sand-clay mixtures	Impervious	3	3	—	—
	Sand and sandy soils		More than 50% of coarse grains <¼ in.	Good gradation	Substantial amounts of all coarse sizes present (up to maximum size in sample)	Just enough clayey fines to bind coarse grains together	SC	Well-graded sand with excellent clay binder	Impervious	4	4	—	—
						"Clean" material (absence of appreciable amount of fines)	SW	Well-graded sands and gravelly sands; little or no fines	Impervious	—	—	3	3[d]
				Poor gradation	Predominantly more of one or more coarse sizes		SP	Poorly graded sands and gravelly sands; little or no fines	Pervious	—	—	4	4[d]
					"Dirty" material (excess of fines)	Fines are silty (nonplastic)	SF[b] Silty	Very silty sands, poorly graded sand-silt mixtures	Semipervious to impervious	6	7	5	—
						Fines are clayey (plastic)	SF[b] Clayey	Clayey sands, poorly graded sand-clay mixtures	Impervious	5	5	—	—

TABLE 5.—(*Continued*)

1		2					3	4	5	6			
Major divisions		FIELD IDENTIFICATION (Excluding material more than 5 in. in diameter)					Group symbol	Soil groups and typical names	Permeability	RELATIVE DESIRABILITY FOR EMBANKMENT USE			
		Visual	Tests on fraction < $\frac{1}{16}$ in. in diameter			Other means of identification							
			Shaking	Breaking	Thread					Homogeneous	Zone 1[c]	Zone 2[c]	Zone 3[c]
Fine-grained soils	Fine-grained soils of low to medium compressibility	More than 50% of sample is fine grains (<0.1 mm[a])	Moderate to fast reaction	Very slight to medium dry strength	Soft weak thread or none	Will form cracks when kneaded while moist	ML	Inorganic silts and very fine sands; rock flour; silty or clayey fine sands with slight plasticity	Semipervious to impervious	8	8	6	—
			Extremely slow reaction or none at all	Medium to high dry strength	Medium to tough thread	Can be kneaded while moist without cracking	CL	Inorganic clays of low to medium plasticity; sandy clays; silty clays; lean clays	Impervious	7	6	—	—
			Slow reaction	Slight to medium dry strength	Soft weak thread	Odor of decay which can be intensified by heating	OL	Organic silts and organic silt-clays of low plasticity	Semipervious to impervious	9	9	—	—
	Fine-grained soils of high compressibility		Slow to moderate reaction	Very slight to medium dry strength	Soft weak thread	Small shiny particles of mica or limestone can be seen	MH	Micaceous or diatomaceous fine sandy or silty soils; elastic silts	Semipervious to impervious	11	11	—	—
			No reaction	High dry strength	Very tough thread	In moist state, can be kneaded for a considerable period without cracking	CH	Inorganic clays of high plasticity; fat clays	Impervious	10	10	—	—
			No reaction	Medium to high dry strength	Soft weak to medium thread	Odor of decay which can be intensified by heating	OH	Organic clays of medium to high plasticity	Impervious	12	12	—	—
Fibrous organic soils of high compressibility		Fibrous, decayed, organic vegetable matter; noticeably compressible					PT	Peat and other highly organic swamp soils	Entirely unsuitable				

[a] 0.1 mm is the smallest visual size. [b] Subdivisions of the GF-groups and SF-groups of the AC system. [c] Zone 1 is inner, impervious zone; Zone 2 is intermediate, semipervious zone; Zone 3 is outer, free-draining zone. [d] If gravelly.

the percentages by weight would be different for each soil group, making it necessary to memorize additional information. Also, these percentages would be inappropriate in field classification, because of the difficulty of estimating weights of materials in various states of compactness by visual inspection. To avoid these objections and still provide a basis for dividing coarse-grained soils into sands and gravels, the writers have rearranged Table 4 for field use. As shown in Table 5, identification of coarse-grained soils (ten of the seventeen groups) may be accomplished by a step by step procedure without requiring the estimation of any fraction other than one half.

Col. 1, Table 4, lists fine-grained soils as "containing little or no coarse-grained material," but (under the heading, "12. Fine-Grained Soil Groups of AC System") the author states:

> "In the transition between coarse-grained and fine-grained soils, usually those soils containing more than 50% of material smaller than 0.1 mm in size (passing No. 150 mesh) are classified as fine-grained soils (ML or CL); and soils containing less than 50% smaller than 0.1 mm in size are classified as coarse grained (SF or SC)."

The writers agree with the latter division between the major soil groups and have used it in Table 5. The same reasoning is used in dividing coarse-grained soils into the gravel (G) and sand (S) groups. Regardless of the amount of fine-grained material present in coarse-grained soil, the latter is classified as gravel or sand, depending on the size of the coarse grains (whether more than 50% of them are larger or smaller than ¼ in.).

Persons who have not had the benefit of considerable laboratory experience in sieve analysis often find it difficult to estimate even the simple fraction, one half. They are inclined to judge quantity by volume, rather than by weight; for example, since the percentage by volume of particles larger than ¼ in. in a soil mass depends on whether the soil is in a loose state or a dense state, accuracy cannot be expected. However, the AC system boundary classifications, suggested by the author, such as GW-SW, or SF (silty)-ML, can be used in all cases where there is any doubt about the percentages.

Table 5 was devised to enable a relatively inexperienced investigator, or one unfamiliar with the AC system, to classify a soil—first, into the coarse-grained division or the fine-grained division, and then, if coarse-grained, into the gravel or sand groups when the soil is typical. For a boundary case in the coarse-grained division, the procedure is to assume that the soil is a gravel and then continue in the chart until the final soil group, say, GC, is reached, using the indicated criteria of gradation, and the amount and character of the fines. Since it could have been assumed that the doubtful coarse-grained soil was a sand, the correct field classification is GC-SC, which follows from the duplicate criteria for the gravel and sand subgroups. Similarly, if there is doubt as to whether a soil should be classified as coarse-grained or fine-grained, the assumption of first one type, and then the other, will result in the proper boundary classification. For fine-grained soils, Professor Casagrande's simple field tests, plus other identification aids, are used to determine the proper soil group. Here, the fraction to be tested is stated in Table 5 as material smaller than $\frac{1}{16}$ in., rather than No. 40 mesh. This is merely a field expedient for separation by hand, which will not affect the resulting classification in the large majority of cases, it is believed.

Together with instructions for performing the shaking, breaking, and thread tests, Table 5 should be sufficient to enable a proper classification in the field without any laboratory equipment. If it is desired, a representative sample can be classified in the laboratory, using the criteria in Table 5 and the Atterberg limits criteria proposed by Professor Casagrande; but it must be recognized that only a very small percentage of the soils examined in the field ever reach a laboratory, and that proper field classification is the primary objective in soils surveys. The writers heartily agree with the author that a complete description of the material is necessary in reporting a soil in a log. However, the facility in identification and classification of soils that must be developed by the use of such a chart as Table 5, is a necessary first step toward that goal.

After a soil is properly classified, as Professor Casagrande has shown, it is possible to indicate the engineering properties typical of the various soil groups and their use in engineering structures. For rolled earth dams, the permeability of a material is of outstanding importance and this property is listed for each group in Table 5. Also, the prospector for embankment materials often desires to know how the soil he has classified compares with other kinds of borrow material that may be available for a homogeneous dam, or for various zones of a zoned dam. To aid in evaluating possible sources of material, the writers have attempted to rate the various soil groups according to their experience, for use in rolled earth dams, considering the permeability, workability, shear strength, and resistance to piping of typical materials. In Table 5, Zone 1 represents the impervious zone, Zone 2, the semipervious zone adjacent to Zone 1 in the dam, and Zone 3, the free-draining outer zone of a rolled earth dam. It is recognized that the numerical ratings are not rigid, but in a qualitative way they will enable the investigator to judge the suitability of a material for rolled earth dam construction.

For final design of a dam, a soil classification alone is entirely insufficient. Laboratory tests must be made on representative samples of the foundation, on the required excavation, and on the proposed borrow areas to determine, among other properties, the permeability, cohesion, angle of internal friction, and compressibility characteristics of the materials. Nevertheless, when adequate soil classifications and descriptions are available in the logs of exploration, the designer has at his disposal timely information which he can use in making preliminary estimates, in determining the extent of additional field investigations needed for final design, and in planning an economical laboratory testing program.

It would be desirable if there were a sufficiently large number of experienced soil engineers to personally perform all the identifications and classifications of soils which are required for earth dam projects, but the fact is that often the investigation is performed by personnel relatively inexperienced in soil mechanics. Under these conditions, the simplest possible descriptive classification system, consistent with adequate utility, appears to be desirable.

The AC system without undue additional refinements is admirably suited for this purpose.

JACOB FELD,[8] M. ASCE.—This authoritative and complete program of soil classification is most timely. An extrapolation of reported experience to engineering problems is required to make the subject of soil mechanics usable in actual practice. To permit such procedure, the soil type must be known, when such experience is a report of someone else's work. A universal adoption of the author's system of classification and, at the same time, a recording of the soil type encountered in each problem reported in the technical literature will make a true wealth of data available in contrast to the present volume of contributions in soil mechanics; most contributions are useless because the authors employ secret codes or untranslatable language in describing the soils encountered and controlled.

The author clearly develops the thesis that soil classification must proceed on the basis of soil action and reaction, and not on the basis of component materials. The former method corresponds to the classification of human beings by their expected reactions, rather than by their height or weight. Textural or geometric soil classifications of granular or noncohesive soils have been reasonably successful, because the interaction of grains, in most cases, is not affected by the gas or liquid occlusions on the grain surfaces. In cohesive soils, there is a great variety of possible soil reactions, with identical textural make-up, depending on the mechanical and colloidal extent of the surface coatings. In addition, textural or sieving classifications do not distinguish between grains of unlike shape.

By magnifying the problem of soil classification, the difficulty of true classification by geometric methods can be easily understood. If a grouping of humans by size (let us say, equivalent volume) is made at a shore resort, and an identical size grouping is made at a winter resort, the two groups are geometrically identical; but they cannot be expected to react alike, because the factor of their clothing has been disregarded. Similarly, as the author described, the kaolin-type clays derived from a single type of feldspar may have particle sizes equivalent to many tough clays, but they react more nearly like a silt than a clay. The particles may be of the same size and, therefore, of equal gravitational adhesion; yet the chemical nature of the grain coverings and the effect on wetness (caused by the kind of electrostatic charge on the grain surfaces) are more important factors in determining the total adhesion and the tightness of possible grain packing, or the cohesion and possible consolidation, respectively.

In soil classification, it is unfortunate that the division between the sand and gravel designations does not agree with the accepted sieve separation in concrete and road base technology. The general agreement, that sand contains all particles passing a No. 4 sieve in all cement and bituminous concrete specifications, seems to require a restudy of this item for soil classification. To make the Airfield Classification (AC) System complete, definitions of the terms used, such as "compressibility" and "dry strength" should be included. It is important to distinguish between compressibility or dilatation as an elastic phenomenon, and changes in shape without change in volume, resulting from plastic flow or viscous deformation.

---

[8] Cons. Engr., New York, N. Y.

The author's recommendation of field identification tests brings to mind a very early treatise on soil classification, presented by John Evelyn before the Royal Society of London almost three hundred years ago (42). Although he was chiefly interested in the agricultural possibilities in artificially prepared soils (as well as natural soils), the recommendations for classifications and field identification are quite similar to recent writings, and are summarized herein for the reader who does not have access to this rare book.

In the "Introduction," Evelyn cites "De Arte Combinatoria" by Athanasius Kircher in which it is stated that there are "* * * no fewer than 179,100,060 different sorts of earths, but only eight or nine are useful to our purpose." Corresponding to modern soil horizons, three strata are identified as: (1) The top layer of "underturf earth," a foot or so deep; (2) the next strata may be "loam, clay, plastic, figuline or smectic"; and (3) the lowest strata may be "chalk, marle, fullers-earth, sandy, gravelly, stony, rock, shelly, coal or mineral." Incidentally, Evelyn states, "The Ancients called them: Creta, Argilla, Smectica, Tophacea, Pulla, Alba, Rufa, Columbina, Macra, Cariosa, Rubrica." He further subdivides the top layer into:

"I. A virgin-earth, black, fat, porous, light and sufficiently tenacious, without any admixture of sand or gravel, which in the lower series may be darkest-gray or tawney, becoming veined with yellow or sometimes red.
II. An obscure color mould, with some loam and sand.
III. A mixture of the former two types with small flints and pebbles.
IV. A totally sandy with a bottom of gravel, rock and 'not seldom' clay."

Then, a general description of soil types follows, which is quoted in full because of its picturesque language:

> "Pure sand is white, black, bluish, red, yellow, harsher and milder and some mere dust in appearance, none of them to be desired alone. But the grey black and ash coloured, or of the travelling kind, volatile and exceedingly light, is the most insipid and worst of all.
>
> "Clays there are of many Kinds—some more pinguid, some more slippery, all of them tenacious of water on the surface where it stagnates without penetration, most of them pernicious and untractable. The blue, white and red clay are all unkind, the stony and looser sort is sometimes tolerable, but the light brick-earth does very well with most fruit trees.
>
> "Loams and brick earth are of several sorts, some approaching to clay, others nearer marle.
>
> "Marle is usually at greater depths, of many colours, all of them unctuous, slippery nature, slackens upon drying after a shower and crumbles into dust."

Evelyn than advised that general textural description of soils may be of less value than classification of properties by the use of the senses. Four tests are described:

(1) By odor or smell—since "Lord Verulam affirms that vegetable odours exist in the soil."

(2) By taste—since the water percolation through soil shows the kind of chemicals therein, "although some say that there is no taste or odour in the best earths."

(3) By touch—fatty and slippery, or gritty, porous and friable, if it sticks to the fingers or dissolves on the tongue. The "best earth is blackish, cuts like butter, sticks not obstinately, is sweet, and becoming mortar when wet, without crusting when dry."

(4) By sight—"grey is pre-eminent, next russett, clear tawney is worse, light and dark coloured are good for nothing, yellowish red is worst of all."

Having analyzed and shown how various basic properties could be identified, Evelyn did not entirely discredit the laboratory technician of his day; he advised the following procedure for the synthesis of desired properties in soils:

> "All these are fit to be known, as contributing to noble and useful experiments, upon due and accurate comparisons, and inquiries from the several particles of their Constitutions, Figures and Modes, as far at least, as we can discover them by the best auxiliaries of microscopes, lotions, strainers, calcinations, triturations and grindings; upon such discovery to judge of their qualities, and by essaying variety of mixtures, and imitating all sorts of mold, foreign or indigen, to compound earths as near as may be resembling the natural, for any special or curious use, and be thereby enabled to alter the genius of Grounds as we see occasion."

He then demonstrated how artificial soils had been made by mixing samples of earths from various sources with types of chemicals and fertilizers, and concluded:

> "Therefore let no man be over-confident, that because some earths are soft, fat and slippery, they may not possibly consist of sands, (of which there are so many Kinds), since 'tis evident, that even all fossile bodies which can be reduced and brought to sands, may by contrition of the particles be rendered so minute, as to emulate the finest earths we have enumerated."

The early French experimenters and theorists in soil problems distinguished only between "loose soils" and "firm soils." However, when M. Chauvelot (43) presented a paper before the French Academy of Sciences in 1783, showing how the lateral earth pressure could be derived from a knowledge of the weight, the internal friction, and the cohesion of a soil without any other description or classification, that learned body announced that the problem of the determination of lateral earth pressure was now susceptible to a rigorous solution. Later, contributors went back to a textural soil classification and M. Mayniel (44), in 1808, reports on his extensive experiments and tables to cover all kinds of soil, namely: Soil, sand, gravel, rubbish or ash, clay and debris fills. Incidentally, Mayniel's division between large and small gravel is denoted by the size of a pigeon's egg.

An anonymous author (45) quotes from the works of Sir Humphrey Davy, classifying the four earths found in soils as: "* * * aluminous (clay), siliceous (flint), calcareous (limestone) and magnesian (not to be mistaken for limestone)." However, a soil is to be classified on how it acts and not on what it contains.

To quote that author:

> "In framing a system of definitions, a soil is to take a particular designation from a particular Kind of earth, not exactly in proportion as that earth may preponderate, or not over others in forming the basis of the soil, but rather in proportion to the influence which a particular Kind of earth, forming part of the sample, has on tillage and vegetation."

That might also be said for the influence on engineering uses of soils.

The need for a universal soil classification is an old one and much credit must be given to the author of this paper in presenting a comprehensive answer to it, in the form of the AC system, which permits and provides a unique name for every type of soil, using expected action and reaction as criteria.

KENNETH S. LANE,[9] ASSOC. M. ASCE.—Of the eight classification systems presented in the paper, the writer has had occasion, since 1937, to work with four—the triangle textural classification, the master curve textural classification, the Public Roads Classification, and the Casagrande system. From this experience the writer can certainly agree with the author's designation of soil classification as the most confused chapter in soil mechanics. Although the author modestly terms the Casagrande system as the Airfield Classification System, the writer prefers to designate it as the Casagrande classification, since it has been widely known as such from its initial publication in 1942. This classification is easily the best of the aforementioned four systems (in their forms as described in the paper) for the airport-highway type of work for which it was originally designed. In its application to dams and levees, it has certain limitations which can be best considered from their relations to the primary requirements of a soil classification system intended for wide usage.

Requirements of a Classification System.—The requirements particularly considered are those for a large and widespread organization handling many projects of all sizes; for less intensive operations, or for work concentrated in a particular region, simpler requirements might suffice. In this type of work, boring logs generally include a classification of the soil into a group or type, plus a word description similar to the descriptive classification covered in the paper (under the heading, "15. Descriptive Soil Classification").

On a large airport, levee, dam, or canal, involving a hundred or more foundation explorations, it is generally not feasible to consult the original boring logs whenever questions arise, so these are usually filed as reference material. Rather, the usefulness of a soil classification system comes from its application to graphic abstracts of boring logs, wherein the most pertinent information is abstracted and placed on boring drawings. Hence, for a large project, this graphic abstract is the form of boring information most used by engineers in designing, by bidders in estimating, and by contractors and field inspectors in planning construction. For this purpose, it is essential that the classification system include some type of shorthand notation to convey maximum information at a glance. All engineering classifications described by the author are so designed, except for one variation of the textural system where the words "some, little, and trace" indicate decreasing percentages of size fractions, and require slow reading for understanding.

---

[9] Chf., Soils and Geology Branch, Garrison Dist., Corps of Engrs., Dept of the Army, Fort Lincoln, Bismarck, N. Dak.

For this type of work, the writer feels that a soil classification system should satisfy the following four requirements:

Requirement A.—It should describe the soil in well-understood terms, which convey an idea of its type and behavior.

1. In addition to soil specialists, the terms used should convey the intended meaning to designing engineers, inspectors, geologists, contractors, and courts of law, which last are the recipients of altogether too many disputes, involving boring information versus foundation conditions encountered.

2. The system should make allowance for regional use of certain terms, particularly meaningful in that region, such as glacial till, buckshot clay, caliche, and others.

Requirement B.—The system should furnish an indication of soil properties and performance.

1. It should permit a crude estimate of permeability, compressibility, and shear strength—the three primary properties of a soil. When combined with other boring information (compactness and water table location), the classification should permit a rough estimate of soil performance, such as frost susceptibility, drainage, settlement, slope stability, wheel load capacity, and compactibility (probable range of compacted density and normally suitable compaction procedure).

2. It should be capable of subdivision. Soil is so complex that no classification system can replace tests. Nevertheless, subdivision can often reduce testing by narrowing the range of properties for soils, at least within a particular region. Approximate correlations of soil properties and performance experiences with the indicators of a classification system often can be facilitated by subdivision, particularly when the testing program extends over a length of time sufficient to accumulate experience from past tests. Furthermore, such correlations, even though rough, are often a "must" for rapid handling of a large number of projects whereas necessity frequently requires that testing be limited, or omitted entirely, for smaller projects.

Requirement C.—It should be applicable from visual examination—both in a simplified form and, with experience, in a more refined form.

1. The simplified form should be suitable for determination by boring inspectors from visual examination.

2. The refined form should be such that: (*a*) It can be assigned by a well-trained classifier or chief boring inspector from visual examination, supplemented by occasional check tests; or (*b*) it can be determined from tests in the case of a less experienced classifier. For the latter case, and also so that different laboratories will arrive at the same result, reasonably definite numerical limits are necessary.

Requirement D.—It should employ a simple system of notation for graphic abstracts of boring logs on drawings recording boring information.

1. The notation should register rapidly in the mind while the reader is scanning and comparing plots of several different borings.

2. The drafting effort should be minimized in the preparation of boring drawings, including those suitable for reduction. These are generally of two types: (*a*) Contract drawings to serve bidders, contractors, and construction engineers, and which also usually form one group of exhibits in any court case; and (*b*) study drawings (usually foundation profiles) to serve soils engineers, geologists, and designing engineers in developing the design treatment and in illustrating its adequacy in reports. Unlike study drawings, contract drawings should be made to avoid interpretation, from the legal standpoint that information furnished to bidders should be basic data—not interpretations.

Requirement A, for naming a soil to cover its type and behavior, has been complicated by the practice of using the soil names, "gravel, sand, silt, and clay," to represent, also, grain-size fractions as arbitrarily delineated by one of the grain-size scales in Fig. 2. The terms, "gravel, sand, silt, and clay," describe common soil types, and like all natural soils, each is usually a composite mixture of from two to four of the size fractions designated as "gravel, sand, silt, and clay sizes." Good definitions of these four soil types have been given by H. A. Mohr, M. ASCE (23), and also in dictionaries; and the terms convey about the same meaning to engineers, geologists, contractors, and courts. The general behavior of each of these four soil types was fairly uniformly understood until soil technicians began to confuse themselves and others by using the same words for size-fraction names. This practice has degenerated to the extent that some laboratories assign a name to a soil from the percentages of different size fractions, according to certain charts or classification triangles like those in Fig. 3; sometimes this is done as a routine matter from a grain-size curve by a classifier who never even sees the sample.

If the size-fraction percentages were a reliable measure of soil behavior, this practice might be condoned; but such is far from the case, as the author has indicated, mentioning the common illustration of plastic lean clay and fine cohesionless silt or rock flour of the same gradation. As one example, samples of Manhattan silt having 20% in the clay-size fraction would be designated in some classifications as silty or lean clay. Since this material becomes quick on the slightest provocation, it would indeed be unfortunate for any contractor (and later for the owner in court) who relied on the cohesion of a true lean clay in attempting to excavate below ground water.

Although frequently of the same gradation, classification of loess as silt ignores the fact (and the consequence) that loess often undergoes a serious rapid settlement from collapse of structure under the influence of seepage, whereas silt is seldom so affected. Because of the angularity of its coarse particles and its greater compactness, glacial till generally has greater shear strength, lower permeability, and higher resistance to excavation than a rounded particle gravel of equal gradation. These and other misleading results obtained with textural classifications have convinced the writer of the necessity for requirement A in naming the soil to describe its type and behavior.

Casagrande Classification.—In most usage to date, the Casagrande classification has generally been applied in the form of the group symbols GW to Pt, as shown in Table 4, Col. 3. When so applied, it is actually a plasticity classification for fine-grained soils, and a textural classification for coarse-grained soils. Since it is based on the position and shape of the entire gradation curve, the textural classification phase is superior to the triangle classification, which is based on only three points of the gradation curve. The shape and position of the entire gradation curve have considerable bearing on important soil properties. This fact is widely recognized and is illustrated by many past attempts at better definition—such as the effective size and uniformity coefficient presented by Allen Hazen; the mean diameter and grade line introduced by Frank B. Campbell, M. ASCE (8); and the mean size, fineness dispersion, and curve shape proposed by D. M. Burmister, Assoc. M. ASCE (46)(47).

Hence, although this textural phase of the Casagrande classification is one of the best of the systems described in the paper, it is subject to the general objections to a textural classification when used alone. For example, such unlike materials as gravel, cohesionless till, talus, and caliche would be classed in group GW, and silt and loess in group ML. From the closing paragraphs of the paper, it is apparent that the author is aware of this deficiency (or possible misapplication) of the system and intends that the group symbol should be supplemented with a soil name. The writer is in agreement with this form of the system, as it then meets requirement A.

For furnishing indicators of soil properties (requirement B), the writer considers the Casagrande classification the best of any yet devised in regard to compressibility and shear strength of fine-grained soils. However, its performance has been relatively poor in indicating the permeability of coarse-grained soils. The writer has experienced ranges in permeability in ratios of from 1 to 10,000 and even from 1 to 100,000 which are considered undesirably wide for a single-class group. (Although many past attempts have shown uncertain relationships between permeability and results of simpler classification tests, the lack of success in deriving these relationships is probably explained by the difficulties encountered in permeability test procedure. With more refined testing techniques, considerable data have been accumulated since 1942 (unpublished) which, it is believed, indicate that such relationships do exist; and, of the simpler classification systems available at present, the data correlate best with a master curve system.)

If subdivisions were feasible, they would probably narrow this permeability range, as well as the range in compacted density, which has also been experienced as relatively widespread in certain groups. After several unsatisfactory attempts at subdivision, the writer has joined in the frequently-voiced comment that the Casagrande classification is not easily adapted to subdivision. In discussing possible expansion of the system in Section 14 of the paper, the author offers a scheme of subdivision, partly by the addition of new groups

(GU, SU, KL, KI, and KH) and partly by refinement using boundary groups (GW–SW, GC–SC, and so on) and by use of subscripts m, c, u, w, and p. When subdivision is practiced, it is preferably accomplished by refinement, rather than by addition, because the latter makes less feasible any later attempts at correlating data gathered prior to expanding a classification system by addition. The proposed expansion of the Casagrande classification to represent a soil as $GF_m - S_wF_c$ (to give, possibly, an extreme case with the binder near the boundary between plastic and nonplastic) would seem to indicate (by its complexity) the difficulties of subdivision.

In respect to requirement C, for determination by visual examination, the Casagrande classification is one of the best, and the author's methods for field identification of fine-grained soils in Section 13 of the paper are worthy of widespread application. However, for checking between different laboratories, it is desirable to have numerical limits for the results of usual classification tests, without the necessity of transmitting a sample for examination. Such limits are also of considerable value for checking classification of a soil by tests, especially in the case of an inexperienced classifier. For fine-grained soils, the Casagrande classification provides such limits, but only to a minor degree for coarse-grained soils.

A good feature of the Casagrande classification is the use of a simple system of notation for boring drawings (requirement D), unless it is complicated by the aforementioned scheme of subdivision. Possibly the best of the simpler systems yet devised for registering quickly in the mind is the system of rectangular and diagonal hatching usually applied with the triangle classification in Fig. 3(*b*), in which heavy symbols indicate the predominant fractions, and lighter symbols denote the two minor fractions of the three main grain-size fractions (48). However, the drafting cost required for suitable reduction is so much greater for this system of hatching (generally requiring inked tracings) that it more than offsets the advantage of quick registering.

QPC Classification.—These difficulties experienced in applying the Casagrande classification, and the greater limitations of the other systems discussed by the author, have led the writer to seek a combination of the better features of several systems. Since 1943, the result has gradually evolved into a combination of soil names from the descriptive classification; a gradation identification from the master curve system started on Quabbin Dam in Massachusetts by Karl R. Kennison and Stanley M. Dore, Members, ASCE (7), and expanded by Frank E. Fahlquist, M. ASCE, and Waldo I. Kenerson into the Providence classification (8); and the plasticity characteristics of the Casagrande classification. The combination, in these three parts, has been tentatively termed the QPC classification after its principal sources (Quabbin, Providence, and Casagrande).

Part 1. Name of Soil Type.—This consists of a soil name, generally with one modifying adjective, selected primarily to describe soil type and behavior rather than gradation. It is chosen from visual examination (preferably on a moist sample) to agree with local semitechnical usage by engineers, geologists, contractors, and courts.

Examples of Soil Names.—The following are typical examples of names, used as applicable to local conditions, with additions as necessary.

Material	Description
Gravel	Variably graded combination of pebbles and sand, sometimes including some silt and clay. Coarser particles, from 6 in. to ⅛ in., at least partly rounded from fluvial action.
Sand	Uniformly graded, bulk of material smaller than ⅛ in., individual particles readily visible to the naked eye.
Silt	Fine grained, lacking plasticity, little or no cohesion when dried, majority of individual grains not visible to the naked eye. Shaking test shows quick or "livery" condition.
Clay	Fine grained, plastic, and cohesive, strong when dried.
Till	Variably graded, unsorted glacial deposit, characterized by angularity of coarser fragments, generally very compact. Indicate whether cohesive or cohesionless (dependent on character of fines).
Loess	Unstratified aeolian deposit, homogeneous powdery soil of silt sizes, structure loose and inclined toward vertical cleavage.
Varved Clay	Alternating thin beds of clay and silt, interstratified, clay-like behavior.
Gumbo	Highly plastic clay, extremely sticky when wet, often a product of weathered shale (localized term).
Talus	Variably sized angular rock fragments, fallen to base of slope, voids seldom all filled.
Decomposed Rock	Structure preserved in place, but chemically disintegrated to soil sizes (with bonding largely destroyed). Also include name of rock.
Coral Sand	Ground-up coral.
Blow Sand (Dune Sand)	Wind-borne, free-moving sand, particles at least partly rounded.
Buckshot Clay	Compact clay, persistently breaking into uniformly small-sized fragments (localized term).
Diatomaceous Earth	Siliceous soil composed chiefly of skeletons of minute organisms.
Topsoil	Weathered surface soil containing organic matter, usually supporting vegetation. Termed loam in some regions.

Examples of Stratum Names Where Grain-Size Number Not Applicable.—

Material	Description
Peat	Partly carbonized vegetable matter, odoriferous, and fibrous. Swamp deposit.
Fill	A man-made deposit. Describe constituents, as cinder fill, rubbish fill, sawdust, ashes, cans, bricks, broken concrete, and so on. Grain-size number used only when soil a major constituent, or when knowledge of gradation desirable, as with slag, cinders, or earth fill.

Examples of Modifying Adjectives.—As frequently used on graphic abstracts of boring logs:

Coarse
Medium
Fine

Gravelly
Sandy
Silty
Clayey

Organic
Micaceous
Cemented

Kaolinitic
Bentonitic
Carbonaceous
Calcareous

Fat } Applied to Clays
Lean }

Generally not more than one adjective is used to supplement the soil name on graphic abstract logs and it is omitted when not particularly significant.

The system is flexible in allowing for the addition of terms of particular regional significance and certain geologic terms well characterizing soil behavior. Devices for obtaining a soil name, such as a classification triangle, have been purposely omitted in order to avoid classification by rote, and its resultant misleading information. For selection of the soil name, guidance of geologic training is helpful; and, for an inexperienced classifier, it might be advisable to add a few remarks to the foregoing definitions to cover behavior similar to those given by H. A. Mohr (23).

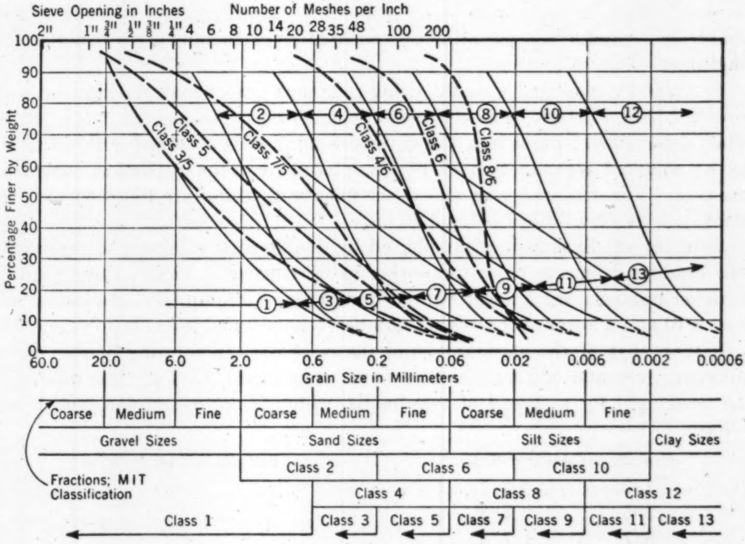


FIG. 8.—LIMITS OF GRAIN-SIZE NUMBER CLASSES

Part 2A. Grain-Size Number.—Part 2A consists of a grain-size number representing position and shape of the entire gradation curve. Thirteen basic classes are recognized as defined by the master curve bands in Fig. 8. Even numbers indicate predominantly uniform gradation; odd numbers indicate predominantly variable gradation.

When a word description is advisable for those unaccustomed to thinking in terms of gradation curves, the thirteen basic classes are defined as follows:

Class	Description of Gradation
1	Graded from gravel to coarse sand sizes with little medium sand.
2	Coarse to medium sand sizes with little gravel and fine sand.
3	Graded from gravel to medium sand sizes with little fine sand.
4	Medium to fine sand sizes with little coarse sand and coarse silt.
5	Graded from gravel to fine-sand sizes with little coarse silt.
6	Fine sand to coarse silt sizes with little medium sand and medium silt.
7	Graded from gravel to coarse silt sizes with little medium silt.
8	Coarse to medium silt sizes with little fine sand and fine silt.
9	Graded from gravel or coarse sand to medium silt sizes with little fine silt.
10	Medium to fine silt sizes with little coarse silt and clay.
11	Graded from gravel or coarse sand to fine silt sizes with little clay.
12	Fine silt to clay sizes with little medium silt and fine clay.
13	Graded from coarse sand to clay sizes with little fine clay.

In these descriptions the terms designate size fractions of the M.I.T. classification. A much better system would be the eventual acceptance of other terms for the size fractions which would not conflict with "gravel," "sand," "silt," and "clay," as soil types. Names from a dead language such as Latin, or the use of Greek letters might be acceptable, or possibly terms such as "coarse," "fines," "superfines," and "colloida." However, any possible terms would need careful consideration for correlation with languages other than English.

Samples are classified visually, using reference jars of materials graded according to the thirteen basic classes, supplemented by occasional check tests. When determined from a test, the grain-size number is obtained with a transparent celluloid reproduction of Fig. 8, placed over the gradation curve in question. For distinguishing visual classifications, these are followed by the letter "v," as "4 v," "7/9 v," and so on.

Primary subdivision is accomplished by double class numbers to represent soils grading from one band to another as illustrated in Fig. 8, thus: "8/6." A higher number over a lower number indicates a more uniform gradation (as class 8/6) and a lower over a higher number a more variable gradation (as class 4/6) than that of the standard band (as class 6). In forming double class numbers, even and odd numbers are never combined. Double class numbers are recognized for a spread of two bands, which is quite common, and is as follows:

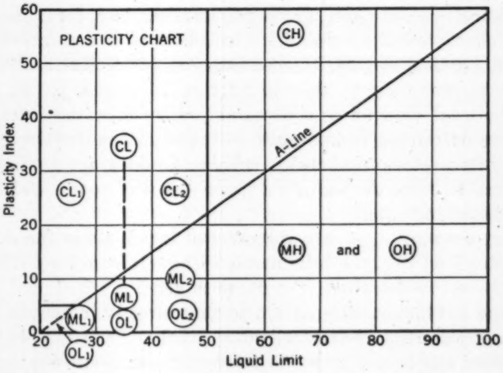
Basically variable gradation		Basically uniform gradation	
1/3	3/1	2/4	4/2
3/5	5/3	4/6	6/4
5/7	7/5	6/8	8/6
7/9	9/7	8/10	10/8
9/11	11/9	10/12	12/10
11/13	13/11		

With this primary subdivision, the thirteen basic classes are expanded to thirty-five grain-size number classes which have been found to cover most needs of the several hundred thousand samples to which this system has been applied.

Rare cases of extremely variable soils grading over a wide range can be handled by recognizing a spread of three bands from lower to higher odd numbers: 3/7, 5/9, 7/11, and so on. However, it is not advisable to reverse this by recognizing a spread in odd numbers of three bands from higher to lower numbers, since the result is one of the uniform basic classes. Thus gradations would not be designated as 7/3 or 11/7; rather these would be noted as classes 2 and 4, respectively.

If further refinement is desired, it may be accomplished by a secondary subdivision to recognize border classes, although the writer has seldom considered this necessary. A border gradation may be defined as one lying near the boundary of two of the adjacent master bands in Fig. 8. Such a gradation is represented as class 4–6 to indicate gradation near the boundary between classes 4 and 6. Other than for attempting close refinement, border classes allow for cases of uncertainty in visual classification, as class 4–6 v.

TABLE 6.—Plasticity-Characteristics Symbols for Fine-Grained Soils with Little or No Coarse Aggregate



Low plasticity When subdivided: L_1 = slight plasticity L_2 = moderate plasticity	L	M C O	Behavior characteristics of silt Behavior characteristics of lean clay Organic content causing plasticity
High plasticity	H	M C O	Behavior characteristics of quite plastic silts, such as micaceous, diatomaceous, or elastic silts Behavior characteristics of fat clay Organic content causing high plasticity, such as organic clays

In such cases of uncertainty, the last number, as "6 v," is chosen as the more probable one. Then, for simplicity, combined class numbers (whether 4/6, 8/6, or 4–6) can always be considered from the standpoint that the last number (in this case, class 6) approximates the gradation—and conservatively so, because it indicates the range of finer sizes. Hence, the thirteen basic classes are the ones to be fixed in the mind. Once these are understood, combined class numbers are easily visualized as variations.

Part 2B. Plasticity Characteristics.—This phase consists of plasticity characteristics symbols taken from the Casagrande system and, likewise, applicable only to fine-grained soils with little or no coarse aggregate. The proper symbol is determined visually as described by the author; or, where Atterberg limit tests are conducted, it is determined from the plasticity chart in Fig. 4. In Table 6 this chart has been redrawn to follow the author's suggestion of dividing the L-group at a liquid limit of 35. However, the writer prefers to retain the original L-group as the simple form (requirement C) and to use the two subdivisions as the refined form designated by subscripts as shown in Table 6.

In application of the QPC system, classification is accomplished visually by examining moist samples—for fine-grained soils, employing the shaking test and examination of plasticity characteristics as the author describes (under the heading, "13. Field Identification of Fine-Grained Soils"). Laboratory tests are employed only when the classifier feels uncertain, or for purposes of checking, or for training an inexperienced classifier.

Boring logs include soil name, grain-size number, and, for a fine-grained soil, the plasticity-characteristics symbol. Also included is a complete description covering color, particle angularity, consistency, plasticity, and dry strength. Such description is similar to that in Section 15 of the paper, except that, considering the legal viewpoint, the writer prefers to avoid interpretation by using adjectives describing compactness. Instead, it is preferred to give basic data without interpretation wherever such basic data are available, as blows per foot required to drive the sampling spoon (sizes of spoon and hammer, and height of fall being indicated).

On contract drawings, graphic abstracts of logs include the symbols aforementioned under Part 2, plus a brief name of the soil from Part 1 for the benefit of bidders. On study drawings, the name is generally included only when it would add some significant information to that already indicated by the grain-size number and plasticity-characteristics symbol. The simpler form (requirement C) includes soil names plus the thirteen basic grain-size number classes with only limited use of border classes. The refined form includes soil names plus grain-size numbers from the thirteen basic classes and from the primary subdivision if applicable, plus plasticity-characteristics symbols.

Summary.—In addition to reasonably satisfying requirements A to D, the features of the QPC system have been demonstrated to be sound from considerable experience with many types of work—airports, highways, dams, levees, and foundations. The use of soil names in Part 1 of this classification follows the oldest system—the descriptive classification. The grain-size numbers (Part 2A) have been used since 1937 with soils from New England to the Gulf Coast, and have been recommended by Joel D. Justin, M. ASCE (49), as well as by others who have given this system a serious trial. As a part of the Casagrande system, the plasticity-characteristics symbols (Part 2B) have been used intensively in airport construction since 1942, and are based on data collected by the author over a much longer period. Taken together, these three phases of the QPC system reasonably cover the results of classification tests for gradation, Atterberg limits, and organic content. For judging the major soil properties, these three classification tests are considered the best currently available when used in conjunction with information on the grain shape and mineral composition. Although additional symbols could be added to cover the last two factors, the writer feels that this can be handled adequately by proper choice of the soil name.

The author is to be congratulated for the excellence of his critical review of the various classification systems, and for devising a classification which is a decided improvement over prior systems. If, from his efforts and those of others, the profession eventually succeeds in evolving a truly adequate system of soil classification, that would, indeed, be a forward step.

GEORGE F. SOWERS,[10] JUN. ASCE.—The purpose of a classification is to make it possible for materials which have similar characteristics to be considered and treated as a group. Those characteristics of immediate interest determine the grouping and the boundaries of the groups. The Airfield Classification (AC) System classifies soils by solid ingredients. Soils whose grains are coarser than 0.1 mm are classified by grain size and grain-size distribution; soils finer than 0.1 mm are classified by the liquid limit and plastic index. These properties serve as guides to the capabilities of the soil as an airfield subgrade, but they may be misleading in other applications, as implied by the author in the "Introduction."

Two properties closely linked with the performance of the soil as a foundation material are not considered in the AC system. These are the water content and the consistency. A soft clay of low "plasticity" may be a worse foundation material than a hard, dried-out clay of high "plasticity." A highly plastic clay is not highly compressible unless it has a high water content. Although the author stresses this fact when he states that compressibility must be compared at the same preconsolidation loads, further qualification should be made. Compressibility is dependent on water content; to ignore this in classification may lead to trouble. The meaning of the terms plasticity and consistency are often confused, and the inclusion of the one factor and the exclusion of the other in the AC system might make the situation worse. To a soils engineer, plasticity refers to the range in water content through which the soil is in a plastic state. However, many engineers and especially contractors and drillers use "plastic clay" to mean just what the name implies—clay in a soft or plastic state. To the man who has to dig the soil, or to build on it, the important point is the consistency of the soil, or its water content with regard to its plastic range. Ignoring these points may be dangerous, especially in the light of the confused terminology.

The greatest difficulty with all classifications is the fact that the soils are grouped according to laboratory tests. The nature of the soil deposit is usually given only secondary consideration, although it is often extremely important. For example, a varved clay consisting of alternate layers of silt and clay probably would be tested in the laboratory as a mixture of both. The classification tests might reveal a sandy or silty clay (perhaps CL) whereas the actual deposit may exhibit entirely different characteristics. Glacial deposits with erratic inclusions of clay, silt, and gravel defy simple classification; their variations in structure are far more important that the characteristics of each individual material.

---

[10] Asst. Prof., Civ. Eng., Georgia School of Technology, Atlanta, Ga.

Loess soils, which cover large areas in the Mississippi Valley, are as hard as concrete in their dry state, but are soft when saturated. These cannot adequately fit into any system of classification that depends on the index tests of soil solids in a disturbed condition. The same holds true for cemented soils, such as the red sands of the south and southwest. The density of natural sand deposits may be more important than the size or size distribution of the grains, and the presence of water only complicates the possibilities.

The discussion of the uses and of the limitations of soil classification systems by the author in the "Introduction" is a sensible approach to this problem. In order to reduce the soil groups to a finite number, many important characteristics must be ignored. Special classifications, such as the AC system applied to airfields, serve an important purpose, but their limitations should be clearly understood. Too often a classification such as this, coupled with such tabulations as Table 4, give some technicians a false sense of security.

The descriptive system in Section 15 has none of these defects because the descriptive terms can be modified to suit conditions at hand. The greatest difficulty with such a classification is a lack of standard terminology. This obstacle need not be condemnatory, for it has been encountered in other fields, such as meteorology and mineralogy. In Section 15, various degrees of each characteristic in the list of descriptive adjectives (as applied to soils, covering such points as consistency, plasticity, and gradation) could be indicated by numbers, similar to Moh's scale of mineral hardness, or to Beaufort's scale of winds. The degrees should be few so as to be distinguishable through the field tests and through the hand identification methods described in the paper, and they should be standardized by appropriate laboratory tests.

The author's plasticity chart (Fig. 4) would be well suited to standardizing the plasticity and the potential compressibility of the fine-grained soils. Consistency could be defined by the unconfined compressive strength as suggested by K. Terzaghi, M. ASCE. On the other hand, engineers are at such great odds as to the meaning of "silt" and "clay" that it probably will be better to use these terms only in a qualitative sense.

In the application of the descriptive method, both the trained soils specialist and the design engineer will have a better picture of the important characteristics of the soil, unhampered by insertion into limited pigeonholes. As Professor Casagrande points out in the "Introduction," "Those who really understand soils can, and often do, apply soil mechanics without any formally accepted classification."

René S. Pulido y Morales,[11] **Esq.**—The subject of soil classification, as stated by the author, is the most confused chapter in the study of soil mechanics. Most people believe erroneously that any soils classification can be used indiscriminately. In order to be successful, the classification must be in accord with the physical properties so that, by using the identification tests, the two most important requisites of compaction and plasticity can be found. It is necessary to know the specific gravity, apparent specific gravity, water content, mechanical analysis, Atterberg limits, and even the elementary chemical tests, so as to establish a relation between the compaction of the soil and the load applied to it.

---

[11] Chf. Engr., Soil Laboratory, Public Works Ministry, Havana, Cuba.

A knowledge of these properties is important, but one must also study the performance of each soil within a given profile for similar work in other places (highways, foundations, dams, and others) so that, once the relations between the tests and their performance are established, the soils can be classified as types.

The research required to correlate the tests with the performance of the finished works can be obtained only after long periods of time. After the completion of a project the tests can be correlated with experience only by extensive research. The first and most important characteristic to be determined is cohesion, which is a function of the resistance of a soil to consolidation. Therefore, it is necessary, and most important, to include the consolidation test in the research program. The granulometric test is also important, and it must be taken into account as well as a measure of the structure as an index type.

The disintegration of igneous rocks will be more or less in an advanced state if the percentage of fine particles is more or less high. Hence, the result of the disintegration state may be determined by the granulometric curve, which is concave upward if the state is advanced, and downward if it proceeds gently. Intermediate grades are represented by a straight line.

The Public Roads (PR) classification has been used to classify for soils for highways. One of the uncertainties of this system is in group A-4 (Table 3), in which, frequently, one finds silty clays with high capillarity; but this is not the general case since silty clays and silty loams, with little cohesion and low capillarity, are also found. Another uncertainty occurs in the elimination of the material retained in the 40-mesh sieve for the determination of the physical properties and also in the exclusion of the physical determinations of the state of compaction of the soil.

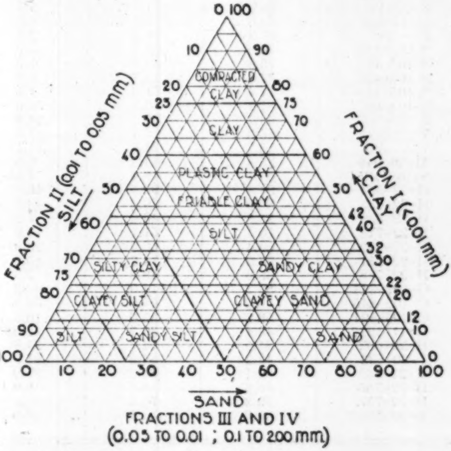


Fig. 9.—Classification by Swiss Engineers

The author reviews many systems of soils classification with great success. The trilinear graphs in Fig. 3 find an interesting comparison in the textural classification of Swiss engineers (50) as illustrated by Fig. 9.

The latest modification of the PR classification, presented for the Highway Research Board to its Committee on Classification of Materials for Subgrades and Granular Type Road (51), is probably the best tool yet devised for the highway engineers. This system eliminates overlapping, subdivides several groups, and introduces the "group index" which is very useful for evaluating the soil within a given group.

In this new classification the silty clay (group A-4 in the previous PR system) has been made A-6, limiting group A-4 to silty materials. The indeterminancy of the previous system has been eliminated by inserting other subgroups.

TABLE 7.—COMPARISON OF PR AND MODIFIED PR CLASSIFICATIONS

Laboratory No.	Percentage passing mesh No. 200	Liquid limit	Plasticity index	PR group	Modified PR
S-108-1662	4.9	24.7	1.9	A-1	A-1-a (0.2)
M-56-1676	37.6	21.8	0.0	A-1-2	A-4 (0.9)
C-102-560	54.4	25.0	8.9	A-2	A-4 (4.3)
H-22-668	34.5	33.1	3.4	A-2	A-2-4 (0.0)
H-22-908	36.6	46.6	8.7	A-2	A-6 (0.0)
H-22-919	36.7	0.0	0.0	A-2	A-4 (0.0)
H-41-1236	29.2	28.0	7.4	A-2	A-2-4 (0.4)
V-85-1319	2.6	30.1	4.6	A-2	A-1-a (0.0
P-14-1381	40.0	32.1	8.5	A-2	A-4 (1.0)
V-70-1401	38.0	20.4	2.6	A-2	A-4 (0.8)
H-25-1448	27.4	34.4	1.3	A-2	A-2-6 (0.6)
P-14-1465	41.8	32.7	11.8	A-2	A-6 (2.0)
M-56-1654	46.8	23.6	3.2	A-2	A-4 (0.3)
H-41-1223	59.4	33.5	7.9	A-2-4	A-4 (5.3)
H-25-1451	26.9	31.6	8.4	A-2-4	A-2-4 (0.3)
M-43-927	33.3	0.0	0.0	A-3	A-3 (0.0)
P-4-806	36.4	26.3	2.7	A-4	A-4 (0.0)
S-105-489	54.6	36.7	5.3	A-4	A-4 (4.5)
S-105-491	75.8	32.5	1.2	A-4	A-4 (8.6)
S-105-492	37.8	25.7	4.6	A-4	A-4 (0.95)
S-105-494	41.4	31.5	7.1	A-4	A-4 (1.4)
S-105-504	81.1	40.1	12.0	A-4	A-6 (9.0)
S-105-506	68.7	39.4	14.8	A-4	A-6 (8.3)
V-72-550	85.3	22.1	17.7	A-4	A-6 (11.4)
V-80-598	15.7	34.6	11.1	A-4	A-2-6 (0.2)
M-59-949	35.8	44.3	9.5	A-5	A-5 (0.0)
S-108-1051	60.1	48.4	7.56	A-5	A-5 (6.4)
H-22-634	94.8	92.5	46.6	A-5	A-7-6 (20.0)
H-22-635	84.9	89.0	43.3	A-5	A-2-5 (0.0)
H-22-636	23.8	49.6	7.8	A-5	A-2-5 (0.0)
H-22-748	70.7	51.1	14.9	A-5	A-7-5 (11.0)
H-22-749	69.4	60.0	23.8	A-5	A-7-5 (17.2)
H-22-751	73.2	74.7	30.8	A-5	A-7-6 (19.8)
H-22-778	63.6	48.2	7.3	A-6	A-5 (7.1)
C-102-844	82.2	114.0	60.3	A-6	A-7-5 (20.0)
C-102-845	8.6	64.8	33.3	A-6	A-2-7 (0.5)
C-109-891	58.2	35.0	10.5	A-6	A-4 (5)
H-41-1222	48.4	39.7	12.9	A-6	A-6 (3.66)
H-22-633	84.0	54.7	21.8	A-6	A-7-5 (15.2)
V-80-593	42.2	84.5	41.9	A-6	A-7-5 (7.1)
H-22-632	82.1	62.0	27.9	A-7	A-7-6 (18.8)
H-22-755	40.8	44.6	8.9	A-7	A-5 (1.2)
H-22-757	39.2	76.8	44.1	A-7	A-7-5 (4.8)
V-80-587	37.6	54.5	22.0	A-7	A-7-5 (15.7)

It is difficult to discuss soil classification for highway engineering purposes without mentioning the California Bearing Ratio (Porter method). This system provides the best way to determine the soil properties for a definite project and, with the help of the modified PR classification, is the more logical method. Table 7 is an example of some soils tests, made by the writer at the Soil Laboratory of the Public Works Department in Havana, Cuba, which have been classified under the PR and the modified PR classification (51) for comparison and also to show the "group index."

In conclusion the writer wishes to congratulate Professor Casagrande on his clear presentation of this soil classification, a problem that is sorely in need of international standardization.

Raymond F. Dawson,[12] M. ASCE.—A stimulating review of the various soil classification systems which have been used is presented in this paper. A new method of classification is also described which will be a real help to engineers working with soils; but, equally important, Professor Casagrande emphasizes the limitations of any soil classification system. Regardless of these cautions many engineers will adopt or continue to use a single classification system for all types of earthwork, and sooner or later this practice will cause difficulty.

A review of the soil investigations published since 1917 reveals an amazing number of classification triangles that have been proposed. It appears that every investigator wished to insure his name for posterity by attaching it to a classification triangle. Actually, these men were attempting to determine all the soils characteristics from the classification triangle, and this was impossible. Instead of seeking new information they continued to juggle old charts, looking for a combination that would be the panacea.

Not until the Atterberg limits were included in the criteria was any real progress made in the classification of soils for engineering purposes. One of the first schemes using the Atterberg limits was the Public Road (PR) classification system, of which the author correctly states (in Section 6): "This classification, therefore, is essentially a grouping applicable to the use of soil in a road surface or as a base beneath a thin bituminous wearing surface." Regardless of this limitation, highway engineers continually use the PR classification system for all types of earthwork construction. Sometime ago, an engineer called the writer's attention to a paper describing a construction method using an A-7 soil in a northern state. He said, "They cannot use that procedure with an A-7 soil; I know, because we tried it." Thereupon he visited the project and found that they were using the method as described. The soil in the northern state had a liquid limit of 42 and a plasticity index of 25, whereas the Texas soil had a liquid limit of 86 and a plasticity index of 50. Both soils were classified as A-7 by the PR system; however, they were two entirely different materials for the construction involved.

The author's Airfield Classification (AC) System is a forward step in the classification of soils for use as subgrades of pavements in airports and similar structures. The AC system will undoubtedly be the standard of classification for subgrade soils in airfield construction.

Another subgrade classification system (51) uses the group index which utilizes both the grading and the Atterberg limits of the soil. To the writer, the group index appears to offer a valuable classification system, although he is of the opinion that modifications of the limit ranges may be necessary. It will be interesting to have the author's discussion of the group index system.

---

[12] Associate Prof., Civ. Eng.; Associate Director, Bureau of Eng. Research, The Univ. of Texas, Austin, Tex.

Professor Casagrande comments (in Section 10) that "Experience has proved that the liquid limit is a good measure of relative compressibility." It is of considerable importance, and further discussion of this statement is desirable.

Engineers must be cautious, or they may be misled by local terminology in describing soil formations. In the Texas gulf coast area, old-time drilling operators call the first clay stratum beneath a sand stratum a "shale" regardless of how soft it may be. Under such conditions the engineer who is unfamiliar with these colloquialisms and who does not examine the cores may not be aware of a serious condition that exists.

D. F. Glynn,[13] Jun. ASCE.—Since a soils classification provides a uniform standard of nomenclature for field identification and for condensing engineering experience with soils, the confusion reviewed by Professor Casagrande's paper is most disturbing. Thus, immediate considerations arising from it concern: "What order could be produced in the existing position?" and "which of the systems described are suitable for incorporation in a coordinated classification?"

Any comprehensive standard classification system would have to meet the needs of agricultural scientists and geologists in addition to civil engineers. Any single system, sufficiently comprehensive to cover all these different interests, would probably be too complex for convenient use. An alternative would be the adoption of a multistage system, the first stage being a simplified field identification and the second a series of detailed functional classifications for each profession. If these were kept as simple logical systems they would form the uniform standards and allow sections with special interests to formulate additional stages to cover their own local interests. It would appear from Professor Casagrande's description that agricultural scientists and geologists already have functional classifications that meet their requirements. Thus the problem is reduced to providing a uniform field identification for general use and to formulating a standard engineering classification.

For a uniform system of field identification, a modified descriptive soil classification would seem very attractive. As a result of wartime cooperation by engineers, geologists, and agricultural scientists in the Southwest Pacific Theater, such a system has already been widely adopted in Australia. This system has sufficiently standardized soil terminology to avoid the disadvantage referred to by Professor Casagrande. It is essentially a field identification system, however, and the ultimate test of the description of any soil is the majority verdict of the experienced field surveyors. In order to keep such opinions up to date and uniform, periodic conferences are held, to which all interested persons are invited. Thus, the formal descriptive standards for adjectives and soil textures are subordinate to field usage and are periodically amended where necessary (52).

An alternative to this simple descriptive standard would be the field identification system based on the Airfield Classification (AC) System. It is difficult for anyone who has not served an apprenticeship with this system to assess its value in practice. Nevertheless, it would seem more complex and less flexible than the simpler descriptive type of classification. As such, it would probably be much harder to reach an international agreement on the basis of this system than with the more widely accepted descriptive classification.

---

[13] Superv. Engr., Soils and Hydr. Testing, Melbourne and Met. Board of Works, Melbourne, Australia.

As a functional engineering classification, the Texas and Civil Aeronautics Administration (CAA) systems would appear too limited in their acceptance. The Public Roads (PR) system is widely used by highway engineers, but in addition to Professor Casagrande's criticism as to the difficulty in keeping abreast of the latest "edition," its basic structure is too closely tied to highway requirements to be a good general system.

From its first appearance in Australia in 1943 the AC system has been regarded by many soils engineers as the most suitable general system. After use by the Corps of Engineers, United States Army, it is probably the most widely adopted of all systems. For these reasons the most recent revision and the discussion on the soils of the various groups are extremely welcome. It impresses one as being simple and easily understood and as following logically from the simpler field description. The principal objection offered in Australia has been due to the lack of definition in the group boundaries. It is suggested, however, that this may be a distinct advantage in that it prevents indiscriminate use by people who have no knowledge of the significance of soil test results.

If the AC system could be accepted as the basic classification to be used for samples, selected from the results of simple descriptive field mapping, it is suggested that the proposed further subdivision of classes should be limited to special applications. The general system, summarized in Fig. 4, would appear sufficient for general use and more detailed subdivision might lead to unnecessary confusion.

Another feature referred to by Professor Casagrande, the grain-size scale, might be also amended to achieve greater uniformity. In Australia an increasing number of laboratories have reverted to the international standard. At a conference of regional officers of the Division of Soils, Council for Scientific and Industrial Research, at Adelaide, Australia, in September, 1946, the sand-silt size boundary of 0.02 mm was discussed (52), in the light of the proposal by R. Glossop and A. W. Skempton (5). Although no change was recommended, it is believed that change to 0.05 mm or 0.06 mm would be readily accepted if sufficiently valid reasons could be offered. On other hand, it is understood that the United States Department of Agriculture is by no means irrevocably committed to 0.05 mm. The time would thus seem ripe for discussion to achieve uniformity on at least this one point.

In view of the proposed international soil conference in Holland, Professor Casagrande's able summary of the present soils classifications is most timely. It is to be hoped that this favorable opportunity for using Professor Casagrande's wide experience in soils mechanics, soils classification, and international cooperation with engineers will not be allowed to pass.

In conclusion, it is desired to thank Professor Casagrande for the comprehensive review of the present state of soils classification and particularly for his discussion on the AC system.

L. F. Cooling,[14] Esq., A. W. Skempton,[15] Esq., and R. Glossop,[16] Esq.—One of the most important, present needs in applied soil mechanics is for a universal acceptance of at least some broad general classification of soils, and in his paper Professor Casagrande has done much to clarify the position in a field where there is admittedly much confusion. The writers met the difficulties involved in this problem a few years ago when they were engaged in preparing drafts for a British code of practice on site investigations. It was soon realized, as stated by the author, that it is not possible to classify all soils into a relatively small number of groups in such a way that the system would adequately meet the requirements of the many divergent problems of applied soil mechanics. However, it was felt that it would be desirable first to adopt a broad classification which would serve as a basis from which, by expansion and amplification within the original framework, the more detailed and specific classification systems required for various practical problems could be built up. For this reason, the approach was to formulate a general basis for the field identification and classification of soils using a system similar to that presented by the author in Section 15. It was considered sufficient for this purpose to take into account two essential characteristics—the size and nature of the particles composing the soil and the strength and structural features of the soil as it exists in the ground. The type of general arrangement suggested is indicated in Table 8.

Six principal types are recognized on the basis of the size and nature of the particles (Col. 1, Table 8), and the simple tests by which these types are distinguished in the field (Col. 2, Table 8) are similar to those described by the author. In this connection the term "dilatancy" corresponds to the property described by the author under "shaking test."

The subdivision of the uniform, well-sorted sands into coarse, medium, and fine is perhaps an elaboration, but it can be readily made and, since the subdivisions can often be associated with important changes in engineering properties, has much to commend it. The division of clays into "lean" and "fat" can be made on the basis of the extent to which the soil exhibits the typical colloidal properties of plasticity, cohesion, and shrinkage. Col. 3, Table 8, gives the more important types of composite soils. These are primarily natural mixtures of two or more of the principal soil types, but mineralogical variations which have an important influence on the soil properties (mica, organic matter, and calcareous material) are also recognized.

Cols. 4 and 5, Table 8, are concerned with the "in-situ" strength and structural characteristics. Since, in most engineering problems, the strength or density of the soil in its natural condition is of controlling importance and since, similarly, the structural features of the deposit (such as laminations and fissures) can be very significant, the writers would stress the importance of their inclusion in any basic classification.

Finally, the color of the soil should be stated since this often has local significance and is helpful for identification. From the terms defined in Table 8, therefore, it is possible to give a fairly accurate picture of a particular soil in a manner that could be generally recognized. The author's suggestion, that with fine-grained soils a note on the effect of remolding should also be included, is a valuable one.

---

[14] In Chg. of Soil Mechanics Div., Bldg. Research Station, Watford, England.

[15] Reader in Soil Mechanics, Univ. of London, London, England.

[16] Engr., Messrs. John Mowlem and Co., London, England.

The writers attach particular significance to the strength classification of clays in all problems except those concerned with near-surface soils, and it would be appreciated if the author would give his opinion on this question. The grouping suggested in Table 8 is based largely on the Boston Building Code with an additional class at each end of the scale. These descriptive terms are implicitly connected with shear strength and, within the writers' experience, they are roughly related in the following manner:

Class	Unconfined compression strength (lb per sq in.)
Very soft	<5
Soft	5 to 10
Firm	10 to 20
Stiff	20 to 40
Hard	>40

From this relation it is possible to develop more specialized classifications for such purposes as foundation bearing capacities and earth pressures.

In cases of near-surface soils and soils used for constructional purposes when the "in-situ" properties are likely to be drastically changed, it is agreed that a classification system leaning more heavily on the composition of the soil is required. Consequently, for site investigations in relation to roads and airfields, the Airfield Classification (AC) System, adopted in 1942 by the United States Engineer Department, was recommended in the aforementioned code of practice.

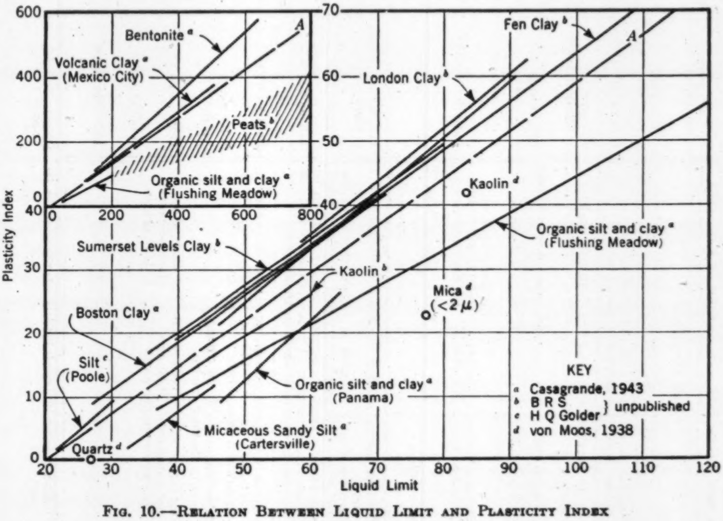


Fig. 10.—Relation Between Liquid Limit and Plasticity Index

TABLE 8.—General Basis for Field Identification and Classification of Soils

Principal Soil Types; Size and Nature of Particles			Composite types	Strength Characteristics		Structural Characteristics	
Divisions (1)	Types (2)	Field identification (3)	(4)	Term (5)	Field test (6)	Term (7)	Field identification (8)
Coarse grained, noncohesive	Stones	Boulders; larger than 8 in. in diameter. Cobbles; mostly between 8 in. and 3 in.	Boulder gravels	Loose	Can be excavated with spade. 2-in. wooden peg can be easily driven.	Homogeneous	Deposit consisting essentially of one type.
	Gravels	Mostly between 3 in. and No. 7 British Standard sieve.	Hoggin[a] sandy gravels				
	Sands	Particles mostly between Nos. 7 and 200 British Standard sieves. Particles visible to naked eye. No cohesion when dry. Uniform Graded Coarse—between Nos. 7 and 25 British Standard sieves. Medium—between Nos. 25 and 72 British Standard sieves. Fine—between Nos. 72 and 200 British Standard sieves.	Silty sands Micaceous sands Lateritic sands Clayey sands	Compact Slightly cemented	Requires pick for excavation. 2-in. wooden peg hard to drive more than few inches. Visual examination. Pick removes soil in lumps which can be abraded with thumb.	Stratified	Alternating layers of varying types.
Fine grained, cohesive	Silts	Particles mostly passing No. 200 British Standard sieve. Particles mostly invisible or barely visible to the naked eye. Some plasticity and exhibits marked dilatancy. Dries moderately quickly and can be dusted off the fingers. Dry lumps possess cohesion but can be powdered easily in the fingers.	Loams (silt, sand, clay) Organic silts Micaceous silts	Soft Firm	Easily molded in fingers. Can be molded by strong pressure in the fingers.	Homogeneous Stratified	Deposit consisting essentially of one type. Alternating layers of varying types.
	Clays	Smooth touch and plastic, no dilatancy. Sticks to the fingers and dries slowly. Shrinks appreciably on drying, usually showing cracks. Dry lumps can be broken but not powdered. They also disintegrate under water Lean clays show these properties to low degree. Fat clays show these properties to high degree.	Sandy clays Silty clays Marls[b] Organic clays Boulder clays Lateritic clays	Very soft Soft Firm Stiff Hard	Exudes between fingers when squeezed in fist. Easily molded in fingers. Can be molded by strong pressure in the fingers. Cannot be molded in fingers Brittle or very tough.	Fissured Intact Homogeneous Stratified Weathered	Breaks into polyhedral fragments along fissure planes. No fissures. Deposits consisting essentially of one type. Alternating layers of varying types. If layers are thin the soil may be described as laminated. Usually exhibits crumb or columnar structure.
Organic	Peats	Fibrous organic material, usually brown or black in color.	Sandy, silty or clayey peats.	Firm Spongy	Fibers compressed together. Very compressible and open structure.	....	....

[a] Hoggin: Sandy gravel with small admixture of clay. [b] Marl: Calcareous clay.

However, it was considered that this classification system could be legitimately regarded as a development and expansion of the basic descriptive soil classification and that it was in the same category as others designed for special engineering purposes, such as foundation bearing capacities, limits of the applicability of various geotechnical processes (5*a*), and so on.

The author's discussion on the properties of fine-grained soils and their field identification is a valuable contribution to the subject. His method of using the liquid and plastic limit values for soil identification by use of the plasticity chart has been followed with interest in Great Britain, and Fig. 10 summarizes results on typical British soils.

A further relationship for fine-grained soils, which seems to the writers to be significant, is that between the liquid limit and the percentage clay fraction—that is, the proportion of particles of a size less than $2\ \mu$. For samples from a given deposit the liquid limit increases with percentage clay fraction and the relation for some typical British soils is given in Fig. 11. Most clays fall within the zone marked "normal clays." However, the kaolins and boulder clays show quite low liquid limits even for fairly high clay contents. This is probably explained by the "inactive" nature of kaolinite compared with the minerals in most clays and by the fact that the particles of the boulder clay have been produced by mechanical comminution and not by weathering. On the other hand, some of the highly weathered clays and organic muds show a high liquid limit even with a comparatively small percentage clay fraction. This may be indicative of a more than usually "active" clay mineral or the presence of unusually fine particles in the clay fraction or, more probably, of a combination of these two effects. In this connection it is interesting to note that the alluvial clay of the River Lea in England has been derived largely from the weathering of the London clay, itself a fairly fat clay, being the product of the weathering and transport of the soil mantle in Eocene times. The indication of the "quality" of the clay fraction which this relationship reveals is considered helpful in assessing the probable characteristics of the soil. As a matter of interest the relationship for seven of the soils given by the author in Table 1, Section 3 are plotted as spot points in Fig. 11. The results agree with the characteristics described with the exception of the diatomaceous earth (No. 5) which, the author states, shows little plasticity although its plasticity index is 42. It would be of interest to know if the diatomaceous earth behaves in an anomalous manner in other respects.

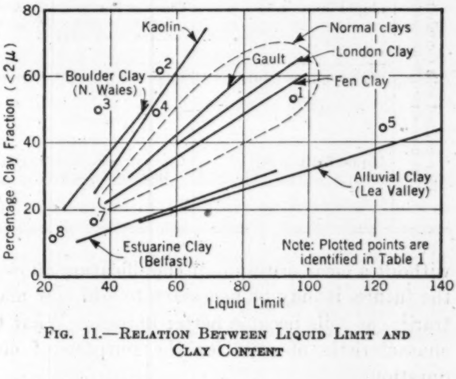


FIG. 11.—RELATION BETWEEN LIQUID LIMIT AND CLAY CONTENT

Milton Vargas,[17] **Jun. ASCE.**—In Section 1 (under the heading, "Development of Systems") Professor Casagrande states that "* * * soil classifications used in civil engineering reflect in their chronological order the development of soil mechanics." This is likewise true of other sciences. Classification systems only reach perfection when the corresponding sciences reach a certain final stage of equilibrium. The existence and success of the Airfield Classification (AC) System shows that this final stage, in the case of soil mechanics, is already in sight.

However, there are some points where explanations are needed. First, in order to be considered perfect, a classification system must be able to divide the things to be classified according to the fundamental characteristics that are significant for the entire class. The AC system begins classifying the soils into two major divisions, according to grain size—the first division referring to coarse-grained soils and the second to fine-grained soils. However, the author was compelled to add a third major division for fibrous organic soils which cannot be classified according to grain-size characteristics. This shows that grain size is a significant characteristic only for a certain class of soils and not generally for all soils. The existence of the third major division without a clear criterion of classification allows the engineer to expect that, in the future, it may be necessary to add new major divisions, more or less arbitrarily, as soils become better known. What then should be the most general characteristic of soils for the purpose of classification? That is an open question.

TABLE 9.—Suggested Subdivision of Soil Groups According to Mineralogical Composition.

1		2		3
Major divisions		Soils groups and subgroups and typical names		Group symbols
Fine-grained soils containing little or no coarse-grained material	Low to medium compressibility Liquid limit < 50	Silts and very fine sands, silty or clayey fine sands, silty clays, slight to low plasticity	Inorganic (except kaolin type)	ML
			Organic	OL
		Clays of low to medium plasticity, sandy clays, silty clays, lean clays	Inorganic (except kaolin type)	CL
			Kaolin type	KL
	High compressibility Liquid limit > 50	Fine sandy and silty soils, elastic silts	Micaceous Diatomaceous Others	MH
		Clays of medium to high plasticity, fat clays	Organic	OH
			Inorganic (except kaolin type)	CH
			Kaolin type	KH

Following the same line of thought, the adopted criterion for separating the major divisions into subdivisions (according to grain-size composition, in the case of coarse-grained soils, and compressibility, or value of liquid limit, in the case of fine-grained composition) appears to be excellent, and the inclusion of intermediate subdivisions among them would do no harm to the system. The same could be said of the groups that divide the major divisions of the coarse-grained soils. However, the groups of the two subdivisions for the fine-grained soils are separated according to two different criteria—grain size and organic matter content. If a new group for the kaolin-type clays is added, as suggested by the author, certainly a third criterion will be introduced—that is, the mineralogical composition.

---

[17] Chf., Soils and Foundations Div., Inst. for Technological Research, Sao Paulo, Brazil.

It would be much simpler, in any future expansion, if the two major divisions of the fine-grained soils were both subdivided into only two groups, that is, the silt group and the clay group, according to grain-size composition. These groups could then be divided into subgroups according to the mineralogical composition and organic matter content, as suggested in Table 9. The systematic framework of the AC system would thus be improved with practically no change. Thus, the way is open for future additions of new groups without the possibility of confusion.

Another requirement for a perfect classification system is that it be suitable for general use under different conditions. A good soil classification must be as useful for the design of airfields as for foundations or tunnels. The author appears to limit his classification system in referring to it as an Airfield Classification System. Its applicability seems to be very much wider.

In his paper the author does not explain whether the group symbols he proposes are intended to be simple initials of the group names, in order that the users may easily remember them, or whether his intentions were to propose symbols of world-wide significance for soils, in the way that chemical symbols represent the chemical elements. In the first case it would be logical that the group symbols should be changed in each language; on the other hand, in order to maintain the universal significance of the symbols they should not be translated. This would be really advantageous for the proper understanding of engineers all over the world—at least as far as soil properties are concerned. The fact that the group symbols are initials of English words would be no impediment when one realizes that the English language is as widely recognized today as Latin was in the days when botanic nomenclature or chemical symbols were established.

DONALD M. BURMISTER,[18] ASSOC. M. ASCE.—A worthwhile contribution to the engineering profession has been made in this critical review of the soil classification systems now in use and in the presentation of the Airfield Classification (AC) System, which served such a useful purpose in the work of the Corps of Engineers, United States Army, during World War II. It is significant that the AC system has been developed on the basis of the important behavior characteristics of soils, and therefore has a broader application than to airfield construction alone.

The greatest practical use of the AC system in the postwar period is in the preliminary exploration, design, and construction stages of projects involving earthwork and foundation problems, where the information obtained is put to immediate and practical use. Those engaged in the soil engineering work on a given project become intimately familiar with the class designations and symbols of the different classes of soils encountered, and they are able to infer accurately the general qualities and behavior characteristics. The group symbols become a convenient tool and expedite the work wherever a large number of samples must be examined and identified. The author has properly warned against the abuse of classification symbols wherein only the group symbol is given without more specific information. Use of symbols alone short-circuits the process and assumes the triple role of identification, description, and classification. After a lapse of some time following the completion of a project such a classification of the soils encountered on a project inevitably becomes more and more vague and less significant as to the actual character and behavior of the soils, because symbols or class designations can never convey a sufficiently precise meaning.

---

[18] Associate Prof. of Civ. Eng., Columbia Univ., New York, N. Y.

For all permanent records of an organization engaged in soil engineering work, as well as for presentation of information in papers and articles in the technical journals, the more precise and significant descriptive classification, such as that discussed by the author, should be used so as to convey a definite meaning to those who have not had the benefit of examining and working with the soils firsthand. The greatest field of progress in soil engineering lies in the accumulation of detailed, accurate information on the physical character and properties of soils and on their behavior under the varied conditions encountered in practice. Thus, a body of authoritative knowledge on soil behavior, which would be of great practical value and use, can be built up, not only in a given organization but also for the profession as a whole. Therefore, in the interest of greatest progress in soil engineering, and of maximum usefulness of data on soil behavior, greater emphasis should be placed on the development and continued use of thoroughness in examining, identifying, and describing soils. These considerations far outweigh any consideration of convenience.

In order to satisfy these essential requirements and needs in the postwar period the base of the AC system should be broadened to make the system more fundamental, and at the same time more precise and significant for all types of foundation and earthwork problems. There are two basic steps in such a process.

The first step is to come to some agreement on the definitions of soil components, which may find general acceptance and which may form a common basis for work in soil engineering. Recommendations have been presented by the Foundations and Soil Mechanics Committee, Civil Engineering Division, American Society for Engineering Education (A.S.E.E.), for the consideration of those interested in soil engineering work (24). These definitions are intended to define terms to be used in identifying and describing soils, and are not a textural classification. The following is quoted from the report of this committee:

> "The definitions of the principal soil components: GRAVEL, SAND, SILT, and CLAY should be based on significant criteria. From an engi-

neering point of view the primary difference between sand and gravel is the size of the constituent grains, which can be recognized visually. The primary differences between sand and silt are that the constituent particles of silt cannot be readily distinguished by the unaided eye and that silt exhibits considerable capillarity. The significant and distinctive difference between silt and clay is that clay exhibits plastic properties and silt does not. In the case of fine-grained soils containing clay the influence of grain size is dominated by the influence of mineralogical and chemical composition. Thus, gravel and sand should be defined on the basis of grain size; sand and silt should be defined on the basis of grain sizes and of capillarity; while silt and clay should be defined on the basis of plasticity. These universally comprehended terms form an adequate and satisfactory basis for the definitions of soil components."

In order to have a workable system of definitions for practical use, it was considered desirable by this committee to define the size limits of the principal soil components and of the coarse, medium, and fine fractions directly in terms of sieve sizes, which convey a definite idea of particle size. The term "CLAY–SOIL" is used instead of "CLAY" in order to be more precise in terminology, because the silt admixture cannot be separated out to determine the proportion of the true clay fraction. The A.S.E.E. recommended definitions are given in Table 10. [Capitalized terms, as "CLAY–SOIL," and soil names are given in quotation marks throughout this discussion to indicate that they are proposed special terminology for describing and naming soils.—Ed.]

TABLE 10.—American Society for Engineering Education Recommendations for Soil Components and Fractions

Principal components (1)	Description (2)	Sieve limit (3)	Sieve Sizes for Subcomponents		
			Coarse (4)	Medium (5)	Fine (6)
BOULDERS and ROCK[a]	Retained on 3-in. sieve	Lower	....	....	3 in.
GRAVEL and STONE[a]	Passes 3-in. sieve; retained on No. 10 sieve	Upper	3 in.	1 in.	⅜ in.
		Lower	1 in.	⅜ in.	No. 10
SAND	Passes No. 10 sieve; retained on No. 200 sieve	Upper	No. 10	No. 30	No. 60
		Lower	No. 30	No. 60	No. 200
SILT	Passes No. 200 sieve; nonplastic, little or no strength when air-dried	Upper	No. 200	....	....
CLAY–SOIL	Passes No. 200 sieve; exhibits plastic properties and clay qualities within a certain range of moisture content; considerable strength when air-dried	Upper	No. 200	....	....

[a] BOULDERS and GRAVEL refer to waterworn material; ROCK and STONE refer to angular fragments.

The second step is to establish a common language for description of soils so that the soils examined and identified can be properly named and described in simple engineering and technical terms. The author points out that at present there is no uniformity in the terms used to describe soils. The descriptive terms, however, should be sufficiently precise and accurate, and at the same time sufficiently flexible in nature, so that the entire range of natural soils can be readily described. The soil name should have a clear, well-defined meaning to enable people who have not had the benefit of examining the soils firsthand to readily understand and to interpret them. These descriptive terms should be suitable, not only for rapid field identification by visual and manual means, but also for the more detailed laboratory identification based on soil tests.

The descriptive terms shown in Table 11, which have been used and found to be satisfactory by the writer, are suggested. Such terms have been used to a considerable extent by field engineers and boring foremen, but without having the precise, well-defined meanings given in Table 11. Although natural soils are composite materials composed of various proportions of the principal soil components in almost infinite variety, recognizable proportions can be defined for practical use. Descriptions of typical soils are given at the end of this discussion to illustrate the use of this terminology in naming soils.

The proportions given in Table 11 cover recognizable ranges, sufficiently obvious for practical purposes. In borderline cases between two proportions either designation is acceptable. The descriptive terms for these recognizable ranges of proportions are used instead of percentages directly because they form a more suitable soil name, which is sufficiently precise for all practical purposes, considering the variation to be expected from sample to sample in the same deposit. The proportions of the gravel and sand components can be readily determined visually; the proportions of the sand and silt components can be determined by feel and by simple manual tests. In the laboratory or in the field office a standard set of bottle samples should be made up for reference. Thus, a person who is not continuously working with soils and identifying them can refer to these standards and check himself, until he has again acquired

TABLE 11.—Descriptive Terms for Cohesionless Soils to be Used in Forming the Soil Name

Soil component (1)	As written in the soil name (2)	Descriptive or qualifying terms as written (3)	Range of proportions (4)
Principal	GRAVEL, SAND, SILT	....	50% or more[a]
		and	35% to 50%[a]
Others	Gravel, Sand, Silt	some	20% to 35%[a]
		little	10% to 20%[a]
		trace	1% to 10%[a]
Subcomponents		coarse to fine	all sizes
		coarse to medium	<10% fine
		medium to fine	<10% coarse
		coarse	<10% medium and fine
		medium	<10% coarse and fine
		fine	<10% coarse and medium

Additional descriptive terms:

(1) Color, grain shape, etc.
(2) Degree of compactness, degree of plasticity.
(3) Inorganic constituents (mica, shells, and foreign matter).
(4) Organic matter (roots, humus, peat, and muck).
(5) Geological origin (alluvial, glacial, wind, beach, swamp, etc., also horizon).

[a] Finer than, or coarser than, the principal soil component.

the sense of identification and of proportions, which really involves seeing and comparing volumes, at the same time giving due consideration to weight and size of grains, and other characteristics.

Such a set of standard bottle samples should be composed of at least fourteen combinations covering the limits of the proportions corresponding to the descriptive terms in Table 11: (*a*) Seven of dry gravel-sand mixtures in the percentage proportions 90–10, 80–20, 65–35, 50–50, 35–65, 20–80, and 10–90; (*b*) three of dry sand of coarse, medium, and fine sizes, respectively; and (*c*) four of moist sand-silt mixtures in the percentage proportions 90–10, 80–20, 65–35, and 50–50. All samples should be at least one pint in volume, and preferably larger.

In examining and identifying soils one should start with the coarsest component and work by stages to the finest component. The proportion of the gravel component and its character as to coarse, medium, and fine fractions should be determined on a fairly large sample. The sand component and finer fractions are then examined to determine the proportion and character of the sand component and of the silt or clay-soil component. Visual means are satisfactory for sands containing not more than a trace of silt. The fine-grained soils or fractions of a soil are more difficult to identify; it is also more important to identify them accurately. Simple manual tests, such as the shaking test for silty soils and the plasticity test (thread rolling test) for clay-soils, can be used satisfactorily for proper identification by adjusting the moisture content to some particular condition. Plasticity and certain clay qualities are the most distinctive and characteristic properties of clay-soils. Therefore, they form a satisfactory and significant basis for identifying and naming these soils. Table 12 lists suggested descriptive terms based on the degree of plasticity of the sand-clay-soil fraction to be used in forming the soil name. These terms have also been used and found to be satisfactory by the writer for a number of years.

TABLE 12.—DESCRIPTIVE NAMES OF CLAY-SOILS BASED ON THE DEGREE OF PLASTICITY[a]

Degree of plasticity (1)	Plastic index (2)	Descriptive name as written (3)	Qualities (4)
Nonplastic	0–1	SILT	Friable
Slight plasticity	1–5	trace Clay	Desirable
Low plasticity	5–10	little Clay	Cohesiveness
Medium plasticity	10–20	CLAY and SILT	Increasingly objectionable plastic displacements and compressibility
High plasticity	20–35	Silty CLAY	
Very high plasticity	>35	CLAY	

[a] Over-all plasticity of sand-clay-soil fraction.

For a field and laboratory identification the over-all plasticity of the sand-silt-clay proportion of the soil as determined by the simple plasticity test is the most significant property, because it has a more direct relation to the behavior characteristics of the soil, and because a separation is not conveniently possible. The gravel sizes can be picked out by hand for the plasticity test.

The range of the degree of over-all plasticity expressed in terms of the plasticity index for each clay-soil type in Table 12 is broadened and adjusted to be more suitable and significant for identifying the soils. The proportion of sand can be determined by the feel of the soil.

The author has given a few examples of descriptive soil classification to illustrate how the AC system can be expanded. The descriptions do not, however, convey a sufficiently definite meaning to a person working in another field of soil engineering, who has not had the benefit of examining the soil firsthand, but who wishes to use the information on the behavior of such soils for the practical purposes of sizing up and judging his own particular problem and for reaching a satisfactory solution. Examples 1, 2, and 3, hereinafter, illustrate some advantages and the general usefulness for all purposes of the descriptive names of soils, based on the terminology in Tables 10, 11, and 12. It is intended that the soil name be put together in a certain definite fashion to emphasize the significant characteristics of the soils examined. The capitalized term in the soil name stands out as the principal component of the soil, so that it can be spotted at a glance. The significant detail then follows, expressed by the qualifying terms giving the proportions of the other soil components, usually in order of their importance. As is frequently necessary one can glance down a boring record, soil profile, or a list of soils from an area, and pick out immediately the types which have certain significant characteristics bearing directly on a given problem. The descriptive names of the soils can be abbreviated somewhat for a large group of similar soils, provided that the upper and lower limits of the group are given in detail.

Example 1.—

(*a*) "Light tan coarse to fine GRAVEL, and coarse to fine SAND, trace Silt." (Maximum size 1½ in.; plus No. 4 sieve, waterworn.)

(*b*) "Light brown coarse to fine GRAVEL, some coarse to fine Sand, trace Silt." (Maximum size 1½ in.; plus No. 4 sieve, waterworn.)

Certain important conclusions can be drawn from the descriptions of these soils based on a general knowledge of the behavior characteristics of different classes of soils. First, the soils will be free draining because there is not more than a trace of silt. Second, the soils ranging from "GRAVEL and SAND" to "GRAVEL, some Sand" (as an abbreviated descriptive classification) represent about the limits of drainable soils that can be considered well-graded. Third, the large gravel content and the large sizes indicate that high supporting capacity can be expected in the compacted condition. The quality of being well-graded is significant only for soils containing a large proportion of gravel, not only because it connotes compactibility to high densities (exceeding 135 lb per cu ft), but also because the close packing and contacts, together with the interlocking and proper bedding of the large particles, provide a high supporting capacity.

Example 2.—

(*a*) "Light brown coarse to fine SAND, trace Silt"; medium compact.

(*b*) "Light brown medium to fine SAND, little Silt"; medium compact.

(*c*) "Light brown medium to fine SAND, some Silt"; loose.

From these descriptions certain significant characteristics are immediately apparent. These soils are nonplastic in character. Soil (*a*) is free draining, has only slight capillarity, and is not susceptible to frost heaving. Soil (*b*) is not as free draining, but can be drained by well points, has some capillarity, and is slightly susceptible to frost heaving with high ground-water levels. Soil (*c*) is more difficult to drain with well points, has rather objectionable capillarity, and has frost heaving characteristics. On the basis of natural compactness, soil (*a*) would be much superior to soil (*c*) in the natural state as to supporting capacity. On the other hand, soils (*b*) and (*c*) would show better compactibility because the silt fills the void spaces and provides a bedding for the sand.

Example 3.—

(*a*) "Brown medium to fine SAND, some Silt, trace Clay"; slight plasticity.

(*b*) "Gray-brown SILT, some medium to fine Sand, little Clay"; low plasticity.

(*c*) "Gray CLAY and SILT, little medium to fine Sand"; medium plasticity.

All these soils are relatively impervious. Soil (*a*) would have fairly good compactibility. The supporting capacity would be only slightly affected by capillary saturation in the compacted state. Soil (*b*) is on the borderline of objectionable plastic displacement characteristics and softening by capillary saturation in the compacted state, particularly with a high ground-water level. Soil (*c*) would compact to a low density and would definitely have objectionable characteristics. However, this soil might have a fairly good supporting capacity in the natural state, if the natural consistency was stiff or better.

The amount of extra effort and time required to identify and describe soils in the more thorough fashion described herein is small when compared to the greater usefulness and value of the information obtained for all purposes, especially to people who wish to use such information to aid the solution of their own particular problems. Such descriptions and soil names, if generally used in boring records and in soil profiles and soil sections, would give far more significant and valuable information to the designing engineer, and, as a part of the contract drawings, to the contractor and construction engineer, than is obtainable at present from the vague and often wholly inaccurate terms generally used. Such information can be interpreted, as noted in the foregoing examples, on the basis of a person's particular experience, into terms of significant behavior characteristics, which would have a more direct bearing on the solution of the practical problems of foundation design and of the practical problems of foundation construction. If a typical soil near the upper limit and another near the lower limit of a group of soils, which have in common similar characteristics, are tested in detail in the laboratory to determine all the pertinent physical properties and strength characteristics, and the results are reported with the descriptive classifications of the whole group, then the range of behavior characteristics to be expected for the entire group can be fairly reliably known. When the actual behavior and performance of the different groups of soils have been observed in sufficient detail in the field and have been correlated with significant soil characteristics, the information would become an important part of a body of authoritative knowledge, which would be extremely useful and valuable to the engineering profession.

M. G. SPANGLER,[19] M. ASCE.—The fundamental purpose of a soil classification system is to provide a language by means of which one person's knowledge of the general characteristics of a particular soil can be conveyed to another person or group in a brief and concise manner, without the necessity of entering into lengthy descriptions and detailed analyses. Since soils are so widely heterogeneous in character, it is not to be expected that one kind of classification system can cover all the possible features of a soil which it may be desirable for one person to convey to another—hence, the desirability of several bases for classification. In engineering work, it is frequently appropriate to use at least several of, or all, four classifications for describing soils, such as: (1) The geological classification, which provides a language for stating the geological history and background of a soil; (2) the textural classification which provides names for describing the relative proportions of sand, silt, and clay size particles in a soil; (3) the pedological system of classification which reveals the nature of the soil profile as affected by the climatic and other environmental conditions under which the profile developed; and (4) an engineering classification that groups soils on the basis of their characteristics which influence engineering performance.

Great care should be exercised not to read into these various classifications more information than they are intended to convey. Particularly, caution is urged against leaning too heavily upon soil classification as a basis for design, since many of the factors that influence design cannot be incorporated into an engineering soil classification. For example, the susceptibility of a road or runway to frost damage is a question that involves not only the properties and characteristics of the soil itself, but climatic and geophysical factors as well.

With reference to the grain-size scales used in various branches of applied soil science, it would be more convenient for civil engineering usage if the arbitrary division between silt size and sand size were established at 0.074 mm (the size of opening of a standard No. 200 sieve) instead of sizes 0.02 mm, 0.05 mm, or 0.06 mm, shown in the various scales quoted by the author in Fig. 2. This is for the purely practical reason that a No. 200 sieve is about the smallest size that can be used in routine soil testing. Sieves with much smaller openings are manufactured, as is well known, but they are much more delicate and expensive and will not wear under rigorous usage as well as the No. 200 sieve. At present, many engineering organizations distinguish between coarse-grained soils and fine-grained soils on the basis of the quantity of material passing the No. 200 sieve, and it would be convenient if this size also marked the boundary indicated by the terms sand size and silt size.

The author mentions the difficulty involved in the fact that textural classifications based on the percentages of sand, silt, and clay size in a soil do not reflect all the physical properties of interest to the civil engineer. The term "texture," as originally used in the United States (53) was not intended to convey any information concerning a soil beyond the relative proportions of the size groups of the material less than 2 mm in diameter. It was not intended to indicate such properties as plasticity, shrinkage, swell, and others. Rather, those additional properties should be expressed by other kinds of classification or by specific test results. The term "texture" is used in some European countries (German *textur*) to embrace properties of soil other than relative amounts of the various size groups, such as soil structure and the form and relative size of the void spaces, but in the United States, so far as this writer is aware, "texture" has always referred only to grain-size relationships. It is the writer's belief that less confusion will ensue if this meaning of "texture" is preserved and emphasized.

---

[19] Research Prof., Civ. Eng. Dept., Iowa State College, Ames, Iowa.

No discussion of triangle textural classification is complete without pointing out that all the information given on an equilateral classification triangle can be shown on a right triangle (54) and that the process of determining the textural class of a soil is much simplified by using the right triangle chart. Since the sum of the percentages of sand, silt, and clay size in the portion of a soil smaller than 2 mm is 100%, the values of only two of these percentages are required to establish a point in an equilateral triangle such as shown in Fig. 3. By virtue of this fact, a right triangle can be constructed, as shown in Fig. 12, on which the percentages of any two of the three size groups can be scaled along the legs of the triangle. The point of intersection of the percentage lines representing the two groups for which the triangle is constructed can be more quickly spotted on such a diagram because of the orthogonal arrangement of the coordinates.

Soils that contain sufficient material larger than 2 mm to be especially noticeable (say, about 8% or 10%) should be qualified by prefixing the term "stony" or "gravelly" to the textural class indicated by the triangle chart, as stated by the author. The principal textural class name of such a soil is determined on the basis of that part of the material smaller than 2 mm. Obviously if a soil contains an appreciable quantity of +2 mm material, the percentage that each of the three smaller sizes bears to the total soil will not be the same as the percentage of these groups in relation to the amount of the sand, silt, and clay only. It is this latter percentage which should be used in the triangle chart classification. The percentages relative to the whole soil can be quickly converted to the percentages of the sand, silt, and clay by multiplying each of the original percentages by the ratio, $\frac{100}{100 - \text{gravel } (\%)}$.

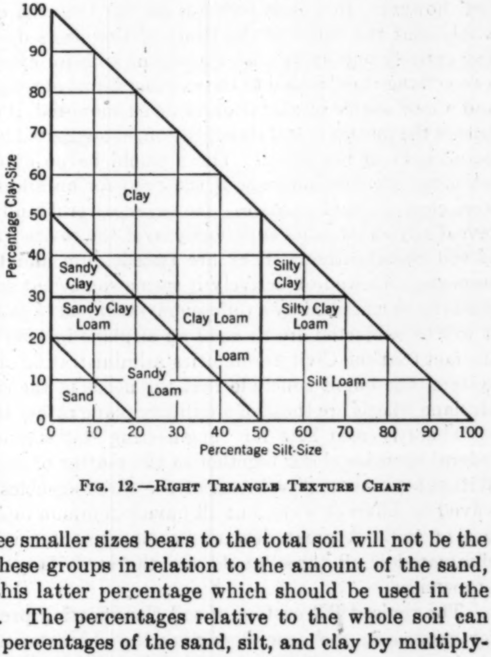


FIG. 12.—RIGHT TRIANGLE TEXTURE CHART

Thus, if the mechanical analysis of a soil shows it to consist of 17% gravel, 26% sand, 36% silt, and 21% clay, the ratio by which the percentages of the three smaller sizes should be increased is 1.2. The new values for these sizes then become 31% sand, 43% silt, and 25% clay. Entering the triangle chart (Fig. 3(*a*)) with these revised figures, the point of intersection falls within the clay loam area and the textural classification of this soil is "gravelly clay loam." If the material larger than 2 mm consists of broken stone fragments instead of rounded gravel particles, the soil would be classed as a "stony clay loam."

If one attempts to use the original percentages of sand, silt, and clay (that is, the percentages referred to the whole soil) three different points of intersection can be obtained on the equilateral triangle chart, depending on which pair of the three sizes is used. In the foregoing example 26% sand and 36% silt gives a point within the area labeled clay; 26% sand and 21% clay gives a point in the silty clay loam area; and 36% silt and 21% clay yields a point in the clay loam area. The necessity for reducing the percentages to the weight of material smaller than 2 mm as a base is apparent.

This writer has no particular criticism relative to the Airfield Classification (AC) System proposed by the author and tentatively adopted by the United States Engineer Department. It is a good system of classification. He does feel, however, that it is unfortunate for the civil engineering profession as a whole that the Office of the Chief of Engineers deemed it necessary to adopt any entirely new system for classifying soils for engineering purposes. Here is a new "language" added to the existing Babel of tongues which must be learned, and a new source of confusion to be surmounted, if an engineer is to avoid pitfalls in the matter of soil classification, as suggested by the author in the opening paragraphs of his paper. There would be no objection to the necessity for learning this new language if the need for an additional classification system were clear cut and positive. However, careful study of the AC system fails to reveal any substantial superiority over the revised Public Roads (PR) System of soil classification. They are tweedledum and tweedledee in many major respects. Viewed objectively it seems somewhat specious to argue that a soil needs to be classified by a different system just because the wheel loads to which it will be subjected are those of an airplane instead of a truck; and, in view of the fact that the Civil Aeronautics Administration also has its own classification system, one might conclude that the needs of the situation are different if the airplane wheels are those of a military plane rather than those of a civil aircraft.

There is need here for "engineering statesmanship" to bring these three federal agencies closer together in the matter of engineering soil classification. Without doubt, the problems of the three agencies are different and must be solved in different ways, but all have a common interest in engineering properties of soil. A great deal of "compartmentation" and confusion could be eliminated if all three would use the same language when discussing those properties.

The revised PR system of soil classification provides an excellent medium for expressing the general engineering characteristics of soils with sufficient clarity for practically all engineering applications and with a degree of precision commensurate with that which can be expected of any classification system.

It has been used by engineers since about 1928, and on an ever widening scale, until nearly all who deal with the soil have a general understanding of the meaning of the symbols A-1, A-4, and others. It is true that local engineering organizations often have had to modify the system to fit their local situations but, by and large, such modifications were readily made within the framework of the basic system, so that the language was essentially preserved.

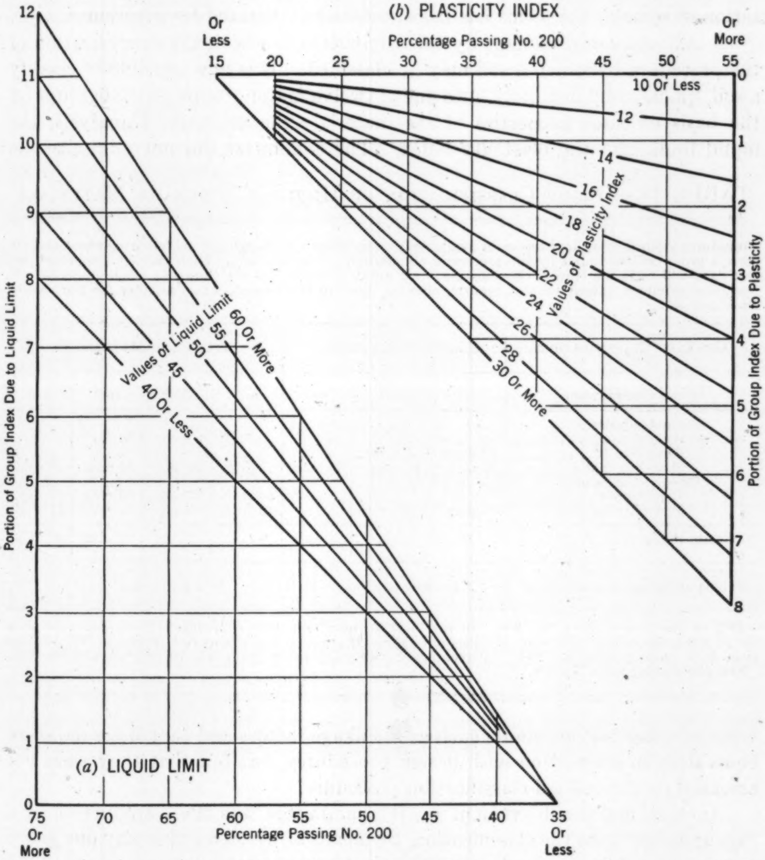


FIG. 13.—GROUP INDEX CHARTS

It is also true that the PR system has undergone a number of revisions since its introduction, which is to be expected in the case of any new tool or device as use demonstrates the need for modification. The author states that the AC system has been revised three times since its inception in 1942. Undoubtedly both systems will undergo further revision as time passes.

The most recent and the most extensive revision of the PR system is that which has been made by a committee of experienced engineers designated by the Highway Research Board, under the chairmanship of Harold Allen, M. ASCE, of the United States Public Roads Administration. The findings of a highway subcommittee of this committee (51) constitute a major revision in the PR system which brings it up to date in the light of many years of experience. This revision retains the basic framework of the system, with changes in detail here and there which are clearly set forth in the report. In other words, the classification language has not been changed, but the meaning and connotation of some of the terms has been modified as dictated by experience.

A major accomplishment by this committee has been the simplification of the procedure by which a soil may be classified. It is now possible to classify a soil, quickly and definitely, into one of twelve major groups and subgroups on the basis of three properties of the material—the mechanical analysis, the liquid limit, and the plasticity index. The committee did not minimize the value of other test constants such as shrinkage factors and centrifuge moisture equivalent in connection with design procedures, but felt that they were not essential in a simplified classification procedure.

TABLE 13.—GROUP CLASSIFICATION OF HIGHWAY SUBGRADE MATERIALS

Procedure: With the required test data available, proceed from left to right in the following chart and the correct group will be found by the process of elimination. The first group from the left into which the test data will fit is the correct classification. All limiting test values are shown as whole numbers. If fractional numbers appear on test reports, convert them to the nearest whole number for purposes of classification.

GENERAL CLASSIFICATION	GRANULAR MATERIALS[a]			SILT-CLAY MATERIALS[b]			
Group Classification	A-1	A-3	A-2	A-4	A-5	A-6	A-7
Sieve Analysis[c]							
Sieve No. 10..........	....	....	....	....	....	....	....
Sieve No. 40..........	50 max.	51 min.	....	....	....	....	....
Sieve No. 200..........	25 max.	10 max.	35 max.	36 min.	36 min.	36 min.	36 min.
Characteristics of:[d]							
Liquid limit..........	....	....	....	40 max.	41 min.	40 max.	41 min.
Plasticity index........	6 max.	nonplastic	....	10 max.	10 max.	11 min.	11 min.
Group index................	....	....	4 max.	8 max.	12 max.	16 max.	20 max.
General rating as a subgrade..	Excellent to good			Fair to poor			

[a] 35% or less passing sieve No. 200. In the "left to right" elimination process it is necessary to place group A-3 before group A-2; but this does not signify that group A-3 is superior to group A-2. [b] More than 35% passing sieve No. 200. [c] Percentage by weight passing sieves No. 10, No. 40, and No. 200. Fraction passing sieve No. 40.

Another major achievement by the committee was the introduction of a "group index" into the classification, by means of which a soil in any one group or subgroup of fine-grained materials may be rated within its own group on the basis of variations in percentages of coarse material, liquid limit, and plasticity index. This group index or "in-group" rating is determined from an empirical formula, which is derived from experience with the original PR classification. It is based on the relationships between the amount of fine-grained material (material passing a No. 200 sieve) in a soil and the liquid limit and plasticity index within the significant ranges of variation of these properties. It can be quickly determined for a given soil by two simple charts which provide a

TABLE 14.—SUBGROUPS CLASSIFICATION OF HIGHWAY SUBGRADE MATERIALS

(Letter M Designates Maximum Values; Letter m Designates Minimum Values)

Procedure: With the required test data available, proceed from left to right in the following chart and the correct group will be found by the process of elimination. The first group from the left into which the test data will fit is the correct classification.

GENERAL CLASSIFICATION	GRANULAR MATERIALS[a]							SILT-CLAY MATERIALS[b]			
Groups	A-1		A-3	A-2				A-4	A-5	A-6	A-7
Subgroups	A-1-a	A-1-b	....	A-2-4	A-2-5	A-2-6	A-2-7	....	....	....	A-7-5 A-7-6
Sieve Analysis:[c]											
Sieve No. 10	50 M	....	....	....	....	....	....	....	....	....	....
Sieve No. 40	30 M	50 M	51 m	....	....	....	....	....	....	....	....
Sieve No. 200	15 M	25 M	10 M	35 M	35 M	35 M	35 M	36 m	36 m	36 m	36 m
Characteristics of:[d]											
Liquid limit	....		....	40 M	41 m	40 M	41 m	40 M	41 m	40 M	41 m
Plasticity index	6 M		nonplastic	10 M	10 M	11 m	11 m	10 M	10 M	11 m	11 m[e]
Group index[f]	0		0	0		4 M		8 M	12 M	16 M	20 M
Usual types of significant constituent materials	Stone fragments, gravel, and sand		Fine sand	Silty or clayey gravel and sand				Silty soils		Clayey soils	
General rating as a subgrade	Excellent to good					Fair to poor					

[a] 35% or less passing sieve No. 200. As in Table 13, group A-3 precedes group A-2; but this does not signify that group A-3 is superior to group A-2. [b] More than 35% passing sieve No. 200. [c] Percentage by weight passing sieves No. 10, No. 40, and No. 200. [d] Fraction passing sieve No. 40. [e] The plasticity index of subgroup A-7-5 is equal to, or less than, the liquid limit −30. The plasticity index of subgroup A-7-6 is greater than the liquid limit −30. [f] The group index is computed by reference to Fig. 13 and is shown in parentheses after the group symbol; thus: A-2-6(5).

Granular Materials—Containing 35% or Less Passing a No. 200 Sieve.—

Group A-1.—The typical material of this group is a well-graded mixture of stone fragments or gravel, coarse sand, fine sand, and a nonplastic or feebly plastic soil binder. However, this group includes also stone fragments, gravel, coarse sand, volcanic cinders, etc., without soil binder.

Subgroup A-1-a includes materials consisting predominantly of stone fragments or gravel, either with or without a well-graded binder of fine material.

Subgroup A-1-b includes those materials consisting predominantly of coarse sand, either with or without a well-graded soil binder.

Group A-3.—The typical material of this group is fine beach sand or fine desert blow sand, without silty or clay fines or with a very small amount of nonplastic silt. The group includes also stream deposited mixtures of poorly graded fine sand and limited amounts of coarse sand and gravel.

Group A-2.—This group includes a wide variety of "granular" materials which are border line between the materials falling in groups A-1 and A-3 and the silt-clay materials of groups A-4, A-5, A-6, and A-7. It includes all materials containing 35% or less passing a No. 200 sieve, which cannot be classified as A-1 or A-3 due to fines content or plasticity, or both, in excess of the limitations for those groups.

Subgroups A-2-4 and A-2-5 include various granular materials containing 35% or less passing a No. 200 sieve and with a minus No. 40 portion having the characteristics of the A-4 and A-5 groups. They include such materials as gravel and coarse sand with silt content or plasticity index in excess of the limitations of group A-1, and fine sand with nonplastic silt content in excess of the limitations of group A-3.

Subgroups A-2-6 and A-2-7 include materials similar to those described under subgroups A-2-4 and A-2-5, except that the fine portion contains plastic clay having the characteristics of the A-6 or A-7 group. The approximate combined effects of plasticity indexes in excess of 10 and percentages passing a No. 200 sieve in excess of 15 are reflected by group index values of from 0 to 4.

Silt-Clay Materials—Containing More Than 35% Passing a No. 200 Sieve.—

Group A-4.—The typical material of this group is a nonplastic or moderately plastic silty soil usually having 75% or more passing a No. 200 sieve. The group includes also mixtures of fine, silty soil and up to 64% of sand and gravel retained on a No. 200 sieve. The group index values range from 1 to 8, with increasing percentages of coarse material being reflected by decreasing group index values.

Group A-5.—The typical material of this group is similar to that described under group A-4, except that it is usually of diatomaceous or micaceous character and may be highly elastic as indicated by the high liquid limit. The group index values range from 1 to 12, with increasing values indicating the combined effect of increasing liquid limits and decreasing percentages of coarse material.

Group A-6.—The typical material of this group is a plastic clay soil usually having 75% or more passing a No. 200 sieve. The group includes also mixtures of fine clayey soil and up to 64% of sand and gravel retained on a No. 200 sieve. Materials of this group usually have a high volume change between wet and dry states. The group index values range from 1 to 16, with increasing values indicating the combined effect of increasing plasticity indexes and decreasing percentages of coarse material.

Group A-7.—The typical material of this group is similar to that described under group A-6, except that it has the high liquid limits characteristics of the A-5 group and may be elastic as well as subject to high volume change. The range of group index values is from 1 to 20, with increasing values indicating the combined effect of increasing liquid limits and plasticity indexes and decreasing percentages of coarse material.

Subgroup A-7-5 includes those materials with moderate plasticity indexes in relation to liquid limit and which may be highly elastic as well as subject to considerable volume change.

Subgroup A-7-6 includes those materials with high plasticity indexes in relation to liquid limit and which are subject to extremely high volume change.

graphical solution for the group index formula. The group index of a soil is reported as a whole number in parentheses after the group or subgroup symbol, as, A-4 (5) or A-6 (12). This method of reporting automatically indicates the "edition" of the PR system which was used to classify a soil.

The classification system embodied in this committee report has been variously called the Highway Research Board system and the group index system. The writer prefers to call it the "Revised PR" system, since that is actually what it is. It is strongly recommended as a complete and comprehensive language for expressing the engineering characteristics of soil material—a system whose terms are widely understood and recognized, which is based on long usage and experience, and which is easy to apply in day to day practice.

The following description of classification groups is extracted from the report, as are Tables 13 and 14 and Fig. 13.

E. W. LANE,[20] M. ASCE.—It is believed that engineers working in soils would be glad to know of the size classification recently proposed by a Sub-Committee on Sediment Terminology of the American Geophysical Union (55) for use in work in sediments as given in Tables 15 and 16. The addition of a discussion of this classification to Professor Casagrande's excellent paper, therefore, seems justified.

TABLE 15.—SCALE OF LARGE PARTICLE SIZES, IN METRIC AND ENGLISH UNITS

Class name (1)	Metric unit (mm) (2)	English unit (in.) (3)
Very large boulders	4,096–2,048	160–80
Large boulders....	2,048–1,024	80–40
Medium boulders..	1,024–512	40–20
Small boulders....	512–256	20–10
Large cobbles.....	256–128	10–5
Small cobbles.....	128–64	5–2.5
Very coarse gravel.	64–32	2.5–1.3
Coarse gravel.....	32–16	1.3–0.6
Medium gravel....	16–8	0.6–0.3
Fine gravel.......	8–4	0.3–0.16
Very fine gravel...	4–2	0.16–0.08

TABLE 16.—SCALE OF SMALL PARTICLE SIZES

Class name	METRIC UNITS	
	(Mm)	(Microns)
(1)	(2)	(3)
Very coarse sand..	2.000 –1.000	2,000–1,000
Coarse sand......	1.000 –0.500	1,000–500
Medium sand.....	0.500 –0.250	500–250
Fine sand........	0.250 –0.125	250–125
Very fine sand....	0.125 –0.062	125–62
Coarse silt.......	0.062 –0.031	62–31
Medium silt......	0.031 –0.016	31–16
Fine silt.........	0.016 –0.008	16–8
Very fine silt.....	0.008 –0.004	8–4
Coarse clay size...	0.004 –0.0020	4–2
Medium clay size.	0.0020–0.0010	2–1
Fine clay size.....	0.0010–0.0005	1–0.5
Very fine clay size.	0.0005–0.00024	0.5–0.24

---

[20] Hydr. Consultant on Chf. Engr.'s Staff, Bureau of Reclamation, Denver, Colo.

The classification is practically a modification of the classification (56) of C. K. Wentworth substituting the term gravel for his terms granule and pebble, subdividing his silt and clay sizes according to the widely used classifications proposed by J. A. Udden (57), and adding subdivisions of Mr. Wentworth's boulder classification. Most of the Wentworth-Udden classification has been widely used among geologists for many years and has been adopted by engineers in many sediment studies, but the granule and pebble classifications were not generally accepted.

The purpose of the Sub-Committee was to obtain agreement among engineers and geologists on a large number of terms in the sediment field, on which there was considerable difference of usage, including that of particle size. The Sub-Committee was composed of engineers and geologists, the engineers being G. H. Matthes, Hon. M. ASCE, L. G. Straub and the late G. C. Dobson, Members, ASCE, C. B. Brown, Assoc. M. ASCE, and the writer. The other members were equally well-known geologists.

The principal advantages of the proposed classification are believed to be as follows: (1) It gives exact subdivisions over the full range of particle sizes of sedimentary materials; (2) the terms used are common ones, and are applied to the range of sizes in common usage; and (3) the lower limit of each size is exactly one half of the upper limit, thus making the division points easy to remember and causing them to plot equally spaced on the semilogarithmic diagrams so widely used to indicate particle size distribution.

This equal size of classes is also a great advantage in the use of statistical methods in treating size ranges. Since the sizes of most sediment samples tend to form straight lines when plotted on logarithmic probability paper, the standard deviation of the logarithm of the sizes indicates the spread of the particle sizes. The composition of a sample can thus be closely represented by two numbers, the median (or geometric mean) and the standard deviation of the logarithm of the size. With these numbers, the composition of a large number of samples can be represented more conveniently than with a large number of size composition curves. It is sometimes possible to analyze a large mass of data by this method in illuminating ways which would be impossible by using the curves. This method of analysis was largely introduced into sedimentary science over a decade ago by W. C. Krumbein and has been widely used. It has more recently been independently introduced into soil mechanics by Donald M. Burmister, Assoc. M. ASCE.

B. K. Hough, Jr.,[21] Assoc. M. ASCE.—One of the many controversial aspects of this significant paper is the distinction between, and the relative importance of, the subject processes of soil classification and soil identification. It is noted that the author gives most attention to classification of soils, said classification being based on soil characteristics which are of various but generally limited interest, and which generally do not include identification. The question is raised herein as to whether identification may not be of major importance in all applications and the basis for more reliable classification and indication of probable soil behavior under service conditions.

---

[21] Associate Prof. of Soil Mechanics, Cornell Univ., Ithaca, N. Y.

It is suggested that such terms as clay and silt, for example, are inadequate for true identification, although they may be sufficient in some applications as classifications. To designate a soil as clay, for instance, is scarcely more satisfactory than to designate one of a number of unidentified construction materials as wood. The differences between clay minerals are as basic and, in some respects, may be as important as the differences between oak and balsa, or pine and cedar. Not only mineralogical, but also chemical distinctions of great importance, notably base exchange capacity (BEC) and type of exchangeable ion, are in some cases of major interest.

Many engineers would agree that identification of soils (especially fine-grained soils) on the basis of their fundamental properties, mineral type, and composition in particular, is sound and desirable in principle but so difficult in practical application as to be of very limited utility, especially in the field. This rather general view may be influenced by misconception of the difficulty of such identification and insufficient knowledge of its practical value. It may now be the case that the practical value can be demonstrated and that identification methods have been simplified to such an extent that in the near future there will be a trend in engineering toward soil identification rather than toward the development of more complex empirical classification systems.

Mineral identification for many years was based on optical methods, which required highly specialized training and experience, usually in the field of geology or physics. For engineering applications this type of analysis has never seemed to be necessary and this may still be true although the increasing evidence of the value of such analysis might justify it. Other methods are available, however, such as the differential thermal analysis method, which can be simplified in application to almost any reasonable degree. Equipment for field use was developed during World War II by the United States Department of Agriculture and is now available for engineering purposes. This equipment can be operated quite satisfactorily by engineers in the junior professional categories. More elaborate equipment for analysis by the same method may be obtained and utilized in soils engineering laboratories without employment of specialists in this field.

The chemical analysis of a soil can also be organized in any reasonably well-equipped laboratory so that it may be conducted by technicians with minimum technical training, and in fact it is now on this basis in many agricultural laboratories. Equipment for field use is also available for certain types of chemical determinations.

In most, if not all soils engineering laboratories, the mechanical analysis is now a standard test and the data from such tests are considered essential in many systems of soil classification. It may well be, however, that in fine-grained soils specific surface is a more significant measurement than gradation or particle diameter, particularly when the latter are determined by such methods as the analysis of sedimentation by the specific gravity hydrometer. This observation is made as an extension of the foregoing comments on chemical analysis, inasmuch as specific surface, although in itself a physical characteristic, is usually indicative of chemical activity. In this respect it is fundamentally preferable to grain-size measurements as the basis for estimating such soil characteristics as permeability, capillarity, plasticity, stability, and many other features which are of engineering interest. Data from hydrometer tests are of little use in such determinations and, in fact, are often accorded no more attention by engineers than a cursory examination despite the considerable cost of performing such tests.

The practical value of the identification of the clay minerals has been more clearly established in recent years. The swelling propensites of the montmorillonite group are generally recognized, although it is still not common practice to make an analysis of mineral types in cases where swelling is of particular importance. Many other correllations between mineral type and engineering behavior may be possible as studies in this area are undertaken. During recent soil tests under the writer's supervision, for example, standard Proctor compaction tests on an unknown clayey soil indicated an optimum moisture content of approximately 30% (dry weight basis). This value considerably exceeded that for any other soil tested by the writer in the course of many years of practical testing experience. Mineralogical analysis revealed the predominant clay mineral to be illite, which with its distinctive potassium bonding may well have characteristics that differ appreciably from those of the more common kaolinite group.

With reference to determination of the type of exchangeable ion, an almost classic example of the practical value of this soil characteristic is the treatment of the Treasure Island (Calif.) Lagoon lining (58) with ocean water to reduce its permeability. None of the soil classification systems described by the author would have indicated the feasibility of this eminently practical and effective treatment. Other examples of the practical importance of chemical reactions in soil in the field of highway engineering are almost too numerous to cite.

In conclusion, it is recommended that efforts be made by engineers to identify soils on the basis of fundamental properties—specifically, mineral type, exchangeable ions, BEC, and specific surface—and that thereafter soils be classified for various purposes. The basic identification will then be valid, no matter what the application or use of the soil may be, and might well serve as the common denominator for the many diversified interests now engaged in soil study and research.

ARTHUR CASAGRANDE,[22] M. ASCE.—The discussions have rounded out the contents of the paper by the addition of valuable material and of many helpful suggestions, for which the writer is grateful.

There is one misunderstanding noticeable in several of the discussions which is caused by the brevity of paper—namely, that the purpose of the Airfield Classification (AC) System is merely to classify the raw material of which the soil is composed, without regard to the engineering properties of the material in its undisturbed state. This interpretation is definitely not the writer's intent. All the data listed in Cols. 1 to 6, Table 4, must be considered as an integral part of the complete classification. Cols. 1, 2, 3, 4, and 6 classify the material in the "disturbed state"—that is, the raw material. In Col. 5 are listed characteristics which refer to the engineering properties of the material in the condition in which it will be used by the engineer. Since Table 4 is prepared specifically for the use of the AC system in airfield projects, the observations and tests listed in Col. 5 are selected for this purpose. When dealing with other types of engineering projects, Col. 5 will obviously require revision. No formal grouping has been proposed for the data which are to be included in Col. 5. However, it is conceivable that, for specific engineering projects, the observations and tests relating to the material in place could be subjected to further classification.

---

[22] Prof. of Soil Mechanics and Foundation Eng., Graduate School of Eng., Harvard Univ., Cambridge, Mass.

Several of the discussions have pointed out that in many cases a geologic classification of the soils should be included in the general classification, and that use should be made of local soil names. The writer agrees with these views and suggests that another column should be added in Table 4 with the heading, "Geologic Description and Local Name." In this connection it should be noted that there is also some danger in the use of local soil names because in different parts of the United States they may refer to materials differing greatly in their engineering characteristics. For example, the term "hardpan" has led to numerous lawsuits in which the lawyers had no difficulty in finding in the literature definitions which suited their individual purposes. It seemed to the writer that the older the quotation the more weight it carried in the Court of Law.

Professor Fadum offers several practical suggestions for promoting agreement on soil terminology. The writer agrees with Professor Fadum that it would be more significant if the Atterberg limits were performed on material smaller than 0.1 mm in diameter, rather than on the fraction passing the No. 40 sieve. Since separation of the fraction smaller than 0.1 mm is cumbersome and time consuming, general acceptance of such a suggestion is not probable until a much faster procedure is developed for the separation of this fraction.

Colonel Stratton reviews the circumstances leading to the development of the AC system, and demonstrates in Fig. 7 how the system was combined with the design procedure for flexible pavements, which was developed under his direction by the Office of the Chief of Engineers, United States Army. Then he reviews the comments made in 1942 by various divisions and districts of the United States Engineer Department—that is, soon after the AC system was adopted as a tentative standard by the Corps of Engineers for military construction. More recently, a modification was developed by the Office of the Chief of Engineers for civil works purposes, based on the experience gained in the use of the AC system since its inception.

Professor Belcher presents a competent discussion on the advantages of the pedological classification. It is true that the thoroughness of exploration on which a pedological classification of a given locality is based results in information which is of value to the engineer, and that much time and effort can be saved by utilizing such available information. However, the writer does not believe that the pedological classification is suitable in general for engineering purposes.

Messrs. Haine and Hilf describe an adaptation of the AC system which is used by the Bureau of Reclamation for use with earth dam projects (59). They emphasize that proper field identification is the primary objective in soil surveys, and found it necessary to subdivide the GF and SF groups according to the type of fines. This step is logical, particularly in connection with the use of soils for the construction of rolled earth dams. In addition, Messrs. Haine and Hilf have rated the various groups of materials according to their usefulness in the design of earth dams—a helpful supplement when using this classification for earth dam projects.

Mr. Feld has brought a welcome diversion into this rather prosaic subject, by reviewing some of the history of soil classification.

Kenneth S. Lane first presents an analysis of the requirements which a classification system should fulfil, and then proposes a new system referred to as QPC (Quabbin, Providence, and Casagrande) classification, in which he combines the advantages of several systems by using a descriptive name, a grain-size number, and the group symbol of the AC system. The writer is in full accord with Mr. Lane's discussion of typical soil names, with emphasis on the geology of the soils. The use of a grain-size number is a desirable supplement in classifying many coarse-grained soils of a great variety, such as are encountered in glacial deposits. In such instances, a man could be trained to estimate the grain-size class by visual inspection. However, when working in areas where coarse-grained soils are rather limited in extent, the necessary skill for a fair differentiation between thirty-five grain-size classes will usually not be available. If high accuracy in classifying the grain-size distribution is needed, one will have to depend on laboratory analyses. Because of Mr. Lane's extensive experience, his recommendations deserve serious consideration by those who wish to use a comprehensive method for classifying soils.

Professor Sowers points out the difficulty resulting from the fact that the term "plastic" is used by builders and drillers to designate the consistency of a soil in its natural state. He also emphasizes the importance of a detailed description of the properties of the undisturbed material and suggests the use of numerical scales to replace the uncertainty of descriptive adjectives. The writer believes that in visual and manual examination only a small number of grades can be distinguished, which are better described by generally understood adjectives. However, there is no question that a quantitative basis should be established for the purpose of standardizing the use of the adjectives.

Mr. Pulido y Morales expresses preference for the modified Public Roads (PR) Classification. He compares (Table 7) the group symbols for forty-four different soils according to the original and the modified PR classification, which to the writer demonstrates the confusion that must result if both classifications, which differ so greatly, make use of the same group symbols.

Because of space limitation the writer must forego a detailed discussion of the group index in the modified PR classification, and of the relationship between the liquid limit and compressibility, which Professor Dawson desires.

Mr. Glynn points out that in Australia a descriptive soil classification has been widely used as the basis of field identification. He suggests that, to prevent confusion, further subdivisions in the AC system should be limited to special applications. The writer is inclined to agree with this suggestion.

Many of the features which are insufficiently presented in the AC system are very well covered in the classification proposed by Messrs. Cooling, Skempton, and Glossop in Table 8. Although specifically adapted to the soil conditions in England, Table 8 may well serve as a general basis for establishing similar classifications in other parts of the world. In Fig. 11, the relationship between the liquid limit and the clay fraction (percentage smaller than 2 microns) is plotted, to illustrate that, for two clays having the same liquid limit, the clay with the more potent colloidal content has the smaller clay fraction. Whether such a relationship will yield information in addition to what can be derived from the plasticity chart, is questioned. Besides, difficulties arise when one also plots the results of materials with high liquid limit which lie well below the A-line, such as organic clays, or micaceous and diatomaceous soils.

Mr. Vargas suggests that the organic and inorganic silts may be grouped together into a major division. The writer questions the value of such a combination, since organic silts and inorganic silts are really quite different types of soils.

Mr. Vargas raises the question as to whether the group symbols, which are abbreviations of English names, should in other languages be changed to the initials of the corresponding names. He answers his question by suggesting that the original symbols should also be retained in other languages. This would certainly prevent confusion. The writer admits that in devising the symbols he has not thought about the possibility that this classification might spread to other countries. The purpose of the group symbols was to give the user something that would not tax his memory. Thus, it would be logical to create new symbols in each language. On the other hand, if this classification should become widely used, it would be desirable to have only one set of symbols which are internationally accepted. There is already one symbol in these groups which is not derived from the English language, namely, the letter, M. Considering that the majority of important references in soil mechanics are, and probably in the future will be, in the English language, and that English is the only language used for the International Soil Mechanics Conferences, the adoption of the present symbols as an international standard would not be too great a hardship. It would mean that, in teaching the classification, the student would have to learn the English names of about one dozen soils. After he has mastered these words, the use of these symbols would be just as understandable to him as if the symbols referred to the names of these soils in his own language.

Professor Burmister proposes to broaden the AC system by a better definition of the principal soil components, and by establishing a common language in describing soils. His discussion contains many valuable suggestions, which merit attention in any future improvements of existing soil classifications.

Professor Spangler has presented a competent defense of the new PR system. By including sufficiently detailed information on this new system, he has contributed a necessary supplement to the paper. Professor Spangler expresses regrets that other organizations deemed it necessary to adopt an entirely new system, when the PR system would have been satisfactory. However, that the original PR system was not satisfactory was admitted tacitly by those who created the new PR system, after the AC system was already in existence. In fact, it seems to the writer that, except for the symbols, the new PR system is closer to the AC system than to the 1929 PR system. Therefore, the writer reiterates that it would have been less confusing and more straightforward if the new PR system had received a new and more logical set of symbols, instead of retaining the original symbols.

Professor Spangler misunderstood the writer's remark regarding the revisions of Table 4 to mean that the AC system has been revised three times since its inception in 1942. Some minor revisions were made in Cols. 12 to 15, Table 4, which are not an inherent part of the AC system.

If none of the existing methods for identifying and classifying soils for engineering purposes makes use of their mineralogical and chemical composition and surface area, the reason is simply that the very complex relationships between these characteristics and the engineering properties of soils have not been established to any useful and practical extent. The writer, however, agrees with Professor Hough that more research along these lines is desirable.

E. W. Lane discusses a new grain-size classification recently proposed by the American Geophysical Union. In soil mechanics there has not developed, so far, any need for such detailed grain-size classifications, and the writer doubts that they will ever be needed.

The writer should like to point out that the soil classifying procedures actually used by the Bureau of Reclamation and by the Corps of Engineers, as well as by the Bureau of Yards and Docks, United States Navy (the latter according to written information received from L. A. Palmer), differ only in minor points. It would seem to the writer feasible and eminently desirable for these three large users of soil mechanics to initiate a cooperative effort, which should eventually lead to a uniform soil classification. If such an effort succeeds, it would constitute the most important step toward resolving order from the chaos of the past.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2352

ESTIMATING DATA FOR RESERVOIR GATES

By Frank L. Boissonnault,[1] Esq.

With Discussion by Messrs. Theodore B. Rights, George R. Latham, D. A. Buzzell, A. E. Niederhoff, Joseph R. Bowman, and Frank L. Boissonnault.

Synopsis

Charts for the determination of weight and cost data on some of the most common forms of water control gates used in conjunction with dams are presented in this paper. The types included are: High-pressure hydraulic operated slide gates, radial crest gates, roller crest gates, drum crest gates, and Broome gates.

There are some published data on weights and costs of gates, but in most cases the exact items included are not known. Where tables of weights are given (in other publications), some gates may include much more conduit lining than others of the same type. Also, some may include considerable auxiliary equipment such as valves and piping, whereas others in the same table do not. In the case of radial gates, sometimes it is not clear whether or not the hoist or even the anchorages and other stationary parts are included. If adequate information is not given, no data are available for easily approximating the necessary addition for a complete estimate. Therefore, the utility of such data is limited, and it is believed that charts, such as those presented in this paper, fill a definite need.

Introduction

The charts in this paper were developed to assemble accurate information on weights and costs of existing gates in convenient graphic form so that they could be compared easily and so that the weight of any gate could be interpolated quickly from the nominal gate dimensions and maximum head on the gate center. The intent is to present all the pertinent data required to enable the engineer to use his own judgment in making an intelligent estimate of weight to fit the individual case, without the necessity of tedious research work each time an estimate is required. To accomplish these objectives with curves, in each chart an expression (involving only the nominal gate dimensions

---

Note.—Published in September, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Engr., U. S. Engr. Office, Los Angeles, Calif.

and maximum head as variables) is developed, some power of which will be as nearly as possible proportional to the weight of the gate. The weights and the costs (in some cases) were then plotted as ordinates against this expression as abscissas. Where the evaluation of the expression involves more than a simple slide rule operation, other curves are given to assist in the evaluation of this function. The dimensions are in feet in all cases.

The various dams mentioned in the text or noted in illustrations are further summarized and identified, as follows:

Dam	Location
Structures of the Bureau of Reclamation—	
Alamogordo	New Mexico
Alcova Tunnel	Wyoming
American Falls	Idaho
Black Canyon	Idaho
Boulder	Colorado River, Nevada
Caballo	New Mexico
Deadwood	Idaho
Easton	Yakima Project, Washington
Friant	Central California
Grand Coulee	Central Washington
Grassy Lake	Wyoming
Guernsey	Wyoming
Island Park	Idaho
Jackson Gulch	...
Madden	Panama Canal Zone
McKay	Northeastern Oregon
Norris	Tennessee
Owyhee	Oregon
Salinas	Southern California
Shoshone	Wyoming
Stony Gorge	Central California
Taylor Park	Colorado
Tieton	Yakima Project, Washington
Structures of the United States Engineer Department—	
Beach City	Ohio
Brea	Southern California
Caddoa	Arkansas River, Colorado
Conchas	New Mexico
Dover	Ohio
Fullerton	Southern California
Hansen	Southern California
Kanawha No. 1	West Virginia
Lytle Creek	Southern California

Dam	Location
Structures of the United States Engineer Department (*Continued*)—	
Mississippi dams 5A, 9, 10, 11, 13, 14, 16, 18, 21, 22, 24, 25, and 26	Upper Mississippi River
Prado	Southern California
Santa Fe	Southern California
Sepulveda	Southern California
Wills Creek	Ohio
Wolf Creek	Cumberland River, Kentucky
Structures of the Appalachian Electric Power Company—	
Claytor Development	Radford, Va.
Structures of the Southern California Edison Company—	
Florence Lake	Southern California
Structures of the City of Pasadena—	
Morris	Southern California

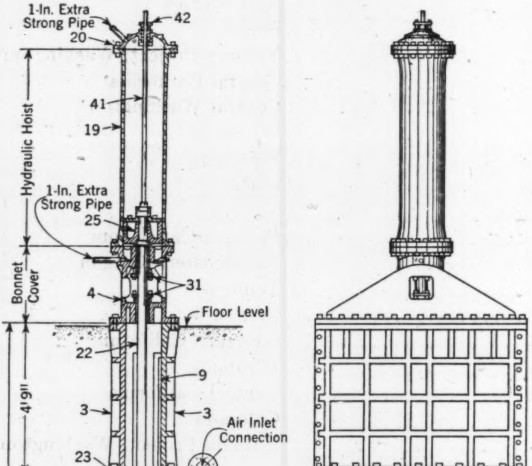

1-In. Extra Strong Pipe
42
20
41
19
Hydraulic Hoist
1-In. Extra Strong Pipe
25
31
4
Bonnet Cover
Floor Level
22
9
4'9"
3
3
Air Inlet Connection
23
10'6½"
By-Pass Alternate Location
10
By-Pass Alternate Location
3'2½"
7
23
8
5'0" (Height of Gate)
5
Center Line, Downstream
Center Line, Upstream
11
3'9"
6
1
2
4'0"
4'0"
5'0"
5'0"
6'8½"

FIG. 1.—ASSEMBLY DRAWING OF A STANDARD HIGH-PRESSURE SLIDE GATE (5 FT BY 6 FT) AS BUILT BY THE BUREAU OF RECLAMATION (SEE ALSO TABLES 1 AND 2)

MAJOR PARTS OF HIGH-PRESSURE SLIDE GATE SHOWN IN FIG. 1

Gate:	
Upstream frame	1
Downstream frame	2
Bonnet	3
Leaf	5
Seats	6, 7, 8, 9, 10, 11
Stuffing glands	31, 42
Bonnet cover	4
Hoist:	
Cylinder Piston—	
Stem extension	19, 25, 41
Stem and stem nuts	22, 23
Upper cylinder head	20
Conduit:	
Lining	(not shown)
Lining transition	(not shown)

DESIGN DATA

Maximum Designed Head, for Gate Leafs, in Feet:	
Cast iron	90
Semisteel	140
Cast steel	250
Pressure in Cylinder, in Pounds Per Square Inch:	
Working pressure	750
Designed pressure	1,000
Coefficient of Friction at Starting:	
Bronze gate seat	0.6

The concrete surrounding conduit and bonnet castings are designed to carry all the load.

HIGH-PRESSURE SLIDE GATES

Fig. 1 shows an assembly drawing of a typical high-pressure slide gate, so called because, in moving, the leaf slides on its sealing strips. The hoist is a hydraulic cylinder and the greatest part of its capacity is required for overcoming the seal friction created by the static pressure of the water against the upstream face of the leaf. The cast-iron conduit lining (not shown) is bolted to the flange on the body or "frame" of the gate and the entire assembly, including the bonnet, is embedded in the masonry structure up to the bonnet cover, which, with the hoist, projects into the gate chamber as shown.

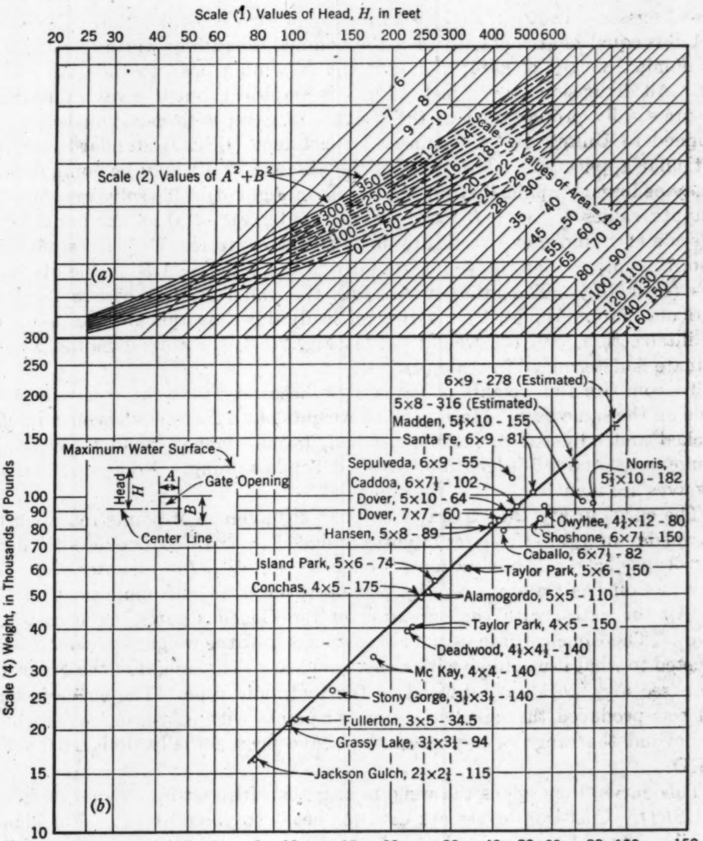


FIG. 2.—WEIGHT OF HIGH-PRESSURE SLIDE GATES WITH HYDRAULIC HOISTS

The weight curve of Fig. 2 is plotted with the weight of the gate as ordinates and the expression—

$$F(W) = \frac{(\text{area})}{10^5} (A^2 H + B^2 H + 6{,}420 \sqrt{H} + 30{,}000) \ldots\ldots\ldots (1)$$

—as abscissas. In Eq. 1, $F(W)$ is a function of the weight W; "(area)" is equal to $A\ B$ is equal to area of gate; A is the nominal breadth or width of gate; B is the nominal height of gate; and H is the maximum head to the gate center line. All dimensions are in feet. Eq. 1 is empirical, but it gives values that are practically proportional to the weights of gates with cast-iron leaves, as designed to United States Engineer Department (USED) standards for the particular application. The value of the abscissa $F(W)$ may be computed or it may be found graphically from the curves in Fig. 2(*a*). To solve graphically, compute values of $(A^2 + B^2)$ and $(A\ B)$. Enter Fig. 2(*a*) at maximum head H (scale (1)), and move vertically down to the curve for $A^2 + B^2$ (scale (2)). From this intersection, move horizontally to the value of $A\ B$ on the diagonal lines (scale (3)). The intersection gives the value of the abscissa for that particular gate. Hence, by moving vertically down on this abscissa value to the intersection with the weight curve (Fig. 2(*b*)), the corresponding weight ordinate is determined in scale (4).

To compare the weights of gates, it is necessary that the comparison be made on the same basis. In Fig. 2 the weights of all gates have been carefully checked and adjusted to include the leaf, frame, bonnet, bonnet cover, hydraulic hoist, and a 5-ft length of cast-iron conduit lining. No hanger, piping, or valves are included.

The curve in Fig. 2(*b*) is drawn through the average of points representing the weights of a range of sizes of gates designed by the USED offices for flood control projects. The notation "Conchas—4 × 5—175" denotes that the gates at Conchas Dam are 4 ft by 5 ft operating at a maximum head of 175 ft. All the gates, with the exception of the Conchas gates, have cast-iron leaves. The Conchas gate leaves are cast steel, but the weight shown has been adjusted to what it would be with a cast-iron leaf. The weight of the Fullerton gates (see Fig. 2(*b*)) was adjusted to the hydraulic type. The curve in Fig. 2(*b*) was produced through two estimated points with cast-iron leaves which are beyond the range of any gates that have been actually designed by the USED.

This curve, then, gives the weights of gates with cast-iron leaves designed by USED. Cast-iron leaves are used for heads to about 100 ft. For higher heads, cast-steel leaves would be used and the gates would be somewhat lighter. However, the unit cost of the steel casting is considerably more than that of cast iron, so that the over-all cost of the gate would be approximately the same as that for the cast-iron leaf gate shown. Therefore, Fig. 2(*b*) gives the weights, for cost purposes, of gates designed to USED specifications, although for heads greater than 100 ft, where the leaf is cast steel, the actual weight will be from 2% to 3% less.

The USED gates shown in Fig. 2 are those for Fullerton, Conchas, Hansen, Dover, Caddoa, Sepulveda, and Santa Fe dams. They were designed for each job, on the basis of a coefficient of sliding friction of 1.00 as set up by the Office of the Chief of Engineers for flood control gates. This coefficient is probably higher than actually exists but, no doubt, was felt to be justified for flood control gates which may not operate under head for years, thus accumulating much corrosion on the sliding surfaces. However, when at last needed during a flood, they must not fail. The normal working pressure in the hoist cylinder is 750 lb per sq in. with a maximum of 1,000 lb per sq in. by the setting of a relief valve.

All other gates shown are designs of the Bureau of Reclamation (USBR). All except Madden, Norris, and Owyhee dams are part of a more or less standard line of slide gates, each size of which is used over the entire range of heads by changing the material in the leaf and applying the proper one of several sizes of hoist cylinders designed to fit mechanically. For heads to 90 ft, cast-iron leaves are used; for heads from 90 ft to 140 ft, semisteel leaves are substituted; and, for heads from 140 ft to 250 ft, cast steel is standard.

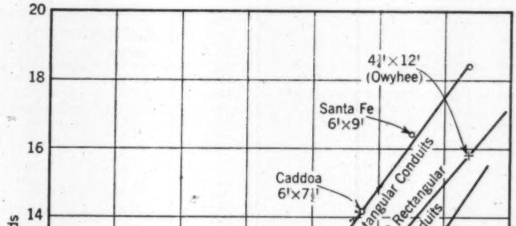


FIG. 3.—WEIGHT OF CAST-IRON CONDUIT SECTIONS

These standard gates are not closely designed for actual heads, therefore; and, when the actual head is used in plotting, they could not be expected to follow a smooth curve as well as if each was designed for an individual condition. Nevertheless, these USBR gates are generally lighter than the flood control gates because the USBR uses a coefficient of sliding friction of 0.60—requiring a hoist and stem only 60% of the capacity of the USED hoists for the same standard working pressure of 750 lb per sq in. In addition, most of the USBR gates shown, have cast steel or semisteel leaves which still further reduce the weight in comparison to the flood control gates. However, the trend of weights in the standard line is roughly proportional to the function $F(W)$ as defined by Eq. 1, with an exponent slightly less than 1.00. On the other hand, the Norris and Madden gates are considerably lighter than would be expected from the trend of the other USBR dams. This deviation is explainable by the fact that the gates were of special size and shape and were specially designed so that the weights are very nearly the most economical for the particular conditions. Also, the normal working pressure on the hoists of these gates is increased to 1,000 lb per sq in., whereas the maximum pressure available for "cracking" the standing friction is at least 50% greater than that of the flood control gates, thereby allowing a further saving in the weight of the hoist for this reason.

TABLE 1.—UNIT COST OF SLIDE GATES, IN CENTS PER POUND

Dam	Location	Year	Number of gates	Size (ft)	Cost bid (¢ per lb)
Hansen	Los Angeles, Calif.	1939	8	5 by 8	14.3
Sepulveda	Los Angeles, Calif.	1940	4	6 by 9	13.4
Stony Gorge	Orleans, Calif.	1926	2	3.5 by 3.5	14
Conchas	New Mexico[a]	1936	12	4 by 5	17.55[b]
Conchas	New Mexico[a]	1939	4	6 by 7.5	12.2[c]
Deadwood	Idaho	1929	2	4.5 by 4.5	15
Alamogordo	New Mexico	1936	2	5 by 5	10
Grand Coulee	Los Angeles, Calif.	1939	..	8.5[d]	15.1
Santa Fe	Los Angeles, Calif.	1946	16	6 by 9	27.25
Bull Shoals	Arkansas	1946	32	5 by 9	31.4[e]

[a] The gates for Conchas Dam were built in Los Angeles. [b] After adjusting this unit cost to west coast delivery, the cost is 16.25¢. [c] This price was too low and should not be considered as a legitimate bid. [d] Ring seal gate, 102 in. in diameter; unit cost is F.O.B. the shop at Los Angeles. [e] After adjusting this unit cost to west coast delivery, the cost is 29¢.

Conduit Lining.—It is anticipated that varying lengths of conduit lining will be required in some cases. Fig. 3 has been developed so that the weight of extra conduit lining can be found. This chart gives weights of 4-ft and 5-ft lengths of rectangular sections and also of 4-ft lengths of circular sections of cast-iron conduit lining. It should be understood that the unit cost used with extra conduit should be lower than the unit cost for the gate. Equations for the weights of conduit sections, in pounds (that is, the equations of the curves in Fig. 3), are: For the 4-ft length of rectangular conduit—

$$W = 1{,}151\,(A + B) - 3{,}445 \qquad (2a)$$

for the 5-ft length of rectangular conduit—

$$W = 1{,}338\,(A + B) - 3{,}943 \qquad (2b)$$

and, for the 4-ft length of circular conduit—

$$W = 370\,D^{1.731} \qquad (2c)$$

Costs.—The approximate shop cost of the gate can be estimated by multiplying the weight by a suitable unit cost for cast iron, machined and fabricated complete. This unit cost will vary with the general economic situation and with conditions in the particular shop. Some idea of the unit costs at which gates of this type have been sold or offered in the past is given in Table 1. The unit cost will also vary with the number of duplicate units involved. Table 2 gives an approximate idea of how the unit cost is thus affected. The table arbitrarily assumes 100% unit cost for the case of ten gates and shows the percentage unit costs for other gates. The data are based on a study of the fixed costs of eleven different jobs as bid by a large manufacturer in the Los Angeles (Calif.) area. This study showed a considerable range between the highest and lowest percentage of fixed costs to total costs when adjusted to a comparable number of gates. The table is arranged to show the effect of this range of variation from the average. In choosing the unit cost factor for a given number of gates, the following general principles should be used as a guide. The fixed costs tend to be higher than average if:

(*a*) The gates are small in size;

(*b*) The costs are quoted when pattern costs are relatively high compared to other costs, as in 1946 postwar conditions; and

(*c*) A large number of gates are involved.

TABLE 2.—VARIATION OF PERCENTAGE UNIT COST OF SLIDE GATES FOR DIFFERENT NUMBERS OF GATES IN CONTRACT

Number of gates involved	RELATIVE FIXED CHARGES IN THE STUDY		
	Lowest	Average	Highest
(1)	(2)	(3)	(4)
1	114	127	134
2	106.3	112	117
3	103.8	106.8	109.7
4	102.5	104.4	106.4
5	101.6	102.9	104.2
6	101	102	102.8
7	100.6	101.3	101.9
8	100.35	100.7	101.1
9	100.1	100.3	100.5
10	100	100	100
11	99.8	99.7	99.6
12	99.7	99.5	99.25
13	99.6	99.3	98.95
14	99.5	99.2	98.7
15	99.4	99	98.5
16	99.4	98.9	98.3
17	99.35	98.8	98.15
18	97.3	98.7	98
19	99.25	98.6	97.8
20	99.2	98.5	97.75
21	99.2	98.45	97.65
22	99.15	98.4	97.6
23	99.1	98.3	97.5
24	99.05	98.25	97.45
25	99.03	98.2	97.4
26	99	98.15	97.35
27	99	98.1	97.3
28	98.95	98.1	97.2
29	98.95	98.05	97.2
30	98.9	98.05	97.15

To obtain the installed cost, transportation cost to the site and erection cost must be added. Erection cost will vary with the location conditions. For average conditions before World War II it was between 3¢ per lb and 4¢ per lb.

RADIAL CREST GATES

The radial or Tainter gate is quite often the most economical type of spillway gate, where it can be used. The length of Tainter gates is usually limited to from 50 ft or 60 ft by the expansion joint spacing in the dam, although gates 80 ft long are in operation at dam No. 24 on the Upper Mississippi project. A service bridge with sufficient headroom is required for the hoisting equipment. Fig. 4, with Tables 3 and 4, gives the weights (including the moving and stationary parts) of a number of radial gates plotted as ordinates against the function:

$$F(W) = \frac{A^2 B H}{10^4} \qquad (3)$$

These weights do not include the hoist and chains. All gates are noncounterweighted for electric hoists except as noted.

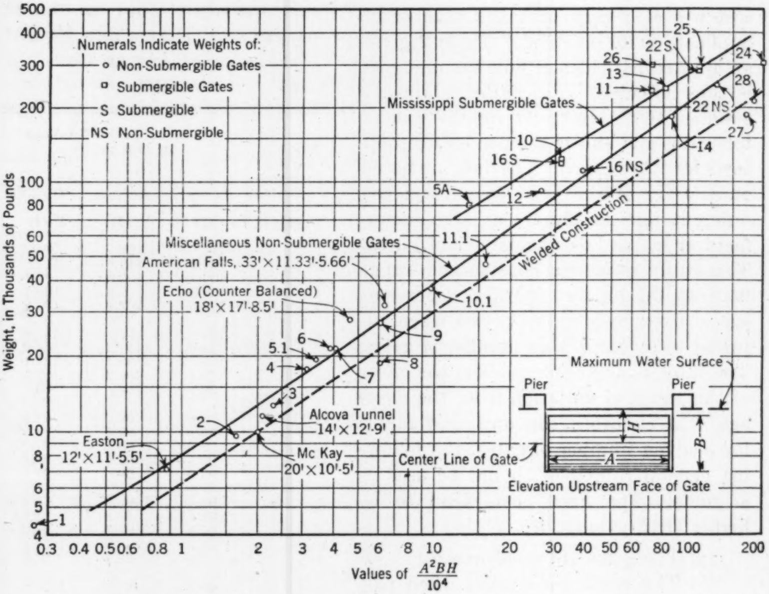


FIG. 4.—WEIGHTS OF RADIAL GATES, MOVING AND STATIONARY PARTS, EXCLUDING HOISTS AND CHAINS

TABLE 3.—UNIT COST OF RADIAL GATES, IN DOLLARS PER HUNDREDWEIGHT, 1939 DATA (PACIFIC COAST DELIVERY[a])

Gates[a]	COMPLETE[b] WITHOUT HOIST AND CHAINS		COMPLETE[b] WITH HOIST AND CHAINS	
	Range	Average	Range	Average
Mississippi gates[c]	\$ 8.70 to \$13.78	\$11.80	\$13.05 to \$17.55	\$13.65
Smaller gates	\$11.00 to \$16.37	\$13.40	\$11.60 to \$21.10[d]	\$16.12

[a] The unit costs in this table are for gates of the conventional riveted construction. The unit costs for welded construction are probably slightly higher. [b] The weights include all moving and stationary parts, with exceptions noted. [c] The freight allowance, on Mississippi gates, from the Mississippi River to the Pacific coast, is \$0.75 per hundredweight. [d] Higher figure due in part to high installation cost.

The information on the Mississippi gates was taken from the contract drawings and specifications and tabulated data made up by the designing offices of the USED. The first twelve items of Table 4 were taken from the work of W. P. Creager and Joel D. Justin,[2] Members, ASCE. Other data were obtained from USBR Specifications 400 to 1,200 and from other publications of USBR.[3,4]

TABLE 4.—IDENTITY, DIMENSIONS, AND UNIT COSTS (DOLLARS PER HUNDREDWEIGHT, WITH HOIST) OF RADIAL GATES AT THE DAMS REPRESENTED BY THE PLOTTED POINTS IN FIG. 4

Point No. (Fig. 1)	Dam[a]	DIMENSIONS (FEET)			Year	1939 unit cost
		A	*B*	*H*		
1.......	....[b]	6	12	6	....	....
8.......	....[b]	13	13	27.5	....	....
2.......	....[b]	15	12.2	6.1	....	....
7.......	....[b]	16	9.5	16.7	....	....
3.......	....[b]	18	12	6	....	....
5.1.....	....[b]	20	13.1	6.5	....	....
4.......	....[b]	25	10	5	....	....
9.......	....[b]	25	14	7	....	....
10.1.....	....[b]	25	14	11.3	....	....
6.......	....[b]	35	8	4	....	....
11.1.....	....[b]	20	20	20	....	....
12.......	....[b]	50	14.7	7.3	....	....
5A.....	5A	35	15	7.5	1935	14.08
16NS....	16[a]	40	22	11	1934	12.09
16S.....	16	40	20	10	1934	....
10.......	10[c]	40	20	10	1935	12.59
14.......	14[a,c]	60	22	11	1937	12.3
11.......	11[c]	60	20	10	1935	13.3
26.......	26[c,d]	40	30	15	1935	13.65
22NS....	22[a,c]	60	27	13.5	1936	16.75
22S.....	22[c]	60	25	12.5	1936	15.97
25.......	25[c]	60	25	12.5	1937	14.87
13.......	13[c]	64	20	10	1937	12.44
28.......	Caddoa[d,e]	64	30	15	1940	15.7
24.......	24[c,d]	80	25	12.5	1938	14
27.......	Wolf Creek[e]	50	37	18.5	1939	16.9

[a] All gates were submergible except 16NS, 14, and 22NS, which are nonsubmergible. [b] These data are from "Hydro-electric Handbook," by W. P. Creager and Joel D. Justin, John Wiley & Sons, Inc., New York, N. Y., 1927, p. 317. Identity of dams and other comparable information were not given. [c] Upper Mississippi River navigation dams. [d] Alloy steel. [e] Welded.

The Mississippi gates are designed for navigation control. For this reason, many of them are submergible—that is, they are arranged for lowering to allow ice and debris to go over the top. Consequently, the submergible gates are heavier than spillway gates, and are shown in a separate curve in Fig. 4.

Except at Caddoa and Wolf Creek dams, all the gates in Fig. 4 are of conventional riveted construction. In the most recent designs, substantial savings have been made in both weight and cost by the use of welded construction. The Caddoa and Wolf Creek gates are of this type. A chart prepared by A. E. Niederhoff,[5] Assoc. M. ASCE, gives the weights of radial gates representative of this type of construction. The dashed weight curve, designated "Welded Construction" in Fig. 4, is plotted to give weights in line with Mr. Niederhoff's chart. Not many gates of welded construction have been built, but the calculated weights of recent designs seem to show fairly close agreement with the "Welded Construction" curve (see Fig. 4). It is possible that better methods of analysis may effect further savings in the weights of this type of gate.

Cost.—The cost of the gate structure can be obtained by taking the weight from the appropriate weight curve and applying a suitable unit cost, estimated from the unit cost data given in Tables 3 and 4—with due regard for the type of construction to be used, location of the installation, and factors peculiar to that particular installation that would affect shop costs or erection costs, or both. The cost of the hoist must be added to obtain the total cost of the gate.

---

[2] "Hydro-electric Handbook," by W. P. Creager and Joel D. Justin, John Wiley & Sons, Inc., New York, N. Y., 1927, p. 317.

[3] "Dams and Control Works," Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo., 1st Ed., 1929.

[4] "Dams and Control Works," Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo., 1938.

[5] "Selection of Tainter Gates," by A. E. Niederhoff, *Western Construction News*, December, 1943, p. 555.

The weight and cost of the hoist can be obtained from the chart in Fig. 5. Since the weight and cost of hoists are plotted against the hoist capacity, it is

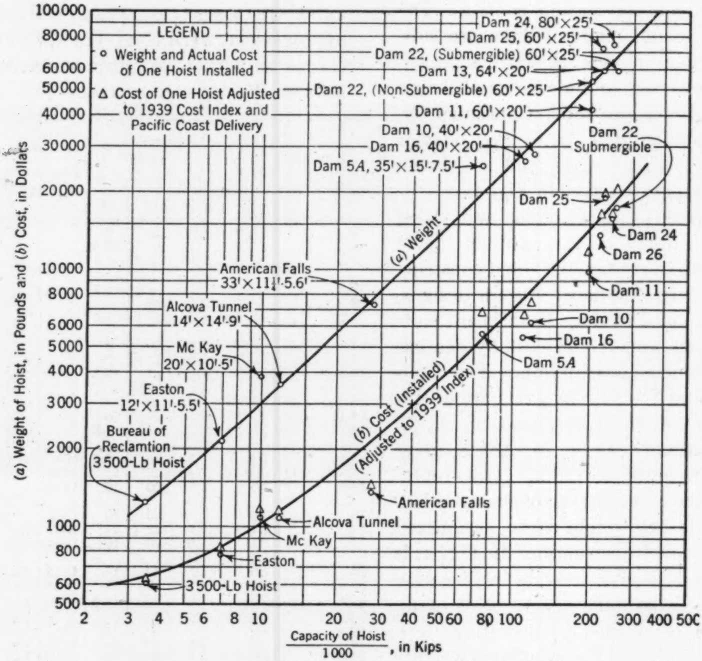


Fig. 5.—Weight and Cost of Radial Gate Hoists

first necessary to estimate the maximum hoist pull required. The hoist pull on the Mississippi gates varies considerably because of differences in ice and silt loading. The weight of the moving part of the leaf on these riveted gates averages about 77% of the total weight, whereas for welded construction the leaf may be nearer to 70% of the total weight as given in Fig. 4. To obtain a rough estimate of the required hoist capacity, it will be on the safe side to take the chain pull due to dead load of leaf, chain, and friction at the point of maximum pull as about equal to the weight of the leaf. To this estimate must be added the ice and silt loading allowed for the particular application. On the Upper Mississippi gates this allowance varies from 30% to 45% of the dead load. In many localities no allowance for ice load is required. The unit cost of the hoist may be taken from the auxiliary curve in Fig. 6.

Roller Gates

The roller gate (sometimes called rolling gate) consists of a steel cylinder with a large pinion at each end, which engages racks built into each abutting pier, and the gate is raised by rolling up these inclined racks. Because of the high section modulus per unit weight, inherent in a cylinder, the roller gate can be built for long spans, thus making a saving in the number of piers required. Also, for the same reason, it is practicable to hoist the gate from one end only. The piers and pier foundations, however, must be larger than those required for radial gates with shorter spans.

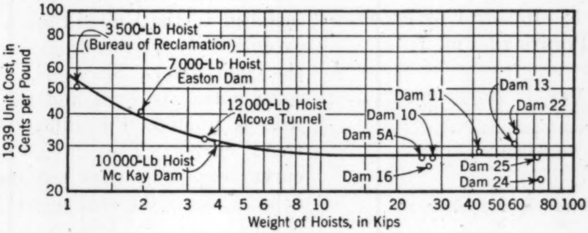


FIG. 6.—AUXILIARY CURVE SHOWING 1939 UNIT COSTS

Although not many roller spillway gates are used, in a complete economic study on a project where a long spillway or wide channel is involved, it is usually necessary or desirable to consider this type of gate against a greater number of radial or other types of gate with much shorter spans. For this reason it is desirable to include weight and cost curves on the roller gate in this paper, although applications (and also available data) are somewhat limited.

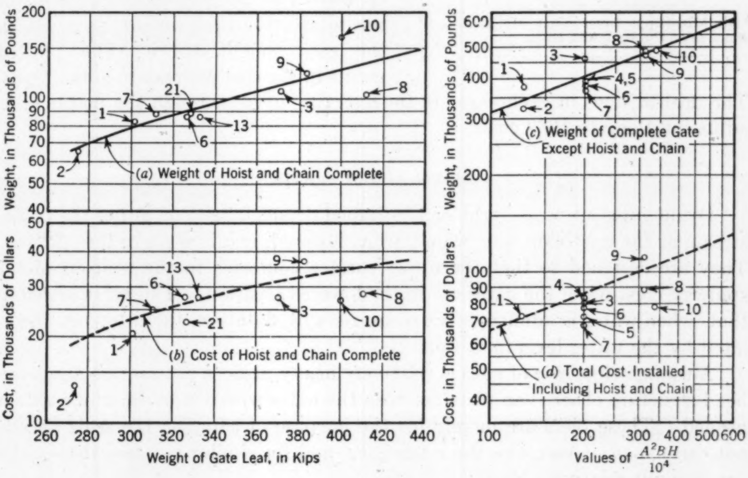


FIG. 7.—WEIGHTS AND COSTS OF ROLLER GATES AND HOISTS

Fig. 7 shows the weights and cost of roller gates and roller gate hoists. The weight and cost of the gates are plotted against the function $F(W)$ as defined by Eq. 3. The identity of the plotted points in Fig. 7, including dimensions (in feet) and other data, are given in Table 5.

TABLE 5.—IDENTITY OF PLOTTED POINTS IN FIG. 7.

Point (Fig. 7)	Dam No.[a]	B (ft)	H (ft)	Year
		(a) WIDTH $A = 80$ FT		
1......	9	20	10	1936
2......	16	20	10	....
3......	26	25	12.5	1935
		(b) WIDTH $A = 100$ FT		
4......	15	20	10	1938
5......	21	20	10	1936
6......	14	20	10	1938
7......	18	20	10	1935
8......	22	25	12.5	1936
9......	25	25	12.5	1937
10......	1	26	13	1935

[a] All Mississippi River dams except No. 10, which is on the Kanawha River.

Except for one dam on the Kanawha River (point No. 10, Table 5) these gates are all on navigation dams on the Upper Mississippi River. They include the majority of gates of this type installed in the United States, with the exception of a number of similar gates on the Ohio River. Since only very large roller gates are built and the range of sizes is limited, it is difficult to draw trend curves. The "weight of gate" curve (Fig. 7(*c*)) does not include the hoist, whereas the cost curve (Fig. 7(*d*)) is for the gate and hoist complete. The plotted points show the actual cost.

The hoist curves show the weight (Fig. 7(*a*)) and cost (Fig. 7(*b*)) of the hoist and chain plotted against the weight of the moving part or leaf. The average weight of the moving parts may be estimated roughly as about 83% of the total structural weight, as given in the gate weight curve. The plotted points in the cost curves are actual costs. The dashed lines in each case are the average curves through the points after adjusting them to 1939 costs.

DRUM CREST GATES

Drum gates are watertight sector-shaped drums hinged on either the downstream or the upstream side, which, when lowered, rest in a pit in the spillway. They are designed so that the upper surface conforms to the contour of the spillway crest when the gates are in the lowered position. The pit is sealed so that, when water is admitted, the drum rises by flotation, the rate being regulated by the water level in the pit.

Although this type of gate is relatively heavy, it does not require any overhead structure other than piers between the gates where more than one gate is needed. These piers are merely for the purpose of carrying the seals and do not carry load. Also, like the roller gate, and for similar reasons, these gates can be built relatively long.

The weights and costs of representative medium and large-size drum crest gates are given in Fig. 8. The weight and costs are plotted as ordinates against the function $F(W)$ as defined by Eq. 3. Weights are for the complete gate including all anchorages, hinge castings, and other stationary parts as well as piping and apparatus for automatic control. The costs are for the complete gate installation, except in the case of Salinas gate. Adjusted unit costs (1939 basis) are as follows:

Dam (Fig. 8)	Cents per pound	Remarks
Tieton	13.2	....
Sepulveda	11.45	Hinged downstream
Florence Lake	25	Long truck haul into high mountains
Black Canyon	15.1	....
Guernsey	13.27	....
Easton	12.2	....
Boulder	10.91	....
Salinas	15.7	Includes 3.1¢ per lb for installation
Grand Coulee	11.78	....
Morris	....	Hinged downstream

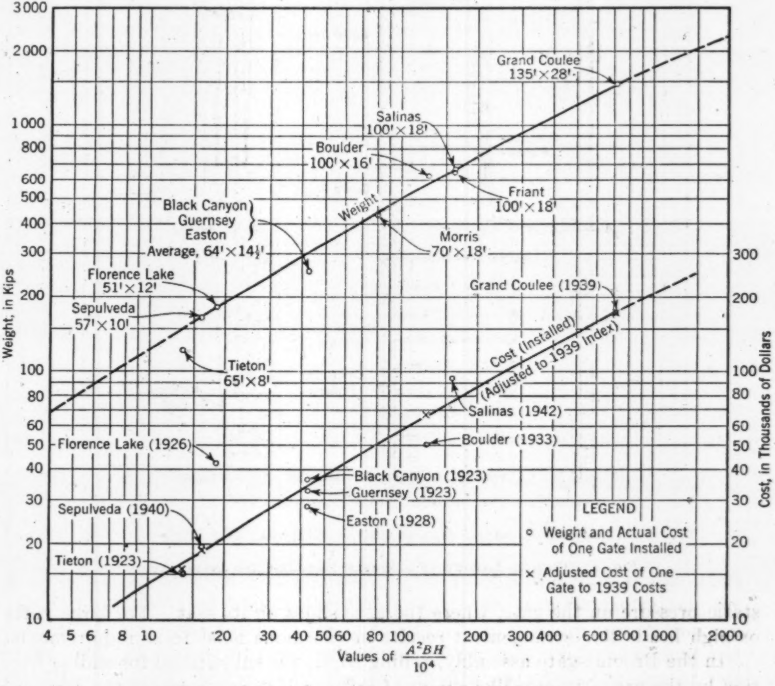


FIG. 8.—WEIGHTS AND COSTS OF DRUM CREST GATES

As indicated, all gates were hinged on the upstream side except those at Sepulveda and Morris dams.

Quite close estimates should be obtained from these data since all the points lie fairly close to the curves. Because of this it seemed justifiable to extrapolate the weight curve at both ends, thus extending the range considerably. However, the extensions are shown by dashed lines to the limits of the page.

Broome-Type Caterpillar Gates

Under the heading, "High-Pressure Slide Gates," it was mentioned that hoisting capacities are required in the order of from 60% to 100% of the entire static pressure on the gate, where the gate slides on its seat. For large gates or high heads this arrangement results in enormous hoist force requirements.

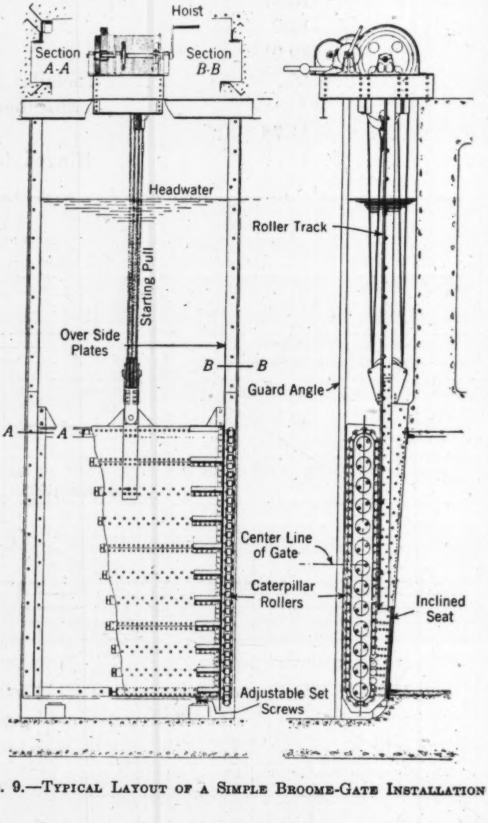

Hoist
Section A-A
Section B-B
Headwater
Starting Pull
Over Side Plates
B
B
A
A
Roller Track
Guard Angle
Center Line of Gate
Caterpillar Rollers
Inclined Seat
Adjustable Set Screws

Fig. 9.—Typical Layout of a Simple Broome-Gate Installation

In the Broome-gate assembly, rolling friction is substituted for sliding friction by the use of tractor-like chains of rollers which roll between the gate leaf and a stationary vertical track on the gate frame. The leaf moves in a vertical direction, therefore; but the seals are inclined at a slight angle from the direction of travel so that when the gate is closed the sealing surfaces at each side just touch. Any movement upward, in opening, lifts the leaf from the seals—eliminating sliding friction on the seal surfaces, with the result that the friction force is reduced to less than 5% of the static pressure. The gate is so designed that there is a sufficient downward component in the hydrostatic pressure on the gate at all times to insure closure, without requiring downward thrust from the hoist. This construction makes possible the use of simple drum hoists and sheaves. Fig. 9 shows a typical layout of the gate.

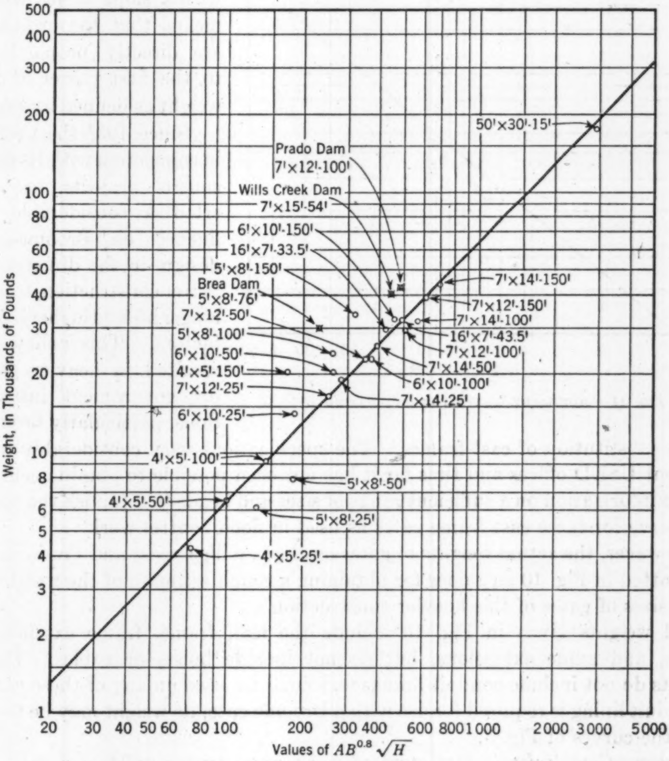


FIG. 10.—WEIGHTS OF BROOME GATES

To obtain an authoritative estimate of the price and weight of the gates made by any individual manufacturer, it is, of course, necessary to submit the requirements and some details of the installation to him before a quotation can be obtained. However, for rough preliminary studies, this is inconvenient and takes time. In Fig. 10 the available weights of a considerable range of this particular make of gates have been plotted as ordinates against the empirical function:

$$F(W) = A\,B^{0.8}\,\sqrt{H} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The auxiliary curve is also given (see Fig. 11) to evaluate $B^{0.8}$ quickly so that values of the abscissa can be obtained by a simple slide rule operation.

In Fig. 10, the points marked by the gate size only, represent a more or less standard line of gates which have been supplied to the hydroelectric industry for many years. The leaves of these gates, and in most cases the frames, are constructed of structural steel. The line drawn through the average of these points is a straight line with a slope of 45° which means that the ordinates are directly proportional to the first power of the weight as defined by Eq. 4.

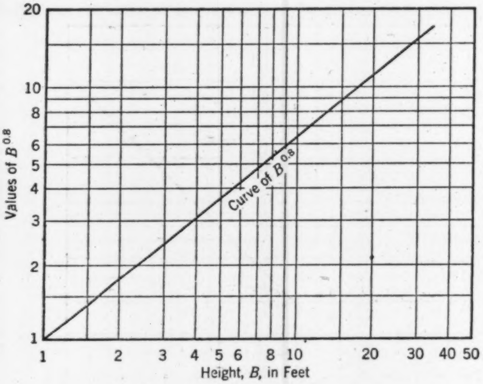


FIG. 11.—AUXILIARY CURVE FOR EVALUATING $B^{0.8}$

Since 1935 the USED, in connection with its flood control projects, has exerted a considerable influence on Broome-gate design, in the direction of better construction, to give longer life and higher safety factors. This policy has resulted in heavier gates for government installations, particularly because of the substitution of cast frames. The specifications vary considerably with different USED offices and thus far it has not been possible to obtain accurate weight information on a sufficient range of sizes and heads to establish the trend of a curve for these cast frame gates as used in flood control work.

However, the actual weights of gates on Brea, Wills Creek, and Prado dams are plotted in Fig. 10 as guides for obtaining a rough estimate of the weight of other sizes of gates of this heavier construction.

All weights given in Fig. 10 include the leaf, frame, frame extensions, guides, and guide extensions, but do not include hoists or cables. These weights do not include conduit linings and none are used on any of these gates. If conduit lining is required for use with a Broome gate, its weight may be taken from the curves of Fig. 3.

Broome-Gate Hoists.—The required hoist capacity for a Broome gate depends, among other conditions, on the displaced volume of the leaf which, for preliminary estimates, is not known. Fig. 12 was developed from a study of the hoist requirements of existing gates. Hoist capacities are plotted against the empirical expression:

$$F(W) = A^{1.173} B^{0.82} H^{0.94} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Auxiliary curves are given for the determination of the fractional powers of the variables (see Fig. 13). Curve No. 2, Fig. 12, shows approximate hoist capacities for the utility line of gates, whereas curve No. 1 is for use on the higher class of gates described. Knowing the required hoist capacity, the weight of the nearest standard hoist can be determined. Each hoist has a considerable range of capacity depending on the roping. The speed will vary inversely as the capacity for the same hoist with different roping.

Costs.—As a rough guide to obtain the cost of Broome gates for government projects, the following unit costs, adjusted to 1939 level, are given:

Gates installed at:	Cents per pound
Ohio Dams (1935)—	
Wills Creek	25.1
Beach City	25.1
Southern California Dams—	
Prado (1939)	20.2
Brea (1940)	23.7

These costs are for gates delivered complete at the site with hoists, but do not include installation. The weights of the gate and hoist should be added together in applying these costs. The installation cost may be taken at from about 3¢ to 4¢ per lb for average conditions and locations close to centers of population.

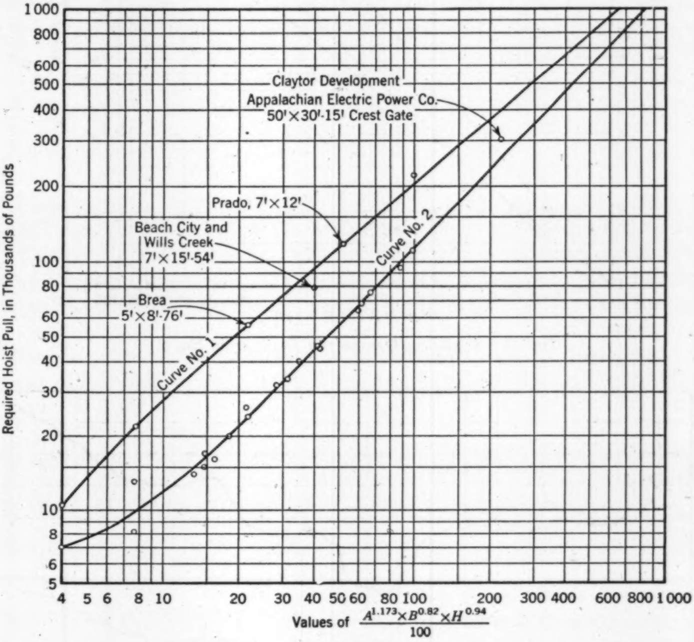


FIG. 12.—WEIGHTS OF HOISTS FOR BROOME GATES

Cost Index.—The cost level of 1939 has been taken as a representative level of prices before the inflation caused by World War II. For convenience in adjusting costs to this base, the curves in Fig. 14 have been plotted from the *Engineering News-Record* Construction and Building Cost Index of April, 1946.

The cost adjustments in Figs. 5 to 8 have been made in accordance with the yearly averages shown in the building cost index curve (Fig. 14(*b*)), since the

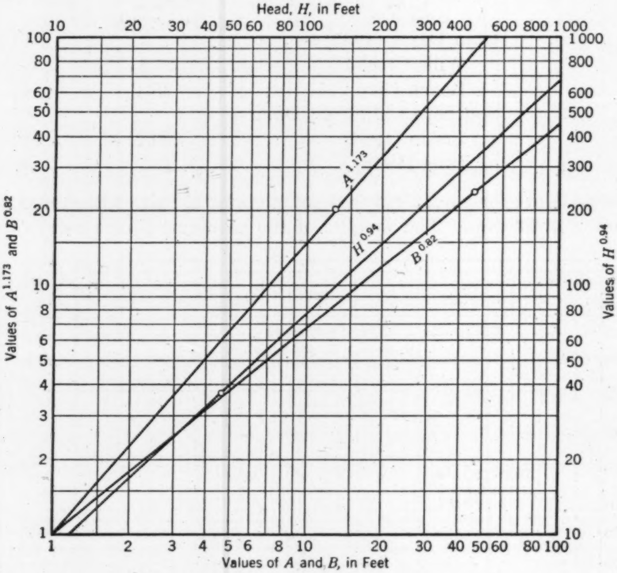


FIG. 13.—AUXILIARY CURVES FOR EVALUATING EQ. 5

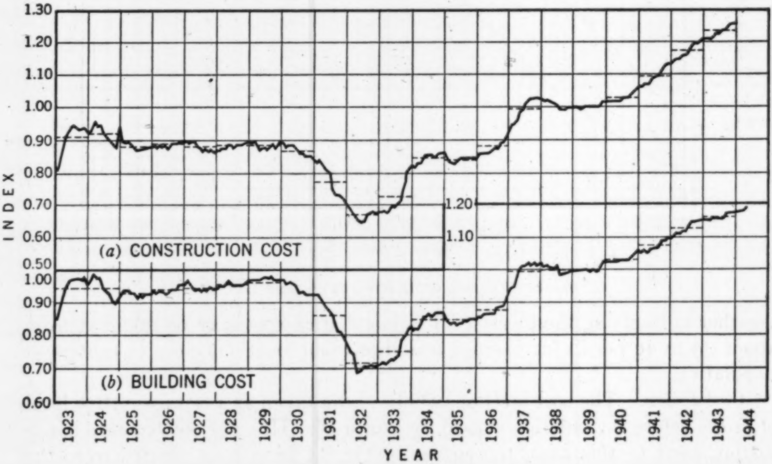


FIG. 14.—COST INDEX CURVES, ASSUMING THE 1939 AVERAGE COSTS EQUAL TO 1.00

best information obtainable seemed to indicate that this curve most nearly represented changes that occurred in shop cost previous to the war. However, a review of bids on both the cast-iron gates and fabricated steel gates received during 1946 definitely indicates that this curve does not reflect the true increase in gate costs for the postwar conditions. Unit costs in 1946 seem to be about double the 1939 costs.

SUMMARY

The estimating charts in this paper can be utilized to advantage in three different ways:

a. As timesavers in preliminary studies to determine the most economic type of gates and the most economical over-all arrangement on the project;

b. As instruments for making consistent preliminary estimates of weights and costs for definite project reports before actual design of gates has been made; and

c. As checks to compare final designs with those compiled previously, to assure the designer that he is not wasting material on the one hand, and as checks against gross error or omission affecting the safety of the structure on the other.

ACKNOWLEDGMENT

Various offices of the United States Engineer Department furnished valuable data for this paper in connection with the radial and roller gates on the Upper Mississippi and Ohio rivers. The writer is also indebted to the Bureau of Reclamation, United States Department of the Interior, both for data furnished directly through the courtesy of the Chief Engineer's Office in Denver, Colo., and for data taken from its various publications and specifications. The courtesy of the Phillips and Davies Company, the Southern California Edison Company, and the Consolidated Steel Corporation in furnishing data on gates is also greatly appreciated.

In particular, the writer acknowledges the helpful suggestions and criticisms given by associates in the Los Angeles Office, USED, and the courtesy of that office in extending permission for publication of these data.

DISCUSSION

Theodore B. Rights,[6] Assoc. M. ASCE.—In this discussion the writer will confine his remarks to radial crest gates. Unless the stress used is noted on the design of the structure a person who is seeking to compare his structure with the one previously designed, is "in the dark." In comparing a newly designed radial gate with an older gate that had a head 5 ft less, the writer was amazed that his designed sections were smaller in size. On investigation he found that, although the skin or face plate of this older gate was designed for 16,000 lb per sq in., the supporting members were only stressed to a maximum of 10,000 lb per sq in., whereas the writer was using 18,000 lb per sq in. throughout. It would be worthwhile if Mr. Boissonnault were to qualify his charts by taking the design stresses into account. For example, the following equation for the weight of the radial or Tainter type of gates[7] makes allowance for unit stress:

$$W = \frac{935\, B\, H_d}{\sigma} \left(A_s^2 + 700\right) \qquad \text{(6a)}$$

and

$$w = (A_s + 2\, B)\, 125 + 225\, L \qquad \text{(6b)}$$

in which, conforming to the notation of the paper: W is the weight of the gate (moving parts), in pounds; w is the weight of the embedded metal, in pounds; L is the length of gate lift, in feet; B is the height of the gate, in feet; H_d is the design head to the bottom of the gate measured in feet; σ is the design stress of the material, in pounds per square inch; and A_s is the clear span width of the gate, in feet.

(Incidentally, it may not be amiss to re-emphasize the widespread misspelling of the word "Tainter." Although the author spells it correctly, it appears as "Taintor" even in some of the most modern technical textbooks. The records of the United States Patent Office for July 6, 1886, show, under "canal lock," that patent No. 344,879 was registered by Jeremiah Burnham Tainter of Menomonie, Wis. The device therein described is the well-known Tainter gate.)

Many of the gates used by the power companies are adaptations of old gates, in which the stresses at the time of their design were 16,000 lb per sq in. One company, specializing in hydroelectric design, does not have a gate in their files that has been designed for 18,000 lb per sq in. When this company receives an order from a client company to design a gate, they search their files, select one of these gates (16,000 lb per sq in), secure "photolithos," and block out titles and elevations. In fact, one of the greatest difficulties in doing this is to detect the old sections that have not been rolled in the past few years. This company should not be overly criticized for using these uneconomical designs as the steel detailed drawings have been made. Photolithos of these details serve the shop just as well as new details, and the cost of making the shop drawings is thereby reduced about 80%, which should give the client company a cheaper job than if a new design with new detail drawings were used. Of course, in order to do this the contract for the gate must be given to the original fabricator.

---

[6] Cons. Engr., Roselle, N. J.

[7] "Engineering For Dams," by W. P. Creager, J. D. Justin, and Julian Hinds, John Wiley & Sons, Inc., New York, N. Y., Vol. III, 1945, p. 906.

The writer has used the charts by A. E. Niederhoff[5] and the results approximate closely the weights obtained from the writer's designs. These weights are for the gate only and not for the anchorage which comprises the larger part of the embedded steel. For an estimate for the larger gates, the writer uses 60% of the gate weight for the weight of the embedded steel. The weight of the anchorage is not directly a function of the gate weight. The location of the pin, both horizontally and vertically, will change the hinge reactions. For identical gate widths and heights, but with different pin locations, different anchorages may be required. For a small hinge reaction, the pier width is usually narrow. Most installations of these small gates have a continuous pin that runs through the pier, so that one arm of one gate and one arm of the adjacent gate rotate on the same pin. This pin is held back by pin plates with anchor rods bolted to them, and the entire assembly is grouted into the pier with only the pin ends cantilevering out. The main anchor rod can be placed in the line of action of the hinge reaction. For gates more than 33 ft wide with a 30-ft head, the piers must be much wider for stability. It would be impracticable to run the pin through the piers, so an anchorage casting placed on cantilever beams is used most frequently. These cantilever beams, both horizontal and vertical, are held down by anchor rods. Since these anchor rods are horizontal and vertical, they must take the components of the hinge reactions, and they are larger, therefore, than the anchor rods placed in the line of the hinge reaction. This type of construction increases the weight of the anchorage considerably. The writer believes that this in part explains the discrepancy between the actual weights and the weights obtained from Eq. 6*a*.[7]

TABLE 6.—WEIGHTS OF RADIAL CREST GATES USED IN POWER DAMS AND WATER SUPPLY DAMS

(Designed for a Stress of Less Than 16,000 Lb per Sq In.)

Gate No.	Date built	FEET				KIPS							
						ACTUAL WEIGHT							
		Radius of skin plate	Height of pin above crest	*A*	*B*	Gate only	Embedded steel	Total	Boissonnault	Niederhoff	Eq. 6*a*; Gate only	Eq. 6*b*; Embedded steel	Total
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
1	1930	32.0	16.0	39.0	30.0	110.0	65.0	175.0	160.0	120.0	117.0	19.1	136.1
2	1948	35.0	18.0	39.0	35.0[a]	145.0	88.0	233.0	200.0	150.0	159.0	21.5	180.5
3	1930	30.0	24.0	34.0	26.0	83.0	....	....	110.0	75.0	73.0	16.7	89.7
4	1929	26.0	14.0	34.0	26.0	62.0	....	....	110.0	75.0	73.0	16.7	89.7
5	1926	26.0	14.0	34.0	24.0	62.0	....	....	92.0	71.0	62.5	15.7	78.2
6	1944	20.0	14.75	33.0	19.0[b]	36.0	11.5	47.5	53.0	42.0	30.3	12.2	42.5
7	1937	24.0	20.0	30.0	16.0[c]	24.0	7.5	31.5	40.0	31.0	21.0	11.3	32.3
8	1925	18.0	7.0	30.0	14.0	28.0	12.5	40.5	36.0	28.0	18.4	10.5	28.9
9	1927	13.0	8.0	24.0	10.0[d]	11.0	1.5	12.5	18.0	14.5	9.0	8.3	17.3
10	1916	20.0	13.0	20.0	13.5[e]	13.0	2.5	15.5	19.0	15.0	11.7	9.0	20.7
11	1913	17.0	9.0	15.0	14.0	9.0	....	....	13.0	12.0	10.5	8.6	19.1

[a] Actual head 40.0 ft. [b] Actual head 17.0 ft. [c] Actual head 15.0 ft. [d] Design head 11.0 ft. [e] Design head 14.0 ft.

The weight of the embedded steel as well as the hoist weight may vary because of the nature of the installation. If a music-note seal (such as that used on some of the Tennessee Valley Authority radial gate installations) is used instead of a wood seal, the friction of the seals is increased and the hoist may have to be increased for this item. In some climates, as stated by the author, the weight of ice and silt on the gate is considerable. The writer has found that it is better actually to compute the hoist requirements, than to use any published data.

The writer further believes that a distinction should be made in this paper between fixed and movable hoists. Some dam installations have been made in the United States, in which a movable or traveling hoist car is used for three or four gates. Of course, a suitable structure must be provided for this movable hoist but, as it is made of structural steel, its cost can more easily be computed.

The writer believes one of the reasons that the welded gates weigh less than the riveted ones is that they, being more modern, have been designed for the higher stress of 18,000 lb per sq in.

The values obtained from the charts in this paper are useful; but it must be remembered that they should be used for estimating purposes, rather than as a rigid control on the design weight. Discussing this point, James S. Bowman, M. ASCE, states:[8]

> "The general layout of a Tainter Gate is a matter of trial and judgment since there is no fixed relation between the length of the arms and the height of the gate. The pins should be above ordinary high water, and the bottom of the gate when raised should clear floating material carried by the extreme flood. The shorter the radius of the face or the greater the inclination of the chord from the vertical, the heavier the gate will be."

Elsewhere it is stated:[9]

> "The radius of curvature affecting the direction of this upward pressure (upward vertical component of the water pressure), also has an important bearing on the relative dead load reaction at the chain and trunnion, on the length and height of the pier, and the weight of the struts."

From the foregoing it can be seen that two gates of identical widths and heads can be of different weights because of the different radii to the face plate and because of the different heights of pins above the crest. A good example of this can be seen in gates 3 and 4, Table 6. With the exception of gate No. 10, the curved skin plate is supported by horizontal channels variably spaced which in turn are supported by vertical ribs that frame into the trunnion arms. Gate No. 10 has the curved skin plate supported on equally spaced vertical ribs which are in turn supported by horizontal beams that frame into the trunnion arms.

For the writer's work, it would be best to have the gate weight separated from the total weight of the gate and the embedded steel. The charts do not give the height of hoist or lift as a variable. In canal work, most gate installations will have approximately the same lift, but in power or water dams, where large flows of water must be passed so as to have a constant head on the turbines or pumps, the lift of gate must be peculiar to that installation.

---

[8] "Handbook of Applied Hydraulics," McGraw-Hill Book Co., Inc., New York and London, 1942, p. 367.

[9] "Manual of Engineering Construction," Canalization, Vol. II, The Engineer School, U. S. E. D., Fort Belvoir, Va., 1940, p. 289.

George R. Latham,[10] Assoc. M. ASCE.—Presented in convenient form, these detailed data based on actual water control gates and hoists should prove of considerable value in estimating costs and preparing preliminary designs. The data have been wisely divided into weights and unit costs to permit the engineer to make proper allowances for the various factors affecting a specific project. The materials included are clearly noted and, as the author states, published data are usually too vague in the material included. In a typical Tainter gate design, the weight of the embedded steel and the anchorage commonly varies from one fourth to one half of the weight of the Tainter gate itself, which indicates the relative importance of the embedded material and of its inclusion in the estimate.

The weights of Tainter gates for private utility projects with which the writer has been associated have been plotted and checked with Fig. 4, and are in close accord with the curve for nonsubmergible gates. However, the weights and costs of the hoists are slightly higher than the values shown in Fig. 5; these would vary somewhat, as the actual hoist construction is dependent upon the requirements of the governing specifications. The data for the Tainter gates and hoists are given separately, thus enabling the designer to determine the relative saving by using a traveling hoist in place of fixed hoists when there are a large number of gates.

TABLE 7.—Cost of Materials per Pound for Tainter Gates and Hoists[a]

Project	Gates				Hoists	
	Size (ft)	Year installed	Cost (cents)	Cost increase (%)	Cost (cents)	Cost increase (%)
(1)	(2)	(3)	(4)	(5)	(6)	(7)
A	30 × 15	1937	11.2	....	18.2	....
		1946	14.1	26	41.0	125
B	34 × 24	1936	7.2	....	22.6	....
		1946	11.0	53	57.0	152
C	39 × 30	1930	5.65	....	....	....
		1948	17.8	215	....	....

[a] Delivered at job site.

For the designer to make a comparison and determine the most desirable gate for a specific project, all suitable types of gates should be considered. This paper covers the usual types of gates with the exception of the common wheeled lift gate with either plain or roller bearings. This popular gate has been proved satisfactory, being used both for crest gates and for submerged intake gates; a pertinent article has been published by G. R. Rich and Ross M. Riegel,[11] Members, ASCE, as well as a discussion by P. E. Gisiger,[12] M. ASCE, on the same topic. Its cost compares favorably with other types, since it is a relatively simple fabricated structural steel gate, and good competition is usually obtained on bids. Published data on costs and weights of lift gates and hoists are also scarce and vague concerning the material included. Information on lift gates in a form similar to that provided in this paper would be of value, and would provide excellent material for a subsequent paper.

---

[10] Structural Engr., Ebasco Services, Inc., New York, N. Y.

In a period when material and labor costs are skyrocketing the estimator is in a very difficult position to prepare a reasonably accurate cost estimate. The writer has had occasion to install duplicate gate and hoist equipment on several projects and it is believed that the costs given in Table 7 may be of interest. To these prices must be added the erection costs, which indicate that the author's closing comment, "Unit costs in 1946 seem to be about double the 1939 costs" does not err, conservatively speaking.

In closing, the writer wishes to commend the author for an excellent paper.

D. A. Buzzell,[13] M. ASCE.—A valuable tool for the estimator's use in computing the probable weight and cost of certain types of gates is provided in this paper. An excellent model is set up for further investigations along the same or related lines.

It is noted that the vertical-lift crest gate is not covered by the author. The vertical-lift crest gate, in widespread use on larger rivers, is usually a fixed-wheel type from 40 ft to 50 ft wide and up to 50 ft high. Such gates usually are operated with one or more traveling gantry cranes; and, for comparison, it is necessary to take account of the costs of both these cranes and the required heavy operating bridge. In other words, a cost study of a dam spillway, for which either Tainter gates with fixed individual hoists or crane-operated vertical-lift gates are considered suitable, necessitates the preparation of two rather complete designs. This problem arises quite frequently in planning spillways and it is obvious that cost and weight curves given in this paper have only limited application to such a comprehensive problem.

The type of hydraulic slide gate shown in Fig. 1 has been considerably modified in the recent design practice of the Corps of Engineers, and higher cylinder working pressures are in common use. The air inlet area has been stepped up from 5% to 10% of the gate area depending on water velocity; the sill depression has been eliminated, and many changes have been made in the gate leaf details. The extent of conduit length protected by liner plates has also been reduced. The net effect of these changes has been a reduction in the over-all weight and cost. All-welded carbon steel designs are being considered for this type of gate.

The Corps of Engineers has developed a new Tainter or radial crest gate with inclined side frames and a box girder anchorage which has effected a marked saving in weight over the values shown in Fig. 4. A recent instance in which Tainter gates already contracted for and designed according to older practice were redesigned according to the new standards furnished decisive proof that the latter design would result in pronounced economy.

---

[11] "Waterways and Gates for Hydro-Electric Plants," by George R. Rich and Ross M. Riegel, *Civil Engineering*, February, 1941, p. 101.

[12] "Economic Design of Waterways and Gates," by P. E. Gisiger, *ibid.*, May, 1941, p. 308.

[13] Office, Chf. of Engrs, Washington, D. C.

The fact that future design practice will not be limited to the types of gates discussed in this paper is indicated by a recently proposed installation of Tainter gates in the outlet conduits of a large dam. There would be single 24-ft by 24-ft gates under 180 ft of head in each of seven outlet tunnels, 23 ft in diameter. The indicated saving over vertical-lift outlet gates was $300,000 per conduit or $2,100,000 for the project.

It is regretted that the study was not extended to include caterpillar gates, a large variety of which are used in flood control outlets and power conduit intakes.

A. E. Niederhoff,[14] M. ASCE.—The tabulations and charts given by Mr. Boissonnault have filled in gaps in data on reservoir gates. It is very commendable that the author has taken the time to assemble this information and to correlate it for use by other members of the profession. As stated in the "Summary," the charts are of immense value in making preliminary estimates quickly and consistently. Any engineer who has worked in the planning department of the United States Army Engineers knows how necessary it is to have charts such as these for reference. Exception is taken, however, to the use of the charts in checking a detailed and finished design.

The charts yield only traditional or historic values and do not take into account advances made in design. If the final, detailed design of reservoir gates had to conform in weight to values from the charts, no improvement would have been made over former designs. New materials and new techniques of fabrication are available which reduce the weights of gates and it is entirely conceivable that other improvements will be made in the future. The trend is away from steel toward noncorrosive light metals.

The text of the article is excellent in content but one important phase has been omitted. If the author had included a discussion on the applicability of the various types of gates, his paper would have been a veritable treatise. Under what conditions does one use a slide gate? In addition to personal preference, there are several other reasons why slide gates are used instead of Broome or submerged Tainter gates. Similarly, drum gates are used on the crests of dams under certain conditions which make the cheaper Tainter gate inapplicable. Not only are economic items involved, but also hydraulic characteristics and operational factors. Perhaps the author will cover these very significant items in his concluding discussion as well as a typical example of the use of his charts.

One important crest gate on dams that was not mentioned is the vertical-lift gate. The writer published a short article on this subject,[15] during the preparation of which it became more and more apparent that the vertical-lift gate is economical for streams having large discharges where ice conditions are unusual. Curves were developed yielding the weight of the gate and the discharge through a gate opening. By the use of formulas the cost of the gate could be obtained, the discharge through a gate could be computed, and the number of gates composing the spillway could be calculated. Through a typographical error in the text the formula for the cost of a gate gave ridiculous results. Using the author's nomenclature and a construction cost index of 116 for the year 1940, this formula should have been:

---

[14] Specification Writer, U. S. Naval Ordnance Test Station, China Lake, Calif.

[15] "Vertical Lift Gate Selection," by A. E. Niederhoff, *Western Construction News*, February, 1944, pp. 73–75.

$$\text{Cost per gate} = 60\left(\frac{A^2 B^2}{2{,}000}\right)^{0.87} \quad\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

It would be of interest to know if the author checks this formula for cost of vertical-lift gates for dams.

The author mentions that welded Tainter gate weights are in fair agreement with the dashed line shown in Fig. 4, and that weights derived therefrom agree with curves published by the writer. He qualifies the use of this curve with the statement (under the heading, "Radial Crest Gates"):

> "Not many gates of welded construction have been built, but the calculated weights of recent designs seem to show fairly close agreement with the "Welded Construction" curve (see Fig. 4)."

Since World War II, when speed of production, economy, and safety dictated that welded steel ships and welded armored steel tanks were necessary, the knowledge of the art of welding has increased. Many ship building concerns used welding equipment exclusively and did not have the tools for doing riveted construction. As a result it is believed that in the future Tainter gates will be welded and Mr. Boissonnault's dashed line curve in Fig. 4 will represent weights that are more currently correct.

The writer wishes to reiterate that, in spite of a thorough checking and rechecking of this fine paper, he has found little cause for complaint. The author has had the time and opportunity to do a fine job of assembling and correlating information and is to be congratulated on his work.

JOSEPH R. BOWMAN,[16] Assoc. M. ASCE.—A great deal of painstaking work in assembling data on gates and hoists and in correlating the several factors contributing to the cost of such equipment is reflected in this paper. The graphical estimating methods presented by the author should have considerable practical value to engineers engaged in the study of water storage and power projects. Although the general methods employed are not new,[2] much of the data presented concerns recently designed equipment, thus reflecting modern trends in engineering practice. The writer endorses the use of graphical methods for estimating purposes in preference to more direct methods because the practical limitations may be shown more clearly in graphical form.

The particular types of gates with which the paper deals have been given excellent treatment, but the field of extensively used modern reservoir gates has not been fully represented. The general field of vertical-lift gates, of which the Broome gate is but one type, should receive further consideration; consequently, much of this discussion will be devoted to gates in this field.

---

[16] Hydr. Engr., Harza Eng. Co., Chicago, Ill.

High-Pressure Slide Gates.—The author has presented a seemingly reliable method of determining the weight of hydraulically operated, high head sluice gates, but Eq. 1 appears to be rather complex for estimating purposes. The writer notes that the weight curve in Fig. 2 has been drawn through points representing only United States Engineer Department (USED) flood control gates, whereas an additional curve, lying somewhat below the author's curve, can be drawn to represent the Bureau of Reclamation (USBR) general-purpose type of high-pressure slide gate.

It is regretted that methods for determination of weights were not presented for high head gates of the ring follower, ring seal, paradox, coaster, and tractor types—as well as for the butterfly, needle, Howell-Bunger, and other types of high-pressure valves. It is hoped that the author may see fit to include supplementary information of this nature in his closing discussion.

The effects of standardization of design on unit prices for the fabrication of gates are well illustrated in Table 2. Although the comparative values shown in this tabulation were derived from prices for high-pressure slide gates, they may be applied with a reasonable degree of accuracy to other types of gates. Further economies are to be realized from standardization of design in the reduction of design costs and other indirect costs.

Tainter Gates.—The writer sees no justification for changing the direction of a gate weight curve to qualify a single (and possibly doubtful) point, as the author has done in the case of the lower end of the middle curve in Fig. 4.

The curves for Tainter gate weights published by A. E. Niederhoff,[5] M. ASCE, replotted in Fig. 4 to express the weights of welded Tainter gates, appear to be quite reliable; inspection of Mr. Niederhoff's article[5] substantiates Mr. Boissonnault's implied belief that the weights expressed by this curve include the combined weights of gate, guides, and anchorages.

The author implies that no relationship can be shown to exist between Tainter gate weights and hoist capacities. However, it can be shown that the majority of points in Figs. 4 and 5 will lie on common curves corresponding to submergible and nonsubmergible types of Tainter gates, respectively, when hoist capacity is plotted against Eq. 3.

Roller and Drum Crest Gates.—Because roller and drum crest gates are usually designed to satisfy the specific requirements of the respective jobs for which they are selected, their weights should be expected to conform to definite characteristic curves. This observation seems to be borne out by the drum gates in Fig. 8, but does not appear to be substantiated by the roller gate points in Fig. 7. Conformity of the drum gate points to a common characteristic curve may be due to the more exacting operational requirements imposed on this type of gate.

Broome Gates.—Figs. 10 to 13, inclusive, for the Broome-type caterpillar gates use parameters considerably more complex than expressed in Eq. 3, but the results obtained through the use of such expressions appear to be more consistent than those obtained by the employment of other methods of analysis. However, the weights of these gates may be indicated approximately in terms of Eq. 3 with sufficient accuracy for preliminary comparison with other types of gates, as will be shown later.

Vertical-Lift Gates in General.—Both the Broome and fixed-wheel types of vertical-lift gates have been used extensively in the low head field as submerged intake gates. Both may be used advantageously as spillway crest gates, particularly where large flows are to be passed over comparatively short regulated spillways; their ability to be raised entirely clear of the water surface gives these gates a further advantage over Tainter gates. Although the Broome gate generally has better seating characteristics than the fixed-wheel gate, the latter has the distinct advantage of considerably lower gate cost. Moreover, the fixed-wheel gate can be split readily into two or more individually handled sections, thereby materially reducing hoist costs with but a small increase in the weight of the complete gate.

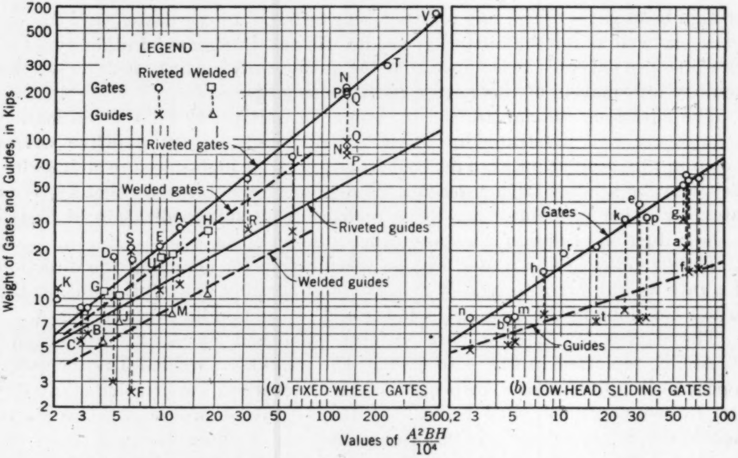


Fig. 15.—Weight of Fixed-Wheel and Low Head Sliding Gates and Guides

Although Broome and fixed-wheel gates may be operated against either full or balanced heads, low head sliding gates are restricted to operation against balanced heads because of their relatively high frictional resistance, and hence are used primarily as bulkheads. Gates of this type have been used on spillway crests as temporary bulkheads, and at hydro plants to close draft tube outlets so that the water passages may be unwatered. Sliding bulkhead gates are frequently used in hydro plant intakes in conjunction with Broome or fixed-wheel gates, thus effecting a saving in intake gate costs.

Fixed-Wheel Gates.—The weights of several fixed-wheel gates and their guides (embedded parts) are shown in Fig. 15(*a*); accompanying dimensional and cost data are given in Table 8. The majority of these gates are of a conventional type of construction, wherein the skin plate is welded to a framework of standard structural shapes with riveted connections. The development of all-welded gates has progressed to an extent which permits their segregation from gates employing older methods of construction. The writer has found it advantageous to be able to make a distinction between gate and guide weights; accordingly, the points and weight curves shown in Fig. 15(*a*) indicate the separate weights of gates and guides for both riveted and welded types of construction. Weight curves developed for fixed-wheel spillway crest gates by Mr. Niederhoff,[15] when replotted in terms of Eq. 3, coincide with the writer's weight curve for riveted gates (exclusive of guides)—thus applying to intake as well as to spillway crest gates.

TABLE 8.—DATA ON FIXED-WHEEL GATES SHOWN IN FIG. 15(*a*)

Gate	No. of gates	DIMENSIONS OF OPENING			Year of purchase	UNIT COSTS (1939 INDEX) ($)	
		A	*B*	*H*		F.O.B. site	Installed
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
A[a,b]	1	14.25	19.33	30.33	1927	0.140	....
B[a,b]	1	11.25	9.00	31.5	1928	0.157	....
C[a]	2	11.7	10.12	21.33	1928	0.065	....
D[a,c]	1	18.0	13.5	10.75	1928	0.070	....
E[a]	2	14.25	14.83	29.6	1930	0.096	....
F[a,c]	10	35.0	10.0	5.0	1930	0.067	....
G[a,d]	3	14.6	13.0	14.5	1931	0.069	....
H[a,d]	3	20.0	20.0	22.3	1936	0.088	....
J[a,d]	6	13.25	14.5	19.75	1936	0.089	....
K[e]	1	10.2	10.2	19.9	1936	....	....
L[a]	2	15.3	39.0	64.5	1939	0.095	....
M[a,d]	4	13.0	16.9	38.0	1948	....	....
N[c,e]	18	40.0	40.0	20.0	1938	0.110	0.130
P[c,e]	22	40.0	40.0	20.0	1938	0.108	0.129
Q[c,e]	18	40.0	40.0	20.0	1939	0.096	0.122
R[e]	1	14.5	22.3	66.9	1942	0.246	0.355
S[e]	1	12.0	12.75	32.6	1942	0.177	0.285
T[c,e]	..	40.0	50.0	25.0	....	....	....
U[d]	2	15.5	15.5	25.75	....	....	....
V[e]	..	50.0	60.0	30.0	....	....	....

[a] Installed on projects designed by consulting engineer. [b] Designed by manufacturers. [c] Spillway crest gates; remainder, power intake gates. [d] All-welded construction. [e] Designed by the Tennessee Valley Authority (TVA); corrosion allowance of 1/16 in. added to skin plates and principal frame members.

The points representing gate K lie considerably above their respective weight curves; the gate was designed for a probable future increase in operating head, and the guides presumably were designed to reduce installation costs at a small increase in first cost. Unfortunately, installation cost data to substantiate the latter conclusion are not available to the writer. The guides of gates N, P, and Q consist of rails mounted on steel towers embedded in the concrete piers;[17] the accomplishments of this design were twofold in that accurate field alinement of the guides was assured at low unit costs of installation.

The writer has indicated no graphical relationship between gate weights and weights or capacities of hoists for fixed-wheel gates, as the methods of application of hoisting equipment in the hydroelectric power field vary considerably. However, in the case of spillway crest gates having wheels equipped with antifriction bearings, the capacity of the hoist should at least equal the weight of the gate plus frictional resistance of about 3% of the total water pressure on the gate. In considering the use of a traveling hoist, the weight of a lifting beam or other device should be added; the weight of a lifting beam depends on its application and may vary from 5% to 25% of the weight of the gate. Hoist capacities may be reduced by sectionalizing gates handled by traveling hoists, or by counterweighting gates lifted by fixed hoists. Hoist capacities for power intake gates vary widely, depending on the method of operation (whether against full or balanced heads) and on the design of the gates themselves. A typical example of the effect of gate design on hoist capacity is given in Table 9, which summarizes a study made to determine the most economical design for gate M in Fig. 15(*a*); this gate is designed for operation against full hydrostatic head, and its wheels are equipped with grease-lubricated roller bearings. Price estimates furnished by a gate manufacturer indicate that the cost of a 16,000-lb gate of built-up plate construction would be approximately the same as the cost of a 19,000-lb gate of standard shape construction. Thus, the cost of gate 3M is greater than that of gate 4M by the cost of ballast required to seat gate 3M.

---

[17] "The Pickwick Landing Project," *Technical Report No. 3*, TVA, U. S. Govt. Printing Office, Washington, D.C., 1941, p. 230.

TABLE 9.—SUMMARY OF ECONOMIC STUDY FOR SELECTION OF DESIGN FOR ALL-WELDED GATE M IN FIG. 15(*a*)

Design	LOCATION OF:		Construction	WEIGHT (LB)		Required hoist capacity (tons)
	Skin plate	Bottom seal		Gate only	Ballast (seating)	
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1M	Upsteam	Downstream	Standard shapes	19,000	29,000	27.5
2M	Upstream	Upstream	Built-up plates	16,000	None	35.0
3M	Downstream	Downstream	Built-up plates	16,000	3,000	20.0
4M	Downstream	Downstream	Standard shapes	19,000	None	20.0

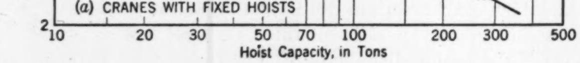


FIG. 16.—WEIGHT OF OUTDOOR GANTRY CRANE GATE HOISTS

Fig. 16 shows approximately the variation of crane weight with hoist capacity (based on project reports for several TVA projects) for outdoor gantry cranes. Installed unit costs (adjusted to 1939 index) for outdoor gantry cranes with trolleys and auxiliary hoists ranged from 20¢ per lb to 30¢ per lb.

Low Head Sliding Gates.—The weights of several low head sliding gates and their guides are shown in Fig. 15(*b*); accompanying dimensional and cost data are given in Table 10. Hoists designed exclusively for handling low head sliding gates operated against balanced heads ordinarily require capacities slightly in excess of the weights of the gates. The excess capacity is equivalent to the weight of a lifting device plus a nominal force required to break the gate away from its sealed position; where sloping guides are used, frictional resistance due to the weight of the gate will be encountered. Gates f and j were sectionalized to reduce hoist costs. Fig. 16 also indicates approximately the variation of crane weight with hoist capacity for outdoor gantry cranes with fixed hoists. Installed unit costs (adjusted to 1939 index) for these cranes ranged from 25¢ per lb. to 40¢ per lb.

TABLE 10.—DATA ON LOW HEAD SLIDING GATES SHOWN IN FIG. 15(*b*)

Gate (1)	No. of gates (2)	OPENING DIMENSIONS *A* (3)	*B* (4)	*H* (5)	Year of purchase (6)	UNIT COSTS (1939 INDEX) ($) F.O.B. site (7)	Installed (8)
a[a,b]........	4	15.3	39.0	64.5	1939	....	....
b[a]........	2	13.5	9.75	25.6	1948	....	....
e[c]........	6	18.0	24.5	37.75	1935	0.112	....
f[c]........	3	25.0	18.0	53.0	1938	0.075	0.088
g[c]........	9	18.66	19.83	80.1	1938	0.088	0.134
h[c]........	3	13.33	12.0	36.0	1939	0.097	0.199
j[c]........	3	25.54	18.0	62.0	1939	0.075	0.096
k[c]........	2	20.5	11.0	52.5	1942	0.104	0.169
m[c]........	2	11.0	7.75	54.1	1942	0.142	0.282
n[c]........	2	10.75	6.50	27.0	1942	0.142	0.236
p[c]........	2	20.5	11.0	71.5	1943	0.122	0.231
r[b]........	..	15.0	30.0	15.0	1940	....	....
s[d]........	..	19.5	14.0	61.4	1947	....	....
t[d]........	2	21.0	14.1	26.0	1947	....	....

[a] Designed by consulting engineer. [b] Power intake gates; remainder, powerhouse draft tube gates. [c] TVA projects. [d] All-welded gates.

Miscellaneous Types of Gates.—During the period between World War I and World War II many types of water control gates were developed; the gates treated in the paper and in this discussion represent the surviving types more commonly used in modern practice. Occasionally, however, it is necessary to resort to special gate equipment adaptable to unusual site conditions. A few noteworthy examples of the use of special gates include the utilization of butterfly spillway crest gates at Exchequer Dam[18] (California), cylinder intake and outlet gates at Hoover Dam[19] (Arizona and Nevada), and a floating ring spillway gate at Owyhee Dam[20] (Oregon). The use of such special types of gates is generally so infrequent that progressive improvements in design and construction make the prediction of their weight and cost a complex problem.

Comparison of Types of Gates.—It is frequently desirable to draw quick comparisons between several types of gates under given dimensional conditions. Accordingly, weight curves for the types of gates so treated in the paper and in this discussion are shown in Fig. 17 as functions of Eq. 3. The weights indicated include the gates and their embedded parts.

---

[18] "Vertical Butterfly Gates on Exchequer Dam," *Engineering News-Record*, August 26, 1926, p. 344.

[19] "Hydraulic Valves and Gates for Boulder Dam—Part I," by P. A. Kinzie, *Mechanical Engineering*, July, 1934, p. 387.

[20] "Floating-Ring Gate and Glory-Hole Spillway on Owyhee Dam," by Lewis G. Smith, *Reclamation Era*, August, 1940, p. 226.

Conclusion.—Mr. Boissonnault has made an important contribution in assembling reliable estimating data for reservoir gates, and it is hoped that the supplementary data given in this discussion will add to the value of the original paper.

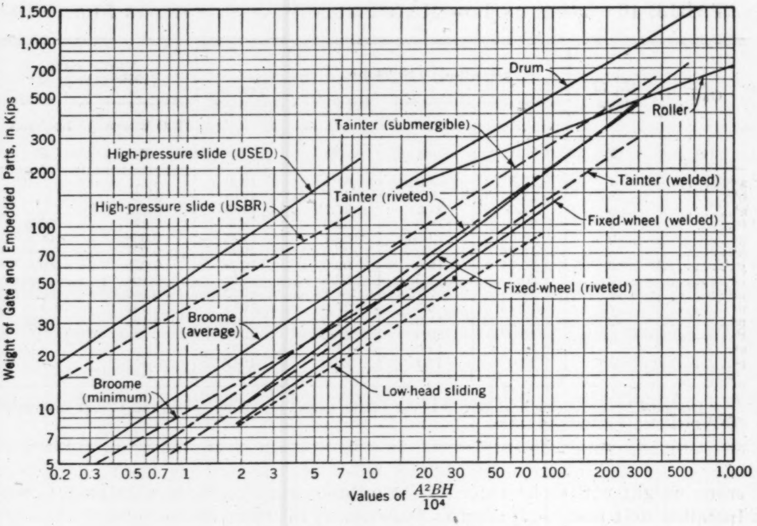


FIG. 17.—COMPARISON OF GATE WEIGHTS

Acknowledgment.—The writer should like to express his gratitude to H. G. Gerdes, M. ASCE, for inspiration derived from his unpublished work on gates, and to Kenneth C. Roberts, James S. Bowman, and Calvin V. Davis, Members, ASCE, and E. Montford Fucik, Assoc. M. ASCE, for their many valuable suggestions.

FRANK L. BOISSONNAULT,[21] ESQ.—The fine reception accorded this paper is very gratifying to the writer and the many commendatory remarks in regard to it, both in the published discussions and in personal correspondence, are greatly appreciated.

Possibly it would be in order to further clarify the method of approach used in the paper, which will serve to answer some of the points brought up in the discussion. The approach is empirical; the data, on as many different gates of the given type and covering as wide a range as possible, were carefully checked to be sure that the same parts were included and that all, therefore, were properly comparable. These data were then correlated graphically by the methods described so that conclusions could be drawn for any set of conditions within the range of the data. The expressions used as abscissas do not purport to be rational formulas for the weight of a gate, but are empirical expressions developed for the purpose of correlation. Therefore, the results are the averages of those which actual practice has shown to exist under present and past states of the art, rather than any individual's idea of what they should be theoretically. Consequently, such details as working stresses, length of radii in Tainter gates, etc., are beyond the scope of these charts, which were developed for making quick preliminary estimates.

---

[21] Engr., U. S. Engr. Office, Los Angeles, Calif.

As brought out in the "Introduction," all the information is given on each point and its position plotted, so that the engineer can use his own judgment in intelligently comparing past practice with the case in hand.

It is believed that too much refinement as to design details is neither warranted nor even desirable because most engineers would rather be a little on the safe side in preliminary estimates. Reliable data of this character, which have sufficient detail and a wide enough range of sizes to establish curves, are very difficult to obtain. The writer has scoured every public source of such information which suggested itself and has used all that could be found in the preparation of the paper. Such data on many other types of gates simply do not exist in the ordinary sources. Neither time nor available data permit the introduction of additional material to the paper at this time.

All the discussions as well as private correspondence addressed to the writer reveal a demand for estimating data on vertical-lift gates in similar form to those presented in the paper. Mr. Bowman's discussion presents some weight and cost data. These, together with Mr. Niederhoff's corrected cost formula and his previous paper on this subject,[15] should be of value in estimating gates of this type.

Mr. Niederhoff takes exception to the use of the charts as "checks to compare final designs with those compiled previously," as suggested in the "Summary" of the paper. In this statement, it was taken for granted that the construction methods and design factors of the new gate would be similar to those used in the chart gates or that the engineer would use his judgment in making suitable allowances for deviations in type of construction. Possibly an example to illustrate the intent of this statement might be in order. Reference is made to the point for the Sepulveda Dam slide (Southern California) gate in Fig. 2(*b*). This is a 6-ft by 9-ft gate with a maximum possible head of only 55 ft, yet its weight is more than 112,000 lb, which makes it the heaviest slide gate of which there is any record. There was nothing in the requirements for this gate which called for special construction. It seems reasonable to believe that, if the designer had then had this chart available to compare the design with previous practice, the plans would have been scrutinized in detail to determine the reasons for the excessive weight. These reasons could easily have been found.

The writer would also like to comment on a few other points raised by Mr. Bowman. Mr. Bowman observes that the function in Eq. 1 used as the abscissa in Fig. 2 for slide gates "appears to be rather complicated for estimating purposes." It was realized that this expression was too complex for a simple slide rule operation. Attention is called to the simple graphical solution of this function in the upper part of Fig. 2, which was developed by the writer for this very reason. It is believed that the complexity of the function is of no moment as long as the result can easily and quickly be obtained graphically, especially when it gives as good results as this one does. Of course, the scale has been reduced in the printing from the original 10½-in. by 12-in. chart so that it is quite small for much practical use. Similar comment applies to Mr. Bowman's remarks relative to the Broome-gate curves.

Mr. Buzzell's comment on a new design for Tainter gates, developed by the Unites States Engineer Corps, is interesting and the writer would appreciate specific information on the particular gates to which he refers. It is noted in this connection that the weight of the Lytle Creek Tainter gate, which was designed by the Los Angeles Office of the U. S. Engineers since the preparation of these charts, is in quite close agreement with the weight which would be predicted by the welded gate curve of Fig. 4. This gate, now under construction, was designed to conform to the then latest Engineer Corps standards, and has inclined side frames as mentioned in Mr. Buzzell's remarks.

Summary.—Most of the data available to the writer have been utilized in the preparation of the paper. This type of material is very difficult to obtain and it is hoped that members of the profession will cooperate to gather information on other types of gates.

As brought out in the foregoing, data in sufficient detail and range on existing gates or finished designs are not available from ordinary sources. Nevertheless, many engineers have information available to them on an individual project with which they are, or have been, connected. The writer would be glad to collect and correlate such data sent to his personal attention. Such an arrangement would soon result in building up a volume of data on a range of gate sizes and types, even though any one contribution in itself would not have much significance.

In closing, the writer wishes to thank the gentlemen participating in the discussion for their interest and their many valuable suggestions.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2353

FRICTION COEFFICIENTS IN A LARGE TUNNEL

By G. H. Hickox,[1] M. ASCE, A. J. Peterka,[2] Assoc. M. ASCE, and R. A. Elder,[3] Jun. ASCE

With Discussion by Messrs. Weston Gavett, W. R. Barrows, Hunter Rouse, Karl R. Kennison, E. J. K. Chapman, Julian Hinds, William P. Creager and Stephen H. Haybrook, G. S. Tapley, J. N. Bradley and S. P. Wing, and G. H. Hickox, A. J. Peterka, and R. A. Elder.

Synopsis

Measurements made to determine friction and roughness coefficients for three different surfaces in the Apalachia Tunnel of the Tennessee Valley Authority are described in this paper. The sections tested were an 18-ft steel pipe coated with bituminous paint, an 18-ft concrete-lined tunnel, and an unlined rock tunnel of 20-ft and 22-ft nominal diameters. Discharges during the tests varied from 975 to 3,210 cu ft per sec.

Methods of reducing the data are described and the results are presented in terms of f and n in the Weisbach and Manning equations, respectively. Also, the surfaces are described in detail and illustrated so that their respective friction and roughness coefficients may be applied to other similar surfaces.

Description of Tunnel

The Apalachia Project of the Tennessee Valley Authority was built primarily as a power project and has been described by H. W. Goodhue, Assoc. M. ASCE, and R. L. Smart and A. A. Meyer,[4] Members, ASCE. The main structures are a comparatively low ponding and diversion dam, a pressure conduit approximately 8.3 miles long, a surge tank, and a powerhouse containing two turbines.

Although the pressure conduit consists mainly of a concrete-lined tunnel 18 ft in diameter, it does have two fairly long reaches of unlined rock which have nominal diameters of 20 ft and 22 ft. Just below the dam there is a comparatively short length of steel pipe 18 ft in diameter. At several other places

---

Note.—Published in April, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Senior Hydr. Engr., TVA Hydr. Laboratory, Norris, Tenn.

[2] Hydr. Engr. Hydr. Laboratory, Bureau of Reclamation, Denver, Colo.

[3] Associate Hydr. Engr., TVA Hydr. Laboratory, Norris, Tenn.

[4] "The Design of Recent TVA Projects: VIII. Apalachia and Ocoee No. 3," by H. W. Goodhue, R. L. Smart, and A. A. Meyer, *Civil Engineering*, October, 1943, pp. 465–468.

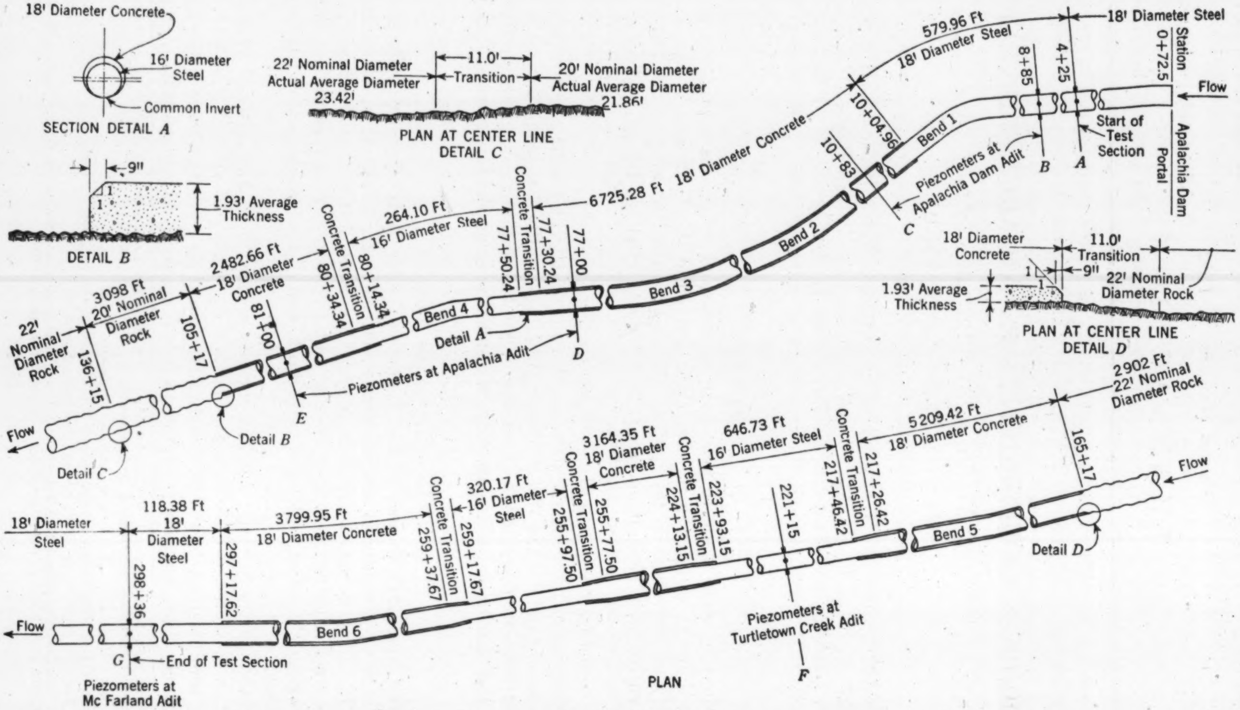

18' Diameter Concrete
16' Diameter Steel
Common Invert
SECTION DETAIL A
22' Nominal Diameter
Actual Average Diameter
23.42'
11.0'
Transition
20' Nominal Diameter
Actual Average Diameter
21.86'
PLAN AT CENTER LINE
DETAIL C
9"
1
1
1.93' Average Thickness
DETAIL B
18' Diameter Steel
Station
0+72.5
Flow
Apalachia Dam
Portal
579.96 Ft
18' Diameter Steel
8+85
4+25
Start of Test Section
A
B
Bend 1
Piezometers at Apalachia Dam Adit
10+04.96
10+83
C
18' Diameter Concrete
6725.28 Ft
Bend 2
Bend 3
77+00
77+30.24
Concrete Transition
77+50.24
D
264.10 Ft
16' Diameter Steel
80+14.34
Concrete Transition
80+34.34
Bend 4
Detail A
Piezometers at Apalachia Adit
2482.66 Ft
18' Diameter Concrete
81+00
E
3098 Ft
20' Nominal Diameter Rock
105+17
22' Nominal Diameter Rock
136+15
Detail B
Detail C
Flow
18' Diameter Concrete
11.0'
Transition
9"
1
1
1.93' Average Thickness
22' Nominal Diameter Rock
PLAN AT CENTER LINE
DETAIL D
2902 Ft
22' Nominal Diameter Rock
165+17
Flow
Detail D
5209.42 Ft
18' Diameter Concrete
217+26.42
Concrete Transition
217+46.42
Bend 5
646.73 Ft
16' Diameter Steel
221+15
Piezometers at Turtletown Creek Adit
F
223+93.15
Concrete Transition
224+13.15
3164.35 Ft
18' Diameter Concrete
255+77.50
Concrete Transition
255+97.50
320.17 Ft
16' Diameter Steel
259+17.67
Concrete Transition
259+37.67
3799.95 Ft
18' Diameter Concrete
Bend 6
297+17.62
118.38 Ft
18' Diameter Steel
298+36
18' Diameter Steel
G
End of Test Section
Piezometers at Mc Farland Adit
Flow
PLAN

Fig. 1.—Plan of Test Section of Apalachia Tunnel Showing Details of Lining and Alinement

where sufficient cover for the pressure tunnel is lacking, short lengths of steel pipe of 16-ft and 18-ft diameters are used. The alinement is nearly straight, but topographical considerations necessitated several bends. In general, the grade is quite flat, sloping toward the powerhouse. Fig. 1 shows a plan of the test reach, which is approximately the upper two thirds of the conduit length. All significant features, such as lengths, diameters, types of linings, and alinement are shown.

Piezometers.—During the construction of the conduit seven sets of piezometers were installed at the locations shown in Fig. 1 so that tests could be made to determine the various hydraulic losses.

Each of these installations consisted of a ring of four piezometers spaced at 90° intervals around the conduit and manifolded together so that an average pressure could be obtained. Where the manifold ring was exposed, as on the steel pipe at the dam, the piezometers were provided with cutoff cocks so that the manifold ring could be drained to prevent damage by freezing.

Fig. 1 shows that the piezometers are so placed that friction loss measurements could be obtained on sections containing three types of conduit: 18-ft steel pipe, 18-ft concrete-lined tunnel, and unlined rock tunnel. It was also possible to measure the loss in one of the bends and in the transitions from the 18-ft tunnels to the 16-ft steel pipe.

TYPES OF CONDUIT SURFACES

The types of surfaces on which friction losses were measured varied widely. Since a clear understanding of the characteristics of these surfaces is necessary for intelligent use of the data, they are described in some detail.

Steel Pipe.—The various steel pipe sections were formed of plates rolled to the proper curvature and butt welded. After the pipes were completed, the inner surfaces were covered with bituminous paint, applied hot with swabs. Each application of the swab covered an area approximately 6 in. wide and 2 ft long and deposited a layer of bituminous material approximately $\frac{1}{16}$ in. deep. The cold steel cooled the paint so rapidly that the resulting surface was quite irregular and included ridges and bumps up to $\frac{1}{8}$ in. high. However, the surface of the paint itself was almost glossy. Fig. 2 shows the interior of one of the steel pipes treated in this manner.

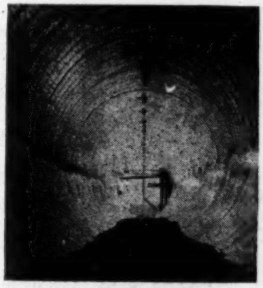


FIG. 2.—INTERIOR OF STEEL PIPE COATED WITH HOT BITUMINOUS PAINT

Between Stations 255+97.50 and 259+17.67 the steel liner was not painted hot because of the lack of ventilation in this section at the time of painting. Instead, the bituminous paint was applied cold in a slightly diluted condition. This produced a smoother surface than that in the other pipe sections.

Concrete Tunnel.—The 18-ft concrete tunnel lining was poured in two stages. An 80° invert section was poured first. The surface of the invert, which was formed by a screed traveling along the top of the longitudinal invert forms, was floated with steel floats immediately after the screed had passed. The quality of the concrete was controlled very carefully so that the best possible surface would result. In addition, concreting was carried on continuously so that no joint irregularities would occur in the invert section.

The arch section was poured against a three-piece steel form, of which the top piece covered a 90° sector and the two side pieces 95° each. Hinged at each joint, these form sections, each 5 ft long, could be collapsed and moved ahead through the other sections. Thus, irregularities due to transverse form joints occurred every 5 ft on the final surface. For convenience in erection, four form sections were kept bolted together as a unit during each change. There is no reason to believe that irregularities at the unit joints are larger than any of the others, because the units were lined up by drift pins in the same manner as were the sections comprising the units. Each piece of each

FIG. 3.—CONCRETE-LINED TUNNEL SECTION AFTER REMOVAL OF STEEL FORMS

FIG. 4.—COMPLETED UNLINED TUNNEL SECTION

section was equipped with an 18-in. by 24-in. inspection door and two or three 2-in. grout pipe holes. Each of these left a slight irregularity on the surface, averaging probably not more than $\frac{1}{32}$ in. high. Some 400 ft of forms were used so that concrete could be placed continuously without resorting to transverse bulkheads.

After the forms were removed, the surface was given a coat of curing compound. This compound disintegrates with time, however, and was probably completely washed off by the time the tests were made. Fig. 3 shows the concrete tunnel lining after the forms were removed and before the curing compound was applied.

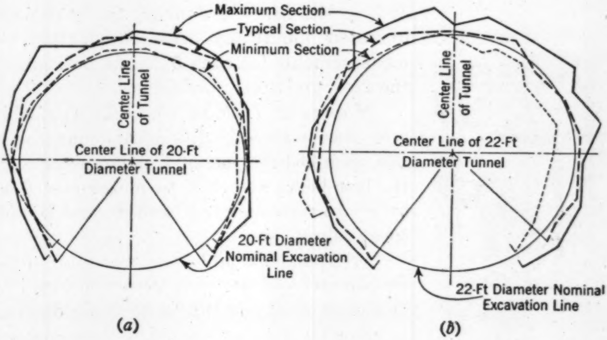

Maximum Section
Typical Section
Minimum Section
Center Line of Tunnel
Center Line of 20-Ft Diameter Tunnel
20-Ft Diameter Nominal Excavation Line
Center Line of Tunnel
Center Line of 22-Ft Diameter Tunnel
22-Ft Diameter Nominal Excavation Line
(a)
(b)

FIG. 5.—MAXIMUM, MINIMUM, AND TYPICAL SECTIONS OF UNLINED ROCK TUNNEL

Unlined Rock Tunnel.—Unlined rock tunnel sections are of two nominal diameters, but are otherwise similar. Headings were advanced by shooting drifts 11 ft long; and, although the nominal diameter was approximated at the beginning of each drift, the actual diameter gradually increased away from the heading. In addition, the unusually hard rock broke quite irregularly, as shown in Fig. 4. Maximum, minimum, and typical sections for both the 20-ft-diameter and 22-ft-diameter unlined sections are given in Fig. 5. However, the bottom of the tunnel had not been cleaned out when the cross sections were measured so that the exact shape of the bottom 80° to 90° is unknown. Areas and perimeters have been determined by measuring the known part and assuming the remainder of the cross section to be similar. Characteristics of the unlined sections are summarized in Table 1.

TABLE 1.—CHARACTERISTICS OF UNLINED ROCK TUNNEL SECTIONS

Nominal diameter (ft)	AREA, IN SQUARE FEET			Diameter, equivalent circle (ft)	HYDRAULIC RADIUS, IN FEET		Average overbreak (ft)
	Maximum	Minimum	Average		Equivalent	Measured	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
20...	419	335	375.3	21.86	5.465	5.36	0.93
22...	504	368	430.8	23.42	5.855	5.68	0.71

FIELD TESTS

The field observations were made at the same time as the turbine index tests. This procedure was possible because both types of tests required that the turbines be operated over a large range of gate openings with each gate position held constant for a sufficient length of time to allow the flow to become steady.

On January 25, 1944, unit 1 was operated alone. Satisfactory pressure measurements were obtained for each 5% gate position between the 45% and 80% positions. On January 26, both units were operated identically with satisfactory pressure measurements for each 5% gate position between the 25% and 100% positions.

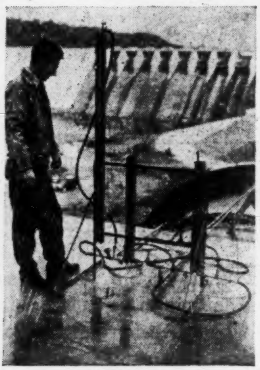


FIG. 6.—OPEN MERCURY MANOMETER (LEFT) AND WATER-AIR DIFFERENTIAL MANOMETERS (RIGHT) AT APALACHIA DAM ADIT

Manometer Installations.—At Apalachia Dam adit, two water-air differential manometers and one open U-tube mercury manometer were used, the last being attached to piezometer C and the others to piezometers A and B, and B and C, respectively.

At Apalachia adit, one water-air differential manometer and one open U-tube mercury manometer were used. In this location the open mercury manometer was connected to piezometer E and the water-air differential manometer to piezometers D and E.

At both Turtletown Creek adit and McFarland adit, single-column 10-ft mercury manometers were used. These manometers, constructed in the shop of the hydraulic laboratory of the Tennessee Valley Authority and carefully calibrated, were made in two sections to facilitate carrying.

FIG. 7.—MANOMETER INSTALLATION (ARROW) ON HILLSIDE AT TURTLETOWN CREEK ADIT

At the Apalachia Dam adit, the pressures were low enough so that the gages could be installed nearly at tunnel level. Fig. 6 shows this installation. It will be noted that each manometer connection was equipped with a blowoff cock so that the piezometer lines could be flushed frequently.

Maximum and minimum pressures at the other three locations were too great to allow the manometers to be installed at tunnel level. Therefore, it was necessary to place them on the hillsides above the tunnel. Fig. 7, which shows the location of the manometer at the Turtletown Creek adit, illustrates the difficulty encountered in finding suitable manometer sites at the proper elevations. Data from all manometers were tied together through the level system established for the tunnel construction.

Connection of the manometers to the tunnel was by $\frac{3}{4}$-in. pipe. The long lines used for most of these connections could have been a source of considerable trouble due to differential temperature effects. Fortunately, however, the tests were conducted on cloudy days and air temperatures were within a few degrees of the water temperature. Thus, it was not necessary to correct for changes in density of the water in the connecting lines.

Observations.—Because each of the four locations was entirely isolated from the others and from the powerhouse, it was necessary to take readings at 1-min intervals continuously throughout the test period to insure a complete record. All observers' watches were carefully compared at the beginning and end of each day's observations and all time readings were corrected to correspond with a master electric clock at the powerhouse. At each location the manometers were operated and read by an experienced member of the hydraulic laboratory's engineering staff.

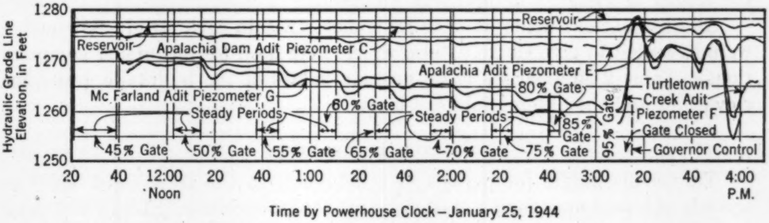


Fig. 8.—Hydraulic Grade Line Elevations for Operation of Turbine Unit 1, Showing Steady Periods for Which Data Were Used in Calculations of Heat Losses

Manometer readings were reduced to elevations of the pressure grade line. These elevations, as well as the differential pressures given by the water-air differential manometers, are shown in Table 2. Reservoir elevations for the test periods were taken from the chart of the headwater recording gage at the dam and are also shown in Table 2.

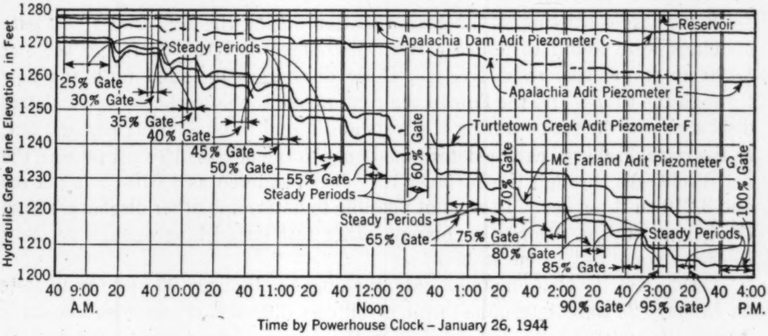


Fig. 9

The records for both days, shown as Fig. 8 and Fig. 9, have numerous breaks in them due to various causes. At some of the sites it was impossible to flush the lines in less than 1 min. Therefore, a break occurred every time this was done. On January 26, rain interfered several times with the recording of measurements, as the data paper became so wet that a legible record could not be kept. A long break occurred in the Turtletown record on January 26 because the top of the mercury column was behind the rubber hose used to connect the two 5-ft gage sections.

Periods during which the turbine wicket gate positions were changed were marked as were also those during which steady flow conditions existed throughout the length of the tunnel. An immediate response in pressure throughout the tunnel to a change in discharge was noted.

All observations made on each manometer within each steady flow period were averaged to give single values, which were used for all further computations. Table 2 summarizes these average values of the hydraulic grade line elevations and the differential head losses.

Tunnel Discharges

Tunnel discharges for periods of steady flow in the river were based on records obtained at a gaging station below the powerhouse. These were supplemented by interpolating discharges based on differential pressures measured on the turbine scroll cases, since steady river flow was only observed at two times during the test period due to the rapidity with which the tests were run. Steady flow conditions existed in the tunnel for an hour or more at gate openings of 25% and 100% on January 26. These periods were long enough so that flow in the river at the gaging station became stable, thus making it possible to determine the river discharges from the gage heights. Discharge measurements made at the station by the United States Geological Survey were carefully examined and replotted to give the best rating curve for the time of the friction measurements.

River stages existing at the gaging station during the 25% and 100% gate openings were found to indicate river discharges of 1,150 cu ft per sec and 3,290 cu ft per sec, respectively. These discharges included both the flow through the tunnel and the flow in the river above the powerhouse. The latter discharge was measured during the tests and found to be 84 cu ft per sec; but, as the river discharge can be determined only to the nearest 10 cu ft per sec, the net discharges through the turbines may be considered as 1,070 cu ft per sec and 3,210 cu ft per sec. It was not possible to determine other discharges from the gaging station records because no other test lasted long enough to allow the river stage to become stable.

Discharges for the remaining gate openings on January 26 and for all tests on January 25 were determined from scroll case differential pressures measured during the index tests. In addition to the differential pressures, the turbine gate openings were recorded in terms of the movement of the servomotor piston that opens the gates. This movement was measured to the nearest $\frac{1}{32}$ in. and is believed reliable.

Known relationships between piston movement, gate opening, discharge, and differential pressure were used to adjust discrepancies in the observed data.

TABLE 2.—ELEVATIONS OF HYDRAULIC GRADE LINE AND MEASURED HEAD LOSSES

Gate opening (%)	GRADE LINE ELEVATION (FEET ABOVE EL. 1200) AT PIEZOMETERS:					MEASURED LOSS, IN FEET, IN SECTIONS:		
	....	C	E	F	G	A-B	B-C	D-E
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Adit[a]	Reservoir	Apalachia Dam	Apalachia	Turtletown Creek	McFarland	18-ft steel pipe	Bend 1	Apalachia transition
(a) JANUARY 25								
45........	77.65	76.96	75.85	....	....	0.047	0.076	0.120
50........	77.75	76.98	75.59	70.75	69.50	0.064	0.096	0.148
55........	77.82	76.99	75.23	69.39	67.91	0.073	0.110	0.177
60........	77.84	76.80	74.77	68.07	66.17	0.089	0.128	0.211
65........	77.91	76.83	74.48	66.80	64.64	0.098	0.145	0.232
70........	77.93	76.79	74.07	65.44	63.04	0.110	0.166	0.260
75........	78.00	76.74	73.75	64.15	61.64	0.123	0.183	0.287
80........	78.01	76.61	73.34	62.92	60.13	....	....	0.312
(b) JANUARY 26								
25........	78.38	77.64	76.29	71.46	70.51	0.051	0.088	0.145
30........	78.47	77.50	75.48	68.76	67.23	0.075	0.131	0.199
35........	78.55	77.31	....	65.13	62.62	0.108	0.188	....
40........	78.58	77.02	73.15	61.10	57.54	0.144	0.241	0.368
45........	78.62	76.72	71.93	57.13	52.98	0.177	0.297	0.450
50........	78.61	76.40	70.62	52.78	47.65	0.210	0.357	0.546
55........	78.66	76.08	69.32	48.61	42.59	0.251	0.411	0.634
60........	78.68	75.78	67.96	....	37.24	0.290	0.475	0.724
65........	78.69	75.48	66.63	39.96	31.94	0.325	0.535	0.820
70........	78.70	75.11	65.33	35.86	27.02	0.363	0.592	0.899
75........	78.70	74.78	64.12	31.81	22.04	0.396	0.645	0.994
80........	78.69	74.46	62.84	27.91	17.28	0.432	0.699	1.066
85........	78.70	74.18	61.75	24.43	13.10	0.458	0.747	1.147
90........	78.69	73.92	60.76	21.28	9.14	0.489	0.788	....
95........	78.67	73.71	....	18.45	5.70	0.521	0.834	..
100........	78.66	73.51	....	16.72	3.68	0.536	0.862	..
100........	78.65	....	59.28	16.69	3.67	....	....	1.315

[a] Names of adit where piezometers were located; or of section where loss was measured.

The actual gate opening depends on the movement of the servomotor piston. In turn, the discharge varies with the gate opening, and the differential pressure with the discharge. Thus, the differential pressure should vary smoothly with the observed movement of the servomotor piston.

It is probable that the measurement of piston travel is more accurate than the measurement of differential pressure. Accordingly, the differential pressure for each gate opening was taken from smooth curves drawn through the points determined by the observed relationship between piston movement and differential pressure. Table 3 summarizes, for each gate opening on January 25 and 26, the piston movement, the observed differential pressure, and the adjusted value taken from the smoothed curve.

The known discharge of 3,210 cu ft per sec for both units at 100% gate opening on January 26 was divided between the two units in proportion to their power outputs, which index test data showed to be 39,790 kw and 40,000

TABLE 3.—GATE POSITIONS, DIFFERENTIAL PRESSURES, AND DISCHARGES

Gate opening (%)	(a) JANUARY 25				(b) JANUARY 26								
	Unit 1				Unit 1			Unit 2			Discharge, in Cubic Feet Per Second		
	Piston movement (in.)	Differential Pressure (In. of Mercury)		Discharge (cu ft per sec)	Piston movement (in.)	Differential Pressure (In. of Mercury)		Piston movement (in.)	Differential Pressure (In. of Mercury)		Unit 1	Unit 2	Total
		Observed	Adjusted			Observed	Adjusted		Observed	Adjusted			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
25....	4.22	1.32	1.32	525	4.25	1.345	1.35	3.19	1.360	1.36	530	530	1,060
30....	4.69	1.99	1.97	640	4.75	1.926	1.90	3.72	1.964	1.99	628	640	1,270
35....	5.19	2.76	2.74	755	5.25	2.545	2.57	4.25	2.726	2.70	730	745	1,475
40....	5.69	3.56	3.56	860	5.72	3.405	3.35	4.78	3.516	3.50	833	850	1,685
45....	6.25	4.60	4.59	975	6.25	4.255	4.25	5.28	4.313	4.31	939	943	1,880
50....	6.75	5.56	5.58	1,075	6.75	5.112	5.12	5.78	5.188	5.19	1,030	1,033	2,060
55....	7.28	6.67	6.72	1,180	7.22	5.938	5.90	6.31	6.163	6.14	1,106	1,124	2,230
60....	7.75	7.77	7.73	1,265	7.72	6.819	6.75	6.81	6.978	6.98	1,182	1,198	2,380
65....	8.25	8.84	8.80	1,350	8.22	7.455	7.58	7.31	7.883	7.83	1,252	1,269	2,520
70....	8.75	9.85	9.87	1,430	8.75	8.500	8.45	7.81	8.528	8.62	1,322	1,331	2,650
75....	9.25	10.93	10.93	1,500	9.25	9.330	9.26	8.28	9.598	9.36	1,383	1,387	2,770
80....	9.75	12.00	11.98	1,575	9.75	10.020	10.03	8.81	9.971	10.15	1,440	1,443	2,880
85....	10.22	13.09	12.95	1,640	10.25	10.620	10.73	9.31	10.905	10.86	1,490	1,495	2,980
90....	10.72	13.82	13.97	1,700	10.72	11.305	11.33	9.81	11.700	11.53	1,530	1,540	3,070
95....	....	....	....	....	11.25	12.110	11.92	10.37	12.253	12.18	1,570	1,582	3,150
100....	....	....	....	....	11.73	12.195	12.36	10.81	12.425	12.61	1,600	1,610	3,210

kw for units 1 and 2, respectively. Assuming that both units were operating at the same head and efficiency, the corresponding discharges were 1,600 cu ft per sec and 1,610 cu ft per sec. Discharges for each unit at other gate openings were then calculated by assuming discharge to be proportional to the square root of the differential pressure. These values for units 1 and 2 on January 26 are also given in Table 3, which shows the calculated discharge for 25% gate opening to be 1,060 cu ft per sec. This compares favorably with the value of 1,070 cu ft per sec obtained from the rating curve. Discharges for unit 1 on January 25 have been calculated from the same relationship and are also given in Table 3.

REDUCTION OF TEST DATA

The hydraulic grade line elevations and differential pressures given in Table 2, together with the discharge rates in Table 3 and the general data shown in Fig. 1, were sufficient for computation of the various losses in the test sections of the tunnel.

Friction losses for each of the three types of surface were expressed in terms of the friction coefficient f in the Weisbach equation:

$$h_f = f \frac{L}{D} \frac{V^2}{2g} \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

and the results are plotted as a function of Reynolds' number **R**, in which

$$\mathbf{R} = \frac{V D}{\nu} \dots\dots\dots\dots\dots\dots\dots\dots (2)$$

In these equations, L is the length of any test reach; D is the diameter of the pipe; V is the average flow velocity; h_f is the energy loss, in feet, over the length L; and ν is the kinematic viscosity of the flowing water. Since air and water were both very close to 60° F during the tests, the value of ν has been taken throughout the calculations as 1.2×10^{-5}, which is its value at that temperature.

Values of the roughness coefficient n in the Manning equation:

$$V = \frac{1.486}{n} R^{\frac{2}{3}} S^{\frac{1}{2}} \dots\dots\dots\dots (3)$$

were also computed. In this equation, V is the average velocity; R is the hydraulic radius; and S is the slope of the energy grade line. These results are shown on plots of n versus V.

Entrance and bend losses were expressed in terms of the velocity head by the equation:

$$H = C \frac{V^2}{2\,g} \dots\dots\dots\dots (4)$$

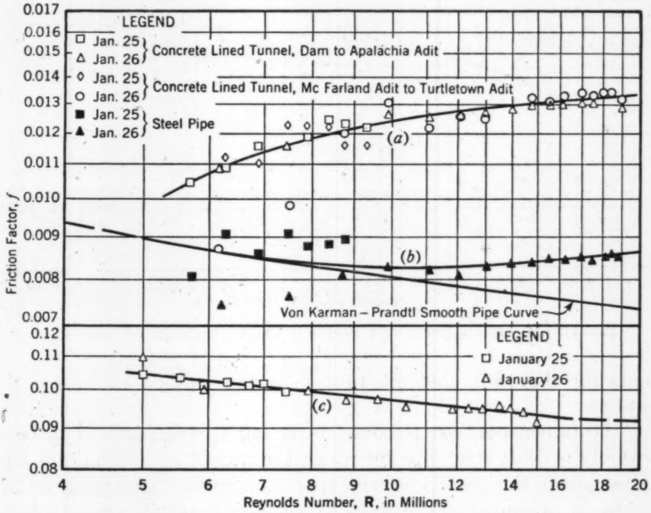


FIG. 10.—PLOT OF FRICTION FACTOR f VERSUS REYNOLDS' NUMBER R
(a) 18-Ft Bituminous-Coated Steel Pipe
(b) 18-Ft Concrete-Lined Tunnel
(c) Unlined Rock Tunnel

in which H, with an appropriate subscript, is the head loss in question; and C, with the same subscript, is the corresponding coefficient.

Losses in the tunnel transitions were expressed in terms of the difference in velocity heads by the equation:

$$H_T = C_T \frac{V^2_1 - V^2_2}{2\,g} \dots\dots\dots\dots (5)$$

In Eq. 5, V_1 and V_2 are the velocities before and after the transition, respectively.

Friction Loss in 18-Ft Steel Pipe.—Friction and roughness coefficients, f and n, for the 18-ft steel pipe were computed by Eqs. 1 and 3 from the discharges given in Table 3, the observed pressure drops given in Col. 7, Table 2, and the dimensions of the test section taken from Fig. 1. These values are given in Table 4 and are plotted as f versus **R** in Fig. 10(*a*) and as n versus V in Fig. 11(*a*).

Preliminary Bend Losses.—The pressure drop measured between piezometers B and C was composed of the loss due to bend 1 and the friction loss in 198 ft of tunnel, of which 120 ft is 18-ft steel pipe and 78 ft is 18-ft concrete-lined tunnel. As both test sections on the 18-ft concrete-lined tunnel contain bends, it was necessary to determine the bend-loss coefficients and the friction coefficients for the 18-ft concrete tunnel by the method of successive approximations.

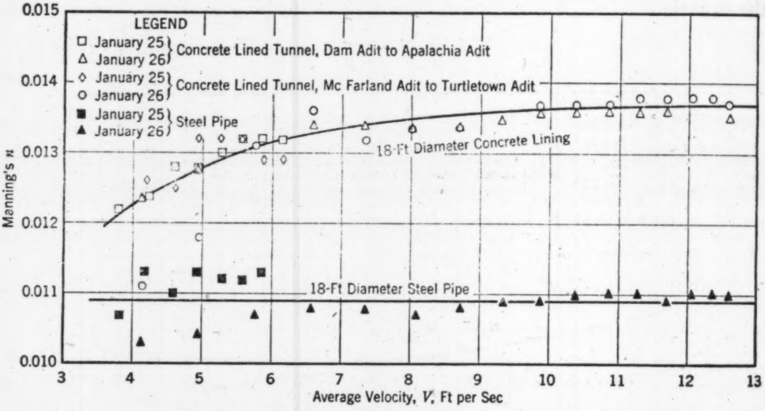


FIG. 11.—MANNING'S ROUGHNESS FACTOR n VERSUS AVERAGE VELOCITY V

First, the tunnel between piezometers B and C was assumed to be composed entirely of 18-ft steel pipe for which the friction coefficient was known. The friction loss in this reach was then approximated by assuming it to be proportional to the measured loss between piezometers A and B. Subtracting this calculated friction loss from the total measured loss gave the loss for bend 1, from which the bend-loss coefficient C_B was computed by Eq. 4. The value of C_B thus determined was applied to bends 2 and 3 between piezometers C and D for determination of n for the concrete pipe. With this coefficient determined, the loss in bend 1 was recalculated. This change in bend loss, however, when applied to bends 2 and 3, did not measurably change f and n as first calculated for the concrete. Table 4 gives the calculated preliminary value of C_B for each rate of discharge; the average preliminary value of C_B for bend 1 is 0.253.

Table 5 lists the six bends included in the test section, together with the deflection angle, the radius, and the diameter of pipe of each. Available information concerning bend losses indicates that C_B varies with the deflection angle and the ratio of bend radius to pipe diameter. Relative values of the coefficient in terms of these variables are given by William P. Creager and Joel D. Justin,[5] Members, ASCE, whose values of K, reflecting the effect of deflection angle, are given in Col. 5. Col. 6 gives values of $\frac{r}{D}$ for each bend and Col. 7 gives the corresponding values of the bend-loss coefficient for a 90° bend. The preliminary values of C_B for bends 1, 2, and 3 are given in Col. 8. For bend 1, this value is found from the calculations of Table 4. For bends 2

TABLE 4.—FRICTION-LOSS COMPUTATIONS

Gate opening (%)	Velocity (ft per sec)	EIGHTEEN-FOOT STEEL PIPE			BEND LOSS[a]		EIGHTEEN-FOOT CONCRETE-LINED TUNNEL[b]				
		Manning's n	Weisbach's f	Reynolds' R	Net loss (ft)	Coefficient, C_B	Computed Loss (Ft)		Manning's n	Weisbach's f	Reynolds' R
							Bend	Friction			
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
					(a) JANUARY 25						
45...	3.83	0.0107	0.00805	5.7×10^6	0.056	0.246	0.11	0.88	0.0122	0.0105	5.7×10^6
50...	4.22	0.0113	0.00903	6.3×10^6	0.068	0 246	0.13	1.11	0.0124	0.0109	6.3×10^6
55...	4.63	0.0110	0.00857	6.9×10^6	0.079	0.237	0.16	1.42	0.0128	0.0116	6.9×10^6
60...	4.97	0.0113	0.00906	7.5×10^6	0.090	0.234	0.18	1.64	0.0128	0.0116	7.5×10^6
65...	5.30	0.0112	0.00877	7.9×10^6	0.103	0.236	0.21	1.91	0.0130	0.0119	7.9×10^6
70...	5.61	0.0112	0.00880	8.4×10^6	0.119	0.244	0.23	2.23	0.0132	0 0124	8.4×10^6
75...	5.89	0.0113	0.00892	8.8×10^6	0.130	0.241	0.26	2.44	0.0132	0.0123	8.8×10^6
80...	6.19	....	....	9.3×10^6	....	....	0.28	2.68	0.0132	0.0122	9.3×10^6
					(b) JANUARY 26						
25...	4.16	0.0103	0.00742	6.2×10^6	0.066	0.246	0.13	1.08	0.0124	0.0109	6.2×10^6
30...	4.99	0.0104	0.00758	7.5×10^6	0.099	0 256	0.18	1.64	0.0128	0.0115	7.5×10^6
35...	5.79	0.0107	0.00810	8.7×10^6	0.142	0 272	0.25	....	....	....	8.7×10^6
40...	6.61	0.0108	0.00829	9.9×10^6	0.179	0.264	0.32	3.18	0.0134	0.0127	9.9×10^6
45...	7.38	0.0108	0.00817	11.1×10^6	0.221	0.261	0.40	3.94	0.0134	0.0126	11.1×10^6
50...	8.09	0.0107	0.00806	12.1×10^6	0.267	0.262	0.49	4.74	0.0134	0.0126	12.1×10^6
55...	8.75	0.0108	0.00825	13.1×10^6	0.303	0.254	0.57	5.56	0.0134	0.0127	13.1×10^6
60...	9.35	0.0109	0 00835	14.0×10^6	0.350	0 258	0 65	6.45	0.0135	0.0129	14.0×10^6
65...	9.89	0.0109	0.00836	14.8×10^6	0.395	0.260	0.73	7.30	0.0136	0.0130	14.8×10^6
70...	10.40	0.0110	0.00844	15.6×10^6	0.436	0.259	0.80	8.08	0.0136	0.0130	15.6×10^6
75...	10.88	0.0110	0.00842	16.3×10^6	0.475	0.258	0.88	8.79	0.0136	0.0130	16.3×10^6
80...	11.31	0.0110	0.00850	17.0×10^6	0.513	0.258	0.95	9.60	0.0136	0.0131	17.0×10^6
85...	11.70	0.0109	0.00841	17.5×10^6	0.550	0.258	1.02	10.26	0.0136	0.0131	17.5×10^6
90...	12.07	0.0110	0.00845	18.1×10^6	0.578	0.255	1.08	....	....	....	18.1×10^6
95...	12.38	0.0110	0.00855	18.5×10^6	0.610	0 256	1.14	....	....	....	18.5×10^6
100...	12.61	0.0110	0.00848	18.9×10^6	0.631	0.256	1.18	11.73	0.0135	0.0129	18.9×10^6
Average	....	0.0109	....	....	....	0.253	....	....	....	....	....

[a] Preliminary computations of bend loss for bend 1. [b] From the dam adit to Apalachia adit.

---

[5] "Hydro-electric Handbook," by William P. Creager and Joel D. Justin, John Wiley & Sons, Inc., New York, N. Y., 1927, pp. 119–120.

and 3, the value of C_B is the preliminary value for bend 1 multiplied by the ratios of appropriate factors in Cols. 5 and 7. For example, $C_{B2} = C_{B1} \times \frac{K_2}{K_1} \times \frac{F_2}{F_1} = 0.253 \times \frac{0.311}{0.571} \times \frac{0.45}{0.28} = 0.222$.

Friction Loss in 18-Ft Concrete-Lined Tunnel, Apalachia Dam Adit to Apalachia Adit.—Between Apalachia Dam adit and Apalachia adit, the tunnel is concrete lined, 18 ft in diameter, and contains two bends. Elevations of the hydraulic grade line at piezometers C and E and the differential pressures between piezometers D and E are given in Table 2. Addition of the differentials between piezometers D and E to the hydraulic grade line elevations for E gave the grade line elevations at D; subtraction of these elevations from those at C gave the friction losses between piezometers C and D.

Losses for bends 2 and 3 were calculated by Eq. 4 from the preliminary values of C_B given in Table 5, and values of n and f were calculated from Eqs. 1 and 3. The results of these calculations are summarized in Table 4. Fig. 10(*b*) shows f plotted against R and Fig. 11 gives values of n in terms of V.

TABLE 5.—BEND-LOSS COEFFICIENTS

Bend No. (1)	Deflection angle (2)	Radius (ft) (3)	Pipe diameter (ft) (4)	Deflection factor, K (5)	Ratio, $\frac{r}{D}$ (6)	Loss coefficient,[a] F (7)	COEFFICIENT C_B		Reported[b] (10)
							Preliminary (8)	Corrected (9)	
1......	34° 15′ 34″	90	18	0.571	5.00	0.28	0.253	0.236	0.160
2......	15° 43′ 05″	250	18	0.311	13.9	0.45	0.222	0.207	0.140
3......	18° 18′ 53″	250	18	0.359	13.9	0.45	0.256	0.239	0.162
4......	12° 40′ 35″	90	16	0.257	5.63	0.30	....	0.114	0.077
5......	4° 41′ 51″	250	18	0.100	13.9	0.45	....	0.066	0.045
6......	8° 43′ 25″	250	18	0.179	13.9	0.45	....	0.119	0.081

[a] For a 90° bend. [b] As reported by William P. Creager and Joel D. Justin,[5] Members, ASCE.

Corrected Bend Losses.—Having determined the roughness coefficients for the concrete-lined tunnel, it was possible to recalculate the loss coefficients for bend 1. This calculation was made by using the average value of n determined for the steel pipe and by taking values of n for the concrete section of the tunnel from a curve drawn through the points determined for that section of the tunnel between Apalachia Dam adit and Apalachia adit. This curve was omitted from Fig. 11 as it differed so slightly from the mean curve shown there that recalculation of the bend losses on the basis of the final curve was unnecessary. From the results of these computations, summarized in Table 6, the average value of C_B for bend 1 was found to be 0.236. Corrected values of C_B for all bends are given in Col. 9, Table 5. These values are within the range commonly used for bend-loss coefficients and they lie between the values given by Messrs. Creager and Justin[5] for "safe" and "probable" values. Col. 10, Table 5, is based on the curve of probable values presented by Messrs. Creager and Justin and is the product of Cols. 5 and 7. These values are less than those based on the measurements at bend 1.

Transition Loss.—The differential manometer between piezometers D and

E at Apalachia adit measured the total head loss due to friction in 96 ft of 18-ft concrete-lined tunnel, 264 ft of steel pipe 16 ft in diameter, and 40 ft of concrete-lined transition sections having an average diameter of 17 ft, and also the loss due to bend 4 and the loss caused by the expansion from the 16-ft diameter to the 18-ft diameter.

TABLE 6.—COMPUTATION OF LOSS COEFFICIENTS

Gate opening (%)	CORRECTED BEND-LOSS CALCULATIONS[a]				TRANSITION-LOSS CALCULATIONS					
	Friction Loss (Ft)		Net bend loss (ft)	Bend coefficient, C_B	Computed Friction Loss (Ft)			Computed bend loss (ft)	Transition Loss	
	Steel pipe	Concrete tunnel			18-ft concrete tunnel	17-ft concrete tunnel	16-ft steel pipe		Net (ft)	Coefficient, C_T
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
					(a) JANUARY 25					
45.....	0.013	0.010	0.053	0.232	0.013	0.008	0.053	0.042	0.004	0.029
50.....	0.015	0.013	0.068	0.246	0.016	0.010	0.064	0.051	0.007	0.042
55.....	0.019	0.016	0.075	0.225	0.020	0.012	0.077	0.061	0.007	0.035
60.....	0.021	0.020	0.087	0.226	0.024	0.014	0.089	0.070	0.014	0.060
65.....	0.024	0.023	0.098	0.224	0.028	0.016	0.101	0.080	0.007	0.027
70.....	0.027	0.026	0.113	0.231	0.032	0.018	0.113	0.090	0.007	0.024
75.....	0.030	0.029	0.124	0.230	0.035	0.020	0.125	0.098	0.009	0.028
80.....	0.033	0.032	....	....	0.039	0.023	0.138	0.109	0.003	0.008
					(b) JANUARY 26					
25.....	0.015	0.013	0.060	0.223	0.016	0.009	0.062	0.049	0.009	0.055
30.....	0.022	0.020	0.089	0.230	0.024	0.014	0.090	0.071	0.000	0.000
35.....	0.029	0.027	0.132	0.253	0.034	0.020	0.121	0.096	....	....
40.....	0.038	0.037	0.166	0.244	0.045	0.026	0.157	0.124	0.016	0.039
45.....	0.047	0.046	0.204	0.241	0.057	0.033	0.196	0.155	0.009	0.018
50.....	0.057	0.057	0.243	0.239	0.070	0.039	0.235	0.186	0.016	0.026
55.....	0.067	0.066	0.278	0.234	0.082	0.047	0.276	0.218	0.011	0.015
60.....	0.076	0.076	0.323	0.238	0.094	0.054	0.314	0.249	0.013	0.016
65.....	0.085	0.086	0.364	0.239	0.106	0.060	0.352	0.278	0.024	0.026
70.....	0.094	0.095	0.403	0.240	0.117	0.066	0.389	0.308	0.019	0.019
75.....	0.103	0.104	0.438	0.238	0.128	0.072	0.425	0.336	0.033	0.030
80.....	0.111	0.113	0.475	0.239	0.139	0.078	0.459	0.364	0.026	0.022
85.....	0.119	0.121	0.507	0.238	0.149	0.084	0.493	0.390	0.031	0.024
90.....	0.127	0.128	0.533	0.235	0.158	0.089	0.522	0.413	....	....
95.....	0.133	0.135	0.566	0.238	0.166	0.094	0.549	0.435	....	....
100.....	0.138	0.140	0.584	0.236	0.173	0.097	0.570	0.451	0.024	0.016
Average	....	....	....	0.236	....	....	....	....	....	0.027

[a] For bend 1.

All these losses except that due to the transition can be calculated from data previously obtained. Values of roughness coefficients for the steel pipe and the concrete-lined tunnel were obtained in the same manner as those for the calculation of corrected bend loss. The difference between the total loss and the sum of the calculated friction and bend losses was taken as the loss due to expansion, the loss caused by the contraction being assumed negligible.

The value of the transition-loss coefficient C_T was calculated by Eq. 5.

The computations are summarized in Table 6, the average value of C_T for all tests being 0.027, which is 40% smaller than the value given by H. W. King,[8] Hon. M. ASCE.

Friction Loss in 18-Ft Concrete-Lined Tunnel, Turtletown Creek Adit to McFarland Adit.—Between Turtletown Creek adit and McFarland adit, the tunnel is largely a concrete-lined tunnel 18 ft in diameter like that between Apalachia Dam adit and Apalachia adit. It should, therefore, have the same friction and roughness coefficients. The computations are somewhat more difficult because the test section includes not only 6,944 ft of the 18-ft concrete lining but also 118 ft of steel pipe 18 ft in diameter, 598 ft of steel pipe 16 ft in diameter, 60 ft of concrete transitions having an average diameter of 17 ft, one contraction, two expansions, and bend 6. In addition, the tunnel does not have the same diameter at the two adits so that the measured drop in the hydraulic grade line is not equal to the drop in the energy grade line.

The drop in the energy grade line can be calculated by adding the velocity head at each piezometer to the elevation of the hydraulic grade line at that piezometer and taking the difference of these sums. Thus, if the energy grade line is represented by E and the hydraulic grade line by H, and the Turtletown Creek and McFarland adits by subscripts T and M, respectively, the drop in the energy grade line is $\Delta E = \left(H_T + \frac{V^2_T}{2g}\right) - \left(H_M + \frac{V^2_M}{2g}\right)$. This can be rewritten as $\Delta E = (H_T - H_M) + \left(\frac{V^2_T}{2g} - \frac{V^2_M}{2g}\right) = (H_T - H_M) + \Delta \frac{V^2}{2g}$. The friction loss in the 18-ft concrete-lined section of the tunnel is this value of ΔE less the other losses listed.

Loss in the steel pipe sections was calculated from Eq. 3 using n as 0.0109. Values of n for the concrete transitions with an average diameter of 17 ft were taken from the preliminary curve previously mentioned. Expansion losses were calculated using 0.027 as the value of C_T, and the loss coefficient for bend 6 was taken from Table 5. Contraction loss was neglected. Values of f and n for the 18-ft concrete were calculated from the net loss, using Eqs. 1 and 3. The computations are summarized in Table 7 and the results are plotted in Figs. 10(*b*) and 11.

Friction Loss in Unlined Rock Tunnel.—Between the Turtletown Creek and McFarland adits the tunnel consists of 3,098 ft of unlined rock tunnel 20 ft in nominal diameter, 2,902 ft of unlined rock tunnel 22 ft in nominal diameter, 7,626 ft of 18-ft concrete-lined tunnel, 20 ft of concrete-lined transition averaging 17 ft in diameter, and 369 ft of 16-ft steel pipe. Losses were also caused by bend 5, by the expansions from 18-ft concrete to 20-ft rock and from 20-ft rock to 22-ft rock, and by the contraction from 20-ft rock to 18-ft concrete. The contractions from 22-ft rock to 20-ft rock and from 18-ft concrete to 16-ft steel are so gradual that losses due to them were neglected.

Energy loss between piezometers E and F was not equal to the fall in the hydraulic grade line between the piezometers because of the smaller diameter, with corresponding higher velocity, at piezometer F. The energy loss was determined, however, by adding the difference between velocity heads at piezometers E and F to the elevation of the hydraulic grade line at piezometer F and subtracting the sum from the elevation of the hydraulic grade line at piezometer E. This process is similar to that employed in finding the energy loss between piezometers F and G. The loss in the unlined rock tunnel was found by subtracting from the total energy loss not only the friction losses in the concrete sections 18 ft and 17 ft in diameter and in the 16-ft steel pipe, but also the losses due to bend 5 and the expansions and contractions.

---

[8] "Handbook of Hydraulics for the Solution of Hydraulic Problems," by H. W. King, McGraw-Hill Book Co., Inc., New York, N. Y., 3d Ed., 1939, p. 191.

TABLE 7.—FRICTION LOSS IN AN 18-FT CONCRETE-LINED TUNNEL; TURTLETOWN CREEK ADIT TO McFARLAND ADIT

Gate opening (%)	COMPUTED FRICTION LOSSES (FT)			Computed transition loss (ft)	Computed bend loss (ft)	Net loss, 18-ft concrete (ft)	Manning's n	Weisbach's f	Reynolds' R
	16-ft steel pipe	18-ft steel pipe	17-ft concrete tunnel						
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
					(a) JANUARY 25				
45......	0.119	0.013	0.011	0.007	0.027	....	....	....	5.7×10^6
50......	0.145	0.015	0.015	0.009	0.033	1.20	0.0126	0.0112	6.3×10^6
55......	0.174	0.018	0.018	0.011	0.040	1.42	0.0125	0.0110	6.9×10^6
60......	0.201	0.021	0.021	0.013	0.046	1.83	0.0132	0.0123	7.5×10^6
65......	0.228	0.024	0.024	0.014	0.052	2.08	0.0132	0.0123	7.9×10^6
70......	0.256	0.027	0.028	0.016	0.058	2.32	0.0132	0.0123	8.4×10^6
75......	0.282	0.030	0.031	0.018	0.064	2.42	0.0129	0.0116	8.8×10^6
80......	0.311	0.033	0.034	0.019	0.071	2.68	0.0129	0.0116	9.3×10^6
					(b) JANUARY 26				
25......	0.141	0.015	0.014	0.009	0.032	0.90	0.0111	0.0087	6.2×10^6
30......	0.202	0.021	0.021	0.013	0.046	1.46	0.0118	0.0098	7.5×10^6
35......	0.272	0.029	0.030	0.017	0.062	2.42	0.0131	0.0120	8.7×10^6
40......	0.355	0.037	0.039	0.022	0.081	3.44	0.0136	0.0131	9.9×10^6
45......	0.442	0.046	0.049	0.028	0.101	3.99	0.0132	0.0122	11.1×10^6
50......	0.531	0.056	0.059	0.033	0.121	4.95	0.0134	0.0126	12.1×10^6
55......	0.624	0.065	0.070	0.039	0.142	5.81	0.0134	0.0126	13.1×10^6
60......	0.710	0.075	0.080	0.044	0.162	....	....	....	14.0×10^6
65......	0.795	0.084	0.090	0.050	0.181	7.74	0.0137	0.0132	14.8×10^6
70......	0.880	0.092	0.100	0.055	0.200	8.53	0.0137	0.0131	15.6×10^6
75......	0.960	0.101	0.109	0.060	0.219	9.43	0.0137	0.0133	16.3×10^6
80......	1.037	0.109	0.118	0.065	0.236	10.27	0.0138	0.0134	17.0×10^6
85......	1.113	0.117	0.126	0.070	0.253	10.94	0.0138	0.0133	17.5×10^6
90......	1.18	0.124	0.134	0.073	0.269	11.72	0.0138	0.0134	18.1×10^6
95......	1.24	0.131	0.141	0.077	0.283	12.31	0.0138	0.0134	18.5×10^6
100......	1.29	0.136	0.146	0.080	0.294	12.58	0.0137	0.0132	18.9×10^6

Friction loss in the 18-ft concrete-lined tunnel was computed from Eq. 3 with the n-values taken from the smooth curve shown in Fig. 11. Friction loss due to the 20 ft of transition was also calculated by this method. In the case of the 16-ft steel pipe, however, the loss was computed with $n = 0.0109$, the value determined for the 18-ft steel pipe. Hydraulic loss due to bend 5 was handled in the same manner as was that for bends 2, 3, and 6, the value of C_B for use in Eq. 4 being taken from Table 5.

Loss in the transition from the 18-ft concrete-lined tunnel to the 20-ft unlined tunnel was calculated by Eq. 5, but the value of C_T to be used is open to some question. On the basis of data supplied by Professor King,[6] C_T was estimated at 0.9. Similarly, based on Professor King's data from the same source and on the value of 0.027 for C_T for the expansion from the 16-ft pipe to the 18-ft concrete tunnel, C_T for the expansion from the 20-ft to the 22-ft rock section was estimated at 0.05.

At the entrance to the 18-ft lined tunnel, the concrete lining has a 9-in. chamfer, as shown in Fig. 1, detail D. Tests on concrete box culverts with a similar beveled entrance[7] indicate that the loss at entrance may be expressed by Eq. 5 with a coefficient of 0.15.

It was not possible to determine roughness coefficients for the 20-ft and 22-ft unlined tunnels separately. However, values of *n* and *f* were computed from Eqs. 3 and 1, assuming that the same values apply equally to both sections. Values of *n* are plotted in Fig. 12 against the average velocity for the two sections. In Fig. 10(*c*), *f* is plotted against a value of **R** based on the average diameters and areas of the two sections. Calculations of these factors are summarized in Table 8. Col. 5, Table 8, includes the bend loss, contraction loss, and expansion losses, all of which were calculated by expressing all velocities in terms of the velocity in the 18-ft tunnel and using that velocity for the calculations.

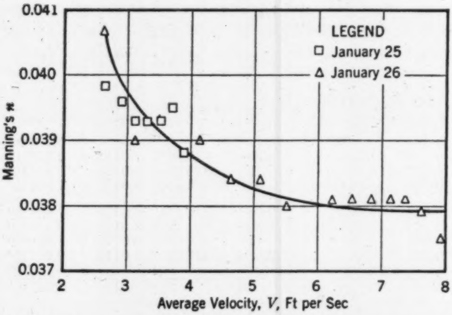


FIG. 12.—MANNING'S ROUGHNESS FACTOR *n* VERSUS AVERAGE VELOCITY *V* FOR UNLINED ROCK TUNNEL

In making the computations of *f* and *n*, the diameter was taken as that of a circle whose area was equivalent to the average area of the section. The hydraulic radius used was one fourth of the equivalent diameter. This assumption is not quite correct because of the irregularity of the tunnel cross sections. For example, the average hydraulic radii of the 20-ft and 22-ft tunnels as measured on the cross sections are actually 5.36 ft and 5.60 ft instead of 5.465 ft and 5.855 ft, respectively. Thus, use of these values in Eq. 1 produces values 2.3% smaller than those given in Table 8. Similarly, when used in Eq. 3, these values give *n*-values 1.5% smaller than those in Table 8. The differences are small, however; and, in view of the uncertainty of actual tunnel shapes (which depend on the character of rock and the manner in which it breaks, as well as on the care taken in measuring the irregular sections), it is believed better to use the simpler assumption of an equivalent circular area and the corresponding hydraulic radius.

SUMMARY

Tests on the Apalachia tunnel were made at discharges between 975 cu ft per sec and 3,210 cu ft per sec. Friction and roughness coefficients were de-

---

[7] "The Flow of Water Through Culverts," *Bulletin No. 1*, Studies in Engineering, Univ. of Iowa, Iowa City, 1926, p. 1119.

TABLE 8.—Friction Losses in Unlined Rock Tunnel

Gate opening (%)	Computed Friction Losses			Computed miscellaneous losses (ft)	Net loss, unlined tunnel (ft)	Manning's n	Weisbach's f	Reynolds'[a] R
	18-ft concrete tunnel	17-ft concrete tunnel	16-ft steel pipe					
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
				(a) January 25				
45	1.02	0.00	0.07	0.14	....	....	....	4.6×10^6
50	1.27	0.00	0.09	0.18	3.13	0.0398	0.104	5.0×10^6
55	1.58	0.01	0.11	0.21	3.73	0.0396	0.103	5.5×10^6
60	1.88	0.01	0.12	0.24	4.22	0.0393	0.101	5.9×10^6
65	2.17	0.01	0.14	0.28	4.82	0.0393	0.102	6.3×10^6
70	2.48	0.01	0.16	0.31	5.37	0.0393	0.101	6.7×10^6
75	2.77	0.01	0.17	0.34	5.98	0.0395	0.102	7.0×10^6
80	3.10	0.01	0.19	0.38	6.38	0.0388	0.099	7.4×10^6
				(b) January 26				
25	1.24	0.00	0.09	0.17	3.17	0.0407	0.109	5.0×10^6
30	1.90	0.01	0.13	0.25	4.20	0.0390	0.100	5.9×10^6
35	2.68	0.01	0.17	0.33	....	....	....	6.9×10^6
40	3.60	0.01	0.22	0.43	7.38	0.0390	0.100	7.9×10^6
45	4.55	0.02	0.27	0.54	8.91	0.0384	0.097	8.8×10^6
50	5.53	0.02	0.33	0.65	10.69	0.0384	0.097	9.6×10^6
55	6.58	0.02	0.39	0.76	12.23	0.0380	0.095	10.4×10^6
60	7.52	0.03	0.44	0.86	....	....	....	11.1×10^6
65	8.55	0.03	0.49	0.97	15.71	0.0381	0.095	11.8×10^6
70	9.45	0.03	0.54	1.07	17.36	0.0381	0.095	12.4×10^6
75	10.34	0.04	0.59	1.17	19.06	0.0381	0.095	12.9×10^6
80	11.19	0.04	0.64	1.26	20.60	0.0381	0.096	13.5×10^6
85	11.98	0.04	0.69	1.35	21.97	0.0381	0.095	13.9×10^6
90	12.73	0.05	0.73	1.44	23.17	0.0379	0.094	14.4×10^6
95	13.40	0.05	0.77	1.52	....	....	....	14.7×10^6
100	13.90	0.05	0.80	1.57	24.78	0.0375	0.092	15.0×10^6

[a] Based on average equivalent diameter of 22.64 ft.

termined for three widely different types of surfaces, including steel coated with bituminous paint, concrete placed against steel forms, and unlined rock. Discharges were based on a rating curve established by current meter measurements supported and supplemented by observations of turbine scroll case differential pressures.

Values of f for the bituminous-coated steel pipe, at values of R below 8×10^6, agree with the curve suggested by Theodor von Kármán,[8,9] M. ASCE, and L. Prandtl[10] for smooth pipe. For greater values of R, the value of f departs from the smooth pipe curve but increases very little in actual value. The average values of f and n are about 0.0085 and 0.0109, respectively.

Values of f and n for the concrete surface placed against steel forms increase with increasing values of V and R, whereas values of f and n for the unlined rock tunnel decrease with increasing R.

---

[8] "Mechanical Similitude and Turbulence," by Theodor von Kármán, *Technical Memorandum No. 611*, National Advisory Committee for Aeronautics, Washington, D. C., 1931.

[9] *Proceedings*, III International Cong. on Technical Mechanics, Stockholm, 1930.

[10] "Neue Ergebnisse der Turbulenzforschung," by L. Prandtl, *Zeitschrift des Vereines Deutscher Ingenieure*, No. 5, 1933.

Bend and transition losses were approximated to enable more accurate calculation of the friction losses in the tunnel. These losses were not recalculated after the final determination of friction loss had been made; but the change, if any, would have been very small. Since these losses are a relatively small part of the total loss, the effect of a small change in the roughness coefficient used in determining them would change them only slightly and the final effect on the friction coefficients for the tunnel surfaces would be insignificant. The coefficients determined are within the range of commonly accepted values.

The data from these tests offer an opportunity to check the validity of the von Kármán equation relating friction coefficient, pipe diameter, and size of roughness. The equation, based on J. Nikuradse's[11,12] experiments on small pipes lined with sand, is

$$\frac{1}{\sqrt{f}} = 1.74 + 2 \log \frac{r}{k} \quad \text{.....................} \quad (6)$$

in which f is the friction coefficient; r is the radius of the pipe; and k is the diameter of the sand grains composing the surface. The equation is applicable only to the region of complete turbulence in which f remains constant with increasing **R**. Although this condition was not quite reached in the tests, the constant value of f may be estimated by inspection of Fig. 10. Table 9 lists the estimated values of f and r for each type of tunnel, together with the corresponding values of $\frac{r}{k}$ and k.

TABLE 9.—COMPUTATION OF SURFACE ROUGHNESS

Tunnel surface (1)	Friction coefficient, f (2)	Pipe radius (ft) (3)	Ratio, $\frac{r}{k}$ (4)	Grain diameter, k (in.) (5)
18-ft steel......	0.0087	9.0	30,900	0.0035
18-ft concrete..	0.0135	9.0	2,690	0.04
22.6-ft rock....	0.091	11.3	6.17	22.0

Calculated values of k are in fair agreement with what would be expected after an inspection of the respective surfaces. The interior of the steel pipe was almost glassy smooth except for the irregularities caused by the method of application. The concrete lining was slightly grainy. Estimates of the mean sand grain diameter, based on visual inspection, varied from 0.03 in. to 0.06 in. The value of 0.04 is within this range. No accurate estimate of average roughness for the unlined rock can be made, but the value of 22 in. does not seem out of line after examination of Figs. 4 and 5. The agreement is the more remarkable when the difference in size between Mr. Nikuradse's apparatus and the Apalachia tunnel is considered.

---

[11] "Widerstandsgesetz und Geschwindigkeitsverteilung von turbulenten Wasserstromung in glatten und rauhen Rohren," *Proceedings*, III International Cong. on Technical Mechanics, Stockholm, 1930.

[12] "Stromungsgesetze in rauhen Rohren," by J. Nikuradse, *Forschungsheft*, Verein Deutscher Ingenieure, Heft 361, 1933.

DISCUSSION

Weston Gavett,[13] Assoc. M. ASCE.—Friction coefficients for large tunnels with three types of wall surface such as described are of great interest.

Apparently the piezometers were located with difficulty, for in some instances several types and sizes of tunnel with bends and transitions are included between piezometers, requiring successive approximations to arrive at the coefficient for each part.

Figs. 11 and 12, showing the relation of Manning's n to V, indicate a straight line for the 18-ft diameter steel pipe but show curves for the 18-ft diameter concrete tunnel and the unlined rock tunnel. A straight line shows that the exponent x in the equation $H_f = C\ V^x$ is 2.00, whereas a curve indicates that x is greater or less than 2.00. As the friction loss for the 18-ft steel pipe, for which $x = 2.00$, was obtained in a straight run between points A and B with no other types, sizes, or bends to complicate the computations, the question arises as to whether the values of x for the 18-ft concrete tunnel and the unlined rock tunnel are, in fact, more and less, respectively, than 2, or whether the variations are due to the assumptions made in arriving at the coefficients.

TABLE 10.—Values of x in $H_f = C\ V^x$

Tunnel section	Friction Loss		x
	Table	Column	
(1)	(2)	(3)	(4)
A-B	2	7	2.00
B-C	2	8	2.09
D-E	2	9	2.01
18-ft concrete	8	2	2.18
Unlined rock	8	6	1.89

TABLE 11.—Computed Roughness Height for Various Conduits

Type of conduit	Length of sections (ft)	Diameter (in.)	Age (yr)	Roughness height, k (in.)
(1)	(2)	(3)	(4)	(5)
Concrete, lock joint	12	51	6.7	0.008
Steel, riveted	7½	72	17.0	0.55
Steel, riveted	7½	72	23.0	0.77
Steel, riveted	7	48	27.4	0.37
Steel, riveted	7	48	39.2	0.894
Steel, riveted	7	42	27.4	0.23–0.40
Steel, riveted	7	42	43.8	0.85
Steel, lock bar	30	72	4.0	0.18
Steel, lock bar	30	74	0.6–9.6	0.028–0.056

By computation, the writer obtained the values of x shown in Table 10, using figures for Q, in cubic feet per second, from Table 3, Cols. 4 and 13, and friction losses from other data in the paper as shown in Cols. 2 and 3, Table 10. It will be noted that the values of x vary only slightly from 2 for the observed data in sections A-B, B-C, and D-E.

Of special interest is the finding that Eq. 6, derived from experiments with 2.5-cm, 5-cm, and 10-cm pipes, gives plausible results for the roughness height k for the 18-ft to 22-ft diameter tunnels. Use of the same equation to compute the roughness height in pipes described elsewhere[14,15] gave the results shown in Table 11. The Nikuradse[11,12] sand size appears to be a practical standard for evaluating wall roughness, comparing different types of pipe, and indicating the growth of roughness height with age. Obviously, many types of roughness are not similar to uniform sand; but some standard is desirable, and any form of roughness may be referred to Nikuradse's sand as an equivalent sand size as was done by H. Schlichting[16] for various forms of artificial roughness.

---

[13] Asst. Engr., Clyde Potts, New York, N. Y.

[14] "Trends in Coefficients of Large Pressure Pipes," by Charles H. Capen, *Journal*, A.W.W.A., January, 1941, p. 1.

[15] Discussion by Weston Gavett of "Trends in Coefficients of Large Pressure Pipes," by Charles H. Capen, *ibid.*, April, 1942, p. 524.

W. R. Barrows,[17] Assoc. M. ASCE.—A part of the Apalachia tunnel was left unlined after an economic investigation had indicated that such a move would be wise. During the course of this investigation, it became necessary to estimate the probable friction loss in the unlined part. A search of the literature produced very little information. To make matters worse, the published data reported widely varying coefficients of friction for apparently similar conditions and gave no clue as to the probable value for the problem at hand. At this point the Aluminum Company of America made available data on the unlined tunnel of its Nantahala project. From these data the value of n in the Manning formula was estimated to be 0.037 for the unlined tunnel. This value compares favorably with the values determined by precise measurements in the Apalachia tunnel and was computed for velocities ranging from 2.18 to 4.28 ft per sec.

The Nantahala tunnel had been recently completed in a rock formation similar to the rock in which the Apalachia tunnel was driven. When it was unwatered after completion of the unit test, it was inspected by a party of engineers familiar with the Apalachia tunnel, who agreed that, in so far as could be observed, there was no difference in either the surface roughness or the manner the rock broke out in the two tunnels. The estimated coefficient for the Nantahala tunnel is slightly lower than that determined for the Apalachia tunnel. The difference could easily have resulted from the fact that the minor losses in the Nantahala tunnel had to be estimated.

Unlined parts of the Nantahala tunnel were driven to three different sections with nominal dimensions of 12 ft wide by 12 ft high, 14 ft wide by 13 ft high, and 13 ft wide by 14 ft high, the theoretical section in each case having an arched roof with a radius equal to half the width and a flat floor connected to the spring line of the arch by vertical sides. Cross sections of the completed tunnel were available and from them the average area and the wetted perimeter of the different sections were determined. The discharge was determined from unit efficiency tests, and the head loss between the forebay and the surge tank was measured.

The experimental values for the coefficient of friction of an unlined rock tunnel presented by Messrs. Hickox, Peterka, and Elder are based on the measured properties of the actual section "adjusted to equivalent circular sections." This point deserves emphasis. The engineer when considering reported values of n for excavations which may deviate considerably from the desired dimensions should assure himself that the values were determined in such a way as to make them applicable to his problem. For instance, assume that the coefficient of friction of an unlined tunnel 12 ft in diameter was determined using the theoretical properties of the section instead of the actual. Such a value applied to a tunnel 30 ft in diameter would result in an estimated head loss equal to approximately two thirds of the actual head loss, assuming equal overbreak. However, the values reported by the authors should be applicable in similar rock tunnels over a wide range of sections.

---

[16] "Experimentelle Untersuchungen zum Rauhigkeitsproblem," by H. Schlichting, *Ingenieur-Archiv*, Vol. VII, No. I, 1936.

[17] Civ. Engr., Albuquerque, N. Mex.

Hunter Rouse,[18] M. ASCE.—Test results that extend the existing range of measured data are welcome in any field of engineering. To the best knowledge of the writer, the resistance measurements presented by the authors of this factual paper go well beyond the available information in two respects. First, the Reynolds numbers are nearly a full logarithmic cycle higher than those of all but a few isolated measurements on large conduits. Second, the absolute roughness of one series far exceeds that of boundary surfaces generally considered in the roughness category. With this apparent two-directional enlargement of the resistance field, several checks upon the matter at once suggest themselves. Are the new results in accord with present knowledge of this problem? If not, do these results indicate errors in present knowledge, or is such knowledge sufficiently sound to disclose inaccuracies in either the data or the authors' analysis?

In 1942 the writer showed[19] that the best available resistance measurements for a wide variety of new and old commercial pipe approximated the transition curve of C. F. Colebrook:[20]

$$\frac{1}{\sqrt{f}} - 2\log\frac{r}{k} = 1.74 - 2\log\left(1 + 18.7\,\frac{r/k}{\mathbf{R}\sqrt{f}}\right) \qquad \text{. . . . . (7)}$$

On the basis of this analysis, the writer prepared a general resistance diagram[19,21] permitting solution for either V or h_f for (presumably) the flow of any fluid at any speed through any size of conduit having a surface comparable to any one of those for which data were analyzed. In this diagram an asphalted surface was listed as having an equivalent sand roughness of $k = 0.005$ in. and a concrete surface of $k = 0.1$ to 0.01 in. The authors' values for the asphalted and concrete linings were 0.0035 in. and 0.04 in., respectively. The writer's diagram included no relative roughness values higher than $k/r = 1/10$, however, for two reasons: First, the effective diameter of a very rough conduit is difficult to define, both practically and theoretically; second, boundary irregularities having a greater relative magnitude than approximately one tenth of the radius begin to represent cross-sectional constrictions rather than surface roughness and cannot significantly be discussed in the same terms. The maximum value listed in this paper is about one sixth.

---

[18] Director, Iowa Inst. of Hydr. Research, State Univ. of Iowa, Iowa City, Iowa.

[19] "Evaluation of Boundary Roughness," by Hunter Rouse, *Proceedings*, 2d Hydraulics Conference, Studies in Eng., *Bulletin No. 27*, Univ. of Iowa, Iowa City, 1943.

[20] "Turbulent Flow in Pipes, with Particular Reference to the Transition Region Between the Smooth and Rough Pipe Laws," by C. F. Colebrook, *Journal*, Inst. C. E., London, Vol. 11, 1938–1939, p. 133.

[21] "Elementary Mechanics of Fluids," by Hunter Rouse, John Wiley & Sons, Inc., New York, N. Y., 1946, p. 211.

The authors restrict their interpretation of the surface-resistance measurements to two primary features—the variation of f with **R**, and the variation of n with **R**. The former, with Fig. 10 the corresponding exhibit, shows little of the actual relationship of the three series to existing generalized treatments; nevertheless, the resulting values of k for the assumed limits of the apparent transition curves are tabulated. Portrayal of n as a function of **R**, furthermore, is hardly consistent with either the true significance of n as a roughness parameter or with the authors' evaluation of k, for most assuredly showing how k (neither more nor less a roughness parameter than n) varied with **R** was not considered.

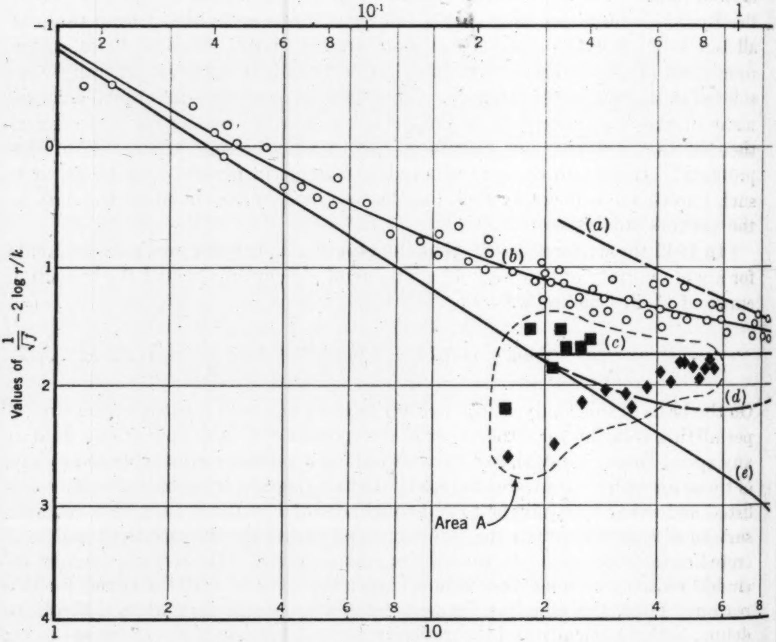
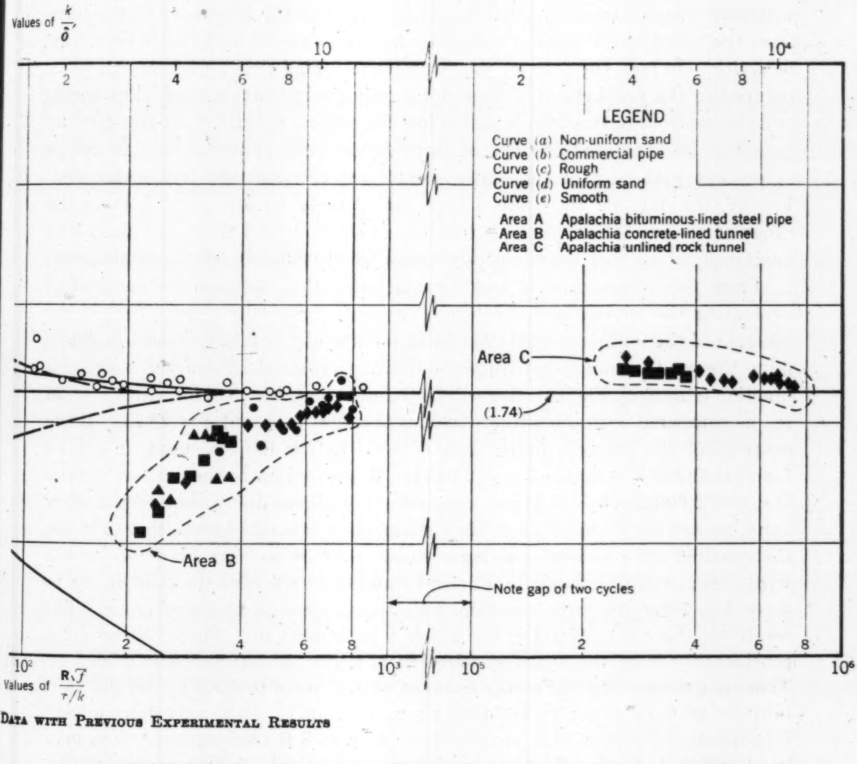


Fig. 13.—Comparison of Apalachia Tunnel

As has long appeared evident,[22] whether the resistance of a conduit boundary is dependent on or independent of **R** is governed by whether the zone of laminar flow at the boundary (that is, the laminar sublayer, for which the nominal thickness δ' is equal to $65.6\, r/\mathrm{R}\sqrt{f}$) encloses and thus renders the surface irregularities ineffective, or is itself disrupted and thus rendered ineffective by the irregularities. In a word, whether the wall behavior is smooth or rough depends on whether the ratio k/δ' is, respectively, much less or much greater than unity. This ratio is evidently proportional to the quantity $(\mathrm{R}\sqrt{f})/(r/k)$. A plot using this parameter for the abscissa scale and the quantity $1/\sqrt{f} - 2 \log r/k$ for the ordinate scale in effect reduces all resistance data to the same common denominator of relative viscosity-versus-roughness action. J. Nikuradse[12] first used this method in correlating his tests on uniform sand roughness, which resulted in the single function for all his data shown by

---

[22] "Modern Conceptions of the Mechanics of Fluid Turbulence," by Hunter Rouse, *Transactions*, ASCE, Vol. 102, 1937, p. 463.



Data with Previous Experimental Results

curve (*d*) of Fig. 13. Extending this study to the conditions of nonuniform sanded surfaces, C. F. Colebrook and C. M. White[23] obtained a gradual displacement of the transition function with increasing nonuniformity; curve (*a*),

---

[23] "Experiments with Fluid-Friction in Roughened Pipes," by C. F. Colebrook and C. M. White, *Proceedings*, Royal Soc. of London, Vol. 161, 1937.

Fig. 13, represents the limiting one of their series. Since most natural roughnesses are essentially nonuniform, it is reasonable that the transition curve for such surfaces should lie between these limits and yet begin at a far lower value of k/δ' than that for uniform surface irregularities; this condition is satisfied by C. F. Colebrook's function shown in Fig. 13 by curve (*b*).

Data previously analyzed by the writer[19] and reproduced in Fig. 13 include new and old pipes varying from 2 in. to 72½ in. in diameter and fabricated of such materials as galvanized iron, wrought iron, cast iron, tarred cast iron, spiral-riveted steel, and wood stave. The three series of the authors' data are likewise reproduced, using the values of r/k given in Tables 4, 7, and 8. It is at once evident that the zone covered by any one series is very small compared with that encompassed by all three; also evident is the departure of the data from the trend of the general transition between smooth and rough boundary behavior. In fact, the data for the bituminous-coated steel pipe (area A), when reduced to the proportions of this plot, no longer indicate a significant trend, and the centroid of this area is well below the general function. A trend of the data for the concrete-lined tunnel (area B) is more apparent, but is not in agreement with any of the earlier functions; the centroid is likewise low. Finally, the data for the unlined rock tunnel (area C) are so far beyond the range of viscous action (note that k is some six thousand times the computed magnitude of δ') that the slight downward trend probably has no significance.

From this comparison it may be concluded that the authors' method of evaluating the relative and absolute roughness is generally unreliable. If the centroid of the unlined tunnel data, with a value of $f = 0.099$, is used to determine the relative roughness from the limiting ordinate 1.74 of the roughness function (curve (*c*), Fig. 13), this will be found to be $r/k = 5.3$, for which $k = 26$ in., as compared with the authors' values of $r/k = 6.17$ and $k = 22$ in. If the centroid of the concrete lining data ($f = 0.0123$) is likewise used, it will be found that $r/k = 4{,}300$ and $k = 0.025$ in., in distinction to the authors' values of $r/k = 2{,}690$ and $k = 0.04$ in. Since the bituminous lining data, on the other hand, lie well within the zone of viscous influence, it is necessary either to locate the centroid on a general resistance diagram (such as that prepared by the writer from the Colebrook function[19,21]) and read r/k directly thereon, or to solve Eq. 7 for the same conditions by the process of trial and error. The result will be $r/k = 170{,}000$, for which $k = 0.00063$ in. These values differ considerably from the authors' selection of $r/k = 30{,}900$ and $k = 0.0035$ in. There is a comparable difference between this value of 0.00063 in. for the 18-ft bituminous-lined steel pipe and the average of 0.005 in. previously determined[20] for asphalted cast-iron pipe varying from 4 in. to 5 ft in diameter, which may be attributable to the effect of pipe joints and methods of surface application.

Although the discrepancies between the authors' and the writer's calculations are small in two of the three cases, it is the general method rather than the numerical results which the writer seeks to emphasize. In no way, however, is it sought to disparage contributions of this nature. Variations such as those indicated by the authors' data have been the rule rather than the exception, even in apparently well-controlled laboratory experiments. Although one cannot expect to duplicate laboratory results under field conditions with respect to either accuracy or over-all range of Reynolds numbers, field corroboration of laboratory analyses are fully as essential as are final acceptance tests for any hydraulic design. The authors are therefore to be heartily commended for their effort to provide several links in the necessary chain of field evidence. The particular merit of the paper as a whole is twofold: First, it gives a clear indication of the desirability of preparing for prototype tests of this nature during the design of every large installation. Second, it gives an equally clear indication of the degree of accuracy which may be expected in predicting the performance of a conduit; surely the use of more than two significant digits is still quite futile.

Karl R. Kennison,[24] M. ASCE.—Much valuable information on coefficients of flow in large diameter tunnels has been assembled in this paper. It occurs to the writer that it will be interesting and instructive to compare this information with that obtained in actual operation over a period of years on the Quabbin Aqueduct tunnel built by the Metropolitan District Water Supply Commission,[25, 26, 27, 28] in Massachusetts. The great length of this tunnel, 24.6 miles, and the uniformity of operating conditions over long intervals of time obviously contribute much to the accuracy of the data and the precision with which the results can be computed. For example, since the filling of Quabbin Reservoir was started late in 1939, the aqueduct has been used for free discharge into Wachusett Reservoir in the following quantities:

Period	Million gal
September 17 to October 28, 1941	20,426
August 6 to September 7, 1943	17,020
June 13 to September 11, 1944	25,425
June 27 to September 22, 1945	22,008
August 26 to September 13, 1946	10,745
January 27 to February 11, 1947	9,660

The tunnel is operated as a pressure tunnel. It was constructed as a horseshoe-shaped grade tunnel, since the internal water pressure is less than the external pressure resulting from the normal ground-water level. This is not strictly true for a short distance at the Wachusett Reservoir end, where a length of 500 ft next to uptake shaft 1 was constructed as a circular pressure tunnel 12 ft 9 in. in diameter, with the lining grouted against the ledge to prevent leakage from the tunnel under the full head of Quabbin Reservoir. This head is effective throughout the entire length of the tunnel as far as the outlet control gates at the top of shaft 1. Another section, 200 ft long at the extreme westerly end next to the Quabbin intake shaft (shaft 12), was similarly constructed as a circular pressure tunnel to prevent excessive leakage from the reservoir into the tunnel whenever the latter is unwatered for inspection purposes by inserting stop logs at shaft 12. With these two exceptions, the horseshoe section is maintained throughout the entire length of 129,413 ft between downtake shaft 12 and uptake shaft 1. These lengths of circular tunnel represent a small percentage of the total and furthermore have the same cross-sectional area. Hence, the fact that these end sections have a slightly greater hydraulic radius can be neglected.

---

[24] Chf. Engr., Construction Div., Met. Dist. Comm., Commonwealth of Massachusetts, Boston, Mass.

[25] "The Metropolitan District Water Supply Tunnel from the Ware River to the Wachusett Reservoir," by Frank E. Winsor, *Journal*, Boston Soc. of Civ. Engrs., November, 1932, p. 461.

[26] "Boston's New Metropolitan Water Supply," by Frank E. Winsor, *Civil Engineering*, June, 1934, p. 283.

[27] "Boston Metropolitan Water Supply Extension," by Karl R. Kennison, *Journal*, New England Water Works Assn., June, 1934, p. 147.

[28] "Ware River Intake Shaft and Diversion Works," by Karl R. Kennison, *Civil Engineering*, August, 1934, p. 388.

The width of the tunnel is 11 ft 0 in. between vertical side walls; the height is 12 ft 9 in. from invert to crown. The cross-sectional area is 127.6 sq ft, and the hydraulic radius is 3.03 ft. Flow is unobstructed except as follows:

(*a*) The downtake waterway in shaft 12 is about 115 ft deep and is crossed by numerous struts spanning the openings in which stop logs and screens are placed. Ordinarily water is drawn from the upper intake, which permits free entrance from the reservoir for a depth of nearly 50 ft from the surface. Water drawn from the lower intake must rise through an uptake shaft before turning to drop into the downtake shaft.

(*b*) Shaft 11-A is an uptake shaft for diverting flood flows from the Ware River into Quabbin Reservoir; hence, there is a large opening in the crown.

(*c*) At the bottom of construction shafts 11, 10, 7, 6, 5, and 3 there is a sizable rectangular opening in the crown and a large sump in the invert.

(*d*) At the bottom of shaft 9 there is a central pier 18 in. wide, in addition to large openings in the crown for access and for placing emergency stop logs in the two 4-ft 9-in. bays, and in addition to the invert sump. There are also two low racks 4 ft 6 in. high in each of the two bays for the purpose of catching any debris that may roll along the invert. These racks are constructed of ½-in. flats, 3 in. on centers. They are placed there merely because the access to this point is convenient because of gravity drainage into Wachusett Reservoir. The net cross-sectional area may well be taken as from about 110 to 115 sq ft.

(*e*) At a point between shaft 9 and shaft 8, there is a 2-ft-wide central pier, which was left after the removal of a heavy steel bulkhead behind which the Quabbin end of the tunnel was constructed while the Wachusett end was first operated for diversions from the Ware River directly to Wachusett Reservoir. The net waterway area is 102.1 sq ft.

(*f*) At shaft 8, a downtake shaft for Ware River floods, there is a considerable enlargement in the area as well as a large sump in the invert crossed by a narrow footbridge.

(*g*) At the bottom of shaft 4, there is a central pier 18 in. wide and the shaft otherwise resembles shaft 9. The net waterway area is 108.5 sq ft. No racks are placed there.

(*h*) At shaft 2, an uptake shaft leading to a spillway at the surface, there is a large opening in the crown as well as the sump in the invert.

(*i*) Shaft 1, the uptake shaft to Wachusett Reservoir, contains a 144-in. by 90-in. venturi meter, by which all the flows are measured. At the top, a branch extends horizontally through two 6-ft control valves in series and discharges under water into a stilling pool through a 5-ft 5½-in.-diameter end nozzle with a waterway area of 23.4 sq ft. The nozzle crown is several feet beneath the surface. Under all the high velocity conditions considered in this discussion, the water is pushed ahead in large eddies, which flow back around the nozzle in such a manner that the static head is practically zero at the nozzle crown (El. 388.2).

By the summer of 1943, Quabbin Reservoir had filled to within 30 ft of the flow line and a continuous draft was made for about a month under conditions of steady uniform flow. On September 3 of that year, fifteen readings were taken, at 1-min intervals, of the water level in a well connected to the bottom of shaft 12. The average level was El. 494.16; the maximum departure from the mean was 0.35 ft. It was estimated that 0.72 ft should be added to this pressure gradient to obtain the true energy gradient, allowing for velocity head and some downstream loss on account of the bend.

On the same day, about 1 hour earlier, ten readings were taken, at 1-min intervals, of the water level in shaft 9. The average level was El. 466.29, and the maximum departure from the mean was 0.09 ft. It was estimated that the true energy gradient, including velocity head, was 467.29. Hence, the loss of head in the 47,098 ft between shafts 12 and 9 was 27.59 ft, which is estimated to be reduced to about 27.3 ft by miscellaneous minor losses occurring at the previously described shaft bottoms en route. Hence, the slope of the hydraulic gradient was 0.580 per thousand.

Four hours earlier on the same day fifteen readings were taken, at 1-min intervals, of the water level in shaft 2. The average level was El. 423.57, and the maximum departure from the mean was 0.20 ft. It was estimated that the true energy gradient, including velocity head, was 424.29. Hence, the loss of head in the 72,413 ft between shafts 9 and 2 was 43.00 ft, which is estimated to be reduced to about 42.0 ft by miscellaneous minor losses occurring past the pier between shafts 9 and 8, and at the other shaft bottoms, including somewhat less than full recovery of velocity head at the downstream end of the piers at shafts 9 and 4. Hence, the slope of the hydraulic gradient was 0.580 per thousand, the same as that in the other stretch upstream. Continuing this same slope for the remaining 9,902 ft to the bottom of uptake shaft 1, the computed energy gradient at that point was El. 418.54, and the difference or loss from this point to the crown of the nozzle in the stilling chamber, El. 388.2, was 30.3 ft. Of this total, 21.4 ft represented velocity head at the discharge end, leaving about 8.9 ft for all the miscellaneous losses in the uptake shaft, including those of the meter and valves. This loss, in feet, equals 0.0000282

TABLE 12.—OPERATING DATA FOR QUABBIN AQUEDUCT TUNNEL

Line No. (1)	Date (2)	Average reservoir elevation[a] (3)	Total head available[b] (ft) (4)	HEAD LOSSES (FT) Velocity head at exit (5)	HEAD LOSSES (FT) Miscellaneous (6)	HEAD LOSSES (FT) Friction (7)	Hydraulic slope (ft per thousand ft) (8)	AVERAGE FLOW Mgd (9)	AVERAGE FLOW Cu ft per sec (10)	Average velocity (ft per sec) (11)	Temperature (°C) (12)	Kutter's n (13)	Hazen-Williams' C (14)
1	September 3, 1943	As described for this day					0.580	561.9	869.9	6.81	17	0.0111	144
2	July 13 to 26, 1944	507.64	119.4	24.2	15.1	80.1	0.619	597.2	925	7.25	17	0.0109	148
3	September 1 to 9, 1944	505.97	117.8	23.8	16.9	77.1	0.596	591.8	916	7.18	16	0.0108	150
4	June 28 to July 20, 1945	519.60	131.4	25.6	18.3	87.5	0.677	614.7	951	7.46	15	0.0110	146
5	September 11 to 21, 1945	519.58	131.4	25.9	18.5	87.0	0.673	617.7	956	7.49	17	0.0109	147
6	September 1 to 12, 1946	528.97	140.8	28.1	17.5	95.2	0.736	643.5	996	7.81	20.5	0.0110	146
7	January 30 to February 10, 1947	528.64	140.4	28.1	17.5	94.8	0.733	643.5	996	7.81	1.0	0.0110	146

[a] Average surface elevation of Quabbin Reservoir. [b] Difference in elevation between Quabbin Reservoir surface and outlet (388.2).

mgd^2, which has been checked by the writer on a number of other occasions as entirely reasonable.

In writing a convenient expression of the loss as a function of the rate of flow, the use of the exponent 2 is justified because of the narrow range within which the rate fluctuates. The loss in getting into the tunnel from Quabbin Reservoir and in the downtake to the bottom of shaft 12 is estimated from excellent data to be 0.0000100 mgd^2 with draft from the upper intake (0.0000160 mgd^2 with draft from the lower intake). The losses are summarized as follows:

Type of head loss	Head, in ft/mgd^2
Reservoir to bottom of shaft 12	0.0000100
Past shaft bottoms and piers	0.0000041
Bottom of shaft 1 to exit	0.0000282
Total miscellaneous	0.0000423
Velocity head at exit	0.0000680
	0.0001103
Friction	0.000235
	0.000345

In confirmation of the previous reference to the inherent accuracy and precision caused by the great length of the tunnel, note that it takes as much as 11% uncertainty in the determination of miscellaneous losses to correspond to 1% difference in the rate of flow. Based on this fairly accurate determination of these miscellaneous losses in terms of the rate of flow, the data for all the long runs have been tabulated and used to determine the probable coefficient, as shown in Table 12. The change in reservoir elevation during any run is so small and at so uniform a rate that no appreciable error is introduced in observing it. The same is true of the steadiness of the rate of flow. Although infiltration of as much as 3 mgd has been noted just after unwatering of the tunnel, leakage under actual operating conditions at full head is considered negligible.

The losses in Col. 6, Table 12, are for drafts from the upper intake in the case of the last two runs, in 1946 and 1947, and from the lower intake in the previous years. In the case of run 2, however, the upper intake was also open providing free access and approximately the same head loss conditions as normal draft from the upper intake at the later high reservoir levels. Col. 12 is based on depth observations in the reservoir in front of the intake.

For convenience, the data in Col. 13 were read from the writer's diagram and those in Col. 14 from the Hazen-Williams slide rule.

The writer has noted, in studying these data, that there appears to be a slight improvement in the coefficient immediately following a long period of flushing out at these high velocities. It should also be pointed out that there is a tendency for air to enter aqueducts of this type and to be carried along the crown. This results in an appreciable lubricating effect, the actual hydraulic radius being somewhat greater than the theoretical and the roughness being actually somewhat greater than indicated by using the theoretical radius. Such differences, however, are very slight.

E. J. K. CHAPMAN,[29] Assoc. M. ASCE.—The results published in this paper are of considerable value and the authors are to be commended for their careful and analytical presentation of the facts. One set of results is of particular interest in that it discloses certain surprising features which would not normally have been expecteed. These are the data given for the 18-ft-diameter concrete-lined tunnel. In the section between the Apalachia Dam adit and the Apalachia adit, the results indicate a variation in Manning's n from 0.0122 to 0.0136 as the velocities increase from 3.8 ft per sec to 12.6 ft per sec. This variation in n is large and seems much greater than could have been anticipated, especially in view of the careful manner in which the tests were conducted. Manning's formula has been well tested in the past in many large tunnels and has proved reliable, and the values of n determined for any given tunnel are usually more consistent. In addition to the variations of n, the top value of 0.0136 is considerably higher than would have been expected from the descriptions of the concrete surfaces given in the paper. In fact, in the section between Turtletown Creek and McFarland adit, a value of 0.0138 was reached and maintained over a fair range of velocity.

It is interesting to note that the value taken for a design basis[4] was 0.0130 and it might well have been hoped that a better value than this would be obtained, especially before the tunnel had seen any service.

The writer is closely associated with James Williamson, who has done much work on the underlying causes of frictional resistance created by turbulent flow in pipes.[30] In the manuscript of an unpublished paper, Mr. Williamson touches on the special conditions occurring in very large pipes with very high *R*/*k*-ratios. He shows how pipes with *R*/*k*-ratios greater than 2,000 can be expected to develop spiral flow which may be of the stable or continuing form, where the pitch is independent of the velocity. In such cases the frictional resistance is expected to be higher than that given by calculations based on normal axial flow.

---

[29] Cons. Engr., Glasgow, Scotland.

[30] "Considerations on Flow in Large Pipes, Conduits, Tunnels, Bends, and Siphons," by James Williamson, *Journal*, Inst. C. E., London, April, 1939, p. 451.

The Apalachia 18-ft-diameter concrete-lined tunnel has an *R*/*k*-ratio of 2,690, or possibly higher. This ratio brings the tunnel into the spiral flow region, and a further unusual feature of the tunnel is that the section is circular, which would allow spiral flow to develop and continue, whereas the more usual horseshoe shape would tend to maintain axial flow.

From these considerations, the writer feels that the probable explanation of the surprisingly high value of Manning's *n* is due to the fact that spiral flow has developed, especially at the higher velocities.

It is noted from the description of the design of the project[30] that the associated Ocoee No. 3 project included a horseshoe-shaped tunnel of somewhat smaller diameter than the Apalachia tunnel. Presumably the workmanship and finish of the concrete surfaces were similar in the two projects. If so, the Ocoee tunnel, in view of its smaller *R*/*k*-ratio in conjunction with its horseshoe section, would probably not present conditions for spiral flow. Thus, frictional loss would be expected to give a more constant and generally lower value of Manning's *n*.

The authors do not publish any data on Ocoee No. 3 in their paper, but possibly some tests were made even if these consisted only of recording the surge tank water levels at steady flows, from which, with suitable allowances, a calculation could be made of the whole loss from reservoir to surge tank and a value of *n* could be determined. If so, it would be extremely interesting to discover if the writer's supposition is borne out by an apparently lower value of *n*.

Probably the only positive method of determining whether spiral flow does occur at Apalachia would be to make pitometer traverses on two or more diameters of the 18-ft pipe and plot the velocity distribution. The velocity contours would clearly indicate whether flow is symmetrical across the whole diameter or not. A suitable place to install appliances for taking these traverses appears to be at the 16-ft-diameter steel pipe crossing over Turtletown Creek. Unfortunately the contraction from 18-ft to 16-ft diameter a short distance upstream will apply an energy effort causing reversion toward uniform velocity, and one could expect at this place only a slight residual indication of spiral flow. Better results would be obtained if it were possible to make the measurements on an 18-ft-diameter section.

It is not often that an opportunity occurs to check a large-scale effect which cannot be determined by laboratory experiment, but the writer feels that such an opportunity exists in the two tunnels referred to and strongly urges that, in the interest of hydraulics, the hydraulic engineers of the Tennessee Valley Authority should re-examine the test results for the two tunnels and make further experiments, if practicable, to determine the effect of spiral flow on resistance to flow.

The pertinent points are:

(1) Does Ocoee No. 3 tunnel, with presumably similar surface conditions to those in the Apalachia tunnel, show a lower coefficient *n* by the Manning formula?

(2) Does spiral flow occur in the Apalachia tunnel and not in the Ocoee No. 3 tunnel?

JULIAN HINDS,[31] M. ASCE.—In this carefully conducted series of experiments, the authors have invaded a size range for circular conduits flowing full on which very little research has been done previously. The results deserve careful study and frank discussion by all engineers interested in the development of better pipe flow data.

Because of the complexity of the layout, the precise allocation of losses to particular reaches of line is not simple, but the complexities are fully presented and the method of allocation is explained in detail. The considerable lengths of the various reaches contributed to dependability.

It is noted that the reduced data are compared both with a "conventional" formula (Manning) and the more recent Reynolds number formulas. This comparison is particularly appropriate because of the efforts being made to unify practice and theory in pipe line design.

The earliest and most fundamental of the formulas used for computing resistance to flow in pipe lines are those proposed by Chézy and Darcy. These formulas are not independent, but are transposable, either into the other. Any table or diagram for finding the coefficients in one of them can be made to yield the coefficients for the other simultaneously. Any difference in regard to convenience of use is inconsequential. The Darcy equation has the esthetic virtue of being nondimensional, for which reason it is preferred in mathematical discussions. Auxiliary methods are required for finding the coefficients in each of these equations.

Supplementing these two basic equations are the conventional Kutter, Manning, Williams-Hazen, Scobey, and other formulas. The Kutter formula is used for finding the Chézy coefficients; and, although it is primarily adapted to open channels, at times it is applied to pipes. The Williams-Hazen and the Scobey formulas are exponential in form and are considered applicable only to pipes. The Manning formula also is exponential and usually is so employed, but is occasionally used for determining the Chézy coefficient—for both open and closed conduits.

All these formulas are empirical, applicable to water only, and may be confidently adopted only within the range of experimental verification. Much work has been done on the development of rational flow formulas for the flow of all fluids in all pipes. Definite progress has been made, but complete coordination between laboratory and field experiments as well as between theory and designing office practice has not been accomplished.[32] If the new formulas are to be applicable to all kinds of fluids, it is essential that they take into account the effect of viscosity. This effect is introduced through the Reynolds number, **R**. (All nomenclature is identical with that used by the authors unless otherwise noted.) The dimensionless factor, *f*, in the Darcy formula (Eq. 1) is used as the other variable; hence, these new equations will be referred to as the $(f-\mathbf{R})$-equations or formulas.

---

[31] Gen. Mgr. and Chf. Engr., The Met. Water Dist. of Southern California, Los Angeles, Calif.

[32] "Comparison of Formulas for Pipe Flow," by Julian Hinds, *Journal*, A.W.W.A., November, 1946, p. 1226.

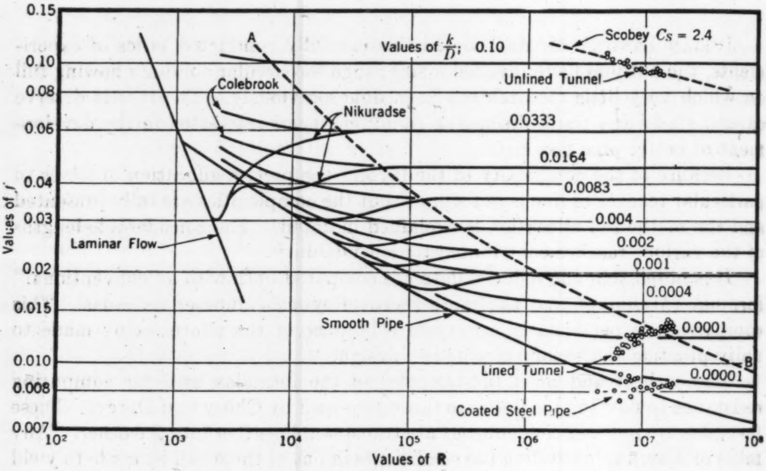


FIG. 14

The relation of these equations to each other, to the authors' data, and to one important collection of previously published data, is illustrated in Fig. 14. This group of curves covers the full range of pipe flow conditions up to $\mathbf{R} = 100{,}000{,}000$, passing from laminar flow on the left through a transition zone to a zone of complete turbulence in the upper right-hand part of Fig. 14. Laminar flow has no connection with the data reported in this paper and need not be discussed.

There are at present three $(f-\mathbf{R})$-equations within the turbulent range bidding for acceptance, each in its particular field. The most thoroughly substantiated of these is the smooth pipe formula credited by the authors to Theodor von Kármán, M. ASCE, and Ludwig Prandtl:

$$\frac{1}{\sqrt{f}} = 2.0 \log \mathbf{R} \sqrt{f} - 0.80 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

The accuracy of Eq. 8 for the flow of water or air through glass or drawn brass pipe was experimentally substantiated by T. D. Stanton and T. F. Pannell for values of **R** up to 500,000.[33] Its conformity to flow in larger pipes and in other types of surfaces has been reasonably well substantiated by others. In 1939 C. F. Colebrook published a diagram showing the applicability of this equation to cast-iron pipe from 4 in. to 60 in. in diameter, provided with spun cement and bituminous linings; he also showed its applicability to an 18-ft concrete-lined tunnel.[20]

---

[33] "Similarity of Motion in Relation to Surface Friction of Fluids," by T. D. Stanton and T. F. Pannell, *Transactions*, Royal Soc. of London A, Vol. 214, 1914, p. 199.

Eq. 8 is presumed to apply to all fluids and is doubtless accurate for actually smooth surfaces beyond all practical requirements. As will subsequently appear, flow in even moderately roughened pipes seems to follow this curve for limited values of **R**.

The smooth pipe curve, Eq. 8, plats as a single line in Fig. 14. It thus serves merely as a boundary for an unlimited multitude of possible flow conditions for which other formulas are required. Eq. 6, likewise reasonably well established, is applicable to flow conditions in the zone of complete turbulence, the upper right-hand part of Fig. 14.

Data substantiating the applicability of this equation to smooth pipes roughened by uniform sand grains glued to the interior surface were published by J. Nikuradse in 1933.[34,35] The results of these experiments classified as to the ratio of sand grain size (k) to pipe diameter (D) are shown in Fig. 14. (The Nikuradse data were transferred to Fig. 14 photostatically. The results are accurate enough for present purposes, but not necessarily exact.) It will be noted that the plotted points follow the smooth pipe curve for varying distances, depending on the relative roughness (k/D), then pull away, continuing downward for a distance and finally rising to a constant value of f. Beyond this point the ($f-\mathbf{R}$)-curve for any specific value of k/D is horizontal. Eq. 6 gives the vertical distribution of the horizontal parts of these lines.

The points at which the platted data assume horizontal directions are not sharply defined, but are approximately delineated by curve A–B, Fig. 14, the equation of which is

$$\frac{1}{\sqrt{f}} = \frac{\mathbf{R}}{200} \times \frac{k}{D} \qquad (9)$$

The region to the right of, and above, this line constitutes the zone of complete turbulence. Within this zone, viscosity has no effect on flow and, for a given pipe, values of Darcy's f and Chézy's C are constant; they are the same for gas, water, gasoline, hot or cold crude oil, or any other fluid. Within the range of applicability of Eq. 6, a diagram of the complicated form of Fig. 14 is unnecessary. Values of both f and C can be taken from a simple diagram or table.

Unfortunately, a large percentage of the flow problems encountered in practice do not fall within this range, but lie in the transition zone. If it could be shown that all rough pipe flows follow the form of Mr. Nikuradse's experimental data, it would be on the side of safety (although not necessarily economical) to ignore the dip shown by him in the transition zone. All rough pipe for all purposes then could be designed in accordance with Eq. 6, except for small values of **R** for which the smooth pipe formula, Eq. 8, controls.

---

[34] "Gesetzmässigkeiten der turbulenten Strömung in glatten Röhren," by J. Nikuradse, *Zeitschrift*, Verein Deutscher Ingenieure, Band 77, 1933, p. 48.

[35] "The Mechanics of Turbulent Flow," by Boris A. Bahkmeteff, Princeton Univ. Press, Princeton, N. J., 1941, p. 32.

However, other experimenters have found that pipes with natural roughness follow a different law in the transition zone. There appears to be a continuous upcurve toward the left, rather than the dip found for the sand-roughened pipe. After analyzing available data, Mr. Colebook proposed a formula covering flow both in the transition zone and in the zone of complete turbulence:[20]

$$\frac{1}{\sqrt{f}} = -2 \log \left(\frac{k/D}{3.7} + \frac{2.51}{\mathbf{R} \sqrt{f}} \right) \qquad (10)$$

Eq. 10 is asymptotic to the smooth pipe curve on the left (in Fig. 14) and to the Nikuradse curves on the right, and is offered by its proponents as a universal rough-pipe flow formula. It is not as formidable as it appears to be. In fact, it is readily adaptable to presentation by diagrams.[36] The chief difficulties with Eq. 10 are lack of final assurance of correctness within the transition zone and lack of an established set of values for k. The relation of the Colebrook curve to the Nikuradse data is shown in Fig. 14. It is not convenient to show data supporting the Colebook equation on this diagram.

The $(f-\mathbf{R})$-equations purport to be at least partly rational and are supported by laboratory experiments. The conventional formulas are empirical and depend for support largely on field tests. Field tests cannot be made with laboratory precision, but a multiplicity of comparisons should show a common trend. To date this has not been demonstrated to be true. Curves based on field experiments frequently cut sharply across theoretical curves, a fact that needs explanation.

The authors' data are platted in Fig. 14 to show their general relation to the $(f-\mathbf{R})$-equations. This platting does not replace, but supplements, the original platting in Fig. 10. Points for the tar-lined steel pipe conform to the smooth pipe curve out to $\mathbf{R} = 10{,}000{,}000$. These points lie too far from the zone of complete turbulence to give a definite indication of the final value of k. If it is assumed that an extension of these experiments would follow the Colebrook trend, then a value of k/D somewhat less than 0.00001 might be anticipated or, say, a value of k less than 0.0002 ft. On the other hand, if it is anticipated that these points will follow the Nikuradse trend, rising to a maximum of 0.00002 or 0.00003 in the vicinity of the line A–B, the value of k would be about 0.0004 ft. The authors arrived at a value of 0.0035 in. or about 0.0003 ft.

These data do not justify the drawing of a conclusion as to the precise value of k for this type of pipe, but they do illustrate the necessity for determining whether flow in such pipe follows the Nikuradse or the Colebrook trend.

The points for the cement-lined tunnel are more consistent within themselves than those for the steel pipe. They cross the curve A–B and appear to level off at about $k/D = 0.0002$ or, say, at a value of k somewhat less than 0.004 ft, which conforms to the authors' value of $k = 0.04$ in. It will be noted that these data definitely follow the Nikuradse rather than the Colebrook trend. If repeatedly confirmed by other field experiments, this fact may have a profound influence on the final selection of a flow formula. However, it will be noted from Fig. 11 that these data also are at distinct variance with the Manning formula, which suggests that a critical review of why these points fall as they do may add something to the over-all understanding of the problem.

---

[36] "Friction Factors for Pipe Flow," by Lewis F. Moody, *Transactions*, A.S.M.E., Vol. 66, 1944, p. 671.

The points for the unlined tunnel are more consistent among themselves than would be expected under the circumstances. They are platted in the upper right-hand corner of Fig. 14 and are far within the zone of complete turbulence. Consequently, they might be expected to lie on a horizontal line; actually, they are distinctly inclined. Their combined center of gravity lies slightly below the $k/D = 0.10$ line as determined by Eq. 10 or 6. The corresponding value of k approximates the authors' estimated value of 22 in. Also, as revealed by Fig. 12, these points show a definite trend away from a constant Manning's n. Strangely, they conform excellently to the Scobey steel pipe formula for a coefficient $C_s = 2.4$. Of course, this value of C_s is entirely outside the range for which Mr. Scobey's formula was proposed and tested and the Scobey formula has no analytical correlation with the rough pipe law; hence, this conformity contributes nothing to the solution of the general problem. Again, a critical examination of why these platted points fail to follow the expected trend should add to the general understanding of the resistance to flow in pipes.

It is hoped that the ideas expressed herein may help in the ultimate evaluation of the results of these and future tests.

William P. Creager,[37] M. ASCE, and Stephen H. Haybrook,[38] Jun. ASCE.—Some excellent experiments to determine the friction-loss coefficients in several types of conduits have been described by the authors. The writers desire to discuss the test results as applied to rock-lined tunnels, about which there is, relatively, a dearth of information.

The writers have advocated that, for consistency of records, tests on open conduit losses be tabulated by the Kutter formula and tests on closed conduit losses by the Hazen-Williams formula for general use where special formulas, such as the Scobey formula, have not been devised for special types of conduits.

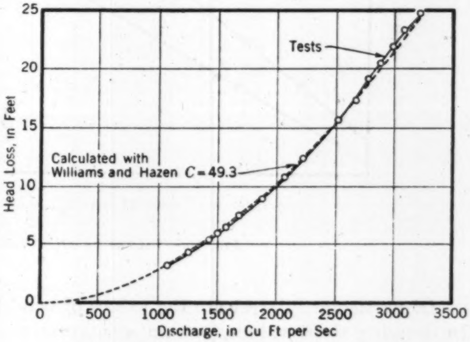


Fig. 15.—Head Losses in Unlined Rock Tunnel

With this in mind, the writers have tabulated in Fig. 15, by a full line, the relation between discharge and losses determined by tests for the Apalachia unlined rock tunnel; and also, by a dotted line, the same data calculated from the Hazen-Williams formula:

---

[37] Cons. Engr., Buffalo, N. Y.

[38] Engr., Justin and Courtney, Philadelphia, Pa.

$$v = 1.32\, C\, R^{0.63}\, S^{0.54} \qquad \text{(11)}$$

using a C-value of 49.3.

The deviation in loss is only 2.5% and the corresponding deviation in C is only about 1.5%. Based on similar comparisons for other unlined rock tunnels, the rock surface trimming in the Apalachia tunnel was well done, it being the opinion of the writers that a Hazen-Williams $C = 50$ is about the best which can be expected and that $C = 45$ is a safe value for average conditions.

G. S. TAPLEY,[39] Assoc. M. ASCE.—The authors are to be commended on their success in determining values of roughness coefficients under complicated conditions for the various types of surface of the structure examined; the values of coefficients obtained confirm, for very large-sized structures, the existing ideas as to the magnitude of such coefficients. The application of Eq. 6 to determine the measure of wall roughness, as shown in Table 9, is interesting.

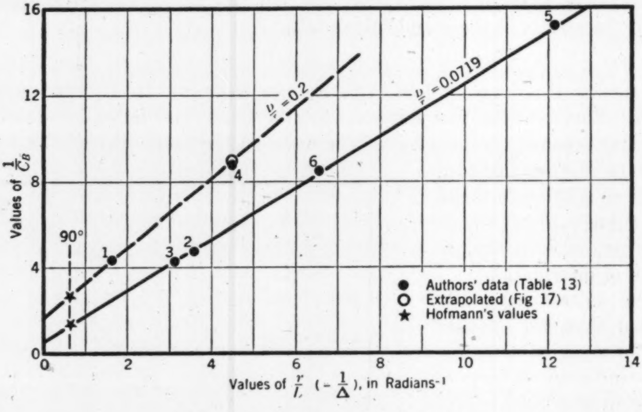


FIG. 16.—VALUES OF $1/C_B$ VERSUS $1/\Delta$

Of particular interest are the values of bend-loss coefficient, C_B, in Table 5. In deriving these values, certain assumptions were made in the preliminary calculations. Perhaps the effect of these assumptions is mirrored to some extent in the final results; for example, the authors' application of Fig. 67 of W. P. Creager and J. D. Justin,[6] Members, ASCE, tacitly assumes that the minimum value of C_B corresponds to $r/D = 5$, and their application of Fig. 68 from the same reference may predetermine the manner of variation of C_B with the central angle. In so far as the authors' final results on bend loss may be considered independent of the assumptions, they furnish a basis for developing data to cover a considerable range of design conditions as shown hereinafter. Data, so developed, can be applied in design practice with less risk of error, perhaps, than can existing data.

---

[39] Associate Civ. Engr., Bureau of Eng., Los Angeles, Calif.

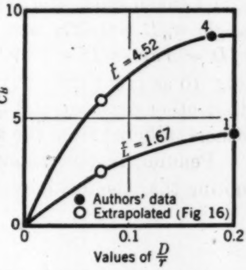


FIG. 17.—VALUES OF $1/C_B$ VERSUS D/r

With reference to Fig. 16, it may be noted that points representing bends 2, 3, 5, and 6 fall along a straight line representing $r/D = 13.9$. For $r/D = 5$, only one point (bend 1) was available. Bend 4, however, has a value of $r/D = 5.63$, and by extrapolating a short distance in Fig. 17, another point representing $r/D = 5$ and $1/\Delta = 4.52$ was obtained—thus furnishing two points through which the locus representing $r/D = 5$ was drawn. Here, $1/\Delta$, of course, equals r/L, where L is the length of curve and Δ is in radians. Calculations are shown in Table 13.

By passing curves through the origin and points representing the data in Fig. 16, Fig. 17 was constructed with maximum points at $D/r = 0.2$ ($r/D = 5$). Passing the curves through the origin in Fig. 17 is equivalent to the assumption that C_B is infinite when Δ is constant and r is infinite while D is finite; L, also, is then infinite.

TABLE 13.—COMPUTATION OF BEND-LOSS ELEMENTS

Bend No. (1)	Deflection angle, Δ (degrees) (2)	r/D (3)	C_B (4)	$\Delta = L/r$ (radians) (5)	$\frac{1}{\Delta} = \frac{r}{L}$ (6)	D/r (7)	$1/C_B$ (8)
1	34.26	5.00	0.236	0.598	1.67	0.200	4.24
2	15.72	13.9	0.207	0.274	3.65	0.0719	4.83
3	18.32	13.9	0.239	0.320	3.13	0.0719	4.18
4	12.68	5.63	0.114	0.221	4.52	0.178	8.78
5	4.70	13.9	0.066	0.082	12.2	0.0719	15.2
6	8.72	13.9	0.119	0.152	6.58	0.0719	8.41
....[a]	90.0	5.0	0.40	1.57	0.637	....	2.50
....[a]	90.0	13.9	0.86	1.57	0.637	....	1.16

[a] "Der Verlust in 90°—Rohrkrümmern mit gleichbleibendem Kreisquerschnitt," by Albert Hofmann, *Mitteilungen*, Hydraulischen Instituts der Technischen Hochschule, München, Heft 3, 1929.

On the other hand, it may be noted in Fig. 16 that, when L is very great, although r/D is finite, C_B is not infinite; in fact, the intercepts on the axis of ordinates are $1/C_B = 0.5$ and $1/C_B = 1.5$, from which $C_B = 2$ and 0.667, respectively (for $r/D = 13.9$ and 5, respectively). In this case C_B has a maximum value depending on the value of r/D. The implication is that for a very long curve of very long radius (but finite pipe diameter) C_B is infinite; whereas, for a very long curve of finite radius and finite pipe diameter (such as in a coil of many loops) the coefficient has a finite value depending on r/D. This seems to suggest that the velocity distribution becomes adjusted to the curvature in the latter case, but continues to be disturbed in the former.

Since the authors' data covered central angles up to only a little less than 35°, it was desired to obtain confirmation of the data in Fig. 16 in the region of larger central angles. A. Hofmann's plotting[40] of the Williams, Hubbell, and Fenkell data referring to a 30-in.-diameter pipe curve with a 90° central angle ($r/L = 0.637$) was used by the writer to calculate values of $1/C_B$ for $r/D = 5$ and $r/D = 13.9$. The calculated values plotted exactly on the loci in Fig. 16 as previously derived from the authors' data. It was found, however, that other experimental data such as that of Balch, Davis, Brightmore, and others, differed from the authors' data considerably.

Pending further confirmation of the data in Fig. 16, it might be worth noting that the loci may be represented by

$$\frac{1}{C_B} = c_1 + c_2 \frac{1}{\Delta} \text{.......................} (12)$$

in which c_1 and c_2 are functions of r/D, and represent, respectively, the intercepts on the axis of ordinates and the slopes of the loci.

The straight-line equation (Eq. 12) can also be represented by the locus of an equilateral hyperbola whose coordinates are Δ and C_B and which has asymptotes at $C_B = 1/c_1$ and $\Delta = \frac{-c_2}{c_1}$. Only that section of the locus for which coordinates are positive is, of course, real. The data are plotted in hyperbolic form in Fig. 18. That the curves pass through the origin in Fig. 18 follows from the fact that the head loss due to the bend (and, therefore, C_B) must be zero when L is zero.

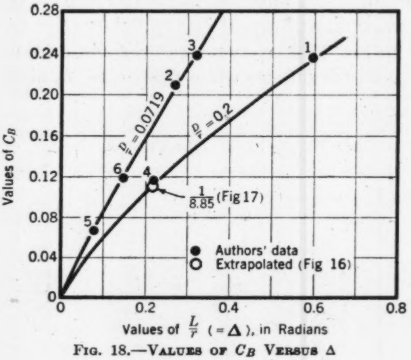


FIG. 18.—VALUES OF C_B VERSUS Δ

Although the authors' development of data on loss of head in bends was auxiliary to the main purpose of their paper, it is believed that the data have some merit as a basis for design material, in a form such as Fig. 16, to which additional lines can be added by the process already described. Needless to say, further confirmation by other experimental data is indicated.

J. N. BRADLEY,[41] ASSOC. M. ASCE, AND S. P. WING,[42] M. ASCE.—The authors and others concerned are to be commended on the meticulous manner in which the planning and the testing were performed on the hydraulic features of the Apalachia Tunnel. In making field tests the usual procedure is to assign to the job a man who is not especially trained in hydraulic testing, with in-

---

[40] "Der Verlust in 90°—Rohrkrümmern mit gleichbleibendem Kreisquerschnitt," by Albert Hofmann, *Mitteilungen*, Hydraulischen Instituts der Technischen Hochschule, München, Heft 3, 1929.

[41] Hydr. Engr., Bureau of Reclamation, Denver, Colo.

[42] Civ. Engr., Bureau of Reclamation, Denver, Colo.

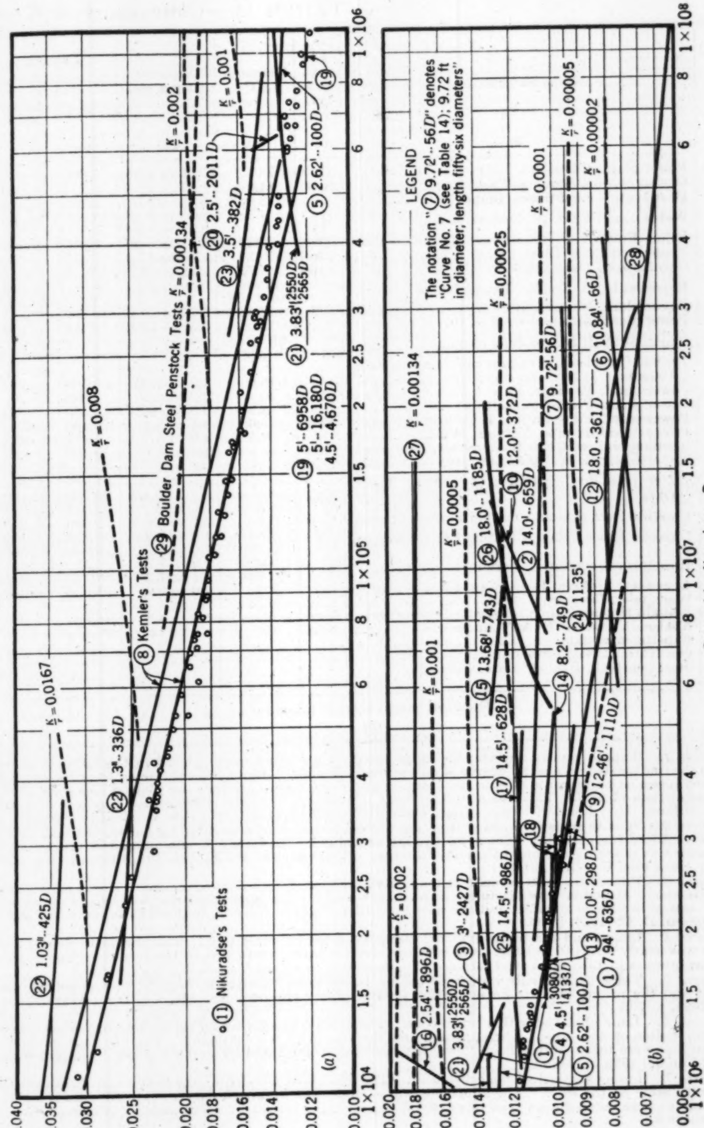


FIG. 19.—FRICTION FACTOR f VERSUS REYNOLDS' NUMBER R FOR LARGE CONCRETE PIPE

TABLE 14.—COMPARISON OF TEST

Curve (see Fig. 19) (1)	Name and location (2)	Age at test (yr) (3)	Diameter (ft) (4)	Velocity (ft per sec) (5)	LENGTH Ft (6)	LENGTH Diameters (7)
1	Power plant, Castelleto, Italy	10	7.94	2.61 to 4.19	5,062	636
2	Chelan station, State of Washington	New	14.0	1.25 to 14.8	9,227	659
3	Deer Flat, Boise Project, Idaho	6	3.0	5.45 to 9.06	7,282	2,427
4	Water conduit No. 10, Denver, Colo	4 1	4.5 4.5	2.58 to 3.50 2.58 to 3.50	18,598 13,902	4,130 3,089
5	Dijon, France	New	2.625	3.01 to 6.59	262.5	100
6	Englewood Dam, Ohio	1 to 4	10.84	17.6 to 42.2	712	66
7	Germantown Dam, Ohio	1 to 4	9.72	20.2 to 36.5	546	56
8	Reported by E. Kemler, Pennsylvania	....	0.009 to 0.438	....	....	1,500 to 2,000
9	Power plant, Livenza, Italy	2	12.46	2.76 to 10.17	13,850	1,111
10	Power plant, Melones, Calif	....	12	7.60 to 13.49	4,469	372
11	Reported by J. Nikuradse, Germany	....	0.03 to 0.238	....	....	....
12	Tunnel, Ontario Power Company, Niagara Falls, Canada	8	18.0	4.0 to 2.00	±6,500	361
13	Power plant, Partidor, Italy	18	10	2.2 to 6.8	2,985	298
14	Power plant, Piavi-Ansiel, Italy	1	8.2	3.28 to 9.31	6,150	769
15	Power plant, Pit Dam No. 1, California	New	13.68	4.7 to 8.2	10,160	743
16	Prosser pressure pipe, Yakima, Wash	4	2.54	4.9 to 5.8	2,276	896
17	Rondout siphon, Catskill Aqueduct, New York	New	14.5	1.6 to 4.8	9,102	628
18	Reported by E. W. Spies, Germany	....	....	....	....	....
19	Water pipe line, Spavinaw Aqueduct, Tulsa, Okla	New	5 5 4.5	2.25 2.25 2.63	34,788 60,998 21,047	6,958 16,180 4,670
20	Umatilla Dam siphon, Umatilla Project, Oregon	New	2.5	3.4 to 3.6	5,026	2,011
21	Umatilla River siphon, Umatilla Project, Oregon	5 2	3.83 3.83	1.4 to 3.2 4.0 to 4.2	9,774 9,831	2,550 2,565
22	Hose-formed conduit, Colorado	New	0.086 0.108	1.9 to 7.2 2.1 to 7.2	36.5	424 336
23	Aqueduct, Victoria, B. C., Canada	2	3.5	1.0 to 2.9	1,336	382
24	Waggitaler, Germany	....	11.35	....	....	....
25	Wallkill siphon, Catskill Aqueduct, New York	New	14.5	1.6 to 4.8	14,300	986
26	Apalachia Tunnel, TVA, Tennessee	New	18	4.2 to 12.6	21,380	1,185
27	Experimental data	....	....	....	....	....
28	von Kármán-Nikuradse data	....	....	....	....	....
29	Steel penstock, Boulder Dam, Colorado	....	....	....	....	....

Description and literary reference

(8)

Curve 1. Smooth cement surface; discharge rated by a current meter placed in the tailrace; about 10% of the line located on a small degree of curvature ($r/D = 30$). About 5% subtracted from the over-all measured losses as an estimate of entrance losses. The writers estimate that the plotted losses are about 1% greater than normal. This was a reliable test. ("Correnti Uniformi entro grandi condotte e grandi canali, Milano," by Giuliode Marchi, Milan, Italy, 1932–1936; Library Data File, U.S.B.R., 91–241.)

Curve 2. Smooth surface resulting from use of steel forms; discharge rated by Gibson method. The section measured was straight. The velocities are probably 3% in error. Combining all errors, the friction factor was probably not more than 4% in error. (*Transactions*, ASCE, Vol. 101, 1936, p. 1409; also Library Data File, U.S.B.R., 91–241.)

Curve 3. Pre-cast in steel forms 6 ft long; discharge rated by color movement, current meter, and Cipoletti weir. All joints were carefully calked on the inside. The alinement was straight. There was a gentle vertical bend near the inlet and one near the outlet (*Bulletin No. 852*, by Fred C. Scobey, U.S.D.A., Washington, D. C., 1924, p. 38.)

Curve 4. Lining and the joints smooth; discharge rated by pitot tube. The alinement of the first section was nearly straight and there was a gentle sinuous curve, vertically, in the second section. This pipe was pre-cast in 12-ft oiled steel forms. (*Engineering News-Record*, April 29, 1926, p. 678.)

Curve 5. These experiments were reported by Bazin to be "perfect in bore." The alinement was straight and the results indicate an unusually smooth pipe. (*Bulletin No. 852*, U.S.D.A., Washington, D.C., 1924, p. 79.)

Curve 6. Lining finished with a brush coat. The finish coat wore off on the bottom but brush marks were still visible on the sides. The approach and the alinement were straight. The inlet was rounded. The readings were taken during flood flows, the discharge being rated by current meter. (*Transactions*, ASCE, Vol. 93, 1929, p. 1588.)

Curve 7. Lining finished with a brush coat. The finish coat wore off on the bottom but brush marks were still visible on the sides. The alinement was straight, the approach curved, and the inlet rounded. The readings were taken during flood flows, the discharge being rated by current meter. (References same as for curve 6.)

Curve 8. Tests by E. Kemler involving observations on 1,500 to 2,000 diameters of brass pipe 0.103 in. to 5.0 in. in diameter. The Nikuradse tests, indicated as plotted points (curve 11, Fig. 19), were included in the data that produced the Kemler curves. (*Transactions*, A.S.M.E., Hydraulics Section, Vol. 55, 1933, p. 7–32.)

Curve 9. Hand-troweled cement finish; discharge rated by current meter placed in the tailrace.

DATA ON LARGE CONCRETE PIPE

Description and literary reference
(8)

There were seven horizontal curves and two large vertical curves. The tests are reliable. (Reference same as for curve 1.)

Curve 10. Some rough spots remained on the surface after the steel forms were removed. This was a poor test, no account being taken of the change in size and shape of the cross section. The alinement was irregular, with six horizontal bends and two vertical bends. The estimated error in the results is ±15%. (Letter dated March 5, 1931, from the Pacific Gas and Electric Company, San Francisco, Calif.)

Curve 11. Smooth pipe; the plotted points designated curve 11 in Fig. 19 denote tests by J. Nikuradse on brass pipe from 0.033 in. in diameter to 3.28 in. in diameter. The equation through these points is good for extrapolation from $R = 30^6$ to 10^8. (*Forschungsheft 356*, Verein Deutscher Ingenieure, 1932, p. 30.)

Curve 12. Steel forms were used and the concrete was rubbed with carborundum brick. The discharge was rated by color movement. The line included five bends on 800-ft radii, with short tangents between. This is equivalent to a curve length of 3,060 ft and a radius of 1,665 ft ($r/D = 92$). The bend loss (±5%) and surge tank losses were not considered. The data used were taken from a curve passing through forty-two observations. (*Bulletin No. 852*, U.S.D.A., Washington, D. C., 1924, p. 83; supplemented by correspondence with the Ontario Power Company in June, 1931, and April, 1935, including a map and a profile.)

Curve 13. Hand-troweled cement finish; discharge rated by current meter placed in the tailrace. This was a very reliable test. The pipe was straight, free from bends and entrance losses, and was equipped with three excellent mercury monometers. The line was built in 1917. (References same as for curve 1.)

Curve 14. Smooth cement surface; generator rating of discharge. It is reported that losses were high because of underestimated intake losses and poor location of the lower piezometer. The writers estimate that these factors make the plotted points ±7% high. (References same as for curve 1.)

Curve 15. On this test oiled forms were used, and a neat cement brush coat. The lining was not smooth. The pipe was probably new at the time of the test and the discharge was rated by a weir below the plant. Surge chamber losses were neglected. The regained velocity head loss h_v was assumed equal to the entrance losses. The invert was placed by hand without forms and it presented a rather rough, uneven appearance. (*Proceedings*, June, 1923, Convention, Pacific Coast Electrical Assn., p. 139; *Engineering News-Record*, October 11, 1923, p. 598; and Library Data File, U.S.B.R., 91-241.)

Curve 16. Surface originally smooth, had become somewhat eroded in 4 years. Discharge was rated by color movement. The line was pre-cast in 4-ft lengths in oiled steel forms. Joints were smooth. The water flowing in this line contains sharp basalt particles which have eroded the bottom of the intake like a sandblast. (*Bulletin No. 852*, U.S.D.A., Washington, D. C., 1924, p. 51). The water enters the pipe in a very turbulent state and the erosion extends 150 ft from the intake. (*Bulletin No. 852*, U.S.D.A., Washington, D. C., 1924, p. 36; also drawing No. 33.19(*b*) in the Denver office of U.S.B.R.)

Curve 17. Use of steel forms in place have resulted in rough joints but with a smooth surface between joints. The line contains a sharp 90° bend and two slight vertical bends in the reach measured. About nineteen diameters upstream there is the last of two bends and constrictions resulting from repair work. The exit head loss was ignored and the results are not consistent. The concrete joints protrude as much as 0.15 ft in places. The effect of the bends was not considered in the computations. (*Bulletin No. 852*, U.S.D.A., Washington, D. C., 1924, p. 81; *Transactions*, ASCE, Vol. 73, September, 1911, pp. 399 and 460; "Catskill Water Supply of New York," by Lazarus White, John Wiley & Sons., Inc., New York, N. Y., 1913, pp. 66 and 73; and *Engineering Record*, January 1, 1910, p. 26; September 17, 1910, p. 312; March 11, 1911, p. 279; and February 28, 1914, p. 240.)

Curve 18. Surface polished. Friction values taken from curve prepared by E. W. Spies. ("Turbulente Stromungon in geraden und gekrummten glotten Rohrleitungen bei hohen Reynolds'schen Zahlen," by Romano Gregorig, Dipl. Masch-Ingenieure, Eidgenossichen Technischen Hochschule in Zurich, No. 695, 1933., D. F. 91-26.)

Curve 19. Pre-cast in oiled steel forms with 12-ft lengths. The joints were smooth and carefully laid. Discharge was rated by venturi meter. In the first test there were twenty-nine bends; in the second, the line was slightly sinuous; and, in the third, the line was nearly straight. (*Engineering News-Record*, May 28, 1925, p. 897.)

Curve 20. Smooth surface; discharge rated by color movement. The line was quite straight in horizontal alinement. Vertical curves were long and gentle. The reach includes five 6-in. valves and three manholes. (*Bulletin No. 852*, U.S.D.A., Washington, D. C., 1924, p. 35.)

Curve 21. Smooth surface. Sections pre-cast in 6-ft steel forms. Discharge was rated by color movement. The alinement was straight. The reach includes eight 6-in. valves, two 6-in. blowoffs, and four 12-in. by 14-in. manholes. The inside surface was painted with a rich cement grout. (*Bulletin No. 852*, U.S.D.A., Washington, D. C., 1924, p. 41.)

Curve 22. Smooth surface; discharge rated by water meter. The conduit in the concrete was formed by a 36.5-ft length of smooth straight rubber hose. (*Technical Memorandum No. 339*, U.S.B.R., Denver, Colo., June 15, 1933.)

Curve 23. Sections pre-cast in oiled steel forms, 4 ft long, steam cured; discharge rated by color movement. The surface is reported as "unusually smooth" but for about half the line is curved gently and no allowance has been made for this curvature in the computations. (*Bulletin No. 852*, U.S.D.A., Washington, D. C., 1924, p. 39.)

Curve 24. Surface formed by troweled gunite. (References same as for curve 18.)

Curve 25. Use of steel forms in place has resulted in rough joints but with a smooth surface between joints. The line contains one sharp 90° bend in the reach measured. The concrete was poured against oiled steel forms but the joints were not smoothed. (*Bulletin No. 852*, U.S.D.A., Washington, D. C., 1924, p. 82; *Engineering Record*, April 2, 1910, p. 460.)

Curve 26. Smooth steel forms were used by the Tennessee Valley Authority (TVA) on the Apalachia Tunnel. Discharge was rated by current meter. The tunnel consists of three types of lining—concrete, steel, and unlined rock. The concrete section contains four long radius bends in the test section. The entire tunnel has a total of six bends.

Curve 27. The solid line marked "curve 27" in Fig. 19(*b*) was plotted from experimental data.

Curve 28. This curve in Fig. 19(*b*) represents the von Kármán-Nikuradse equation: $1/f^{0.5} = 0.8 + 2 \log(R f^{0.5})$.

Curve 29. Extrapolated from data for the 13-ft steel penstock tests at Boulder Dam, introducing the value of $f = 0.0177$ in the formula: $\frac{1}{f^{0.5}} = -2 \log \left(\frac{K}{7.4\,r} + \frac{2.51}{R\,f^{0.5}} \right)$. (*Transactions*, ASCE, Vol. 109, 1944, p. 59.) This formula was derived by C. F. Colebrook and C. M. White to cover the transition from smooth pipe to rough pipe ($K/r = 0.00134$).

structions to acquire help in the vicinity and to perform a definite series of tests in the shortest possible time. As a result, instrument taps are placed in the most convenient points along the pipe rather than in the more significant locations, and difficulties are often encountered with the work performed by the untrained personnel. Information resulting from the tests is too often questionable and is at times contradictory. Tests on prototype structures are a bountiful investment when performed with care and a thorough understanding of the problem involved.

In the case of the Apalachia Tunnel tests, pressure taps were installed in significant locations along the tunnel at the time of construction and well-trained personnel performed the test work and analyzed the data. The results should be very dependable.

The writers and others of the Bureau of Reclamation have collected and compiled test data on friction coefficients for large pipe lines for a number of years. It is felt that some of this information should be of value to the engineering profession as a whole. For example, data showing the relation of the friction coefficient to Reynolds' number are presented in Fig. 19. The data contained in Fig. 19 represent results from concrete pipe to 18 ft in diameter, velocities as high as 42 ft per sec, and Reynolds' numbers approaching 30,000,000.

As each curve in Fig. 19 is numbered, by referring to Table 14, information can be found concerning the diameter, velocity, length, age of pipe, character of surface, and estimated reliability of each test. Also listed in Table 14 are the references to published papers and bulletins on original work from which the information in Fig. 19 was obtained. For the sake of clarity, the actual test points have been omitted. In most cases these did not fall directly on the line as drawn but fell to both sides with considerable variation in some cases. The lines represent average values.

Superimposed in Fig. 19(*a*) are the results of the Nikuradse laboratory tests on smooth brass pipe and on brass pipe artificially roughened with sand; the Kemler boundary of tests on smooth brass pipe; and a line representing the von Kármán and Nikuradse equation for friction coefficients in smooth pipes for large Reynolds' numbers. Also included in Fig. 19 (designated as curve 26) are the results on the concrete-lined section of the Apalachia Tunnel tests. These were obtained from Fig. 10(*a*).

In looking over this graphical presentation of nearly all the apparently reliable hydraulic friction tests of pressure conduits surfaced with concrete and including pipes from 1 in. to 18 ft in diameter, one is first impressed by the wide range of coefficients found at any given Reynolds' number. For example, at a Reynolds' number of 10,000,000, the coefficient varied from 0.008 in the Ontario Power Company test (curve 12) to about 0.0125 in the Apalachia tests (curve 26)—a range of about 25% from the mean value. Whether this variation represented a real difference in the character of the concrete surfaces in the two cases or whether it represented errors in the measurement of hydraulic losses or in computations made to allow for bend or entrance effects is not known. The concrete surface of the Ontario conduit is known to have been exceptionally smooth, but without a quantitative measure of the roughness, such as has been developed by mechanical engineers to measure the smoothness of bearings, it would appear useless to speculate on the reasons for the differences shown. In the circumstances, it would appear necessary for hydraulic engineers, in their computations, to allow for the extreme range of possible friction coefficients. Since this is an economically wasteful procedure, the need for the development of some quantitative laboratory methods of measuring roughness seems evident.

The second feature of the curves in Fig. 19 is the general upward slope of the lines showing the results of tests of individual pipes. An upward slope means that friction losses are increasing at a greater rate than the square of the velocity. This is in accord with the Colebrook and White[43] transition law for the change between smooth and rough surfaces and is indicated in Fig. 19 by the family of $\frac{K}{r}$-curves. Apparently in the larger pipe and at the economic velocities usual in hydraulic design, most flow falls within the transition classification. The plotted data should serve as a warning that, where large diversion tunnels are to be designed to function at high velocities, friction losses may prove to be greater than would be anticipated from low velocity tests. The writers would appreciate comments from the authors on this characteristic of the data presented in the paper.

G. H. HICKOX,[44] M. ASCE, AND A. J. PETERKA[45] AND R. A. ELDER,[46] ASSOC. MEMBERS, ASCE.—The interest shown in this paper, as evidenced by the questions asked, the criticisms made, and the new data presented, is gratifying to the authors.

Mr. Barrows raises an important point when he emphasizes the authors' use of "equivalent circular sections." This method of computation was chosen because it provided a simple relationship between cross-sectional area and hydraulic radius. Although the actual hydraulic radius was a trifle smaller than that obtained by assuming an equivalent circular area, the amount of the difference is small and depends on the variability of the overbreak. The irregularities of overbreak can scarcely be known in advance, but some estimate of the average diameter is always available. Lacking other knowledge, the designer will assume an ideal section for his calculations and will thus utilize the data in the same manner as for the calculations of the paper. It follows that the actual areas, including overbreak, should be used instead of the nominal area. Failure to observe this precaution may result in considerable error. Thus, the designer must estimate the probable overbreak for the particular rock he is working in, and for this particular type of excavation, and must add this overbreak to the nominal diameter.

Messrs. Bradley and Wing are to be commended for the data of Fig. 19 and the accompanying descriptions of tunnel surfaces. This chart, in a much more complete form, was available to the authors, by courtesy of the Bureau of Reclamation, before the Apalachia tunnel tests were made. It is unfortunate that limitations of space prevented showing the experimental points in Fig. 19 and it should be noted that many of the curves presented are supported by scattered data. It was in part the uncertainty caused by these scattered results and by the conflicting nature of the curves that prompted the tests reported in this paper.

---

[43] "Turbulent Flow in Pipes, with Particular Reference to the Transition Region Between the Smooth and Rough Pipe Laws," by C. F. Colebrook, *Journal*, Inst. C. E., London, Vol. 11, 1938–1939, pp. 133–136.

[44] Associate Director, Eng. Experiment Station, Univ. of Tennessee, Knoxville, Tenn.

[45] Hydr. Engr., Hydr. Laboratory, Bureau of Reclamation, Denver, Colo.

[46] Associate Hydr. Engr., TVA Hydr. Laboratory, Norris Tenn.

The authors are indebted to Mr. Hinds for his clear understanding and elucidation of the practical problems surrounding the use of the (f − **R**)-equation for fluid resistance. The nature of the transition region still seems to be very imperfectly understood and much additional work remains to be done, both in laboratories and in the field, before the differences between the Nikuradse and the Colebrook types of transition are fully understood. It seems significant that, although the Apalachia tests were carefully made, they do not conform to either the Nikuradse or the Colebrook type of transition. In the meantime, designers should be aware of the possibilities implied by the data, as suggested by Messrs. Bradley and Wing, and should design with caution.

The tests on the unlined rock tunnel seem to indicate that in spite of the apparent good results the Nikuradse equation for f in terms of k/r for the completely turbulent region may break down for large values of k/r by the introduction of other effects as indicated by Professor Rouse. His suggestion that perhaps the irregularities of the tunnel wall constituted constrictions of the flow area is certainly worth consideration. The unlined rock tunnel was a succession of enlargements and constrictions caused by the method of driving it. The tunnel was at approximately its nominal diameter at each heading, and enlarged gradually to the next heading, with a very irregular surface as described in the paper and shown in Fig. 4. There seems to be little relationship between the size of the projecting rocks and the thickness δ' of the turbulent boundary layer. Perhaps the agreement with the Nikuradse equation is only fortuitous.

Mr. Chapman's suggestion that spiral flow may account for the high value of f in the concrete-lined tunnel is interesting. The cost of preparing for, and conducting, tests to investigate this point would be high because of the size of the conduit, the pressures in it, and its relative inaccessibility. No comparable data are available for the Ocoee No. 3 tunnel.

Mr. Gavett remarked on the successive approximations required by the piezometer locations. The location of the piezometers was a compromise between a complete set of pressure measurements such as might have been made in the laboratory and the practicalities of the field situation. Extensive installations are expensive and it was necessary to set up as few as would yield usable results. Observations were also a matter of serious consideration. The locations were isolated and access was over roads distinctly mountainous in character, some of which had been closed to public use and were no longer being maintained. The distance by such roads was several miles between points less than 7,000 ft apart along the tunnel. The distribution of personnel and equipment to these sites and the reading of gages at 1-min intervals for a continuous 8-hr period in rainy midwinter weather presented difficulties not encountered in the laboratory. Under these conditions it seemed much more reasonable to make a number of approximations than to obtain complete data.

With regard to Mr. Gavett's question about the exponent x in the equation $H_f = C\ V^x$, most of the available data indicate that x is 2 when the Reynolds number is large enough so that complete turbulence exists. For the part of the roughness curve lying on the smooth pipe curve, the exponent is less than 2. In the transition from "smooth" to "rough" flow, it may be either less than or greater than 2, depending on conditions that do not seem to be well understood. J. Nikuradse found a section of the transition from smooth pipe flow to complete turbulence where x was greater than 2. C. F. Colebrook found, however, that for commercial pipe x was everywhere less than 2 in the transition zone. In this field more data are needed, for unfortunately most flow occurs in the transition region, where knowledge of friction factors is least certain. This point is well brought out by Mr. Hinds in his excellent discussion.

The variation of Manning's n with velocity, mentioned by Messrs. Chapman and Hinds, is inherent in the equation whenever the frictional resistance does not vary with the square of the velocity. As a matter of fact, variation with the square of the velocity seldom occurs except for fully developed turbulence, which is beyond the range of most practical applications. Figs. 14 and 19 show the type of variation that may be expected. The surprising feature is not that Manning's equation does not fit these data, but that its inadequacy had not been discovered long ago.

It is hoped that, by the presentation of the data, use of the (f — **R**)-relationship for tunnel friction calculations may be stimulated. The empirical equations of Manning, Scobey, Kutter, and Hazen-Williams are useful within the region of experiment, but there is considerable doubt that the coefficients are characteristic of the surfaces. These equations (except Kutter's) are exponentials and plot as straight lines on logarithmic paper. Consequently, the particular value of the exponent of S (or V, depending on the form of the equation) can be correct only for a relatively small range of velocities as indicated by the curves of Figs. 14 and 19.

The authors do not quite agree with Professor Rouse in his interpretation of the test results. His position appears to be that the Apalachia tunnel data should conform to the curves of Fig. 13. Curve (d) of this figure was obtained from Nikuradse's data on uniformly sanded surfaces, and curve (b), from Colebrook's data on certain commercial surfaces. Mr. Colebrook and C. M. White showed that the transition curve depended on the character of the surface. The authors see no reason why the data presented by them should conform to any of the transition curves previously published. Of far greater significance is the fact that, in all cases, the data for each type of surface tend unmistakably to approach the line representing a value of $\frac{1}{\sqrt{f}} - 2 \log \frac{r}{k} = 1.74$. The scattering of the data for the asphalt-lined tunnel at low Reynolds numbers is accounted for by the fact that the measured head losses were very small. The squares (Fig. 13) represent a maximum head loss of 0.123 ft; and the first four diamonds, a maximum head loss of 0.144 ft. The losses for the concrete-lined tunnel varied from 0.88 ft to 12.58 ft; and for the unlined rock, from 3.13 ft to 24.78 ft. These data scatter much less. The only inconsistency in them is that they lie outside the range limited by the curves of Fig. 13. This is not unexpected because the surfaces tested probably differed from those represented in Fig. 13.

Since the data for all surfaces approach a value of 1.74 for $\frac{1}{\sqrt{f}} - 2 \log \frac{r}{k}$ as the Reynolds number increases, it seems clearly logical to estimate the apparent constant value of f in Fig. 10 for use in calculating k by Eq. 6. The results thus obtained are those of Table 9. There seems to be no more significance in the centroid of "Areas" A, B, and C of Fig. 13, as used by Professor Rouse, than in a centroid of Nikuradse's test data. Further, the similarity in the values of k for the asphalted surface, determined by Mr. Colebrook and the authors as 0.005 in. and 0.0035 in., respectively, cannot be ignored. Professor Rouse's value of $k = 0.00063$ in. needs explanation.

The authors believe that the test data have significance but not necessarily that described by the Colebrook transition function. These data have been presented in the hope that study of the transition region will be stimulated and that an explanation will eventually be evolved which will include the apparent deviations of the present contribution.

Professor Rouse's remarks concerning portrayal of Manning's n as a function of the Reynolds number appear irrelevant. At no point in the paper was n shown in that relationship except in the tables, where it might equally have been considered to be a function of gate opening or head loss. In spite of the fact that leading hydraulicians are turning toward the representation of surface roughness and frictional resistance in terms of f and the Reynolds number, there are still a great many designers who prefer to use empirical relationships such as Scobey's and Manning's equations. It was for this reason that Manning's n was shown as a function of velocity in Figs. 11 and 12. These plots show clearly that Manning's n is not a constant function of the surface and that is the reason they were included. It is true that the velocity for each tunnel is proportional to the Reynolds number. However, those engineers who still use Manning's equation, and for whose benefit Figs. 11 and 12 were included, are not likely to interpret the data in terms of the Reynolds number in spite of the proportionality that exists.

Mr. Tapley's discussion of bend losses is interesting and points out the extremely unsatisfactory state of knowledge in this field. In the tests made by the authors, the only independent determination of bend loss was at bend 1. Losses at the other bends were calculated in Table 5 by application of the data given by Mr. Creager and Joel D. Justin,[5] M. ASCE, in their Figs. 67 and 68. This procedure was followed to provide as good an estimate as possible of the loss chargeable to friction. The determination of bend loss is one of the weak points in the paper, but fortunately the bend losses were relatively small and errors in their determination were consequently unimportant. In addition to the discussion by Mr. Tapley, it might be pointed out that tests by Albert Hofmann[40] also indicated that the bend loss was materially affected by the surface roughness of the bend, the bends with the rougher surfaces, surprisingly enough, having the lower bend-loss coefficients. This serves further to emphasize the need for definitive experimentation in this field.

The data presented by Mr. Kennison have been converted to terms of f and the Reynolds number as listed in Table 15. In making these calculations

TABLE 15.—CONVERSION OF DATA FROM TABLE 12

Line[a] No.	Q (cu ft per sec)	V (ft per sec)	s ($\times 10^3$)	f	Temperature (degrees C)	R ($\times 10^{-6}$)
1......	869.9	6.81	0.580	0.0103	17	7.43
2......	925	7.25	0.619	0.0096	17	7.91
3......	916	7.18	0.596	0.0095	16	7.63
4......	951.	7.46	0.677	0.0100	15	7.72
5......	956	7.49	0.673	0.0099	17	8.17
6......	996	7.81	0.736	0.0099	20.5	9.30
7......	996	7.81	0.733	0.0099	1.0	5.33

[a] Corresponds to Col. 1, Table 12.

the cross section was assumed to be an equivalent circle with the given area of 127.6 sq ft. The values of f are unusually close to the smooth pipe curve and it would be of interest to know the condition of this surface. The points are plotted in Fig. 20.

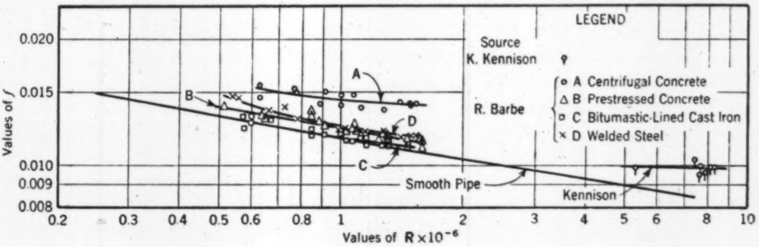


FIG. 20.—FRICTION FACTOR VERSUS THE REYNOLDS NUMBER FOR DATA OF K. KENNISON, M. ASCE, AND R. BARBE

Additional data on surface roughness of commercial pipes have recently been published,[47] and, in the belief that they have not come to the attention of the profession generally, are presented here for convenience of reference. The pipes were all 800 mm (31.5 in.) in nominal diameter and the test length was 200 m (656 ft). The results are shown in Fig. 20, together with the curve for smooth pipe flow. The surfaces are as follows:

A. Centrifugal concrete, of good average quality (new); 4-m (13.06-ft) lengths, with male and female ends, joined with aquarium cement, afterward smoothed on the inside, making a perfectly smooth joint.

B. Pre-stressed concrete (Freyssinet system); 6-m (19.85-ft) lengths with an inside surface formed during manufacture by very smooth helicoidal bands

---

[47] "La Mesure dans un Laboratoire des pertes de charge de conduits industrielles," by R. Barbe, *La Houille Blanche*, Mai-Juin, 1947, pp. 191–203.

48 mm wide with a pitch of 15 m. The bands were very smooth but were separated by fairly rough grooves about 5 mm wide and 1 mm deep. The joints were made by placing the sections end to end and sealing them with flexible rubber rings.

C. Cast iron with smooth bitumastic coating applied by a centrifugal process (new); 5-m (16.4-ft) lengths, very smooth joints, bell and spigot, with poured lead seal, smoothed inside.

D. Welded steel (new); 6-m (19.85-ft) lengths, each length composed of three sheets welded end to end and rolled. There were one longitudinal and two transverse welds in relief on the interior or each section. The joints were the same as for surface *B*.

It will be noted that the bitumastic-lined cast iron is the smoothest, and agrees very closely with the smooth pipe curve.

The authors wish to extend their sincere thanks to all who took the trouble to read and comment on a paper which, after all, merely added to the many conflicting data already existing without offering any explanation.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

TRANSACTIONS

Paper No. 2354

ANALYSIS OF STEPPED-COLUMN MILL BENTS

By Daniel S. Ling,[1] M. ASCE

With Discussion by Messrs. Ralph E. Spaulding, E. I. Fiesenheiser, Walter J. Gray, L. J. Mensch, Arthur R. Graves, Erik Sollid, A. Floris, Charles W. Dohn, and Daniel S. Ling.

Synopsis

The "moment-distribution" technique is applied to the analysis of stepped-column mill bents, in this paper. Curves for obtaining the necessary distribution coefficients and the fixed-end moments are included. The method permits evaluating the relative resistance, to horizontal forces, of columns having different stiffnesses due to length, size, or other factors. Several examples of two-column and three-column bents are included to describe the method fully.

Although this method will require more time than that of obtaining moments from an assumed point of contraflexure, the additional time would seem to be justified, because of the difficulty in estimating the location of such a point of contraflexure accurately for stepped columns.

I. Introduction

In the design of mill buildings, especially where the crane clearances require considerable height, the horizontal forces from wind and crane thrust are important factors. In short buildings, it is simple enough to transmit these forces to the ground through a bracing system to the braced end bents of the building. However, with the trend toward larger and larger industrial buildings, the length increases so that one or more expansion joints are required; and horizontal forces must be resisted by each individual bent. Often, as a result, the bending stresses in the columns become greater than the direct stresses, and the "column" essentially becomes more of a vertical beam than a column from the standpoint of stress analysis.

The simple and more familiar method of analysis for such "columns" is to determine the bending moments from an assumed point of contraflexure. With a column of constant section from top to bottom, the location of this point can be assumed with reasonable accuracy under any set of assumptions regarding the base and top fixations. However, for a column supporting a crane runway,

---

Note.—Published in October, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Associate of Paul S. Calkins, Cons. Engr., Detroit, Mich.

a "stepped section" is considerably more economical, in most cases. Immediately, the question arises as to the location of the point of contraflexure; inaccuracy as to this assumption may cause considerable error in the final result.

It is hoped that this paper will present a more satisfactory method for the analysis of stepped-column mill bents. The loadings given in the examples are used as illustrations of the method, and not as recommendations. Admittedly, these loadings are not the nearest approach to the probable applied conditions, but they do illustrate the various types of design loads as used by designers.

The method of successive approximations, or so-called "moment distribution," is used without explanation, to determine the true end moments, since this method is probably familiar to most engineers and can be found in various textbooks and reference books.

In particular, the writer has used a set of unpublished mimeographed notes, by L. C. Maugh, Assoc. M. ASCE, entitled "The Analysis of Continuous Framed Structures," as a reference in preparing this material. The discussion of the basic theory is given briefly herein as Example 1. For a more detailed discussion of the theory, reference can be made to "Statically Indeterminate Structures,"[2] also by Professor Maugh.

II. General Conditions and Design Assumptions

With the increasing necessity for the greatest economy in the use of steel and the resultant use of higher working stresses, it has been more than ever important to eliminate as many uncertainties as possible in structural design. Since these columns are essentially beams, they should be treated as such. The analysis of a beam of variable moment of inertia is generally understood; it is not difficult, although the work involved does require considerable time and is somewhat tedious. The writer used this general method, with good results, on several industrial buildings erected during World War II. To save time in the analysis of beams of variable moment of inertia, a series of curves is presented which reduces the work to that required for a continuous beam of two spans with the translation of one joint. By the well-known method of successive approximations introduced by Hardy Cross, M. ASCE,[3] the analysis is not a lengthy problem.

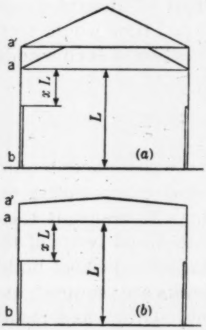


Fig. 1

As with any design problem, certain assumptions must be made to simplify the problem and to save time. Some of these assumptions are known to be erroneous. However, the error involved is so small as to be safely ignored. In this method of analysis the following assumptions were made (see Fig. 1):

a. Hooke's law applies.

b. The principle of superposition applies.

---

[2] "Statically Indeterminate Structures," by L. C. Maugh, John Wiley & Sons, Inc., New York, N. Y., 1946.

[3] *Transactions*, ASCE, Vol. 96, 1932, p. 1.

c. In the deflection of a bent, point a and point a′ move the same amount—that is, point a′ is always directly above point a. (Unless the truss is subject to considerable deflection, this is a safe assumption.)

d. The column is simply supported at points a and a′—that is, the truss or knee brace connections create no effective fixity at these individual points. Since the column is usually quite rigid in relation to the truss or knee brace members, this is a safe assumption.

e. Both columns deflect the same amount—that is, any change in length of roof trusses caused by horizontal forces is too small to be significant.

f. The relation between the moments of inertia of the upper and lower parts of the column is known. (This relation can be approximated readily and quite closely after some experience; but if the preliminary design indicates too large an error the analysis can be corrected quickly with a new ratio of moments.)

(It can be argued that, in effect, the column is fixed at the top. If points a and a′ are close together, it is true that there can be little deflection between these points; but, if this distance is larger, the deflection may be considerable. Assumption *d* is more general and therefore safer than the assumption of full fixity at point a.)

The explanation of such a design method is shown more readily by examples, and this device will be used in this paper. For greater ease in illustrating the principles of the method, the following loading conditions were used in the examples although it is realized that this procedure does not represent true wind action:

(1) Beam bending in a column, from wind, occurs entirely in the windward column;

(2) Wind force is assumed uniformly distributed over the entire length of a column; and

(3) Horizontal force from a crane is applied entirely to one column of a bent.

Symbols and Notation.—The letter symbols in this paper are defined where they first appear, in the text or by illustration, and are assembled for convenience of reference in Appendix I.

III. ILLUSTRATIVE EXAMPLE

Example 1.—Consider a column of a symmetrical bent similar to those in Fig. 1 and fixed at the base. If I is the moment of inertia of the main section of the column and I_o is that of the smaller (reference) section above the step, let $I/I_o = 8$ and $x = 0.3$. The general slope-deflection equations, for moments in a beam (ab, Fig. 1) with a variable moment of inertia, are:

$$M_{ab} = \frac{E\,I_o}{L}\left[C_1\,\theta_a + C_2\,\theta_b - (C_1 + C_2)\,\frac{\Delta}{L}\right] + M_{f,ab} \ldots\ldots\ldots (1a)$$

and

$$M_{ba} = \frac{E\,I_o}{L}\left[C_2\,\theta_a + C_3\,\theta_b - (C_2 + C_3)\,\frac{\Delta}{L}\right] + M_{f,ba} \ldots\ldots\ldots (1b)$$

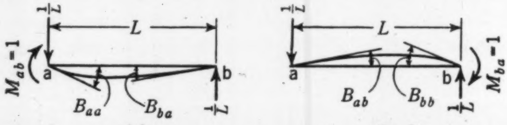
In Eqs. 1, E is the modulus of elasticity; C is the slope-deflection coefficient; θ is the angular rotation, with subscripts a or b denoting point of rotation; Δ is the relative displacement of one end of the beam to the other end; M_f is the fixed-end moment; M_{ab} equals moment in joint a of member ab; and M_{ba} equals moment at joint b of member ab. If the beam is hinged at point a, moment M_{ab} is zero. Solving for θ_a in Eq. 1a and substituting in Eq. 1b:

$$M_{ba} = \frac{E\,I_o}{L}\left(C_3 - \frac{C_2^2}{C_1}\right)\left(\theta_b - \frac{\Delta}{L}\right) + M_{f,ba} - \frac{C_2}{C_1} M_{f,ab} \ldots\ldots (2a)$$

Similarly, if the beam is hinged at point b:

$$M_{ab} = \frac{E\,I_o}{L}\left(C_1 - \frac{C_2^2}{C_3}\right)\left(\theta_a - \frac{\Delta}{L}\right) + M_{f,ab} - \frac{C_2}{C_3} M_{f,ba} \ldots\ldots (2b)$$

The relation between the end rotations and the end moments can be determined by obtaining B_{aa} and B_{ba}, the angles of rotation at joints a and b, respectively, caused in each case by a unit moment at joint a; and B_{ab} and B_{bb}, the angles of rotation at joints a and b, respectively, caused by a unit moment at joint b (Fig. 2). For any end moments M_{ab} and M_{ba}, then:



(a) (b)

FIG. 2

$$M_{ab} = \frac{B_{bb}}{\beta}\,\theta_a + \frac{B_{ba}}{\beta}\,\theta_b \ldots\ldots\ldots\ldots\ldots\ldots (3a)$$

and

$$M_{ba} = \frac{B_{ba}}{\beta}\,\theta_a + \frac{B_{aa}}{\beta}\,\theta_b \ldots\ldots\ldots\ldots\ldots\ldots (3b)$$

in which

$$\beta = B_{aa}\,B_{bb} - B_{ba}^2 \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Comparing Eqs. 3 with Eqs. 1, with $\frac{E\,I_o}{L} = K_o$:

$$C_1\,K_o = \frac{B_{bb}}{\beta} \ldots\ldots\ldots\ldots\ldots\ldots (5a)$$

$$C_2\,K_o = \frac{B_{ba}}{\beta} \ldots\ldots\ldots\ldots\ldots\ldots (5b)$$

and

$$C_3\,K_o = \frac{B_{aa}}{\beta} \ldots\ldots\ldots\ldots\ldots\ldots (5c)$$

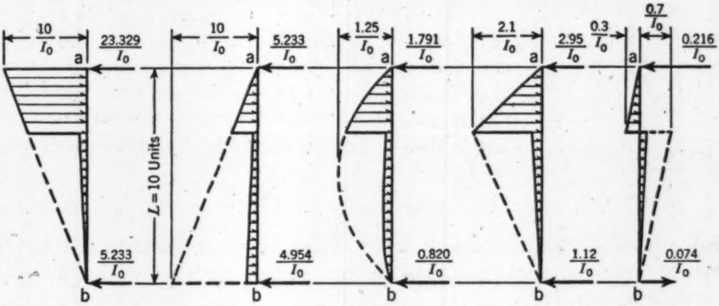
In Eqs. 5, $\Delta = 0$ and fixed-end moments are temporarily ignored. Having determined the coefficients, the fixed-end moments can be determined for any applied loads by computing the angular rotations α_a and α_b of the end tangents of the beam considered simply supported. The fixed-end moment must prevent any rotation of the end tangents; therefore,

$$M_{f,ab} = C_1 K_o (-\alpha_a) + C_2 K_o (-\alpha_b) \ldots\ldots\ldots\ldots (6a)$$

and

$$M_{f,ba} = C_2 K_o (-\alpha_a) + C_3 K_o (-\alpha_b) \ldots\ldots\ldots\ldots (6b)$$

The angles in Eqs. 5 can readily be determined by the conjugate beam method, by determining the reactions of a beam loaded with the $\frac{M}{I}$-diagram (since E is constant). Figs. 3(*a*) and 3(*b*) illustrate the use of that method for this example. An end moment of ten units is applied at end a and the length L is divided into ten units. For the $\frac{M}{I}$-diagram applied as a load the reaction at end a is $23.329/I_o$, and at end b it is $5.233/I_o$. Since these values are for an end moment of ten units and a length of ten units, the end rotation B_{aa} at end a is $\frac{23.329}{100} \times \frac{L}{I_o}$ or $0.23329 \frac{L}{I_o}$; and the rotation B_{ba} at end b is $0.05233 \frac{L}{I_o}$. In the same way, by applying an end moment of ten units at end b it is found that $B_{ab} = 0.05233 \frac{L}{I_o}$ and $B_{bb} = 0.04954 \frac{L}{I_o}$. Substituting these values in Eq. 4: $\beta = \frac{0.23329}{K_o} \times \frac{0.04954}{K_o} - \left(\frac{0.05233}{K_o}\right)^2 = \frac{0.00881}{(K_o)^2}$. Then (see Eq. 5*a*) $C_1 K_o = \frac{0.04954}{K_o} \times \frac{(K_o)^2}{0.00881}$. The stiffness ratio K_o cancels and $C_1 = 5.63$. Likewise $C_2 = \frac{0.05233}{0.00881} = 5.95$; and $C_3 = \frac{0.23329}{0.00881} = 26.50$. The carry-over factor from end a to end b is $\frac{C_2}{C_1} = \frac{5.95}{5.63} = 1.06$; and from end b to end a the carry-over factor $\frac{C_2}{C_3} = \frac{5.95}{26.50} = 0.22$. Substituting these values in Eqs. 6:



(*a*) (*b*) (*c*) (*d*) (*e*)

FIG. 3

$$M_{f,ab} = K_o (-5.63\, \alpha_a - 5.95\, \alpha_b) \ldots\ldots\ldots\ldots (7a)$$

and

$$M_{f,ba} = K_o\,(-\,5.95\,\alpha_a - 26.50\,\alpha_b) \qquad \text{(7b)}$$

As shown in Fig. 4, the loadings on this column which produce bending are of three types—a uniform load of 0.5 kips per ft, a concentrated load (at the change in section) of 8 kips, and an eccentric load of 120 kips. The upper load of 32 kips will also produce some bending, particularly in the upper section; however, this will not be especially severe, since it is near the center line of the lower section. In this example, clockwise resisting moments are designated "plus (+)" and counter-clockwise resisting moments, "minus (−)"; the number within a box indicates the ratio of the moment distributed to that side of joint; the number at an arrow between joints indicates the "carry-over" to the adjacent joint in the direction of the arrow; and, at the completion of single distribution at a joint, a single line is drawn under the numbers and after the final distribution (unbalanced factor of negligible size) a double line is drawn under the numbers and a summation is made.

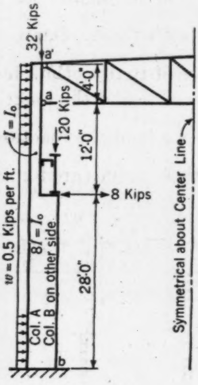


FIG. 4

Fig. 3(c) shows the $\frac{M}{I}$-diagram for a unit uniform load applied to a beam of length L. As before, the end reactions give the end rotations or, in this case, angles α_a and α_b. Substituting the values of the end reactions in Eqs. 7:

$$M_{f,ab} = 0.0516\,W\,L \qquad \text{(8a)}$$

For $W = 20$ kips and $L = 40$ ft, Eq. 8a yields $M_{f,ab} = 0.0516 \times 20 \times 40 = 41.3$ ft-kips; and $M_{f,ba} = 0.1126\,W\,L = 0.1126 \times 20 \times 40 = 90.1$ ft-kips.

Fig. 3(d) shows the $\frac{M}{I}$-diagram for a unit load applied 0.3 L from end a; thus:

$$M_{f,ab} = 0.0994\,P\,L \qquad \text{(8b)}$$

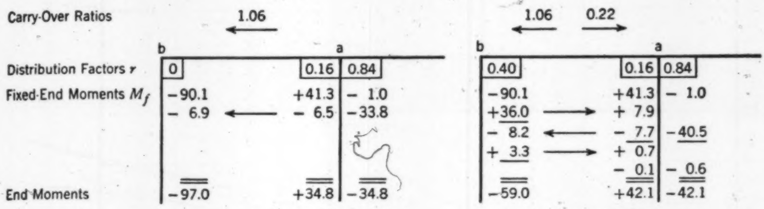
For $P = 8$ kips and $L = 40$ ft, Eq. 8b yields $M_{f,ab} = 0.0994 \times 8 \times 40 = 31.8$ ft-kips and $M_{f,ba} = -\,0.1210\,P\,L = -\,0.1210 \times 8 \times 40 = -\,38.7$ ft-kips.

For a unit moment applied 0.3 L from end a, Fig. 3(e) shows the $\frac{M}{I}$-diagram. Again substituting in Eq. 7:

$$M_{f,ab} = +\,0.166\,M_x \qquad \text{(8c)}$$

For $M_x = 120$ ft-kips Eq. 8c yields $M_{f,ab} = 0.166 \times 120 = +\,19.9$ ft-kips and $M_{f,ba} = +\,0.325\,M_x = +\,39.0$ ft-kips.

For the column under consideration—with $C_1 = 5.63$; $C_2 = 5.95$; and $C_3 = 26.50$—the carry-over ratios from points a to b and from points b to a, respectively, are: $\frac{C_2}{C_3} = \frac{5.95}{26.50} = 0.22$; and $\frac{C_2}{C_1} = \frac{5.95}{5.63} = 1.06$. Computations for the distribution factors r at point a, Fig. 4, are shown in Table 1; and fixed-end moments for the several types of load are shown in Table 2. The unbalanced moments at end a are distributed as shown in Fig. 5(a), assuming full fixity at the base; and the shear H_b is computed as $\frac{0.5 \times 40}{2} + \frac{97.0 - 34.8}{40} = 11.56 \leftarrow$ kips.


Carry-Over Ratios
1.06
1.06
0.22
b
a
b
a
Distribution Factors r
0
0.16
0.84
0.40
0.16
0.84
Fixed-End Moments M_f
−90.1
+41.3
− 1.0
− 6.9
− 6.5
−33.8
End Moments
−97.0
+34.8
−34.8
−90.1
+41.3
− 1.0
+36.0
+ 7.9
− 8.2
− 7.7
−40.5
+ 3.3
+ 0.7
− 0.1
− 0.6
−59.0
+42.1
−42.1

(a) UNBALANCED MOMENTS AT POINT a

(b) CONCENTRATED LOAD OF 8 KIPS

FIG. 5.—DISTRIBUTION OF MOMENTS, EXAMPLE 1

In a like manner distributing the fixed-end moments (in ft-kips) for the 8-kip concentrated load produces: $M_{ba} = -44.1$, $M_{ab} = +26.7$, $M_{aa'} = 26.7$ and (in kips) $H_b = 2.84$. For the 120-kip eccentric load: $M_{ba} = +35.6$, $M_{ab} = +16.7$, and $M_{aa'} = -16.7$. If only 60% fixity is assumed at point b the procedure is similar (see Fig. 5); and H_b becomes $10.0 + \frac{59.0 - 42.1}{40} = 10.42 \leftarrow$ kips.

TABLE 1.—DISTRIBUTION FACTORS, r, AT POINT a, FIG. 4

Member	C	K	$C\,K$	r
ab......	5.63	$I_e/40$	$0.141\ I_e$	0.16
aa′......	3.00	$I_e/4$	$0.750\ I_e$	0.84
Σ........	....	....	$0.891\ I_e$	....

The horizontal forces, of course, cause a translation of the top of the column in relation to the base or, conversely (since by assumption c—points a and a′ move the same amount), point b moves some distance Δ in relation to points a and a′. This translation, naturally, causes additional end moments in the column. These moments are not readily determined directly but can be obtained easily as follows:

TABLE 2.—FIXED-END MOMENTS, EXAMPLE 1, IN FOOT-KIPS

Loading	$M_{f\ aa'}$	$M_{f,ab}$	$M_{f,ba}$
Uniformly Distributed:			
0.5 kips per ft..........	$0.5 \times 4^2/8 = 1.0$	41.3	−90.1
Point Loads:			
8 kips concentrated...	....	31.8	−38.7
120 kips eccentric......	....	19.9	39.0

Assume a value of $E\,\Delta$; in this case, assume $E\,\Delta = 40$ units. From Eqs. 1 the end moments for this translation, only, are:

$$M_{f,ab} = K\left[(C_1 + C_2)\frac{E\,\Delta}{L}\right] = (5.63 + 5.95)\frac{40}{40} = 11.58 \text{ ft-kips } (K = 1)$$

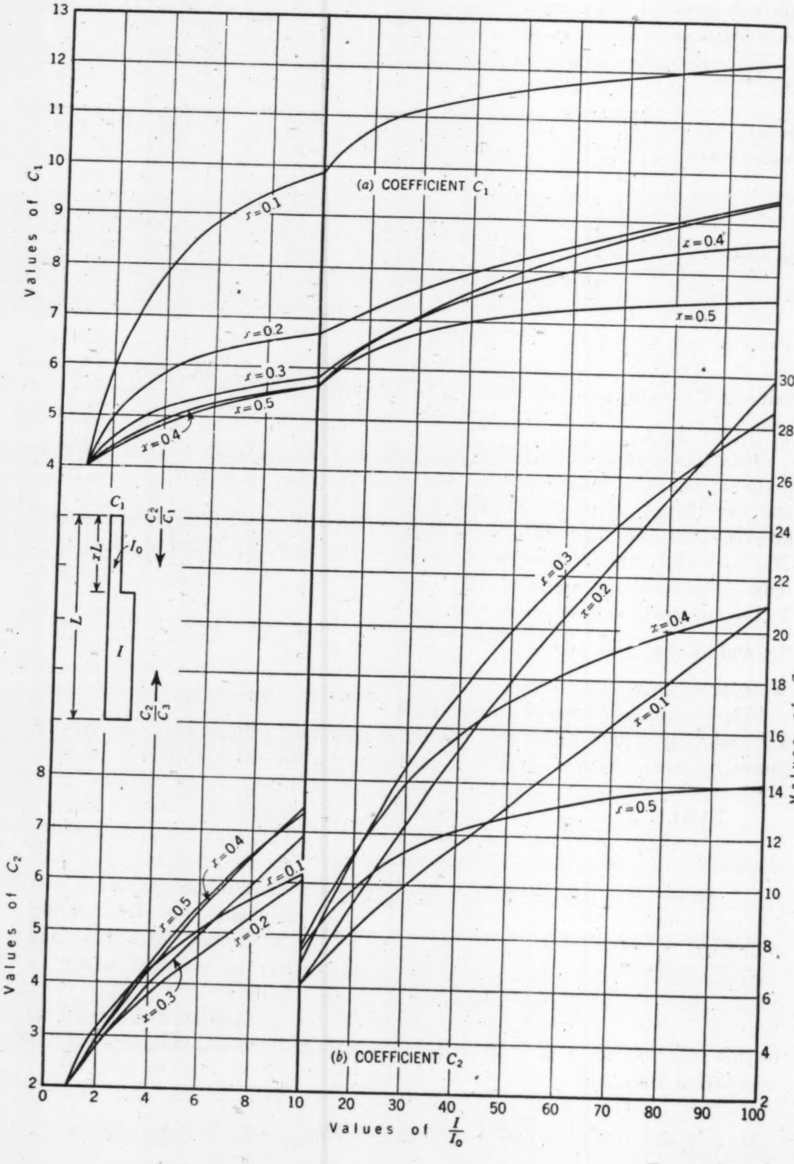


FIG. 6.—CURVES OF

and

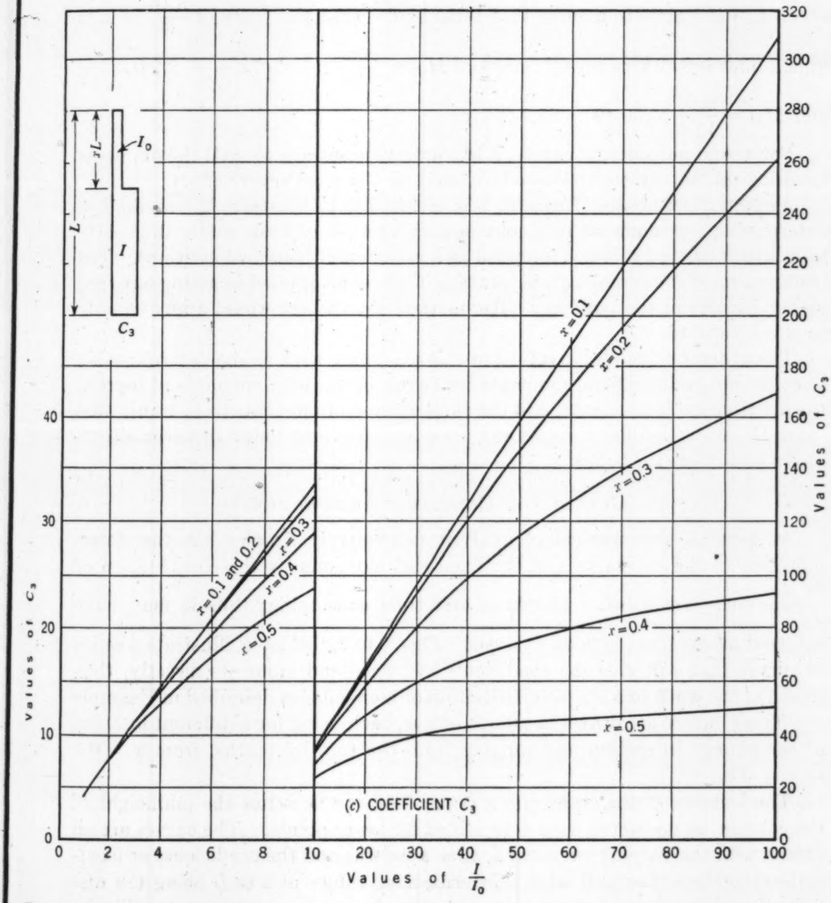
$$M_{f,ba} = K\left[(C_2 + C_3)\frac{E\,\Delta}{L}\right] = (5.95 + 26.50)\,\frac{40}{40} = 32.45 \text{ ft-kips.}$$

By distribution, as before, the final moments, in foot-kips, become:

$$M_{ab} = -11.58 + 0.16 \times 11.58 = -9.73$$
$$M_{ba} = -32.45 + 1.06 \times 0.16 \times 11.58 = -30.49$$

and, in kips:

$$H_b = \frac{30.49 + 9.73}{40} = 1.01$$



(c) COEFFICIENT C_3

Distribution Coefficients

For the uniform horizontal load on the column, and with no translation considered, it was shown that the shear at point b on the windward column was 11.56 kips. Since the total force assumed acting on the column was 22 kips a force of $22 - 11.56 = 10.44$ kips is the force creating deflection in the bent. Assuming that both columns resist this deflection equally, the force resisted by each column is 5.22 kips. Then the end moment M_{ab} for this 5.22 kips is $\frac{5.22}{1.01} \times -9.73 = -50.5$ ft-kips and M_{ba} is $\frac{5.22}{1.01} \times -30.49 = -157.6$ ft-kips.

For the 8-kip concentrated load the force causing bent deflection is $8 - 2.84 = 5.16$ kips. Since the crane thrust is spread to adjacent bents by the roof bracing, it is logical to assume that three bents resist this force or $\frac{5.16}{3} = 1.72$ kips. This is 0.86 kips per column; and $M_{ab} = \frac{0.86}{1.01} \times -9.73 = -8.2$ ft-kips; and $M_{ba} = \frac{0.86}{1.01} \times 30.49 = 25.7$ ft-kips.

For design purposes, of course, it is now only necessary to total the moments from logical loading conditions which produce the most severe effect.

As stated previously, Example 1 is given only to illustrate the method of determining the moments in a column from any set of loads under the conditions assumed, and it is not intended as a criterion of loading conditions. For instance, most codes require a percentage of the wind applied as acting outward on the lee side of the building. The method shown here would apply equally to this condition.

The general principles involved in this example—such as the determination of constants and fixed-end moments for beams of variable moments of inertia, the distribution of moments, and the computation of moments from translation—are no doubt familiar to most engineers and are to be found in various textbooks and reference books.

IV. Curves for Constants and Moments

In general, this method of analysis is relatively simple; but the determination of the constants and the fixed-end moments by the construction of $\frac{M}{I}$-diagrams is somewhat laborious and time consuming; thus it may have deterred others from such an analysis. Figs. 6 to 9, inclusive, illustrate a series of curves that will give the coefficients and fixed-end moments directly, thus reducing the work to a simple distribution of moments, as described in Example 1. These curves are drawn in groups of five, each being for a different location of the change in section and ranging from the top, by tenths, from $x = 0.1$ to $x = 0.5$.

The location of the crane girder is not likely to be below the midheight of the column, so no curves were determined for lower points. The curves are all drawn with the varying values of I/I_o as abscissas and the coefficients or fixed-end moments as the ordinates. Intermediate values of x ($x\,L$ being the distance from the top of column section to the change in section) can readily be

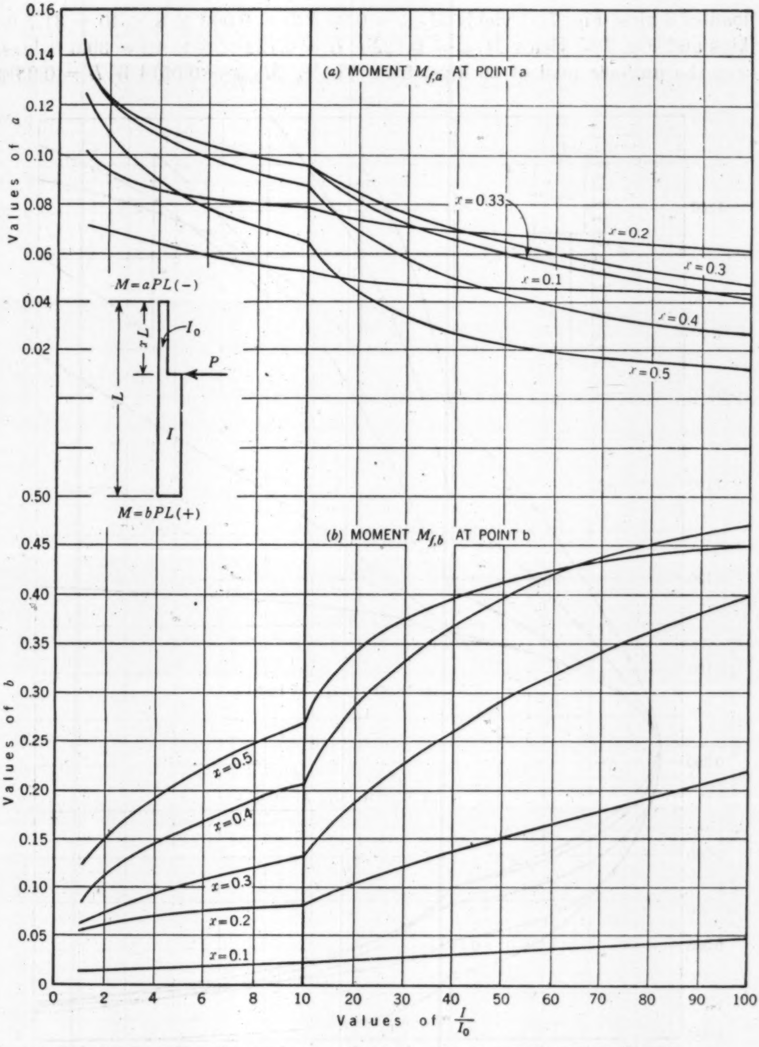


FIG. 7.—FIXED-END MOMENTS FOR A FORCE APPLIED AT THE CHANGE OF SECTION

interpolated. In two of the series of curves (Figs. 7(*a*) and 9(*a*)) it was found desirable to add a curve for $x = 0.33$.

Example 2. Use of Curves.—For the known values of $I/I_o = 8$ and $x = 0.3$, Fig. 6 reveals that $C_1 = 5.65$, $C_2 = 5.95$, and $C_3 = 26.5$. Thus, the "carry-over" from point a to point b $= \frac{5.95}{5.65} = 1.05$. Similarly, for the horizontal load of 8 kips, Fig. 7(*a*) yields $M_{f,ab} = 0.99\ P\ L = 0.099 \times 8 \times 40 = 31.7$ ft-kips and Fig. 7(*b*) yields $M_{f,ba} = 0.122\ P\ L = 0.122 \times 8 \times 40 = 39.0$ ft-kips. For the uniform load of 20 kips, from Fig. 8, $M_{f,ab} = 0.0514\ W\ L = 0.0514$

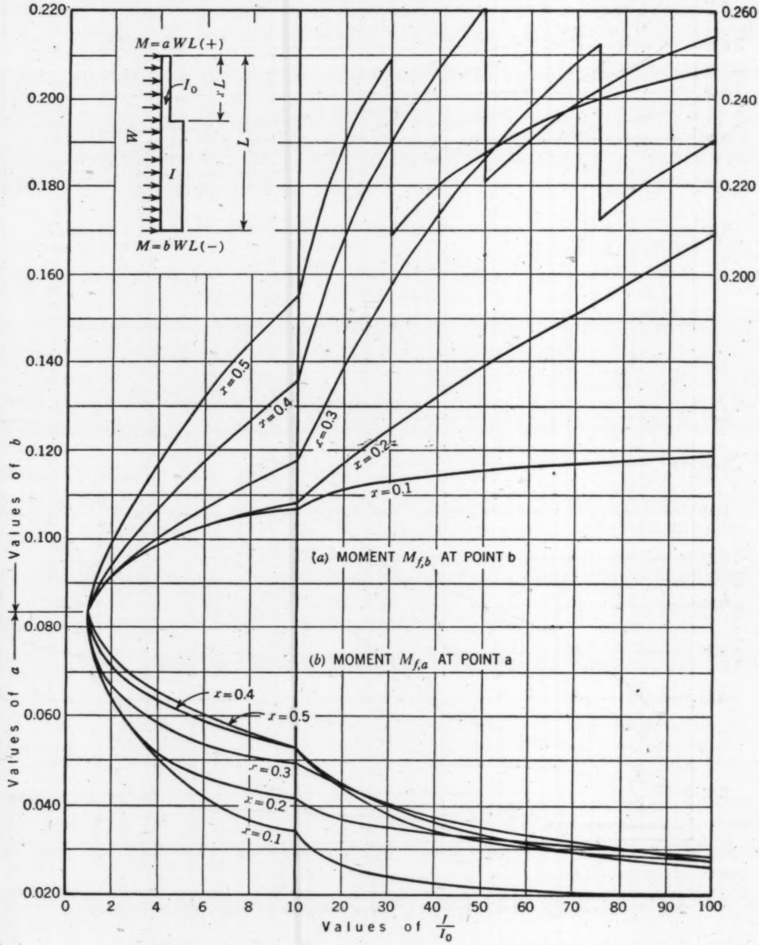


FIG. 8.—FIXED-END MOMENTS FOR UNIFORMLY DISTRIBUTED HORIZONTAL LOAD

$\times 20 \times 40 = 41.1$ ft-kips and $M_{f,ba} = 0.1125\ W\ L = 0.1125 \times 20 \times 40 = 90.0$ ft-kips. For the applied moment of 120 ft-kips from Fig. 9, $M_{f,ab} = 0.167\ M_1 = 0.167 \times 120 = 20.0$ ft-kips and $M_{f,ba} = 0.323\ M_1 = 0.323 \times 120 = 38.8$

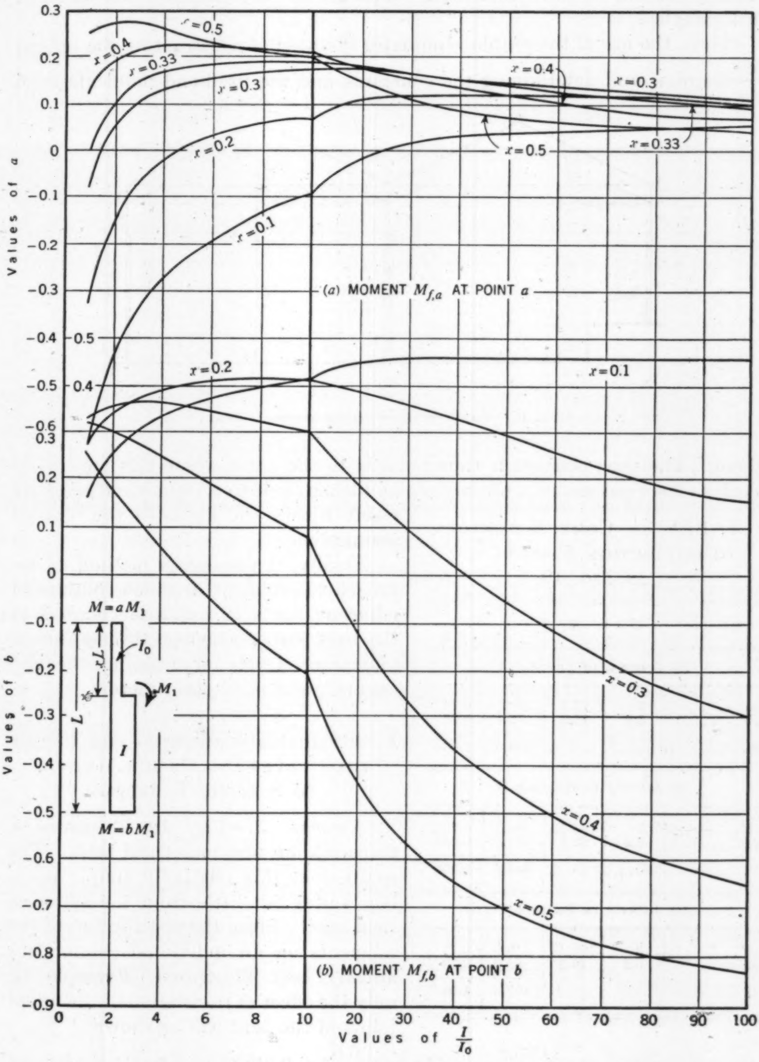


FIG. 9.—FIXED-END MOMENTS FOR A MOMENT COUPLE (AN ECCENTRIC LOAD) APPLIED AT THE CHANGE OF SECTION

ft-kips. It will be noted that these values are all very close to those obtained in Example 2.

Thus, the use of the graphs eliminates the need of constructing the several $\frac{M}{I}$-diagrams and determining the constants and the fixed-end moments from them. The time element is thus reduced to the point where this method of analysis becomes entirely practicable and lends itself to a logical and accurate result within the limits of the original assumption.

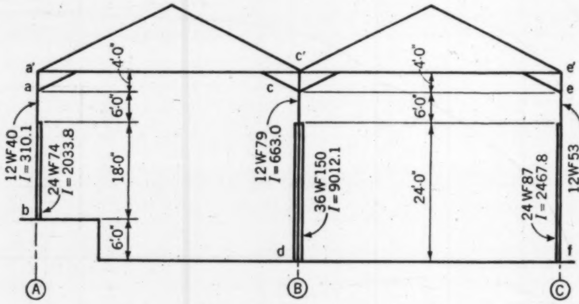


FIG. 10.—THREE-COLUMN UNSYMMETRICAL BENT

Finally, it presents a method of approach to problems in which columns in adjacent bents or opposite columns in the bent are of varying stiffness due to mezzanine levels of framing, heavier column section, or other factors.

TABLE 3.—COMPUTATION OF DISTRIBUTION FACTORS *r*, IN FIG. 10

Member	*C*	*K*	*C K*	*r*
(*a*) JOINT a IN SECTION A				
aa'	3	77.5	232.5	0.75
ab	5.9	12.9	76.1	0.25
Σ	....	....	308.6	1.00
(*b*) JOINT c IN SECTION B				
cc'	3	165.75	497.2	0.765
cd	6.9	22.1	152.5	0.235
Σ	....	....	649.7	1.000
(*c*) JOINT e IN SECTION C				
ee'	3	106.55	319.6	0.78
ef	6.2	14.2	88.0	0.22
Σ	....	....	407.6	1.00

V. APPLICATIONS SHOWING THE USE OF THE CURVES AND DESIGN METHOD TO SPECIFIC PROBLEMS

Example 3.—Fig. 10 illustrates a three-column unsymmetrical bent. The solution of this relatively simple problem varies but little from the symmetrical bent. Since the distribution of the moments on an individual column has already been illustrated (Example 1), only the effect of translation of the upper joints of the bent will be shown.

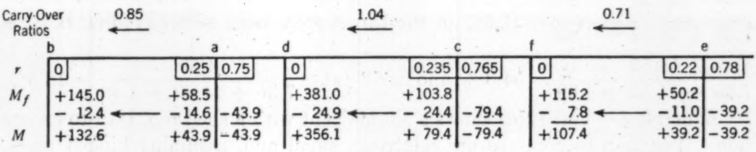
For line A: $\frac{I}{I_o} = \frac{2,033.8}{310.1} = 6.55$; $x = \frac{6.0}{24.0} = 0.25$; $C_1 = 5.9$; $C_2 = 5.0$; and $C_3 = 22.0$ (from Fig. 6). Carry-over ratios are: $\frac{5}{22.0} = 0.23$ and $\frac{5}{5.9} = 0.85$.

Also, $K_{aa'} = \frac{310}{4} = 77.5$; $K_{ab} = \frac{310}{24} = 12.9$; and the distribution factors are computed as in Table 3(*a*). For $E\,\Delta = 10$:

$$M_{f,ab} = \frac{(5.9 + 5.0) \times 12.9 \times 10}{24} = 58.5 \text{ ft-kips}$$

and

$$M_{f,ba} = \frac{(5.0 + 22.0) \times 12.9 \times 10}{24} = 145.0 \text{ ft-kips}$$


Carry-Over Ratios
0.85
1.04
0.71
b
a
d
c
f
e
r
0
0.25
0.75
0
0.235
0.765
0
0.22
0.78
M_f
+145.0
− 12.4
+132.6
+58.5
−14.6
+43.9
−43.9
−43.9
+381.0
− 24.9
+356.1
+103.8
− 24.4
+ 79.4
−79.4
−79.4
+115.2
− 7.8
+107.4
+50.2
−11.0
+39.2
−39.2
−39.2
M

(*a*) SECTION A (*b*) SECTION B (*c*) SECTION C

Fig. 11.—Distribution of Moments, Example 3, Fig. 10

These values of fixed-end moment are distributed as in Fig. 11(*a*), with the result that the shear at point b is computed as $\frac{132.6 + 43.9}{24} = 7.35$ kips.

For line B: $\frac{I}{I_o} = \frac{9{,}012.1}{6{,}630} = 13.6$; $x = \frac{6.0}{30.0} = 0.2$; $C_1 = 6.9$; $C_2 = 7.2$; and $C_3 = 44.5$. Carry-over ratios are: $\frac{7.2}{44.5} = 0.16$ and $\frac{7.2}{6.9} = 1.04$. Also, $K_{cc'} = \frac{663}{4} = 165.75$; $K_{cd} = \frac{663}{30} = 22.1$; and the distribution factors are computed as in Table 3(*b*). For $E\,\Delta = 10$:

$$M_{f,cd} = \frac{(6.9 + 7.2) \times 22.1 \times 10}{30} = 103.8 \text{ ft-kips}$$

and

$$M_{f,dc} = \frac{(7.2 + 44.5) \times 22.1 \times 10}{30} = 381.0 \text{ ft-kips}$$

As for line A, these values of fixed-end moment are distributed in Fig. 11(*b*), the resulting shear at point d being $\frac{356.1 + 79.4}{30} = 14.53$ kips.

For line C: $\frac{I}{I_o} = \frac{2{,}467.8}{426.2} = 5.8$; $x = 0.2$; $C_1 = 6.2$; $C_2 = 4.4$; and $C_3 = 20.0$. Carry-over ratios are: $\frac{4.4}{20.0} = 0.22$ and $\frac{4.4}{6.2} = 0.71$. Also, $K_{ee'} = \frac{426.2}{4} = 106.55$; $K_{ef} = \frac{426.2}{30} = 14.2$; and the distribution factors are computed as in Table 3(*c*).

For $E\,\Delta = 10$:

$$M_{f,ef} = \frac{(6.2 + 4.4) \times 14.2 \times 10}{30} = 50.2 \text{ ft-kips}$$

and

$$M_{f,fe} = \frac{(4.4 + 20.0) \times 14.2 \times 10}{30} = 115.4 \text{ ft-kips}$$

These values of fixed-end moment are distributed as in Fig. 11(*c*), the shear at point f being $\frac{107.4 + 39.2}{30} = 4.89$ kips.

From the foregoing it is then evident that column line A takes $\frac{7.35}{7.35 + 14.53 + 4.89} = 27.5\%$ of the thrust from bent sidesway; line B takes $\frac{14.53}{7.35 + 14.53 + 4.89} = 54.3\%$; and line C takes $\frac{4.89}{7.35 + 14.53 + 4.89} = 18.3\%$.

Example 4.—Fig. 12(*a*) shows a simple bent with a shed roof lean-to on one side. The lean-to columns are relatively small and, assuming hinged bases, they provide no horizontal stiffness to the bent. However, since joint e can move only horizontally (the column prevents vertical translation) the stiffness of the lean-to roof truss insures that although joints c and d can deflect horizontally, they deflect the same distance. Thus, the column translation (E-line) in this case, will be a combination of two types as shown in Figs. 12(*b*) and 12(*c*), and the total stiffness of these columns is much increased.

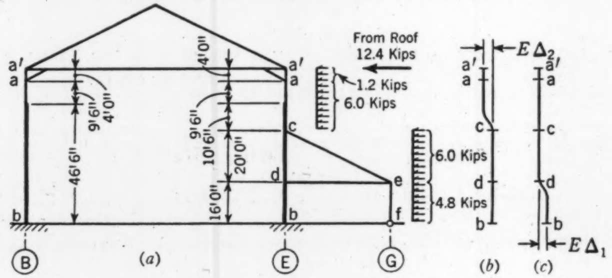


FIG. 12.—SIMPLE BENT WITH LEAN-TO

A somewhat different method is required to analyze this bent. First, assuming that $E\,\Delta_2 = 0$, assign an arbitrary value to $E\,\Delta_1$ and distribute the moments produced. Then express the shears in terms of $E\,\Delta_1$, or since E is a constant, it can be dropped. Next, with $E\,\Delta_1 = 0$, assign a value to $E\,\Delta_2$, distribute the moments, and express the shears in terms of Δ_2. The total shear in the column between any two joints is the sum of these two shear terms. Thus knowing the force system which produces translation, equations can be set up for the solution of Δ_1 and Δ_2. However, since column B also offers resistance to this force system, the method used in Example 1 must be applied. For horizontal forces acting only on column B, the shears in column E produced by translation are the same between points a and c and between points d and b. Assume any value for this shear, and solve for Δ_1 and Δ_2. Then the total deflection at point a is the sum of these for this particular shear. Since the tops of columns B and E move the same amount (assumption *e*), determine the shear in the B columns for the same Δ-value and the relative stiffness of the two lines of columns is proportionate to these shears.

TABLE 4.—COMPUTATION OF DISTRIBUTION FACTORS *r* FOR COLUMN B, FIG. 12

Member	*C*	*K*	*C K*	*r*
aa'.........	3	107.5	322.5	0.85
ab.........	7.3	7.7	56.2	0.15
Σ..........	....	....	787.7	1.00

To apply the foregoing to a numerical example, first consider column B, Fig. 2, with: $I_o = 430; I = 3{,}130; \frac{I}{I_o} = 7.3;$ and $x = \frac{9.5}{56} = 0.17$. From Fig. 6: $C_1 = 7.3$; $C_2 = 5.2$; $C_3 = 24.5$; and the carry-over factor is $\frac{5.2}{7.3} = 0.71$. Also, $K_{aa'} = \frac{430}{4} = 107.5$; $K_{ab} = \frac{430}{56} = 7.7$; and the distribution factors are computed as in Table 4. For $E\,\Delta = 10$:

$$M_{f,ab} = \frac{10\,K_o\,(C_1 + C_2)}{L}$$

$$= \frac{10 \times 7.7\,(7.3 + 5.2)}{56} = 17.2 \text{ ft-kips}$$

and

$$M_{f,ba} = \frac{10\,K_o\,(C_2 + C_3)}{L}$$

$$= \frac{10 \times 7.7\,(5.2 + 24.5)}{56} = 40.8 \text{ ft-kips}$$

These values of fixed-end moment are distributed as in Fig. 13, the shear at point b, Fig. 12, being $\frac{39.0 + 14.6}{56} = 0.96$ kips.

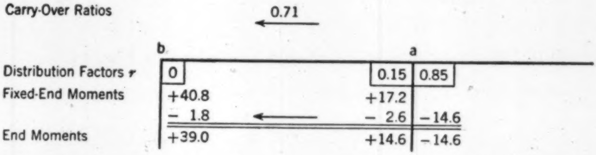


FIG. 13.—DISTRIBUTION OF MOMENTS IN COLUMN B, FIG. 12 (EXAMPLE 4)

TABLE 5.—COMPUTATION OF DISTRIBUTION FACTORS *r* FOR COLUMN E, FIG. 12

Member	*C*	*K*	*C K*	*r*
(*a*) JOINT a				
aa'.........	3	107.5	322.5	0.74
ac.........	5.3	21.5	114.0	0.26
Σ..........	....	....	436.5	1.00
(*b*) JOINT c				
ca.........	20	21.5	430.0	0.41
cd.........	4	156.5	626.0	0.59
Σ..........	....	....	1,056	1.00
(*c*) JOINT d				
dc.........	4	156.5	626.0	0.445
db.........	4	195.6	782.4	0.655
Σ..........	....	....	1,408.4	1.00

For E line: Assume I_o and I the same as for column B, with $I/I_o = 7.3$. Furthermore: $x = \frac{9.5}{20} = 0.475$; and (see Fig. 6) $C_1 = 5.3$; $C_2 = 6.1$; and

$C_3 = 20.0$. Carry-over ratios are: $\frac{6.1}{5.3} = 1.15$; and $\frac{6.1}{20} = 0.305$. Finally: $K_{aa'} = 107.5$; $K_{ac} = \frac{430}{20} = 21.5$; $K_{cd} = \frac{3{,}130}{20} = 156.5$; and $K_{db} = \frac{3{,}130}{16} = 195.6$. Distribution factors are computed as in Table 5.

For $E\,\Delta_1 = 1$:

$$M_{f,bd} = M_{f,db} = \frac{6\,K\,E\,\Delta}{L} = \frac{6 \times 195.6 \times 1}{16} = 73.4 \text{ ft-kips}$$

and, for $E\,\Delta_2 = 1$:

$$M_{f,ac} = \frac{(C_1 + C_2)\,K\,E\,\Delta}{L} = \frac{(5.3 + 6.1) \times 21.5 \times 1}{20} = 12.3 \text{ ft-kips}$$

and

$$M_{f,ca} = \frac{(C_2 + C_3)\,K\,E\,\Delta}{L} = \frac{(6.1 + 20.0) \times 21.5 \times 1}{20} = 28.0 \text{ ft-kips}$$

These values of fixed-end moment are distributed as in Fig. 14.

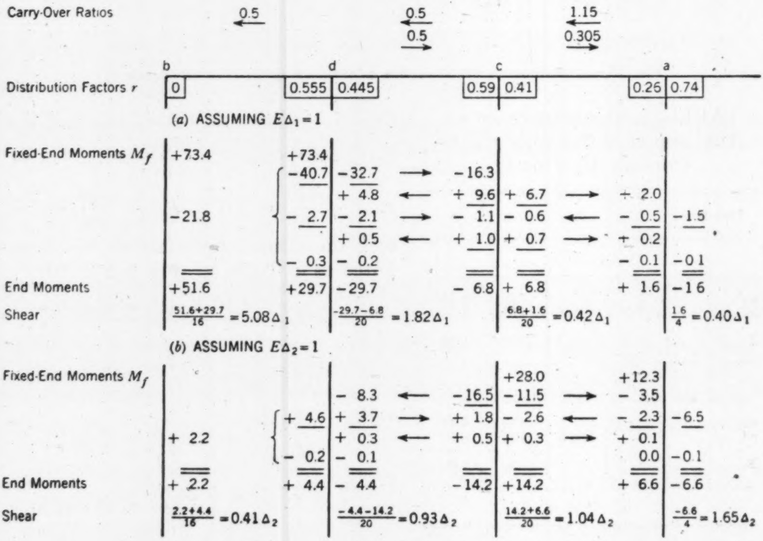

Carry-Over Ratios
0.5
0.5
0.5
1.15
0.305
b
d
c
a
Distribution Factors r
0
0.555
0.445
0.59
0.41
0.26
0.74
(a) ASSUMING EΔ1=1
Fixed-End Moments Mf
+73.4
+73.4
-40.7
-32.7
-16.3
+ 4.8
+ 9.6
+ 6.7
+ 2.0
-21.8
- 2.7
- 2.1
- 1.1
- 0.6
- 0.5
-1.5
+ 0.5
+ 1.0
+ 0.7
+ 0.2
- 0.3
- 0.2
- 0.1
-0.1
End Moments
+51.6
+29.7
-29.7
- 6.8
+ 6.8
+ 1.6
-1.6
Shear
(51.6+29.7)/16 = 5.08Δ1
(-29.7-6.8)/20 = 1.82Δ1
(6.8+1.6)/20 = 0.42Δ1
1.6/4 = 0.40Δ1
(b) ASSUMING EΔ2=1
Fixed-End Moments Mf
+28.0
+12.3
- 8.3
-16.5
-11.5
- 3.5
+ 4.6
+ 3.7
+ 1.8
- 2.6
- 2.3
-6.5
+ 2.2
+ 0.3
+ 0.5
+ 0.3
+ 0.1
- 0.2
- 0.1
0.0
-0.1
End Moments
+ 2.2
+ 4.4
- 4.4
-14.2
+14.2
+ 6.6
-6.6
Shear
(2.2+4.4)/16 = 0.41Δ2
(-4.4-14.2)/20 = 0.93Δ2
(14.2+6.6)/20 = 1.04Δ2
-6.6/4 = 1.65Δ2

FIG. 14.—DISTRIBUTION OF MOMENTS IN COLUMN E, FIG. 12 (EXAMPLE 4)

Assume that $E\,\Delta_1 + E\,\Delta_2$ is caused by forces which produce shears in lengths ac and db each equal to 1 kip (forces at points a and a′ total 1 kip and reaction at point b equals 1 kip) then:

$$5.08\,\Delta_1 + 0.41\,\Delta_2 = 1 \dots\dots\dots\dots\dots\dots\dots (9a)$$

and

$$0.42\,\Delta_1 + 1.04\,\Delta_2 = 1 \quad \text{.................... (9b)}$$

Multiplying Eq. 9b by $\frac{5.08}{0.42}$, it becomes:

$$5.08\,\Delta_1 + 12.58\,\Delta_2 = 12.1 \quad \text{.................. (9c)}$$

Solving Eqs. 9a and 9c simultaneously: $\Delta_2 = 0.913$; and $\Delta_1 = 0.124$. Consequently, $\Delta_1 + \Delta_2 = 1.037$.

Since both columns are assumed to deflect the same amount at the top, Δ for column B would be 1.037 to correspond to the foregoing values. The total shear is distributed in the bent as follows:

Deflection	Computation	Shear	
Column B....................	$\frac{0.96}{10} \times 1.037$		0.100
Column E:			
$\Delta_1 = 0.124$.............	5.08×0.124	0.63	
$\Delta_2 = 0.913$.............	0.41×0.913	0.37	
$\Delta_1 + \Delta_2 = 1.037$.............	....		1.00
Total shear in bent..........	....		1.10

Column E then resists $\frac{1.00}{1.10}$ or 91% of the horizontal thrust and column B resists $\frac{0.10}{1.10}$ or 9%.

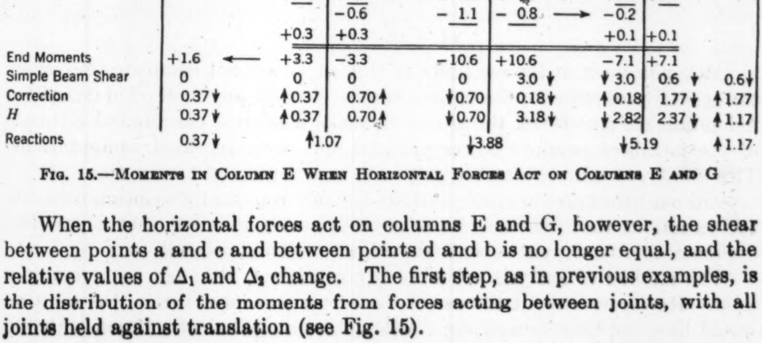

Carry-Over Ratios
0.5
0.5
0.5
1.15
0.305
b
d
c
a
a′
Distribution Factors r
0
0.555
0.445
0.59
0.41
0.26
0.74
Fixed-End Moments M
+16.3
−7.0
+0.6
+ 1.9
+1.7
+4.7
−5.4
−10.7
− 7.5
−2.3
+3.0
+2.4
+ 1.2
+ 0.7
+0.6
+1.7
−0.6
− 1.1
− 0.8
−0.2
+0.3
+0.3
+0.1
+0.1
End Moments
+1.6
+3.3
−3.3
−10.6
+10.6
−7.1
+7.1
Simple Beam Shear
0
0
0
0
3.0 ↓
↓ 3.0
0.6 ↓
0.6↓
Correction
0.37↓
↑0.37
0.70↑
↓0.70
0.18↓
↑ 0.18
1.77↓
↑1.77
H
0.37↓
↑0.37
0.70↑
↓0.70
3.18↓
↓ 2.82
2.37↓
↑1.17
Reactions
0.37↓
↑1.07
↓3.88
↓5.19
↑1.17

FIG. 15.—MOMENTS IN COLUMN E WHEN HORIZONTAL FORCES ACT ON COLUMNS E AND G

When the horizontal forces act on columns E and G, however, the shear between points a and c and between points d and b is no longer equal, and the relative values of Δ_1 and Δ_2 change. The first step, as in previous examples, is the distribution of the moments from forces acting between joints, with all joints held against translation (see Fig. 15).

As a first step in the solution of the auxiliary force system, assume that all resistance to translation is taken by the E-line columns. The force system on the E-line columns then consists of forces at joints a, a′, c, and d equal and opposite to the reactions determined in Fig. 15 plus forces applied directly to the joints such as wind forces on the roof. Since the base of column G is hinged, half the uniform load of 4.8 kips, shown acting on it, is assumed to go to base f and half to the top point e. This half at point e plus the 6 kips acting on the lean-to roof are distributed between points c and d on column E, Fig. 12. Without another set of calculations (translating points c and d, distributing the moments produced, and determining the shears) the exact distribution between these two joints is not known, but such distribution is not needed for the present problem; only the shear is needed between points d and b, and between points a and c. The shear between points a and c is $5.19 - 1.17 = 4.02$ kips plus 12.4 kips from the roof, or a total of 16.42 kips. The shear between points d and b is increased by $3.88 - 1.07 + 6.0 + \frac{4.8}{2}$, or 27.63 kips. Since there is no relative translation between joints c and d the shear is not needed.

To determine the relative values of Δ_1 and Δ_2, set up formulas similar to Eqs. 9, except that the shear values are different:

$$5.08\,\Delta_1 + 0.41\,\Delta_2 = 27.63 \qquad (10a)$$

and

$$0.42\,\Delta_1 + 1.04\,\Delta_2 = 16.42 \qquad (10b)$$

Multiplying Eq. 10*b* by $\frac{5.08}{0.42}$, it becomes:

$$5.08\,\Delta_1 + 12.59\,\Delta_2 = 198.8 \qquad (10c)$$

As in Eqs. 9, solve Eqs. 10*a* and 10*c* simultaneously, and determine: $\Delta_2 = 14.05$; and $\Delta_1 = 4.30$. Consequently, $\Delta_1 + \Delta_2 = 18.35$.

For a Δ-value on the B-line columns of 18.35 units, the shear would be $\frac{18.35}{10} \times 0.96$ or 1.76 kips. The E-line columns thus would take $\frac{27.63}{27.63 + 1.76} = 94.2\%$ of the thrust and the B-line columns only 5.8%.

VI. CONCLUSION

Although most of the examples in this paper are not complete they show the general procedure for the various cases. As with any method of design for indeterminate structures, the ability to make a relative close initial estimate of the members required is very helpful and saves repeated computations. This ability, of course, comes largely with experience and observation.

Although the types of problems described in this paper by no means exhausts the variations that are likely to occur in practice, it is felt that a practical approach to other forms of bents should not be found difficult.

The method of analysis probably is not original, although not commonly used. However, without the use of the curves presented herein, the solution would be quite time consuming. Although this method of solution with the curves will require more time than that of assuming a point of contraflexure and solving as a statically determinate structure, the greater assurance of a reasonably accurate analysis should justify this extra time.

ACKNOWLEDGMENT

Figs. 6 to 9 were prepared by the writer for his own use in structural design and were original in form and development. L. T. Evans, Assoc. M. ASCE, in the revised edition of "Rigid Frames,"[4] has similar curves for all but the uniform load condition. His curves do not, however, cover the range of those presented. It appears that the curves by Mr. Evans were developed somewhat earlier than those presented herein.

The paper is based on data assembled and presented as a thesis[5] to the University of Michigan, Ann Arbor, Mich., in 1943, in partial fulfilment of the requirements for the professional degree of Civil Engineer.

APPENDIX I. NOTATION

The following letter symbols, used in the paper and in its discussion, conform essentially to American Standard Letter Symbols for Mechanics, Structural Engineering and Testing Materials (ASA—Z10a—1932), prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1932:

a = a nondimensional coefficient of end moment at the top of a stepped column (see Fig. 7);

B = angle of rotation caused by a unit torque; thus—B_{ab} = rotation at joint a caused by a unit torque applied at joint b;

b = a nondimensional coefficient of end moment at the bottom of a stepped column (see Fig. 7);

C = a coefficient, with suitable subscript, used in slope-deflection equations; thus:

  C_1 = the distribution coefficient for point a for given values of I/I_o and x in Fig. 6(a);

  C_2 = the coefficient (see Fig. 6(b)) which, with C_1 or C_3, gives the carry-over ratio (the carry-over ratio from point a to point b is C_2/C_1 and from point b to point a, is C_2/C_3);

  C_3 = the distribution coefficient for point b, the bottom of the column, for given values of I/I_o and x in Fig. 6(c);

E = modulus of elasticity;

I = moment of inertia of a column section, the moment of inertia for the reference section (the upper and smaller section of a stepped column) being designated I_o;

---

[4] "Rigid Frames," by L. T. Evans, Edwards Bros., Ann Arbor, Mich., 1938.

[5] "Analysis of Mill Bents with Stepped Columns by the Method of Successive Approximations," by Daniel S. Ling, thesis presented to the Univ. of Michigan, Ann Arbor, in 1943, in partial fulfilment of the requirements for the professional degree of Civil Engineer.

K = stiffness ratio $E\,I/L$ used in moment distribution, the stiffness ratio of the upper and smaller section of a stepped column being designated $K_o = E\,I_o/L$:

L = length of member;

M = bending moment; the subscript f denotes "fixed end"; other subscripts refer to a designated joint and span—that is, in general, $M_{f,ba}$ equals fixed-end moment at joint b in member ab;

P = a point load or concentrated load;

r = distribution factor;

W = total weight, force, or pressure;

w = weight, force, or pressure per linear foot;

x = a ratio of length;

α = angular rotation of the end tangents of a deformed beam, considered simply supported;

β = a substitution constant, function of the unit rotation B;

Δ = linear displacement of one end of a beam with respect to the other end; and

θ = angular rotation of a beam.

DISCUSSION

Ralph E. Spaulding,[6] M. ASCE.—Any clear-cut rapid method of determining the stresses in stepped-column mill bents is of interest to the designing engineer. The author is to be complimented on his derivation of such a method; however, the writer differs with Mr. Ling in his application of Fig. 4 to this method. These differences involve: (1) Determination of x for $x\,L$; and (2) point of application of the horizontal thrust from the bridge crane.

(1) When analyzing any member of discontinuous cross section in bending (such as stepped-back columns), it must be recognized that the effective moment of inertia does not change abruptly when the cross section so changes, even in the case of a symmetrical column, such as the author's column B, Fig. 10. The larger value of I is not effective right up to the junction with the smaller cross section. There is a transition period between the end of the smaller cross section and some point farther along in the larger cross section.

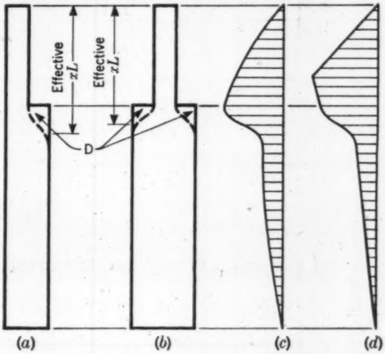


Fig. 16

Referring to Figs. 16(*a*) and 16(*b*), when stepped-back members are subjected to bending, photoelastic and strain gage tests reveal that adjacent to the smaller part of the member, the corners D, of the wider part of the member are "dead areas." The effective cross section for $\frac{M}{E\,I}$-analysis approximately follows the dotted lines. Much time may be saved in computations by using only two moments of inertia for the entire length of the member (such as the author's values of I and I_o); but to do this without introducing undue inaccuracy, it is necessary to make a scientific guess as to the proper value of the effective length $x\,L$. It is always greater than the net length of the smaller section; and, comparatively, it is much greater for the eccentric column (Fig. 16(*a*)) than for the concentric column (Fig. 16(*b*)). The $\frac{M}{I}$-diagram for unit uniform load would be as shown in Fig. 16(*c*) instead of Fig. 3(*c*). This does not in any way detract from the value of the author's curve sheets—as long as the proper effective value of distance $x\,L$ is used.

(2) The effective point of application of the side thrust from a bridge crane is at the level of the top of the crane rail. Assuming that the height of the crane

---

[6] Cons. Engr., Jacksonville, Fla.

beam and the crane rail in Fig. 4 is 2 ft, the applied load of 8 kips would be 10 ft from point a instead of 12 ft. The $\frac{M}{I}$-diagram for this case would be as indicated in Fig. 16(*d*) rather than as shown in Fig. 3(*d*).

E. I. FIESENHEISER,[7] M. ASCE.—The use of moment distribution and the balancing of shear produced by sidesway provide a convenient solution for stresses in the mill building bent. An alternative method for determining the

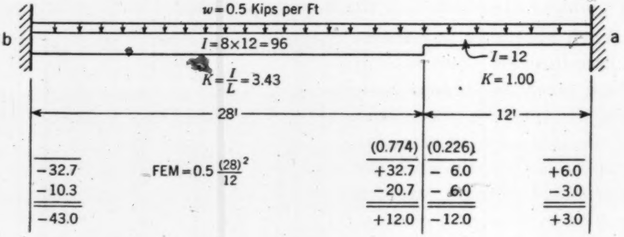

w = 0.5 Kips per Ft
b
a
I = 8×12 = 96
I = 12
K = I/L = 3.43
K = 1.00
28'
12'
(0.774)
(0.226)
−32.7
−10.3
−43.0
FEM = 0.5 (28)²/12
+32.7
−20.7
+12.0
− 6.0
− 6.0
−12.0
+6.0
−3.0
+3.0

FIG. 17

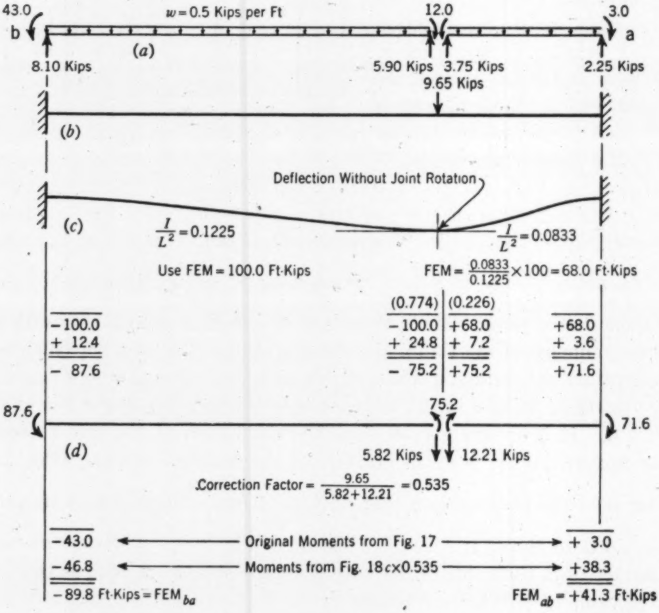

43.0
b
w = 0.5 Kips per Ft
12.0
3.0
a
(a)
8.10 Kips
5.90 Kips
3.75 Kips
2.25 Kips
9.65 Kips
(b)
Deflection Without Joint Rotation
(c)
I/L² = 0.1225
I/L² = 0.0833
Use FEM = 100.0 Ft-Kips
FEM = 0.0833/0.1225 × 100 = 68.0 Ft-Kips
(0.774)
(0.226)
−100.0
+ 12.4
− 87.6
− 100.0
+ 24.8
− 75.2
+68.0
+ 7.2
+75.2
+68.0
+ 3.6
+71.6
87.6
75.2
71.6
(d)
5.82 Kips
12.21 Kips
Correction Factor = 9.65/(5.82+12.21) = 0.535
−43.0
Original Moments from Fig. 17
+ 3.0
−46.8
Moments from Fig. 18c×0.535
+38.3
−89.8 Ft-Kips = FEM_ba
FEM_ab = +41.3 Ft-Kips

FIG. 18

---

[7] Associate Prof. of Civ. Eng., Illinois Inst. of Technology, and Consultant, Inst. for Nuclear Studies, Univ. of Chicago, Chicago, Ill.

fixed-end moments in the stepped column involves only shear and moment distribution for the stepped member itself. This method is convenient and does not require the use of moment diagrams. Since it may be of interest to some engineers, it is presented herein.

The procedure is to assume the stepped column to be a combination of two separate members. Fixed-end moments due to loads on the separate members are first computed and balanced, and the resulting unbalanced shear (or joint restraint), acting at the junction of the two members, is then calculated. The shear is balanced by application of an equal and opposite force at this joint.

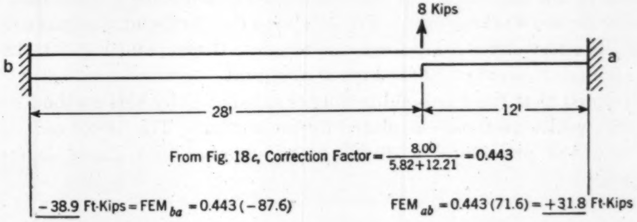

8 Kips
b
a
28'
12'
From Fig. 18c, Correction Factor = $\frac{8.00}{5.82+12.21}$ = 0.443
– 38.9 Ft-Kips = FEM$_{ba}$ = 0.443 (– 87.6)
FEM$_{ab}$ = 0.443 (71.6) = +31.8 Ft-Kips

Fig. 19

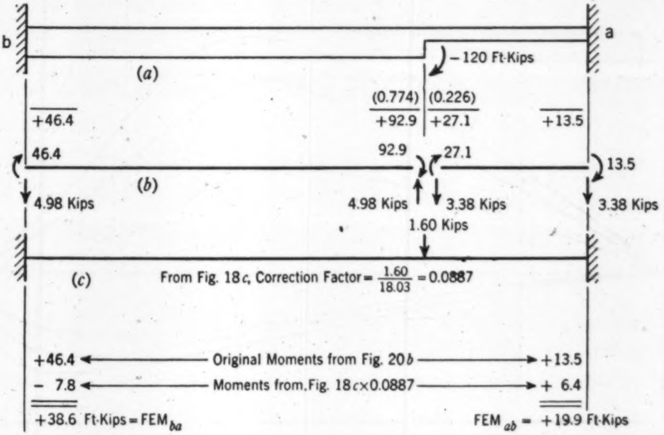

b
a
(a)
– 120 Ft-Kips
(0.774)
(0.226)
+46.4
+92.9
+27.1
+13.5
46.4
92.9
27.1
13.5
(b)
4.98 Kips
4.98 Kips
3.38 Kips
3.38 Kips
1.60 Kips
(c)
From Fig. 18c, Correction Factor = $\frac{1.60}{18.03}$ = 0.0887
+46.4 ← Original Moments from Fig. 20b → +13.5
– 7.8 ← Moments from Fig. 18c×0.0887 → + 6.4
+38.6 Ft-Kips = FEM$_{ba}$
FEM$_{ab}$ = +19.9 Ft-Kips

Fig. 20

In calculating the effect of the applied force the joint is allowed to deflect without rotation. In this case the applied moments must be proportional to I/L^2 for the two segments of the stepped column and then balanced. Shears are computed and balanced by use of the proper correction factor. To obtain the final fixed-end moments, the moments caused by the loads acting on the separate members and those due to the shear balance are added.

The alternative method is illustrated by the calculations shown in Figs. 17, 18, 19, and 20. These calculations are made for the problem in Example 1, in which the ratio of moments of inertia is given as 8 to 1. Since only relative values are significant, moments of inertia of 96 and 12 are assumed. Figs. 17 and 18 show the calculation of the fixed-end moments for a wind load of 0.5 kips per ft on the 40-ft column. Fig. 19 shows the fixed-end moment computation for the effect of the 8-kip force due to crane thrust; and Fig. 20, the effect of the applied moment of 120 ft-kips at the joint.

It is noted that the fixed-end moments calculated by this method compare reasonably well with those calculated by the author. The use of moment diagrams and the procedure of dividing the member into many segments is unnecessary.

Walter J. Gray,[8] **Jun. ASCE.**—The sets of factors presented in this paper permit rapid analysis of the examples. However, they are not sufficiently complete for general use. Additional curves should be presented to show:

1. Fixed-end moments for concentrated side loads at other positions than at crane level, permitting a direct solution of the case of shed roof thrust as shown in Fig. 12.

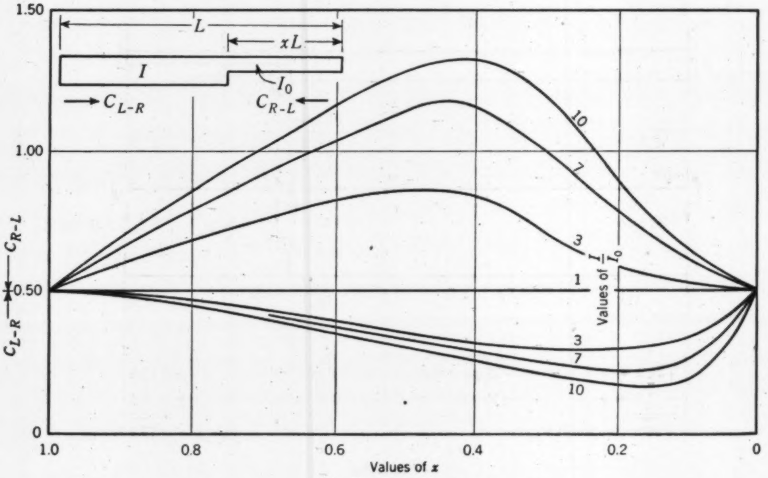


Fig. 21.—Carry-Over Factors for Stepped Member

2. Distribution factors, permitting a direct evaluation of the effects of truss restraint and lower floor restraint.

---

[8] Civ. Engr., Public Utilities Comm., City of San Francisco, San Francisco, Calif.

3. Fixed-end moments, indicating the effects of eccentric application of loads on the column, although these might be approximated from fixed-end moments for concentrated side loads at positions other than at crane level.

The carry-over factor curves should be redrawn to give a direct reading in one set of curves, thus avoiding the use of two factors. Fig. 21 is such a set of direct reading carry-over factor curves with a limiting value of $I/I_o = 10$, as given in an unpublished paper written by the writer in 1942 on "Analysis of Stepped Crane Columns by Moment Distribution," which is on file in the Engineering Societies Library at New York, N. Y. Values were calculated by the column analogy[9] procedure of Hardy Cross, Hon. M. ASCE.

The design of the stepped column is not complete, however, even though the combined stresses due to maximum moments and thrusts are computed, since they must be compared with the allowable combined stress in a stepped column. This latter value requires consideration of the slenderness ratio of the column before an allowable axial stress value can be determined. Thomas C. Shedd,[10] M. ASCE, suggests using an "equivalent" slenderness ratio of $L_1/R_1 + L_2/R_2$.

L. J. **Mensch**,[11] M. ASCE.—There was a dearth of technical literature on the analysis of mill bents by the theory of elasticity until about 1913 when a treatise appeared on the subject[12] in full panoply by W. Gehler, which showed the analysis of each of the bents in Fig. 22 for nearly ten loading conditions. Simple working formulas, clearly arranged, were given for each case—formulas not equaled by later authors in the ease of arrangement. This enabled engineers to design such bents in a very short time. A number of writers refigured some of the cases.[13,14,15] Had the author compared his results with any of those in the references suggested, he would have found his assumptions incorrect, as the moments at the foot of the columns are only one half the correct amount.

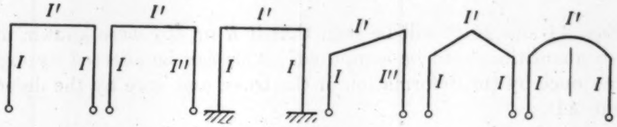


Fig. 22

---

[9] "The Column Analogy," by Hardy Cross, *Bulletin No. 215*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., October, 1930.

[10] "Structural Design in Steel, "by Thomas C. Shedd, John Wiley & Sons, Inc., New York, N. Y., 1934, p. 395.

[11] Gen. Contractor, Evanston, Ill.

[12] "Der Rahmen," by W. Gehler, W. Ernst und Sohn, Berlin, 1913.

[13] "Analysis of Statically Indeterminate Structures by the Slope Deflection Method," by W. M. Wilson, F. E. Richart, and Camillo Weiss, *Bulletin No. 108*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., 1918.

[14] "The Structural Engineers Handbook; Data for the Design and Construction of Steel Bridges and Buildings," by Miles S. Ketchum, McGraw-Hill Book Co., Inc., New York, N. Y., 3d Ed., 1924.

[15] "Rahmen Formeln," by Adolf Kleinlogel, W. Ernst und Sohn, Berlin, Vol. 8, 1939.

It is true that no formulas are available for bents with stepped columns and that the theory of elasticity would require the solution of a structure three times indeterminate complicated by the presence of a roof truss; this is a task even beyond the capacity of a large engineering organization, when it has to design perhaps five bents each for five loading conditions.

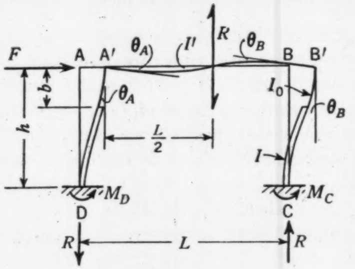


FIG. 23

Many years ago the writer stumbled on a solution that gives values within a small percentage of the theoretical results. It is based on the fact that the truss is from two times to ten times stiffer than the columns, and on the fundamental loading condition by a horizontal force at the top of one column. A symmetrical bent will deform as shown in Fig. 23. By inspection,

$$\theta_A = \theta_B \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

$$M_A = R\frac{L}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

$$AA' = BB' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

$$M_D = M_C = h\frac{F}{2} - R\frac{L}{2} = \frac{F\,h}{2} - M_A \ldots\ldots\ldots\ldots (14)$$

and

$$F\,h = M_D + M_C + R\,L = 2\,M_D + R\,L \ldots\ldots\ldots\ldots (15)$$

From Eqs. 14 and 15 it will be seen that if R or M_A were known, all other unknown quantities could be computed. This can be effected by expressing θ_A twice—once by the deformation of the truss, and once by the deformation of column AD.

Assuming the truss to have parallel chords and the moment of inertia, I',

$$\theta_A = \frac{R\,L^2}{12\,E\,I'} = \frac{M_A\,L}{6\,E\,I'} \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

From the column deformation for a uniform column section,

$$\theta_A = \int_0^h \frac{M\,dx}{E\,I} = \frac{\frac{x\,F}{2}\,dx - M_A\,dx}{E\,I} = \frac{\frac{F\,h^2}{4} - M_A\,h}{E\,I} \ldots\ldots (17a)$$

For stepped columns the column deformation is

$$\theta_A = \int_0^b \frac{\frac{F}{2}\,x\,dx - M_A\,dx}{E\,I_0} + \int_b^h \frac{\frac{F}{2}\,x\,dx - M_A\,dx}{E\,I} \ldots\ldots\ldots (17b)$$

Four particular combinations of values for stepped columns will be considered, as follows:

Combination	Values of I/I_0	Values of b
1	5	$h/3$
2	5	$h/4$
3	10	$h/3$
4	10	$h/4$

For these combinations, respectively,

$$\theta_A = \frac{13}{36}\frac{F\,h^2}{E\,I} - \frac{7}{3}\frac{M_A\,h}{E\,I} \tag{18a}$$

$$\theta_A = \frac{10}{32}\frac{F\,h^2}{E\,I} - 2\,\frac{M_A\,h}{E\,I} \tag{18b}$$

$$\theta_A = \frac{1}{2}\frac{F\,h^2}{E\,I} - 4\,\frac{M_A\,h}{E\,I} \tag{18c}$$

and

$$\theta_A = \frac{25}{64}\frac{F\,h^2}{E\,I} - \frac{13}{4}\frac{M_A\,h}{E\,I} \tag{18d}$$

Denoting $I'/L \div I/h$ by n, and eliminating θ_A by combining Eq. 16 with Eqs. 17*a* and 18, for a column of uniform section,

$$M_A = \frac{F\,h}{4}\left(\frac{6\,n}{1 + 6\,n}\right) = 0.242\,F\,h \tag{19}$$

For the four combinations of I/I_0 and b previously given,

$$M_A = \frac{13}{84}\,F\,h\left(\frac{14\,n}{1 + 14\,n}\right) = 0.1525\,F\,h \tag{20a}$$

$$M_A = \frac{5}{32}\,F\,h\left(\frac{12\,n}{1 + 12\,n}\right) = 0.1535\,F\,h \tag{20b}$$

$$M_A = \frac{1}{8}\,F\,h\left(\frac{24\,n}{1 + 24\,n}\right) = 0.124\,F\,h \tag{20c}$$

and

$$M_A = \frac{25}{208}\,F\,h\left(\frac{19.5\,n}{1 + 19.5\,n}\right) = 0.1185\,F\,h \tag{20d}$$

Inasmuch as n generally has a value of 5 or more, the fraction containing n in Eqs. 19 and 20 is nearly equal to 1; the approximate values were determined for $n = 5$. By Eq. 14, it can easily be proved that, for uniform columns,

$$M_D = 0.258\,F\,h \tag{21}$$

and for the four combinations of I/I_0 and b previously given,

$$M_D = 0.3475\,F\,h \tag{22a}$$

$$M_D = 0.3465\,F\,h \tag{22b}$$

$$M_D = 0.376\,F\,h \tag{22c}$$

and

$$M_D = 0.3815\,F\,h \tag{22d}$$

The results given by Eqs. 21 and 22 enable solution for other loading conditions, as follows:

Assume a column of uniform section to be affected by a horizontal wind load, w, per linear foot. A relaxation type method is used. If the column is not only fixed at point D, but is also held fixed at the top of point A, the end moments are $w\,h^2/12$ and the end shears are $w\,h/2$. If point A is released, the shear $w\,h/2$ will act like the force F in the fundamental case and

$$M_D = \frac{w\,h^2}{12} + 0.258\,f\,h = \frac{w\,h^2}{12} + 0.129\,w\,h^2 = 0.212\,w\,h^2 \ldots\ldots (23)$$

The shear at the bottom of the column is

$$V = \frac{w\,h}{2} + \frac{F}{2} = \frac{3}{4}\,w\,h \ldots\ldots (24)$$

In the case of a stepped column, for example, when $I/I_0 = 10$ and $b = h/3$, the first step is to compute the moments at the ends of the column, considered as a beam uniformly loaded (say, by the slope-deflection method), from which

$$M_D = 0.123\,w\,h^2 \ldots\ldots (25a)$$

and

$$M_A = 0.052\,w\,h^2 \ldots\ldots (25b)$$

and the shears at points D and A are

$$V_D = \frac{4}{7}\,w\,h \ldots\ldots (26a)$$

and

$$V_A = \frac{3}{7}\,w\,h \ldots\ldots (26b)$$

The final values at point D are

$$M_D = 0.123\,w\,h^2 + (0.376 \times w\,h^2)\,\frac{3}{7} = 0.284\,w\,h^2 \ldots\ldots (27a)$$

and

$$V_D = \frac{4\,w\,h}{7} + \frac{3\,w\,h}{14} = \frac{11\,w\,h}{14} \ldots\ldots (27b)$$

Similarly, for $I/I_0 = 5$, and $b = h/3$,

$$M_D = 0.277\,w\,h^2 \ldots\ldots (28a)$$

and

$$V_D = 0.77\,w\,h \ldots\ldots (28b)$$

Other loading conditions may be found by the same short cut. For uniform vertical loads on the truss, the point of inflection of the columns will be found to be: For $I/I_0 = 5$, between $0.04\,h$ and $0.06\,h$ below the midheight; for $I/I_0 = 10$, between $0.06\,h$ and $0.02\,h$ above the midpoint; and for a column of uniform section, $h/3$ above point D.

It is assumed in the foregoing analyses that the foot of the column is fixed. In many bents the columns rest on walls or piers, often 20 ft high. These supports are affected by the moments M_D and by the shears, and must be designed accordingly.

There is a question as to what value of I' to use for a Fink truss of a depth equal to perhaps one fifth of the span. The moment of inertia will be greater than that of a parallel chord truss having a depth of about one tenth that of the span; and the foregoing formulas can be used safely, as long as a stiff connection is provided between truss and columns by a brace capable of transferring the moments occurring at the top of the columns.

ARTHUR R. GRAVES,[16] M. ASCE.—The chief contribution of this paper to engineering and technical knowledge lies in Figs. 6, 7, 8, and 9, which may be used in calculating the fixed-end moments for three kinds of loading, and the distribution coefficients and carry-over ratios for any kind of loading. All, of course, are restricted to members with only one abrupt change of section, but their use is not limited to the design of stepped columns. The mill building designer needs such timesaving devices as these because as soon as a project is decided on, the owner, thinking of his anticipated profits, considers every day a loss of one day's profits until the mill is working—the larger the mill, the greater is the urge for speedy construction.

The foundations must be started first and this cannot be done until the column bents are designed. For columns with a constant moment of inertia, the points of contraflexure may be assumed and the horizontal shears distributed over the columns with comparatively little work, yet with satisfactory results. For stepped columns, however, the results from this approximate method will, in most cases, be far from those obtained by more exact methods. An indication of how serious these variations can be is shown by the range in values of the various functions given in Figs. 6, 7, 8, and 9. Another more convincing demonstration may be seen in some of the buildings in which the columns must be plumbed, or in which the crane runway must be lined up every few years, because the actual moments on the base of the columns are much more than those calculated by the approximate method.

In most cases, a more exact analysis of the column bent takes so long that the designer cannot delay the foundations long enough to make it. With the use of these or similar charts (Figs. 6, 7, 8, and 9) the labor and the time of calculation are reduced so that it is practicable to make a closer analysis of the typical bents, which, in turn, afford a better base on which to approximate the irregular bents, if time does not permit a closer analysis of them. It is possible for each designer to make up his own charts, but it is not practicable because of the time and labor required. The writer has a different set of charts, serving the same general purpose, which were developed over a period of more than ten years.

Even with charts such as the author's (Figs. 6, 7, 8, and 9), there is still much work to be done in analyzing column bents, and for most common mill bents the method outlined in this paper may be shortened without too great an inaccuracy in the final results. This method takes into account the deflection of the column between the knee braces and the roof truss. In the most common mill bents this refinement is not as important as it seems at first. The chief effect of this deflection of the column is that it makes the bent more limber and increases the horizontal deflection. If this deflection is assumed not to exist, two more charts may be plotted which will further reduce the amount of work. One chart should locate the point of contraflexure of the bent under translation forces. The other chart should give a stiffness factor to be used in distributing the translation forces among the columns in the bent.

---

[16] Senior Designer, Fabricated Steel Constr., Bethlehem Steel Co., Bethlehem, Pa.

With this alternate method the fixed-end moment at the bottom of the column will be less than that obtained by the original method outlined in the paper. The moment from translation will be greater, however, and, when the fixed-end moment and the translation moment are added together, the resulting moments from the two methods will differ by a small percentage.

The fixed-end moment at the top of the column will be greater with the alternate method and so will the moment from translation for wind load and crane thrust. However, these moments are always opposite in sign and their difference is changed much less than either of the two moments. The fact that the fixed-end moment and the translation moment at the top of the column are opposite in sign makes it possible to vary the calculated moment at this point, due to crane thrust of eccentric load, by varying the number of bents over which the translation effect is assumed to be distributed. With cranes having heavy trolleys the forces involved are considerable. In this connection it is also possible to have the unloaded columns of a bent of just the right stiffness so that the translation moment is equal and opposite in sign to the fixed-end moment at the top of the loaded column. This would result in a calculated moment of zero. To take care of such possibilities it is advisable that no column be designed for less than 50% of the fixed-end moment at the top of the column.

In the case of eccentric loads the signs of the fixed-end moments at both top and bottom of the column will vary with different ratios of the moments of inertia and with different ratios of the length of the upper part of the column to the entire length of the column. For one column the fixed-end moments may have the same sign as the translation moment; for the next column the reverse may be true. This variation in the fixed-end moment is clearly shown in Fig. 9. When adding fixed-end moments from eccentric loads to the translation moments the sign of each moment should be carefully checked.

It should be noted that the author's charts, given in Figs. 6, 7, 8, and 9, were published previously by L. C. Maugh,[2] M. ASCE.

Erik Sollid,[17] Esq.—The work that the author has done in connection with the design of stepped columns is time saving and valuable. Curves similar to the ones presented by the author are in use in the design office of the Bureau of Reclamation (USBR). They were prepared chiefly by L. W. Crandell and H. L. Neve.[18] In addition to curves similar to those shown by Mr. Ling, the USBR memorandum includes a set of curves giving the fixed-end moments for a concentrated load anywhere along the member having both ends fixed, or having the heavier end fixed and the smaller end hinged. It was found that by letting each curve represent a different ratio of

$$m = \sqrt[3]{\frac{I}{I_0}} \qquad (29)$$

---

[17] Engr., Bureau of Reclamation, U. S. Dept. of Interior, Denver, Colo.

[18] "Concrete Building Design Data," by L. W. Crandell, George Evans, Erik Sollid, H. L. Neve, and Samuel Judd under the direction of L. N. McClellan, *Technical Memorandum No. 629*, U. S. Bureau of Reclamation, Denver, Colo., November, 1943.

—instead of the ratio x as used by the author, more orderly curves were obtained—that is, curves that did not have the tendency to intersect or twist around one another as is especially apparent in Figs. 6, 8, and 9.

It is dangerous for a designer to accept any design curves or diagrams, however, without the satisfaction that they are correct. Therefore, the writer wishes to call attention to a simple method for spot checking the diagrams without resorting to the more laborious methods of moment area or column analogy. As seen from Fig. 9, the signs of the fixed-end moments for a moment applied at the change in section varies both with the $\frac{I}{I_0}$-ratio and the value of x, and it is impossible to determine the signs for the fixed-end moments by inspection. In the spot check discussed herein, however, it is a simple matter to determine the direction of the end moments. The method consists of treating the member as a continuous beam with one redundant support.[19] Instead of using the author's sign convention, the writer prefers the following one: If the moment at the end of a member tends to cause clockwise rotation of the adjacent joint into which the member frames, that moment will be called positive (+); conversely, a moment tending to cause counterclockwise rotation is negative (−).

In the following examples a member (as shown in Fig. 24) will be used. This member is similar to the column ab in Fig. 4.[19a]

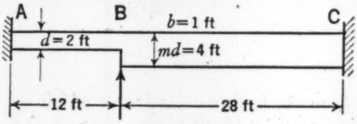


FIG. 24

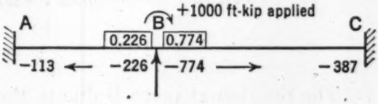


FIG. 25

Tentatively the beam will be supported at point B (Fig. 24), thus defining a continuous beam of two uniform sections. The frame properties of these beams are then as follows: $I_{AB} = \frac{b\,d^3}{12} = \frac{1 \times 2^3}{12} = 0.666$ and $I_{BC} = \frac{1 \times 4^3}{12} = 5.333$.

Case I.—Assuming points A, B, and C fixed, the stiffness ratio equals $K_{AB} = \frac{I}{L} = \frac{0.666}{12} = 0.0555$ and $K_{BC} = \frac{5.333}{28} = 0.19$. The distribution factors around joint B are $r_{BA} = \frac{0.0555}{0.0555 + 0.19} = 0.226$ and $r_{BC} = \frac{0.19}{0.2455} = 0.774$.

Case II.—Assuming point A hinged and points B and C fixed, $K_{BA} = 0.0555 \times 0.75 = 0.0416$ and $K_{BC} = 0.19$, as for Case I. The distribution factors are $r_{BA} = \frac{0.0416}{0.0416 + 0.19} = 0.18$ and $r_{BC} = \frac{0.19}{0.2316} = 0.82$.

Example 1.—Determine the fixed-end moments due to an applied moment at point B (Fig. 24), the change in section.

In Fig. 25 only one distribution around joint B is necessary to find the moments created by an applied load of 1,000 ft-kips at point B. Distribution factors are shown in rectangular boxes and moment carry-overs are indicated by arrows. The reaction on the support at point B will be upward and equal to $-(226 + 113)\frac{1}{12} + (774 + 387)\frac{1}{28} = 13.2$ kips. The true fixed-end moments will now be the moments shown at ends A and C in Fig. 25, plus the moments caused by an upward force of 13.2 kips at the change in section. To find the moments resulting from this force, hold joint B against rotation and give it an upward deflection of $\Delta = 10{,}000\, z$. The quantity 10,000 is chosen arbitrarily and z is the correction factor. The fixed-end moments in spans AB and BC resulting from this deflection are

---

[19] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932.

$$M_{AB} = M_{BA} = \frac{6\,E\,I}{L^2} = \frac{6 \times 0.666}{12^2}\, 10.000\, z\, E = -\,278\, z\, E$$

and

$$M_{BC} = M_{CB} = \frac{6 \times 5.333}{28^2}\, 10.000\, z\, E = +\,408\, z\, E$$

Next, let joint B rotate and distribute the moments as shown in Fig. 26.

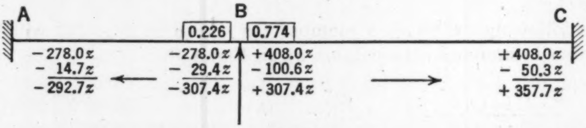


Fig. 26

The reaction at point B due to these moments is downward and equal to 73.8 z. Setting this reaction equal to the reaction of 13.2 kips, caused by the moments in Fig. 25, gives $z = \frac{13.2}{73.8} = 0.179$. The correct fixed-end moments caused by a moment of 1,000 ft-kips, applied as shown in Fig. 25, and without any support at point B, are now

$$M_{AC} = -\,113 - 292.7 \times 0.179 = -\,165.4 \text{ ft-kips}$$

and

$$M_{CA} = -\,387 + 357.7 \times 0.179 = -\,323.0 \text{ ft-kips}$$

For an applied moment of − 120 ft-kips at the change in section, the fixed-end moments are

$$M_{AC} = -\,165.4 \times \frac{-120}{1{,}000} = +\,19.9 \text{ ft-kips}$$

and

$$M_{CA} = -\,323.0 \times \frac{-120}{1{,}000} = +\,38.8 \text{ ft-kips}$$

which agree with the results given by the author in the paragraphs containing Eqs. 8.

Example 2.—Determine the fixed-end moment generated by a concentrated transverse load of 8 kips at the change in section, as shown in Fig. 4. In Fig. 26 the moments have already been obtained for an arbitrary deflection at point B. It is recalled that these moments produced a downward reaction of 73.8 z. The moment resulting from the 8-kip load is then simply $\frac{8}{73.8} = 0.108$ times the moments in Fig. 26, or

$$M_{AC} = -292.8 \times 0.108 = -31.6 \text{ ft-kips}$$

and

$$M_{CA} = +357.7 \times 0.108 = +38.6 \text{ ft-kips}$$

This compares to 31.8 and 38.7, respectively, as given by the author—a very good agreement. Finding carry-over factors and stiffness factors is equally simple. By definition, the carry-over factor is the ratio between the resulting moment at the fixed end of a beam to the moment applied at the other end of the beam. The stiffness factor of a member is equal to the moment required to turn the free end of the member through a unit rotation when the other end is held fixed or free. These are not the general definitions of the stiffness and carry-over factors, but they suffice for the present purpose.

Example 3.—Find the carry-over factor from end A to end C, in Fig. 24. This is done merely by applying a unit moment at end A, considering it as hinged. After distributing around joint B, the moments are as shown in Fig. 27.

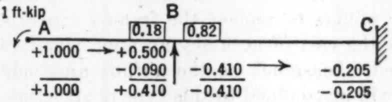


Fig. 27

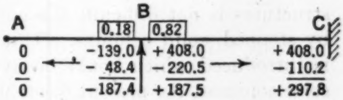


Fig. 28

The reaction at point B is upward and equal to $\frac{1 + 0.41}{12} + \frac{0.41 + 0.205}{28} = 0.14$. In this case, with joint A hinged, the moments generated by an arbitrary upward deflection at point B of $\Delta = 10.000\ z$, after joint B has been allowed to rotate, are as shown in Fig. 28. Moments are in terms of z, the correction factor.

The reaction at point B is downward and equal to $\frac{187.4\ z}{12} + \frac{187.5 + 297.8\ z}{28} = 32.9\ z$. Equating this reaction to the reaction of 0.14, from Fig. 27, gives $z = \frac{0.14}{32.9} = 0.00426$; and the moment at end C is

$$M_{CA} = -0.205 + 297.8 \times 0.00426 = +1.055$$

Since a unit moment was applied at end A, the moment at end C is equal to the carry-over factor. The author gives 1.06—which is also a very good agreement.

The stiffness factor is found in a similar way, but in this case, end A is held rigid while the distribution and the deflection correction are made. The final moment at end A will now be equal to the ratio between the stiffness of the beam of constant section AB, to the stepped beam AC. A more direct way would be to apply the stiffness factor of the constant section AB as a moment at end A. Then after completing the foregoing distribution and deflection correction, the final moment at end A would be the stiffness factor for the stepped beam AC. The correctness of this follows directly from the definition.

It is believed that the foregoing discussion will help a designer in gaining confidence in the design curves, and especially will help in checking as to the direction of the end moments in a beam with a moment applied somewhere in the span—usually a difficult and confusing problem.

A. FLORIS,[20] ESQ.—At great length well-known expressions are derived by well-known methods and applied to the analysis of stepped-column mill bents in this paper. In the mind of a critical reader doubts will arise as to the scope, usefulness, and value of such efforts. Since the introduction of relaxation methods the importance of tables and charts in the analysis of rigid frames and continuous beams with variable moments of inertia has been unduly exaggerated. The integrations involved in the evaluation of various constants can be replaced by summations, and results can be obtained in a comparatively short time. The great advantage of the summation process lies in the fact that it is applicable to all conceivable cases; its use is not restricted to the number or completeness of available tables and diagrams.

Although the application of the moment-distribution method to framed structures is not difficult, the author prefers to replace the framed supports by stepped, solid columns. Thus, at the cost of accuracy an approximation is introduced which could be avoided. Inasmuch as the author uses only solid columns, the present discussion will be confined to this type of structure.

By using one of the general methods of mechanics, all the necessary expressions of the relaxation methods can be derived, quite simply, as follows: Adapting the notation of the paper, let

$$C_2 = \lambda = \frac{1}{L^2}\int_o^L \frac{(L - x)\,x}{I_x}\,dx \ldots\ldots\ldots\ldots\ldots \tag{30a}$$

$$C_1 = \zeta = \frac{1}{L^2}\int_o^L \frac{x^2}{I_x}\,dx \ldots\ldots\ldots\ldots\ldots \tag{30b}$$

and

$$C_3 = \rho = \frac{1}{L^2}\int_o^L \frac{(L - x)^2}{I_x}\,dx \ldots\ldots\ldots\ldots\ldots \tag{30c}$$

Then

$$\phi = C_1\,C_3 - C^2_2 \ldots\ldots\ldots\ldots\ldots \tag{31}$$

For the joint rotations, the stiffness factors are as follows:

At the far end—

$$\Lambda = \frac{C_2}{\phi} \ldots\ldots\ldots\ldots\ldots \tag{32a}$$

---

[20] Los Angeles, Calif.

and at the near end—

$$Z = \frac{C_1}{\phi}; \quad P = \frac{C_3}{\phi} \qquad (32b)$$

For sidesway or bar rotations, the stiffness factors for the far and near ends are, respectively,

$$Y = \frac{C_1 + C_2}{\phi}; \quad \Omega = \frac{C_3 + C_2}{\phi} \qquad (33)$$

Carry-over factors for joint a and joint b (Fig. 29) are C_2/C_1 and C_2/C_3, respectively.

In addition, the fixed-end moments are expressed by

$$M_{f,A} = \frac{1}{\phi L}(F_a C_1 - F_b C_2) \qquad (34a)$$

and

$$M_{f,B} = \frac{1}{\phi L}(F_b C_3 - F_a C_2) \qquad (34b)$$

in which

$$F_b = \int_o^L \frac{M_{ox}\, x}{I_x}\, dx \qquad (35a)$$

and

$$F_a = \int_o^L \frac{M_{ox}\,(L - x)}{I_x}\, dx \qquad (35b)$$

In Eqs. 35, M_{ox} is the bending moment for the given loading of the beam on two supports.

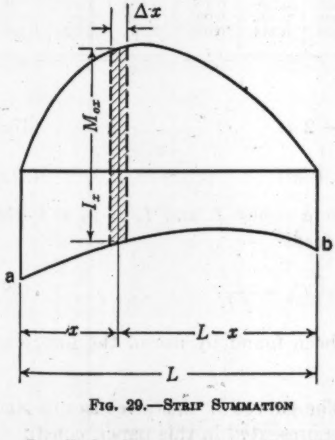


Fig. 29.—Strip Summation

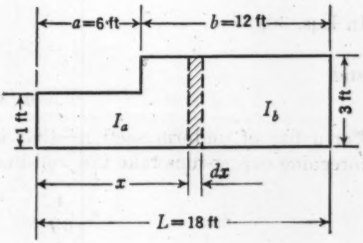


Fig. 30.—Stepped Bar

In all the foregoing equations C_3 will be equal to C_1 if the beams or columns are symmetrical. The bending-moment diagram produced by an external load is shown in Fig. 29. Dividing this area into a convenient number of equal strips Δx, and replacing the integrals in Eqs. 30 and Eqs. 35 by sums, all the necessary quantities can be calculated. The same is true for unloaded beams and columns, in which case $M_{ox} = 0$. Eqs. 30 and Eqs. 35 are applicable to all cases including stepped columns, the simplest of all.

For comparison the constants of a solid column, 1 ft thick and 18 ft long (Fig. 30), are given in Table 6. In case 2 the column was divided into nine equal strips with $\Delta x = 2$ ft, and by means of Fig. 29, the column factors determined. Case 3 gives the basic expressions C_2, C_1, and C_3 taken from Figs. 6 to 9, and the remaining quantities were calculated. The great discrepancy between values for case 1 and those of the author (case 3) raises a fair question as to the reliability of these charts. In order to check the accuracy of the summation method, the integrals of the basic quantities have been evaluated. The results are

$$C_2 = \frac{1}{6\,L^2}\left(\frac{\delta}{I_a} + \frac{L^3 - \delta}{I_b}\right) \quad \text{(36a)}$$

$$C_1 = \frac{1}{3\,L^2}\left(\frac{a^3}{I_a} + \frac{L^3 - a^3}{I_b}\right) \quad \text{(36b)}$$

and

$$C_3 = \frac{1}{3\,L^2}\left(\frac{\gamma}{I_a} + \frac{L^3 - \gamma}{I_b}\right) \quad \text{(36c)}$$

TABLE 6.—COMPARISON OF RELAXATION FACTORS

Case	C_2	C_1	C_3	ϕ	Λ	Z	P	Y	Ω	$\frac{C_2}{C_1}$	$\frac{C_2}{C_3}$
1......	10.30	5.04	51.40	153	0.0674	0.0329	0.336	0.100	0.403	2.05	0.205
2......	10.44	5.17	51.59	158	0.0662	0.0328	0.327	0.099	0.392	2.02	0.203
3......	13.50	6.70	70.00	287	0.0472	0.0234	0.244	0.070	0.291	2.02	0.193

In Eqs. 36,

$$\delta = a^2\,(3\,L - 2\,a) \quad \text{(37a)}$$

and

$$\gamma = a\,(3\,b\,L + a^2) \quad \text{(37b)}$$

For a bar of uniform section—that is, for $a = b = L$ and $I_a = I_b = I$, the foregoing expressions take the well-known form:

$$C_2 = \frac{1}{6\,I}\,; \qquad C_1 = C_3 = \frac{1}{3\,I} \quad \text{(38)}$$

The values for case 1, Table 6, have been found by use of the foregoing expressions.

Charles W. Dohn,[21] **Jun. ASCE.**—The curves of elastic constants and moments for stepped-column mill bents, as presented in this paper, constitute a worthy addition to design knowledge. It is well to note attempts to apply rational methods to problems which, heretofore, were solved by making many broad assumptions. When the final sections chosen are no different from those obtained by prior methods, the more rational analysis serves to justify the many broad assumptions of the prior methods.

---

[21] Engr., New York, N. Y.

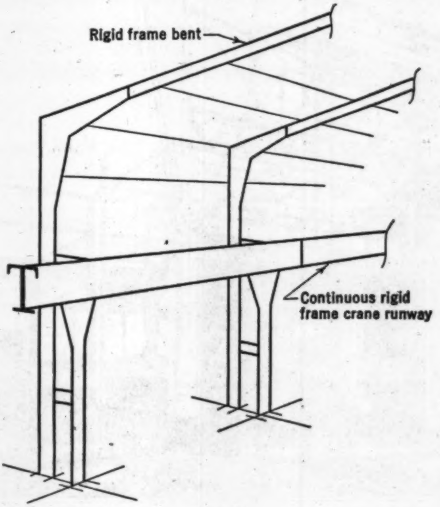

Rigid frame bent
Continuous rigid frame crane runway

FIG. 31

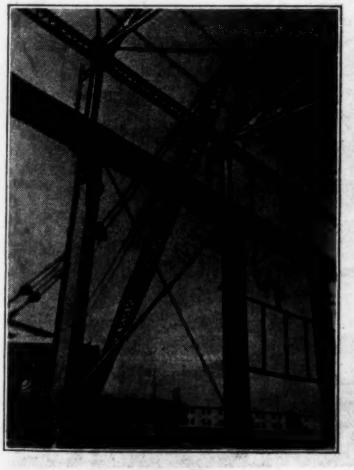


FIG. 32

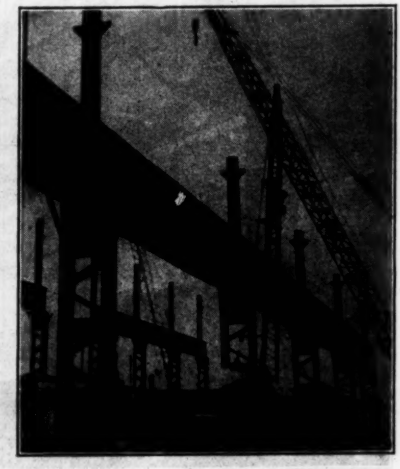


FIG. 33

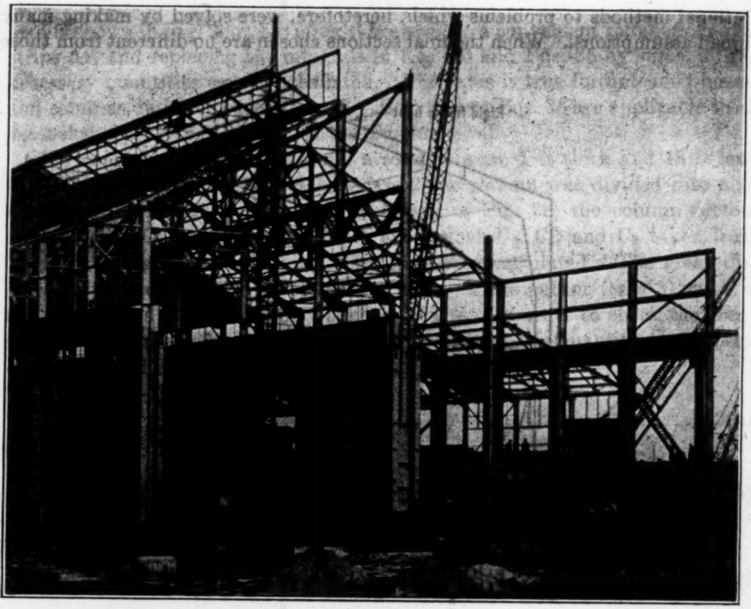


FIG. 34

FIG. 35

The author limits the value of his elastic constants and moments in the designs presented by making assumption *c* (under the heading, "II. General Conditions and Design Assumptions"), which assumes no end rotation of the truss. The statement is made that, unless the truss is subject to considerable deflection, this is a safe assumption. If the span length is rather long, as in the usual wide mill building, under vertical load the truss will exert considerable moment on the small upper stepped-column section, and this plus other loading conditions has been found critical in previous designs with rigid top connections of which the discusser has knowledge. The position of the end diagonal, in the truss shown in Fig. 4, would cause excessive stress in the lower chord of the end panel. If the slope is reversed, thus bringing the end diagonal into the lower chord point at the column, a better stress condition prevails for a rigid joint of this nature. This is especially true for long spans. With further reference to Fig. 4, the author makes the statement:

> "The upper load of 32 kips will also produce some bending, particularly in the upper section; however, this will not be especially severe, since it is near the center line of the lower section."

True, it will not be a severe loading for the lower section; however, it is an extremely severe loading for the upper section when considered in conjunction with moments induced by other loading conditions. The upper step is limited by space requirements in its ability to resist bending. By making the neutral axis of the diagonal and the chord intersect at the neutral axis of the upper column section, the eccentricity of the end reaction may be eliminated, and moment on the upper column section may be reduced.

Example 1 would be more valuable if the complete analysis for all the eccentric loads shown in Fig. 4 were included. The 120-kip eccentric load will cause a translation of the column at the point of application of the couple—hence, the writer is not convinced that the method of superposition applies to Fig. 3(*e*). Inasmuch as bending is combined with axial compression, the deflection changes the action of the axial forces, and the latter produce not only axial compression, but also some additional bending. Small deflections may produce considerable changes in the action of forces.

The space requirements for clearance at the ends of the bridge crane and the difficulty of determining maximum moment conditions for a stepped column as part of a continuous and rigid structure subjected to various independent loads contribute to make the independent crane column economical of design, material, and construction. The diagrammatic sketch (Fig. 31) depicts a mill building with a continuous roof bent connected to a continuous rigid frame crane runway. The connections between the bent and the runway do not transmit vertical or bending forces, but may be utilized to absorb lateral shocks from the crane. Such a design is rigid, positive, economical, and more easily fabricated and erected than the stepped-column mill bent design. In the design of a rigid frame crane runway, such as indicated in Fig. 31, the method presented by R. C. Brumfield,[22] M. ASCE, would be applicable. The arrangement shown in Fig. 31 is, furthermore, unique, as it requires less longitudinal and lateral bracing than a stepped column. Aside from economy of material, this is an advantage at the working level. Argument might be presented that the stepped-column mill bent may also be made rigid with the crane girder. When the stepped column is made rigid with the girder and the roof, the resulting maximum stress conditions are not positively determinable without expenditure of excessive effort, and it is doubtful if such a design can match in economy one which is proportioned for the independent load conditions rather than an unfavorable combination of these dissimilar loading conditions. The independent column is not necessarily much less compact; it can be used economically with wider and variable width bays; and it is more economical of crane girder and bracing material than the stepped-column mill bent when used in rigid frame designs. The independence of the building and the crane runway may increase the value of both. Either the building or the runway can be removed or altered independently to accommodate use, and the building is adaptable to the addition of multiple aisles. It would be debatable which type of column is best in resisting seismic forces. The independent system can be made as stiff or as flexible as requirements dictate; the stepped column is somewhat constrained in dimension and ill proportioned to resist lateral forces as part of a rigid frame.

---

[22] "Moving Loads on Restrained Beams and Frames," by R. C. Brumfield, *Transactions*, ASCE, Vol. 111, 1946, p. 805.

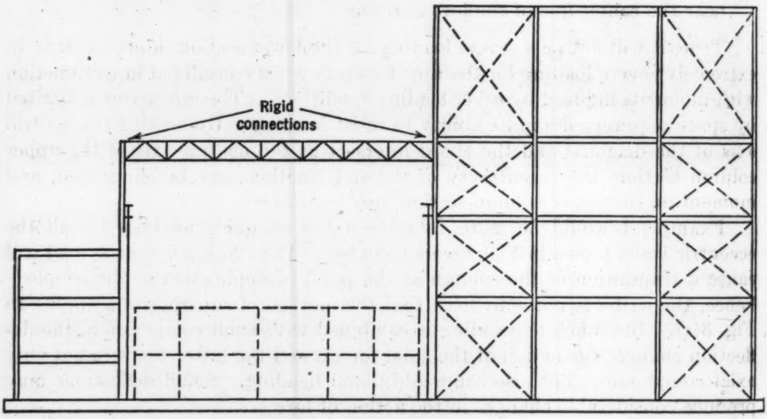


Fig. 36.—Power Station Bent

The mill buildings shown in Figs. 32, 33, 34, and 35 were erected several years ago and represented good practice then. At present, and particularly in the future, continuous frames and rigid connections will vie more for better practice where good foundation conditions are available. The illustrations are presented in the hope of forming a more complete picture of typical engineering problems in selection and analysis of stepped-column designs wherein the author's elastic constants may be applied.

Fig. 32 shows a typical stepped column in a mill building. Note the longitudinal and lateral bracing and the rigid roof connections, not completed at the time the photograph was taken. The girders are not continuous and the bays are not very wide. Brackets are provided for knee braces to transmit longitudinal forces from the crane girder to the column, and thence to the bracing. These knee braces had not been added at the time the photograph was taken. The inside face of the column is a very deep channel section riveted to the lower column section. Brackets and a heavy billet are provided for the crane girder seat.

Fig. 33 illustrates another version of the stepped column wherein two stub columns support two crane runways and a roof column. Note that the bays are wide, and that the crane girders are deep, noncontinuous, and nonrigid. Fig. 34 shows a heavier service, mill building group. Note the maze of bracing and the massiveness of the columns, as well as the deep crane girders, that result with simple spans and relatively nonrigid frames. (The high peak roof truss is not well adapted to use as a rigid frame member.) A stairway has been located so as to utilize part of the "dead" space between the columns. The mill building columns in Figs. 32, 33, and 34 present opportunity in their design for the elastic constants presented by the author.

Fig. 35 illustrates a multiple-aisle mill building with varying bay sizes, and with crane runways at different levels. The independent crane column was used for this difficult framing problem; however, the crane runway spans are neither continuous nor rigid, and the high peaked roof truss is not properly part of a proportioned rigid frame.

Mr. Ling's elastic constants for a stepped column would be useful in the design of a power station bent such as shown in Fig. 36. In this design the roof truss is rigidly connected to the upper crane column in the turbine room. The boiler house is braced for lateral forces. A study of moment connections was made for the boiler house but they were not used because they were not required as much as for, say, an office or apartment building. The bracing in the boiler house thus absorbs the lateral and sway forces from the crane-girder bent over the turbine room.

DANIEL S. LING,[23] M. ASCE.—The various discussions of this paper are greatly appreciated and the writer sincerely hopes that the original paper, clarified and enlarged on by the discussions, will be of practical use to the structural engineer.

Mr. Spaulding's objections are entirely valid. It was realized when the curves were originally developed that actually a gradual, rather than an abrupt, transition occurs at the column step. For most design problems, the error is not believed to be large, but if the designer prefers he can readily use the modified "x L," suggested by Mr. Spaulding.

Referring to the point of application of the side thrust from the crane, again, the writer must agree with Mr. Spaulding. This factor was considered when the curves were prepared but no ready solution was apparent without the complication of another variable, as the point of application of load in relation to the column step would vary with each size of crane girder. It was felt, however, that because of the stiffening effect of the usual crane girder connection the error involved would not be too large. Mr. Gray has suggested that fixed-end moment curves for other points of application would be desirable and Mr. Sollid has stated that such curves are available in a publication of the United States Bureau of Reclamation.

---

[23] Cons. Engr., Detroit, Mich.

Mr. Gray's statement (2) is not clear. Distribution factors around any joint are simply the ratio of $(C\,K)$-values of the various members at the joint (see Table 1); and the use of curves would seem neither possible for the great variety of different conditions, nor of any practical benefit. Curves such as those suggested in statement (3) would, no doubt, have some use but would involve many series because of the variation not only in the point of application, but also in the terms x and I/I_o. The need for curves for carry-over factors is a matter of personal preference. Since the moments from translation derive from $(C_1 + C_2)\,\frac{E\,\Delta K}{L}$, the writer feels that a C_2-curve is preferable to a carry-over factor curve and has, in fact, often been forced to determine C_2-values from the carry-over curves in the series on haunched members published by the Portland Cement Association.

As to the statement that the design is not complete, the writer has noted that a complete design was not shown, but simply the determination of moments from bent action. The completion from that point should be no problem to the experienced designer and it was felt would add no worthwhile addition to the paper.

Mr. Mensch refers to the publications presenting formulas for bent analysis. Often, such bents will fit a case under consideration, but all too frequently the building to be designed does not seem to fit any of the cases and stepped column conditions are not included. Frankly, the writer has always been hesitant to use the sometimes lengthy formula because of great possibility of error due to misinterpretation, involved calculations, and hesitancy to accept a formula of whose accuracy he is not personally certain and which would take much time to check. The latter could, of course, be stated of the curves in Figs. 6 to 9, but a few spot checks are not too difficult.

Mr. Mensch's statement that the writer's moments at the foot of the columns are in error is not explained; certainly, it is at variance with Mr. Sollid's concurrence with the writer. In the examples given in this paper, the writer essentially assumed infinite stiffness of the roof truss. Although this assumption is, admittedly, apt to be considerably in error in many cases, it is readily possible for any assumed K-value of the truss to rotate around a "joint" midway between points a and a′, making use of the curves for determining C-values and fixed-end moments. The writer questions whether the usual roof truss connection to the column is adequate to develop the full stiffness of the truss.

In the time available the writer has not had the opportunity to study fully Mr. Mensch's method of solution. It seems applicable only to symmetrical bents and the writer does feel the greatest merit of his suggested solutions is for use with unsymmetrical bents.

Professor Fiesenheiser suggests a method of temporarily assuming a support at the column step. This suggestion is entirely feasible; and, if no curves for stepped columns were available, would be much easier than construction of the various $\frac{M}{E\,I}$-curves needed for a stepped section. With the use of the curves presented in the paper it would seem that Professor Fiesenheiser's method is somewhat longer, and especially so when bent effect is considered. It does, however, present another useful possibility to the designer when special conditions arise, and could adequately serve to handle the case of the applied loads or moments at points other than the column step. After moments for the applied forces were determined, the effects of bent translation would be found by the method presented in the paper.

Mr. Sollid presents the same general method as that given by Professor Fiesenheiser and suggests it as a spot check for the curves. His statement indicates that *Technical Memorandum No. 629*[18] should be a desirable addition to the reference library.

It is agreed, as Mr. Graves states, that the curves can be used for any type of problem for stepped members other than columns. The writer has knowledge, however, of unpublished curves that are more applicable to stepped beams or slabs. With reference to the moment at the top of the column, the designer should, of course, consider the worst possible combination of loads. Perhaps, Mr. Graves' suggestion of not less than 50% of the fixed-end moment has merit.

Mr. Graves' statement is correct; the curves in Figs. 6 to 9, inclusive, are included in "Statically Indeterminate Structures."[2] Permission had been granted to L. C. Maugh, M. ASCE, to include the curves in his book and because of wartime difficulties and restrictions the book was published prior to the publication of this paper.

If, by "solid" columns, Mr. Floris means a full, rectangular section, he appears to have misinterpreted this paper. The curves are applicable to any section for which the moments of inertia can be determined and the examples in the paper show standard rolled beam sections. The inference regarding integration is not understood. Actually, the writer's curves were prepared by summations from the $\frac{M}{E\,I}$-diagrams, essentially the same process as Mr. Floris' method.

Mr. Floris' C-factors are not the same as the writer's and, therefore, are not comparable. The difference is readily noted in Eq. 38, in which all the C-values include I; whereas, in the slope-deflection equations for uniform section, $C_1 = C_3 = 4$ and $C_2 = 2$. In view of Mr. Floris' statement regarding accuracy, the point should be made that the writer determined, by the conjugate beam method, the slope-deflection coefficient for the case of Fig. 30 in which x (Fig. 1) $= 0.33$ and $\frac{I}{I_o} = \frac{3^3}{1^3} = 27$. The values are: $C_1 = 6.93$, $C_2 = 13.6$, and $C^3 = 68.3$—which, considering the small scale of the curves, are in good agreement with the values determined (case 3, Table 7) by Mr. Floris from the curves.

For a correct comparison of Mr. Floris' factors with those of the writer, the term Λ is the writer's C_2 multiplied by K_o, $Z = C_1 K_o$, $P = C_3 K_o$, $Y = (C_1 + C_2) K_o$, and $\Omega = (C_3 + C_2) K_o$. For the column of Fig. 30, using foot units, $I_o = \frac{1}{12}$, $L = 18$, or $K_o = \frac{1}{12 \times 18} = \frac{1}{216}$. With this multiplying factor, case 3, Table 7, should read $\Lambda = 0.062$, $Z = 0.031$, $P = 0.0324$, $Y = 0.098$, and $\Omega = 0.386$; and, if the values obtained from the $\frac{M}{EI}$-curve were used, the values for case 4 would be: $C_2 = 13.67$, $C_1 = 6.93$, $C_3 = 68.3$, $K_o = \frac{1}{216}$, $\Lambda = 0.0633$, $Z = 0.0321$, $P = 0.316$, $Y = 0.096$, $\Omega = 0.380$, $C_2/C_1 = 1.97$, and $C_2/C_3 = 0.200$. It is thus noted that Mr. Floris' "great discrepancy" does not exist. It would appear that Mr. Floris is simply presenting another, somewhat similar method of obtaining stiffness factors and fixed-end moments, which is as time consuming as the method used by the writer in preparing the curves presented in this paper.

Mr. Dohn's statements regarding design conditions will not be challenged. It is realized that one cannot generalize greatly with respect to design assumptions because what may not be too important in one case, may, indeed, be serious in another. In the main, the writer wished to present his curves for use where applicable.

The argument of stepped columns versus independent columns is, to a great extent, a matter of opinion; but the writer has felt that the objection to stepped columns simply on the basis of difficulty of analysis should not hold. Actually, in many designs using independent columns, the crane columns are so thoroughly connected to the building column with diaphragms or lacing that essentially a stepped column results. The use of knee braces on crane girders is, to the writer, most questionable. Too often such knee braces are used without any consideration being given to the bending produced in the column from the deflection of the crane girder, with consequent overstressing of the column.

The writer wishes to thank all the discussers of the paper and is pleased that several of them feel that the curves presented will be of use to the designer.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2355

TESTS OF TIMBER STRUCTURES FROM GOLDEN GATE INTERNATIONAL EXPOSITION

BY A COMMITTEE OF THE SAN FRANCISCO (CALIF.) SECTION, ASCE, ON TIMBER TEST PROGRAM

COMMITTEE

HAROLD B. HAMMILL, *Chairman*

HENRY J. DEGENKOLB
HENRY D. DEWELL (deceased March 20, 1946)
JOHN J. GOULD
LEON H. NISHKIAN

WITH DISCUSSION BY MESSRS. JAMES H. CARR, JR., HOWARD J. HANSEN, E. G. STERN, MELVIN W. JACKSON, CHARLES MACKINTOSH, SIDNEY NOVICK, FRANK J. HANRAHAN, WALTER J. RYAN, WALTER M. PRATT, AND COMMITTEE OF THE SAN FRANCISCO (CALIF.) SECTION, ASCE, ON TIMBER TEST PROGRAM.

NOTE.—Published in May, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

CONTENTS

PAGE

Synopsis 1125

A. Foreword 1126

B. History 1127

C. General Features 1130

  I. Operations and Computations 1130

  II. Safety Factor 1133

  III. Nomenclature 1134

D. Detailed Report of Tests 1137

  I. Truss Tests 1137

  II. Joint Tests 1187

  III. Column Tests 1221

  IV. Plywood Tests 1223

SYNOPSIS

Seventeen trusses, thirty-seven joints of structural frames, and three 15-ft struts, all of which had served for some 2½ years as structural parts of the Golden Gate International Exposition buildings at San Francisco, Calif., were removed from these buildings when they were dismantled, stored for some weeks in the open air, and later were tested for failure in a large testing frame constructed especially for this purpose. Also, one hundred and three laboratory specimens, constructed to find the lateral resistance of nails in plywood and the shearing and buckling strength of plywood panels, were tested.

The trusses were of various types, spans, and sizes. Thirteen were entirely of timber. Seven of these had connections framed with split rings or shear plates, and the other six had connections of volts and washers. Four of the seventeen trusses were of the rod and block type, with compression diagonals of wood and with tension members of steel rods. The thirty-seven joints with shear plates, split rings, and bolts were of various types: Seventeen with members connected by steel gusset plates and shear plates, one with plywood gusset plates and bolts, two joints and four splices with split rings, six bolted spliced joints of girders resisting moment and shear, two butt-block end joints of trusses, three joints with steel straps and pins, two bolted joints with steel straps, and four laboratory-built tension splices or compression splices with plywood splice plates and split rings. One spaced column, two T-section columns and struts, eighty-two nail fastenings in plywood, three plywood panels in shear, and eighteen plywood panels subjected to buckling were also tested.

Loads were applied with calibrated hydraulic jacks. Deflections of each truss as a whole, as well as deformations of its component parts, were measured during the progressive loadings. Likewise, deflections and deformations were measured in the various joints, columns, and plywood tests. Numerous photographs illustrate the features of interest in the tests.

Loads and stresses were computed after each test. For example, the strength or weakness of each truss and its parts was investigated theoretically to correlate test phenomena and structural theory.

A. FOREWORD

This report presents: Tests and studies of the load-carrying capacity, strength, deflection, and deformation of members of full-sized wood trusses; similar tests of joints of wood trusses and structural frames, splices of wood girders, and built-up wood columns; tests of the buckling and shearing strength of plywood panels; and tests of the lateral strength of nail fastenings in plywood. All the trusses had supported the roofs or floors of the Golden Gate International Exposition buildings for 2 or more years, and were made available for testing when the buildings were dismantled in 1941. All test specimens were taken intact from these buildings, or were fabricated of material from them. Most of the trusses were framed with modern timber connectors (that is, split rings and shear plates) although some had bolted joints without connectors and others were framed with web members of steel rods and wood struts.

These tests and their analyses are believed to be unique in presenting a comprehensive study of actual wood trusses and structural members, not designed and constructed as test specimens. The structural members had functioned as parts of buildings for 2½ years; they were generally of good material; they had been designed by competent structural engineers; they had been built under ordinary field circumstances and under varying conditions of inspection; and, finally, they were tested as taken from the buildings with such defects as normally exist, without tightening loose bolts and (except in a few cases specifically noted, where trusses were damaged while being taken down) without correcting any deficiencies resulting from service.

The original purpose was to take advantage of an opportunity seldom available; namely—(1) to test entire trusses in place in the buildings, under normal conditions of use as nearly as possible with the various defects of workmanship accompanying commercial practice; (2) to study the relation between laboratory tests of individual members or joints and their combinations as found in framed structures; (3) to determine whether factors had been overlooked in design, construction, inspection, or maintenance on which more information was needed; (4) to note the effect of shrinkage and other factors that develop in service; and (5) to test individual joints with multiple connectors and with several members forming concentric or eccentric connections.

It became necessary, however, to demolish many of the buildings before it was possible to test their trusses in place. As a substitute, although not so desirable, the individual trusses were tested as isolated specimens in a specially constructed testing frame. Some of these trusses were of types in which the influence of tension perpendicular to the grain was an important factor. These tests revealed for the first time the importance of considering this stress combined with horizontal shear or other stress in eccentrically connected joints.

Neither the assumed loads nor the unit stresses for which the trusses were designed are known to the committee. The trusses and the structures all functioned satisfactorily. As a result, it is entirely possible that, in the case of some of the critical conditions found in testing these joints and trusses to ultimate load, the designing engineer may have refrained from utilizing the full strength of a member or joint, fully realizing that eccentricities (for example) produced additional stress which required reduction of stress from direct axial loading.

Phenomena revealed by the tests and subsequent analysis and discussions of these phenomena may be noted as follows:

1. The inadequacy of small individual tests as a premise for determining the behavior and strength of full-sized structures;
2. The importance of considering the effect of combined stresses and the need for further tests and more data on this subject;
3. The importance of adequate supervision or inspection to assure accuracy of fabrication and erection;
4. The tendency of a truss chord to rotate when tension members are connected on one side and compression members on the other, resulting in bending or bowing of web members flatwise;
5. The need of keeping adjacent faces of members in contact by tightening nuts on bolts when it has been necessary to use lumber that has not been seasoned to the moisture content to which it will come during service;
6. The desirability of reducing end checking and splitting;
7. The beneficial effect of cross bolts or stitch bolts, or threaded spikes or dowels, or some kind of adjustable strapping in offsetting the effect of splitting;
8. The importance, where shear plates are used, of avoiding oversized bolt holes, either by error or by reaming to permit passage of bolts through holes bored improperly in the wood;
9. The importance of straight end cuts, without reentrant angles; and
10. The importance of ample end distances in tension members for ring connectors and shear plates.

The main objective of the tests was to determine a factor of safety for the structures tested, for comparison with factors of safety for individual test specimens. This objective was not realized fully, and any comparison of the ratios of ultimate loads as related to assumed safe working loads is misleading unless examined in the light of the material presented in Section C–II under "Safety Factor" and with due attention to individual test comments.

Of the twenty tests made, on sixteen trusses, the results show that: Five failed because of rupture of the wood members at joint details; six failed at bolted joints; and nine failed at connector joints, because of types or combinations of stresses not anticipated.

B. HISTORY

In October, 1940, preliminary work was started on the Timber Test Program at Treasure Island in San Francisco. The program was conceived by John J. Gould, M. ASCE, who had been chief structural engineer of the Exposition, and it was adopted by the San Francisco Section of the Society through the interest and recommendation of the late Frederick H. Fowler (Past-President, ASCE), Harold B. Hammill (M. ASCE), and Walter Dreyer (M. ASCE)—all Past-Presidents of the San Francisco Section. The sponsorship of the University of California (at Berkeley) was obtained through the cooperation of C. T. Wiskocil, M. ASCE, and W. C. Pomeroy, registrar of the university, then acting as administrative officer of the Works Progress Administration (WPA) for the university. The San Francisco Section appointed a Timber Testing Committee with L. H. Nishkian, M. ASCE, as chairman. When he resigned the chairmanship because of press of private work and extended absence from San Francisco, Mr. Hammill succeeded him. The other members of the committee were Mr. Gould and the late Henry D. Dewell, M. ASCE, and Henry J. Degenkolb, Assoc. M. ASCE. Mr. Gould was retained as technical director of the program by the committee, which functioned in cooperation with the various timber industries who furnished the funds required for the work. Mr. Degenkolb was appointed assistant technical director. Byron L. Nishkian, Jun. ASCE, assisted Messrs. Gould and Degenkolb in the first few months of the program. Mr. Gould, in a part-time capacity and Mr. Degenkolb in a full-time capacity, directed all the technical and administrative phases of the program and prepared the preliminary report, which was edited by Mr. Dewell and then reviewed by the entire committee.

Industries that contributed to the program include the West Coast Lumbermen's Association; National Lumber Manufacturers' Association; Douglas Fir Plywood Association; Timber Engineering Company, Inc., of Washington, D.C.; Timber Engineering Company of California; Timber Structures of Portland, Ore.; Summerbell Roof Structures of Los Angeles, Calif.; the Western Pine Manufacturing Company of Spokane, Wash.; Henry Mill and Timber Company of Tacoma, Wash.; and the Northwest Bolt and Nut Company of Seattle, Wash. The following were instrumental in the organization of the project and kept in close touch with all progress: The late Chester J. Hogue, M. ASCE, in charge of technical service, West Coast Lumbermen's Association; A. C. Horner, Assoc. M. ASCE, consulting engineer, Timber Engineering Company of California; and the late J. E. Mackie, Assoc. M. ASCE, then western manager, National Lumber Manufacturers' Association.

The San Francisco Section and the committee owe these three gentlemen particular gratitude, for they more than once were instrumental in securing additional funds when such financial assistance was sorely needed. Finally, Messrs. Mackie and Horner were generous in supplying the services of their office staffs in the preparation of this report. The Division of Timber Mechanics of the Forest Products Laboratory in Madison, Wis., was very helpful in giving consultation, in answering inquiries, and in reviewing this report.

The late John A. Newlin, M. ASCE, formerly chief of the Division of Timber Mechanics of the Forest Products Laboratory, made a special trip to San Francisco for the purpose of witnessing the tests and advising with the committee. Because his death intervened, the report has not had the benefit of his criticism, which would have been of great value because of Mr. Newlin's knowledge of timber mechanics and wide experience in timber testing. Fortunately, notes made by him have been materially helpful.

Preliminary work began in October, 1941, and consisted principally of the selection of specimens, investigation of methods of testing, application for WPA assistance, and general review of the condition of the Exposition buildings for availability of test specimens. Many studies and drawings were made by the technical director and his assistant to determine the most advantageous and economical tests and procedures. Unfortunately, various setbacks beyond the control of the committee hindered progress. Some of the buildings were demolished by the wrecking contractors for the Exposition before any arrangements could be made for securing the required specimens. This circumstance was partly offset by the cooperation of the contractors, who donated or allowed various specimens to be taken in return for payment of labor and equipment costs for the extra trouble involved in setting these specimens aside.

The proposal of the United States Navy to take over all of Treasure Island and its buildings caused several revisions in the list of specimens considered necessary for a satisfactory fulfilment of the program. Later, the Navy did take over the island, necessitating the dismantling of the laboratory built for the test program and removal of its equipment to a new location in San Francisco. Finally, the WPA assistance was secured later than had been anticipated. However, to counteract several of those difficulties, the program was fortunate in having the splendid cooperation of Clarence E. Seage, Assoc. M. ASCE, who was chief engineer of the California Commission for the Exposition and who was instrumental in securing many specimens. Ralph G. Wadsworth, M. ASCE, then deputy WPA administrator for Northern California, was especially helpful in the organization of the project and in expediting the work on the many necessary preliminary details.

On February 17, 1941, the Timber Test Program, known as a WPA project, was opened officially. For several months the major part of the project was the collection of specimens. It had been hoped that these could be assembled simultaneously with the testing. However, the rapidity of the demolition made it necessary to obtain the specimens during this work and to delay the testing. Simultaneously, material was furnished to two senior students of the University of California for a special study in timber supervised by the technical director and his assistant. In May, 1941, tests were made on the Cavalcade cantilever structure; and, in June, a building of 2,800 sq ft was constructed, with a 500,000-lb testing frame, all as described in this report.

During the last few days of June, 1941, the first of the trusses was tested—the San Francisco Building truss; and, shortly thereafter, in turn, the Chinese Building trusses, Alameda Contra-Costa Counties Building trusses, Cavalcade truss, French Building truss, Merry-Go-Round trusses, and two rigid-frame legs from the Brazil Building. At this time, because of the national emergency of World War II, it became necessary to move all timber testing equipment and specimens from Treasure Island to San Francisco. In this emergency, entailing unexpected expenses, a building was loaned to the committee by the City of San Francisco, through the cooperation of the late A. D. Wilder, M. ASCE, then director of public works, and James H. Turner, M. ASCE, then civil engineer of the Public Utilities Commission and later manager and chief engineer of the Hetch Hetchy Water Supply, Power and Utilities Engineering Bureau. There was not enough room in this building, however, for all the specimens and equipment, but through the courtesy of J. F. Marias and T. Twohig, then president and rental manager, respectively, of the State Board of Harbor Commissioners, space was obtained at the Ferry Building for additional storage.

Considerable effort and time were expended in making arrangements for moving, and in settling the new building, after which the remaining tests were conducted. Simultaneously, the office staff began and completed the compilation of the data on which this report is based.

Much credit is due Messrs. Degenkolb and Byron Nishkian for the excellent and full photographic record of the tests, only a part of which appears in this report.

C. GENERAL FEATURES

I. Operations and Computations

Equipment.—The necessity of strict economy affected not only the equipment but also the methods of testing. Although the supply of WPA labor was sufficient, cash reserves were at a minimum. The equipment as planned had to be capable of testing 70-ft trusses, individual joints of other structural frames, plywood panels, and 30-ft columns. In the tests of trusses, the loads had to be applied at various points to simulate actual loading conditions. To meet these varied requirements, a large testing structure, or testing frame of 500,000-lb capacity—one that could be used partly or entirely for all the required operations—was designed by the technical staff and then built with WPA labor, using timbers from the dismantled Exposition buildings, connected with 4-in. split rings and $\frac{3}{4}$-in. bolts.

Hydraulic jacks were used for applying, measuring, and controlling the loads. A pump, flexible hose, two 20-ton jacks, and a 50-ton jack were purchased, along with a large gage for measuring the oil pressure in the jacks. Two 200-ton jacks were loaned to the project by D. J. and T. Sullivan, contractors for moving structures, for testing the heavy specimens. All this equipment was checked and calibrated at the testing laboratory of Abbot A. Hanks, Inc., of San Francisco. Calibration curves were plotted, and studies were made on various factors that might affect their accuracy. Among these factors were the effect of eccentric loading of the plungers, the direction of movement of the plunger, and the effect of using different hose lengths between the pump and the rams.

In loading specimens, care was exercised in centering jacks and loads. As a result, it was felt that the applied loads were measured well within 3% of their true values. A slight source of possible error was that the pressure gage read only to divisions of 25 lb per sq in. for a range of 10,000 lb per sq in. of oil pressure in the jacks. In the truss tests, two or three jacks were used simultaneously with one pump and gage to provide several loading points on a truss, and dividing beams spread the load over several panel points as desired. For this purpose, special Y-connections provided several sources of oil pressure for the jacks.

Twenty gages with Ames dials or Starrett dials, reading directly to 0.001 in., with a range of 0.3 in., were used. In the tests on the trusses, it was desired to read many more deformation points than there were dial gages. A special gage was developed for this purpose consisting of two small metal angles, one fastened to each side of the deformation point. Small ball points were attached on the outside of the outstanding legs of the angles, and the distance between points was then measured with a micrometer. By using the rachet of the micrometer, a uniform pressure was placed on the ball points at each reading so that any deformations due to bending of the angle legs would be constant. These gages had the advantage that, with the micrometers used, they had a range of about 1 in., reading to 0.001 in. By using vernier calipers, this range could be extended to about 5 in., reading to 1/128 in. and to any desired larger range by measuring with a ruler fitted with the heads of two carpenter's squares. The principal disadvantage of this special gage was the time consumed in making the required number of readings. To offset this disadvantage, in several places where an accuracy of 0.01 in. was considered sufficient, simpler gages were made, using a multiplying arm with a large scale. These measuring devices were calibrated with the dial gages and had a range of about 2½ in.

Testing Procedure.—In most of the tests, the actual readings were made by WPA workers, none of whom had previous experience in this field. They were under the direct supervision of the assistant technical director—the only engineer present at all the tests, who followed through all the operations. These assistants were picked with the cooperation of WPA officials, and were then taught to read the various instruments. All required computations were reduced to such simple forms that it was possible to have practically all operations, both in the field and in the office, performed by this staff, despite the fact that its members were only unskilled workers.

Because of the large number of points to be read at each increment of load, the load was applied to a predetermined value, and then held for a sufficient period to read the various gages. Loading was then resumed, and at the next predetermined load point the process was repeated. These reading intervals lasted from a few minutes or less to a half hour or more, depending on the number of gages and other factors. Errors in reading occasionally occurred, because of the inexperience of the workers. When discrepancies were found in the final analysis, the more logical reading was taken or the questionable reading was not used in any computations. In still other cases, the questionable reading has been given in the detailed report with a note of caution indicating the possibility of error.

In timber testing, a variation in results is usually introduced by changing the speed by which a specimen is loaded. Because of the peculiar equipment used in this program, it was not possible to employ a uniform speed; nor were the rates of loading specifically determined. Since a structure is seldom, if ever, loaded at a uniform speed, the value of the test results should not be discounted because of the methods used. In all cases, the times of reading and loading were recorded; and, in this report, the total elapsed time is indicated on the individual test.

Truss Tests.—Trusses were mounted on the frame with their longitudinal planes horizontal—in other words, in a flat position. They were suspended from the roof by hangers composed of two planks, 2 in. by 8 in., nailed to form a T-section. The web of each T-section was nailed to the roof rafters overhead and the flange was nailed to the truss. These roof rafters were then tied down to the frame of the building with plumber's tape, so that the hangers also acted as struts to prevent members of the truss from buckling upward under load. The truss was thus supported at varying spacings, usually from about 4 ft to 8 ft—depending on its weight, dimensions of top chord, locations of gages, and other factors. These hangers offered practically no resistance to the horizontal deflections of the truss specimens under load, for they could be moved easily by hand.

Truss loads were applied in varying increments up to the predetermined point at which the dial gages were removed. Loads were then increased until failure occurred, only the deflection and micrometer gages being read. On several trusses, failure occurred earlier than expected, while the gages were still in place. Reactions were taken on blocks that simulated the actual restraining conditions in the structure from which the truss was removed.

Joint Tests.—The testing frame was designed with the thought of testing full-sized joints of structural frames as economically as possible. In the past, it has sometimes been the custom to test two identical joints at a time, back to back. This method was not considered desirable nor practical in this program; and, in those joints which consisted of three or more members meeting at a common intersection, a procedure was used as shown in Fig. 1. The load was applied through the hydraulic jack, putting compression in one of the members, and was resisted by another member in compression, blocked tightly against the testing frame. In some specimens, a screw jack was installed on this second compression member to help adjust the specimen and to reduce possible secondary stresses. The third member necessary to preserve equilibrium was put in tension by the following procedure. Before testing, large steel plates were fastened to it with bolts or shear plates to provide the necessary tension connection; and, after placing the joint in the testing frame, these plates were connected to two 2-in. steel rods, which were in turn fastened to

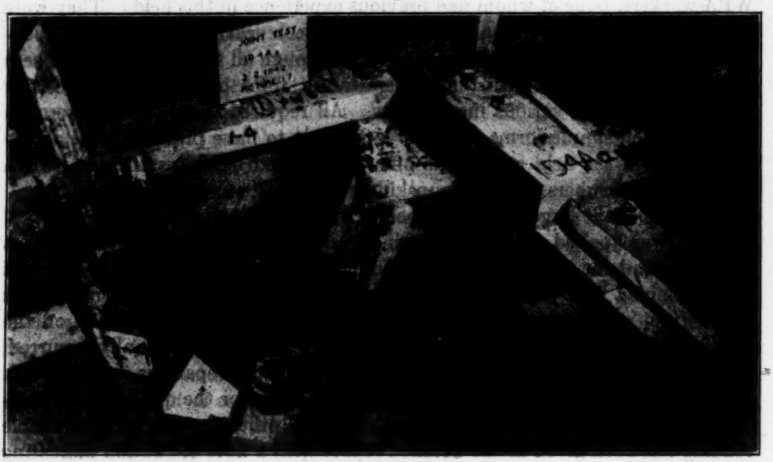


FIG. 1.—TYPICAL FAILURE OF TENSION DIAGONALS OF SHEAR PLATED JOINTS (JOINT J5)

the testing frame. One exception to this procedure was joint J13 in which all the members were put in compression. Gages were installed to measure the various deformations. In almost all the joints tested, additional gages were installed to measure the angular change between members, as a check on any possible secondary stresses that might be induced by the deformation of the joint. In no case were the movements such that these secondary stresses influenced the tests.

Plywood Tests.—The tests of the strength of nails in plywood in resistance to lateral forces, and of plywood in resistance to shear and buckling, were made in the testing frame, as described later in Section D–IV.

Reduction of Data and Calculations.—After the field data were typed and mimeographed, they were reduced, calculations were made as necessary, and stress-strain curves were plotted. Stresses in members of trusses and joints were calculated, making use of forms and with the usual assumptions for the actions of structures under load. In the case of a truss with eccentric joints, stresses were first calculated by assuming pin-ended concentric connections at the panel working points. Then corrections were made, as necessary, to the stress in any members for change of slope and the increased shear in the panel caused by eccentric connection. Stresses in these connections were calculated on an approximate basis only, as a more theoretically correct analysis would have involved unnecessarily laborious computations on structures statically indeterminate to many degrees.

One feature of the mechanics of recording the results of the tests was the use of photography. All specimens were photographed before and during the tests, at failure, and at the disassembly of the specimen to record all possible visual features. These photographs and their accompanying notes form an indispensable record. In the event that other investigators review this work, the field notes should be used with the photographs. For the information of those interested, the data observed, with the curves, field notes, photographs, calculated reductions, etc., were filed at the San Francisco office of the National Lumber Manufacturers' Association, the Engineering Societies Library in New York, N.Y., and the Forest Products Laboratory in Madison, Wis. This report includes only a small percentage of the total data available.

II. Safety Factor

One of the principal results expected from these tests was a ratio between loads causing failure in the specimens tested and the calculated allowable load that the specimen, in its weakest part, might be expected to sustain safely. Such ratios, customarily designated as "factors of safety" are likely to be misleading in a composite structure if compared to factors of safety of isolated test specimens. Quite different results may be expected, for instance, when a joint is tested (1) as an isolated specimen in a testing machine, (2) as a part of a truss tested by itself, or (3) as a part of a roof assembly in a building where the assembly as a whole may provide additional stability. Furthermore, factors of safety used in determining safe working load for nails, bolts, and connectors and for timber are derived from different considerations (ultimate load, proportional limit, deformation, etc.) and hence are not properly comparable.

For these reasons the term "factor of safety" has not been used in this report. Instead a ratio, designated by R, is used in presenting test results. In discussing the tests, factors R are given, which are ratios between the computed member stress at ultimate loads and the usually accepted design capacities (allowable loads), which are further defined in each test or group of tests. These allowable loads, where possible, are based on the recommendations of the Forest Products Laboratory or those of the Timber Engineering Company, Inc. (TECO), in its 1939 manual.[1]

It should be emphasized that, in most tests, failure occurred in a manner neither expected nor previously investigated and the factors R are based on the nearest available criterion usually used by designers. This criterion is clearly defined in each case. For example, in one truss of the North Square Building, the factor R is based on the allowable connector capacities, although failure was clearly due to tension perpendicular to the grain.

III. Nomenclature

Notation.—The following letter symbols, used in this report, and in its discussions, conform essentially to American Standard Letter Symbols for Mechanics, Structural Engineering and Testing Materials (ASA—Z10a—1932) prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1932:

A = section area; in Table 26, the symbols A to Q are introduced as key letters for typical dimensions indicated in Figs. 84 and 85;

d = depth of a beam;

E = modulus of elasticity in tension and compression;

e = eccentric distances;

F = total stress in a member, imposed by external loads; F_A = total stress in member A, etc.;

K = a factor derived from the curves of Fig. 83(c);

L = length of span or height of a column; height of a panel;

P = an applied load:

- P_b = allowable load on bolts;
- P_c = allowable load on connectors;
- P_l = proportional limit load;
- P_{bl} = proportional limit load on bolts;
- P_{cl} = proportional limit load on connectors;
- P_{wa} = allowable load on wood;

R = ratio between the computed stress at test loads and the usually accepted design capacities (allowable loads):

- R_{ba} = ratio R between maximum test load and allowable load on bolts;
- R_{bl} = ratio R between proportional limit load and allowable load on bolts;
- R_{ca} = ratio R between maximum test load and allowable load on connectors;

---

[1] "Timber Connector Construction," Manual of the Timber Eng. Co., Inc., Washington, D. C., 1939.

R_{cl} = ratio R between proportional limit load and allowable load on connectors;

R_{wa} = ratio R between maximum test load and allowable load on wood;

R_{wl} = ratio R between proportional limit load and allowable load on wood;

R_l = ratio between computed stress at the proportional limit loads and the usually accepted design capacities (allowable loads);

r = least radius of gyration;

s = stress per unit area; s_{wa} = allowable unit stress in wood;

W = total applied load on truss;

w = applied load on truss, in pounds per linear foot;

X = an unknown total force;

Δ = total deformation or displacement; and

θ = angle of twist; also the slope of the grain in relation to the axis of a member.

Explanation of Terms and Abbreviations.—

1. Allowable Stresses in Timber.—The allowable load W on a structural member of either one or more pieces, when determined by the strength of the timber, using the recommended unit working stresses s of the Forest Products Laboratory is termed "wood allowable load," and is abbreviated P_{wa}. The unit working stresses for selected structural Douglas fir (the grade used in practically all cases) are:

Description	Stress, s_{wa} (lb per sq in.)
Bending or tension parallel to the grain....	1,600
Longitudinal Shear Stress:	
In beams..............................	100
In joints..........................	150
Compressive Stress:	
With the grain......................	1,200
Across the grain....................	350

On surfaces inclined at an angle θ with the direction of the grain, the allowable working stress s_{wa} in compression is expressed by the Hankinson formula as given subsequently in Eq. 1.

2. Allowable Stresses on Bolts.—The allowable load on a structural member of either one or more pieces when determined by the safe load on the bolts in a bolted connection, as computed in accordance with the recommendations of the Forest Products Laboratory,[2] is hereinafter termed "bolt allowable load" and is abbreviated P_b.

3. Allowable Loads on Connectors.—The allowable load on a structural member, of either one or more pieces, determined by the recommended standard design loads given by TECO,[1] is hereinafter termed "connector allowable load," and is abbreviated P_c.

---

[2] "Wood Handbook," Forest Products Laboratory, Forest Service, U.S.D.A., Madison, Wis., 1940.

4. Compression Diagonals.—When used in the description of a joint test, the designations "compression diagonal" and "tension diagonal" refer to the nature of stress in the diagonal during the test, and not necessarily to stress function in the structure from which it was taken.

5. Proportional Limit.—The load at the proportional limit, as judged by the load-deformation curves, is termed hereinafter the proportional limit load and is abbreviated P_l.

6. Maximum Stress Ratio.—The ratio of the maximum stress in a member or connection, to a calculated allowable load is hereinafter termed R_{wa}, R_{ba}, and R_{ca}, respectively.

7. Proportional Limit Ratio.—In a manner similar to item 6, the ratios of the proportional limit load to the allowable loads are R_{wl}, R_{bl}, and R_{cl}, respectively.

8. Malleable Iron Washers.—The designation MIW means "malleable iron washer"; for $\frac{7}{8}$-in. bolts, $3\frac{1}{2}$ in. in diameter and $\frac{7}{16}$-in. thick; and, for $\frac{3}{4}$-in. bolts, 3 in. in diameter and $\frac{7}{16}$-in. thick.

9. Identification of Structures.—The Exposition buildings from which the trusses tested were taken, and their abbreviations when applied to trusses, are identified in this report as follows:

Building	Abbreviation
Alameda-Contra Costa Counties Building....	AC
Brazil Building..........................	B
Cavalcade...............................	C
Chinese Building.........................	Ch
French Building..........................	F
Merry-Go-Round...........................	M
North-Square Court.......................	NS
San Francisco Building...................	SF

10. Connectors.—Split rings or shear plates, used for connecting timbers in various ways, are termed "connectors."

11. Member.—A structural element that extends between two panel points of a truss or structural frame and acts primarily in resistance to direct or axial tension or compression is a "member." The term includes those structural elements of test specimens which had so functioned in the structural frames from which they were taken. A member consisting of one piece or one "stick" of solid cross section—as a member composed of one 6-in. by 10-in. section—is variously termed: "One-piece," "one stick," or "single member." Similarly, a member consisting of two separate pieces or two separate "sticks"—as a member composed of two 3-in. by 10-in. pieces—is variously termed: "Two-piece," "two stick," or "double" member.

12. Single Shear.—A member has its end connection, or connections, in single shear when its connectors are in one face of the member only, or when bolts or nails attach the member to a connecting member, splice pad, or gusset plate by only one face.

13. Double Shear.—A member has its end connection, or connections, in double shear when its connectors are in two opposite faces of the member and

are equally divided in number between each face; or when bolts or nails attach the member to a connecting member, splice pads, or gusset plates by two opposite faces.

14. Loose Bolts.—Examination of the bolting of the specimens of this test program led to the conclusion that all bolts had been inserted in slightly oversized holes. Nevertheless, the bolts were found to be tight in their assemblies—that is, they could not be turned by hand. Shrinkage of the wood, from its green condition at the time of framing to its seasoned condition at the time of testing, had caused the sides of each wood member, through which a bolt passed, to draw away from the washers, or from the gusset plate, or plates, to which the member was connected. This, then, is the meaning of such phrases and terms as "the bolts were loose," and "loose bolts."

15. Wood Handbook.—Wood Handbook refers to the "Wood Handbook"[2] prepared by the Forest Products Laboratory of the United States Department of Agriculture.

D. DETAILED REPORT OF TESTS

I. Truss Tests

Alameda-Contra Costa Counties Building (AC) *Trusses.*—For purposes of identification the truss numbers of this building are prefixed by the letters TAC.

These tests were of four trusses from the AC Building, with compression diagonals and tension rods and, except for minor variations in depth of daps and cuts, of identical construction. They were 39 ft 4½ in., center to center of span, about 6 ft deep, and had 6-in. by 8-in. top and bottom chords substantially parallel. Fig. 2 shows the detail. Except for truss TAC–3 the trusses were in good condition. This truss, when tested, was found to have been injured either during erecting or dismantling.

Workmanship appeared to have been good; daps and cuts were well fitted. Bolts at the splices were generally tight in their holes, although spaces between truss members and splice pad indicated that shrinkage of from about ⅛ in. to ¼ in. had occurred. Some of the washers were loose, but the bolts themselves could not be moved by hand. Timbers consisted of Douglas fir of select structural grade, and the trusses were tested with an average moisture content of 12%. The end butt blocks at joints L1 and L7 were in good condition, and no checks were observed in the horizontal plane of theoretical maximum shear—for example, on plane A1-A2, Fig. 3. A sound knot was found in this plane in truss TAC–4.

Testing.—As usual, trusses were tested in a horizontal position, the loads being applied as concentrations at the upper panel point as shown in Fig. 2. Deformations, particularly those of the end joints, were measured by gages. Referring to Fig. 4, a typical installation at one end joint: Gage A measures the horizontal slip between block and chord; the stick between points X and Y is fastened to butt blocks by using blocks at the ends, so that gage B gives a measure of the bending deformations of the block as a beam between points X and Y as well as of the butt blocks; gage C measures movements of the projected end of the diagonal perpendicular to its axis—the movement being measured with respect to the bottom chord at the working point; gage D measures deformations parallel to the diagonal, between points D and Z as the stick between these two points is fastened to the diagonal only, thus projecting movements to gages C and D; gages E, F, and G measure deformations

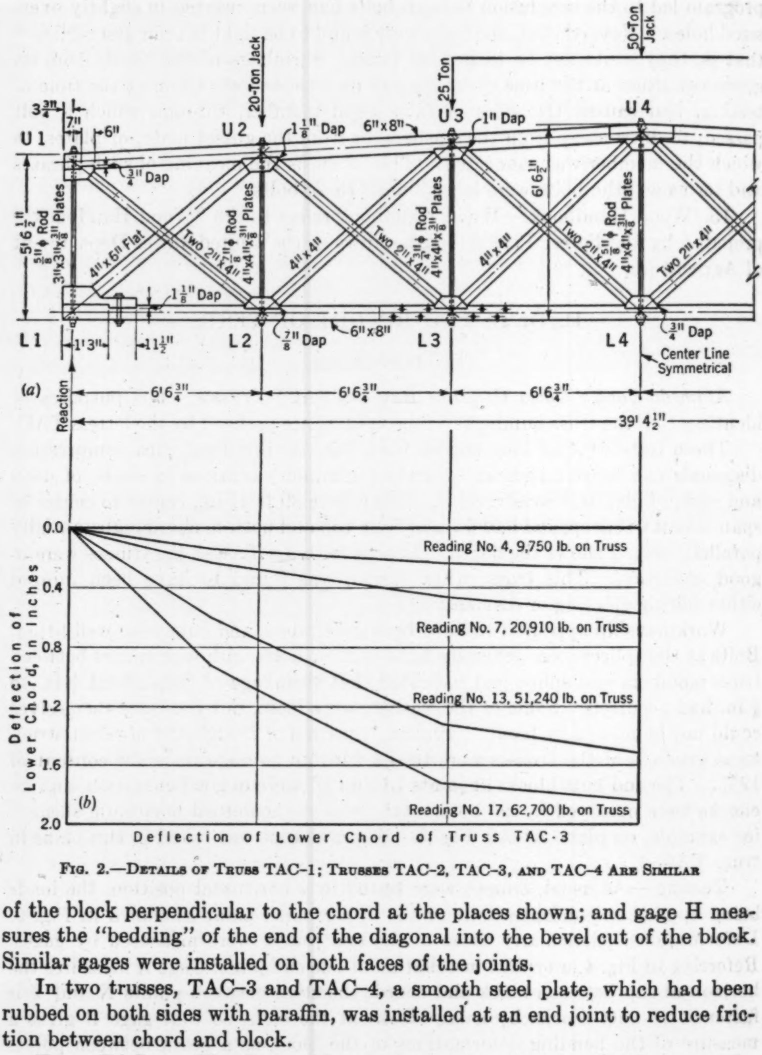

50-Ton Jack
20-Ton Jack
25 Ton
U 1
U 2
U 3
U 4
$3\frac{3}{4}''$
7''
6''
$\frac{3}{4}''$ Dap
$1\frac{1}{8}''$ Dap
6''x8''
1'' Dap
$5'6\frac{1}{2}''$
$\frac{5}{8}''$ ϕ Rod
$3''\times3''\times\frac{3}{8}''$ Plates
4''x6'' Flat
Two $2''\times4''$
$\frac{7}{8}''$ ϕ Rod
$4''\times4''\times\frac{3}{8}''$ Plates
4''x4''
Two $2''\times4''$
$\frac{3}{4}''$ ϕ Rod
$4''\times4''\times\frac{3}{8}''$ Plates
4''x4''
$6'6\frac{1}{2}''$
Two $2''\times4''$
$\frac{5}{8}''$ ϕ Rod
$4''\times4''\times\frac{3}{8}''$ Plates
Two 2''x4''
$1\frac{1}{8}''$ Dap
L 1
L 2
L 3
L 4
1'3''
$11\frac{1}{2}''$
$\frac{7}{8}''$ Dap
6''x8''
$\frac{3}{4}''$ Dap
Center Line Symmetrical
(a)
Reaction
$6'6\frac{3}{4}''$
$6'6\frac{3}{4}''$
$6'6\frac{3}{4}''$
$39'4\frac{1}{2}''$
0.0
0.4
0.8
1.2
1.6
2.0
Deflection of Lower Chord, in Inches
Reading No. 4, 9,750 lb. on Truss
Reading No. 7, 20,910 lb. on Truss
Reading No. 13, 51,240 lb. on Truss
Reading No. 17, 62,700 lb. on Truss
(b)
Deflection of Lower Chord of Truss TAC-3

FIG. 2.—DETAILS OF TRUSS TAC–1; TRUSSES TAC–2, TAC–3, AND TAC–4 ARE SIMILAR

of the block perpendicular to the chord at the places shown; and gage H measures the "bedding" of the end of the diagonal into the bevel cut of the block. Similar gages were installed on both faces of the joints.

In two trusses, TAC–3 and TAC–4, a smooth steel plate, which had been rubbed on both sides with paraffin, was installed at an end joint to reduce friction between chord and block.

Truss TAC–1.—The first evidence of overstress in truss TAC–1 was observed at joints L7 and U2 at the butt blocks, where the end diagonals were bearing on their beveled cuts. At joint U2, the block showed small splits and about ½-in. embedment of the diagonal. The load in member L_1U_2 at this time was about 45 kips, giving a bearing stress on the butt block of about 2,000 lb per sq in., although this indentation of ½ in. did not cause the load on the truss to decrease. Final failure occurred very suddenly in shearing off the table of the butt block at joint L7, as shown in Fig. 5. The shearing plane was A1-A2, Fig. 3. The load in member U6–L7 was 49 kips, and the calculated shearing stress was 474 lb per sq in., neglecting friction.

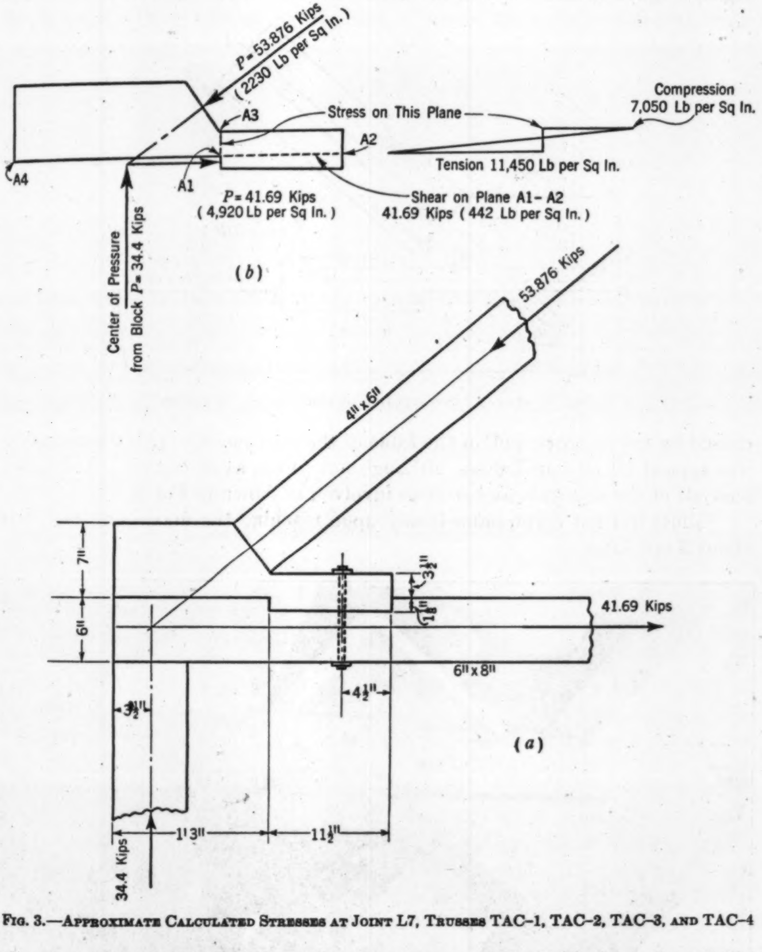

P = 53.876 Kips
(2230 Lb per Sq In.)
A3
Stress on This Plane
Compression
7,050 Lb per Sq In.
A2
Tension 11,450 Lb per Sq In.
A4
A1
P = 41.69 Kips
(4,920 Lb per Sq In.)
Shear on Plane A1 - A2
41.69 Kips (442 Lb per Sq In.)
Center of Pressure
from Block P = 34.4 Kips
(b)
53.876 Kips
4" x 6"
7"
6"
3½"
1⅛"
41.69 Kips
6" x 8"
4½"
3½"
(a)
1' 3"
11½"
34.4 Kips

FIG. 3.—APPROXIMATE CALCULATED STRESSES AT JOINT L7, TRUSSES TAC-1, TAC-2, TAC-3, AND TAC-4

Truss TAC-2.—Testing on this truss was begun in the afternoon and continued until evening when loads were released, and was resumed the following morning, when failure occurred, as in truss TAC–1, at joint L7 in shearing off the table of the end butt block. Before failure, as shown by the arrow in Fig. 6, a space opened between the inner end of the butt block and the chord, caused by the eccentric pull of the table of the butt block. This phenomenon was typical for all four trusses, although not to the same degree. A detailed analysis of the moments and stresses involved is shown in Fig. 3.

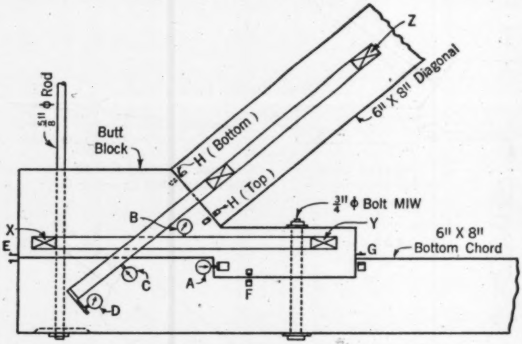

Z
6" X 8" Diagonal
5/8" φ Rod
Butt Block
H (Bottom)
H (Top)
3/4" φ Bolt MIW
Y
6" X 8" Bottom Chord
G
X
E
B
C
A
F
D

FIG. 4.—TYPICAL INSTALLATION OF GAGES AT AN END JOINT

Failure did not occur immediately upon reaching the maximum load, but about 2 min later.

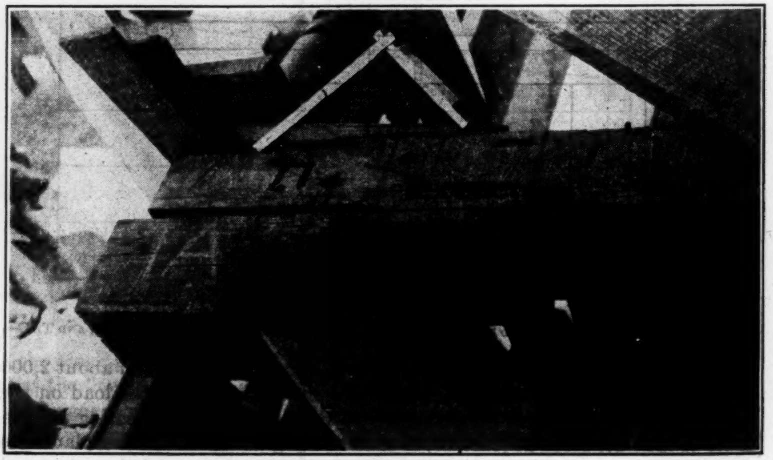


FIG. 5.—FAILURE OF BUTT BLOCK AT AN END JOINT

Truss TAC–3.—As previously noted, a paraffined steel plate was installed at the end joint of panel point L7 in truss TAC–3. Loading was started on a

Saturday; and, at a total load of 49.7 kips on the truss, the lower chord failed in tension between two panel points. The load was released and examination indicated the chord had suffered a previous injury. The chord was repaired by splicing. On the following Monday afternoon testing was resumed, when failure occurred at joint L7 after the maximum load had been applied for about 6 min. The failure was similar to those described for trusses TAC–1 and TAC–2.

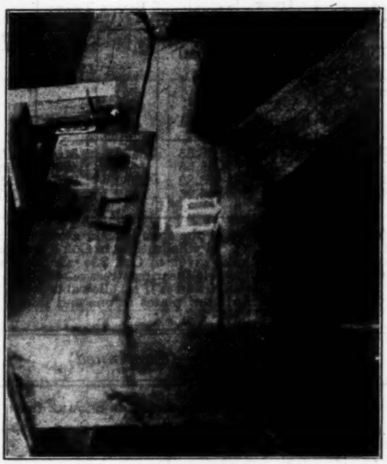


FIG. 6.—BENDING OF BUTT BLOCK PREVIOUS TO FAILURE IN SHEAR

Truss TAC–4.—A paraffined plate was also installed in the end joint of panel point L1 in truss TAC–4. Failure occurred at this joint about 5 min after the maximum load had been applied in a manner similar to that previously described.

Results of Tests, Trusses TAC.—Results of these tests are listed in Table 1. Fig. 2(*b*) includes the deflection curves for the lower chord of one of the trusses, for four different loadings. Fig. 7 shows center deflection of the upper and lower chords of truss TAC–1. Fig. 8 shows the opening of the lower chord splices at L3 and L5; Fig. 9 shows the embedment of the end diagonals in the bevel cuts of blocks at U6; Fig. 10 shows the deformation at gage D; Fig. 11 shows the slip of butt blocks with respect to the chord at gage A. Six curves are presented in Fig. 12, showing deformations at gages on butt blocks of joints L1 and L7 on truss TAC–3, the location of the gages being shown in Fig. 4.

Computations based on a unit shear stress of 150 lb per sq in. in the end butt block indicate that the trusses were designed for an allowable load of about 660 lb per lin ft. Trusses TAC–1 and TAC–2 failed at R_{wo} = 3.15 and 2.95, respectively. The two trusses on which paraffined steel plates were installed failed at R_{wo} = 2.55 and 2.0.

TABLE 1.—SUMMARY OF TEST RESULTS, STRESSES IN THE TRUSSES OF THE ALAMEDA-CONTRA COSTA BUILDING (AC)

Truss	CENTER DEFLECTION			Truss failed at panel point (Fig. 2)	Maximum load on truss[c] (kips)	SHEARING STRESS s (LB PER SQ IN.) ON BLOCK THAT:		Ratio R_{wa} ([d])	TOTAL TIME ELAPSED		Deflection of end blocks at allowable load[e]
	Under load of 600 lb per lin ft[a]	At Loads of Nearly:[b]				Failed	Did not fail		Hr	Min	
		Load (kips)	Deflection								
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
TAC-1..	1/790	62.70	1/255	L7	68.55	474	510	3.15	3	50	1/1,030
TAC-2..	1/665	60.73	1/257	L7	68.55	443	430	2.95	2	35[f]	1/980
TAC-3[g]..	1/774	45.25	1/378	L7[h]	58.82	382	385	2.55	2	55[f]	1/7,200
TAC-4[g]..	1/755	37.85	1/468	L7[h]	45.25	297	326	2.00	2	10	1/4,600

[a] Center deflections (expressed in fractions of the span length) at a computed allowable truss load of 600 lb per lin ft, measured on lower chords. [b] Center deflections in Col. 3 (expressed in fractions of the span length) at ultimate loads near (not at) the loads given in Col. 2. [c] Concentrated loads applied at panel points U3 to U6, inclusive. [d] Based on an allowable shearing stress s_{wa} of 150 lb per sq in. [e] Deflections of end blocks (expressed in fractions of the span X and Y), at allowable loads, under secondary bending. [f] On two days. [g] These trusses had paraffined steel plates in one end joint. [h] Failed at end joint with paraffined steel plates.

TABLE 1.—(*Continued*)

Truss	DEFLECTION OF END BLOCKS UNDER SECONDARY BENDING[i]		STRESSES, IN POUNDS PER SQUARE INCH						AVERAGE ULTIMATE SHEARS IN END BLOCKS THAT FAILED	
			Computed Maximum in the Block							
			Direct Tension[j]		Bending		Total[k]			
	Load (kips)	Deflection	At s_{wa}	At ultimate load	At s_{wa}	At ultimate load	At s_{wa}	At ultimate load	Lb per sq in.	R_{wa}
	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)
TAC-1..	32.04	1/670	750	2,340	3,260	10,200	4,010	12,540	457[l]	3.1
TAC-2..	46.26	1/555	740	2,320	3,280	10,300	4,020	12,260	....	....
TAC-3..	45.25	1/1,220	700	1,890	3,310	8,910	4,010	10,800	340[h, l]	2.25
TAC-4..	37.85	1/930	740	1,550	3,590	7,500	4,330	9,050	....	....

[i] Expressed as fractions of the span between points X and Y, Fig. 4. Deflections in Col. 13 are the highest readings when the load on the truss was as given in Col. 12. [j] Parallel to the grain. [k] Computed maximum tension plus bending stress in the block at the plane A_1–A_2. [l] The difference between these two stresses (117 lb per sq in.) corresponds to a coefficient of friction of 0.315.

Loads at failure of the tables of the butt blocks (at plane A1-A2, Fig. 3) produced calculated shearing stresses of 297 lb per sq in., 382 lb per sq in., 443 lb per sq in., and 474 lb per sq in. The first two values of s_s are for the two trusses in which paraffined steel plates were used, the average shearing stress being 340 lb per sq in. Where plates were not installed, the average ultimate shearing stress was 458 lb per sq in. This increased strength, amounting to 35%, may be explained by the frictional resistance on plane A1-A4 in trusses TAC-1 and TAC-2, relieving the shearing stresses on plane A1-A2.

Failures in the blocks at these relatively low unit stresses were probably due to the abrupt change in section at point A1, resulting in a combination of high bending and shearing stresses occurring simultaneously in the vertical plane at point A1.

Fig. 3 presents an analysis of the stresses in the end butt block of truss TAC–2 at the time of failure. In the analyses it has been assumed that the bolt holding down the block was loose and therefore did not act in tension to reduce the bending moments in the vertical plane at point A1. Actually, the washers under the heads of these bolts were so loose that they did not become tight even in the tests. The bolts could not be moved by hand, however, so that some slight tension may have been present through friction on the surfaces of the bolts. If any such tension did exist, it would have resulted in somewhat reduced bending stresses in the butt block.

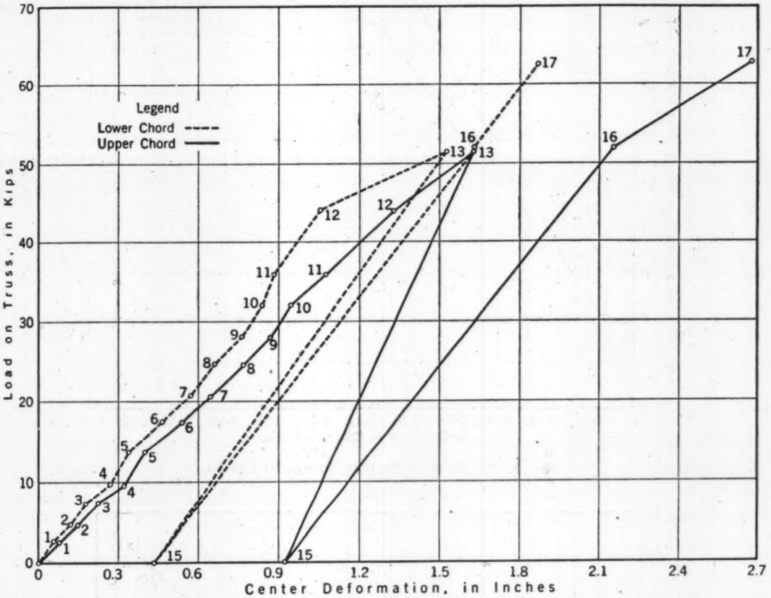


Fig. 7.—Center Deflection, Truss TAC–1

The tests of the two trusses in which the end joints were modified by the introduction of paraffined steel plates seem to indicate that friction plays an important part in the resistance of this type of end joint. However, the tests are not sufficient in number to justify reliance on friction in timber design.

The tests also confirm what good judgment would have dictated—in the end butt blocks, top and bottom cuts should not be made in the same vertical plane. They also seem to show that abrupt changes in cross section of wood members reduce the strength more than is accounted for by present design practice and that, when such abrupt changes cannot be avoided, lower unit working stresses should be used. Further tests on joints in which a combination of shearing stress, direct stress, and bending stress is involved are much needed for clarification.

The effect of shrinkage causing increased stresses in timber joints is illustrated in these tests where the bolts in butt blocks became ineffective in tension, thus introducing bending and causing local flexural stresses.

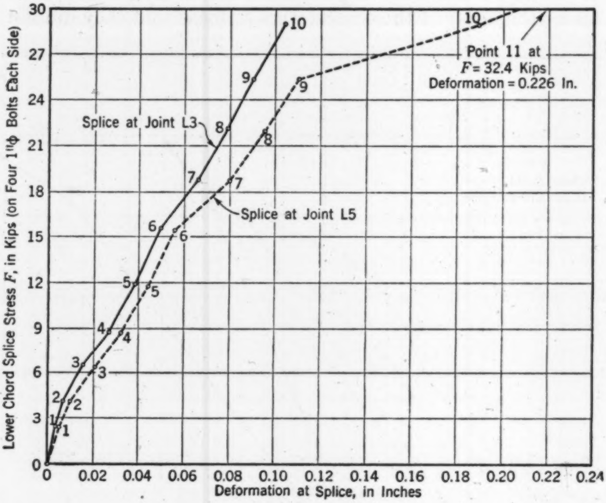


FIG. 8.—OPENING OF SPLICES, JOINTS L3 AND L5

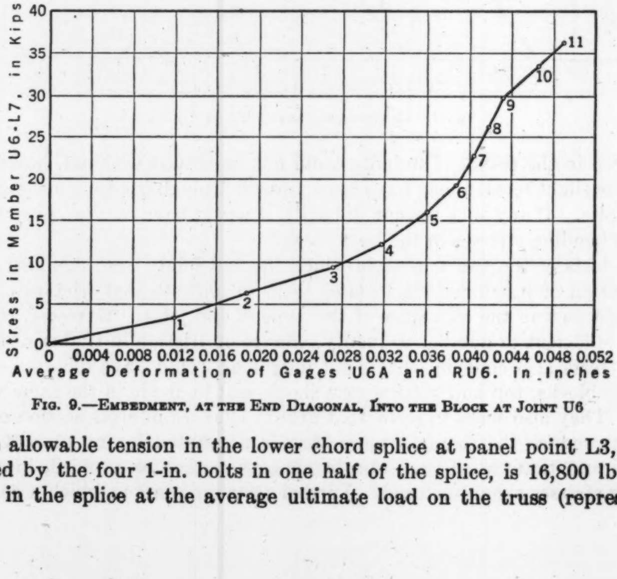


FIG. 9.—EMBEDMENT, AT THE END DIAGONAL, INTO THE BLOCK AT JOINT U6

The allowable tension in the lower chord splice at panel point L3, as determined by the four 1-in. bolts in one half of the splice, is 16,800 lb. The tension in the splice at the average ultimate load on the truss (representing failure of the end butt blocks in shear and not failure of the splice) was 60,000 lb. In the splice bolts, therefore, R_{ba} would have a value of 3.6; and, since the splice evidenced no weakness, the potential value of R_{wa} or R_{ba} would be greater. In trusses TAC–1 and TAC–2, which sustained the maximum ultimate loads (68,500 lb), the value of R_{ba} would be 4.1.

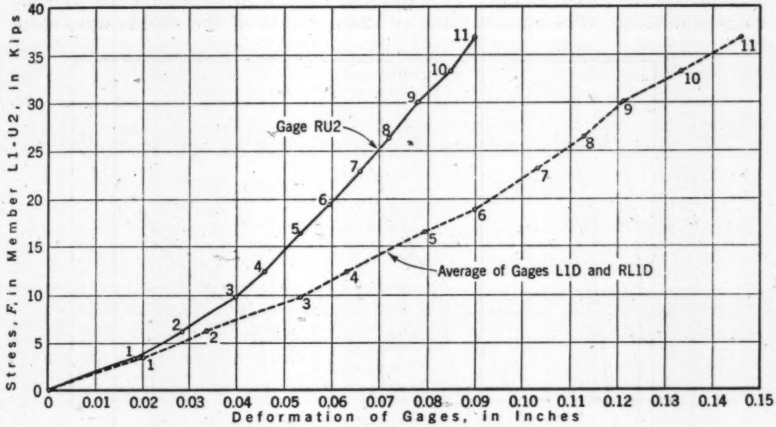


FIG. 10.—TOTAL AXIAL DEFORMATION OF DIAGONAL WITH RESPECT TO THE CHORDS AT PANEL POINTS L1 AND U2

This series of tests furnishes another illustration of the need for tests of full-sized timber trusses and structures since only in such way can combinations of different kinds of stresses show their true effect.

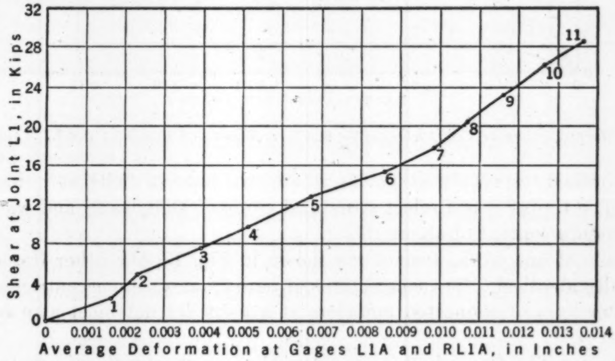


FIG. 11.—SLIP OF BUTT BLOCKS WITH RESPECT TO THE CHORD AT GAGE A

Brazil Building (B) *Trusses.*—Two shallow trusses (perhaps, more accurately, two trussed frames) about 46.5 ft long and averaging 2 ft 10 in. deep were obtained from the Brazil Building and tested. Originally they formed the vertical legs of a rigid-frame bent. Unfortunately, the building had been dismantled carelessly; and, although the two frames, when carefully examined, appeared to be in good condition, the possibility exists that they had been damaged. No evidence of such damage, however, was disclosed by the tests.

The chords of the frames were formed of three 12-in. planks each, with the web members of two 3-in. by 6-in. timbers, connected to the chords by split rings and bolts. The outside 2-in. by 12-in. planks of the chords were quite badly checked, especially at points where two lines of bolts and rings were used. The timber was a select structural grade of Douglas fir and was tested at a moisture content of about 15%.

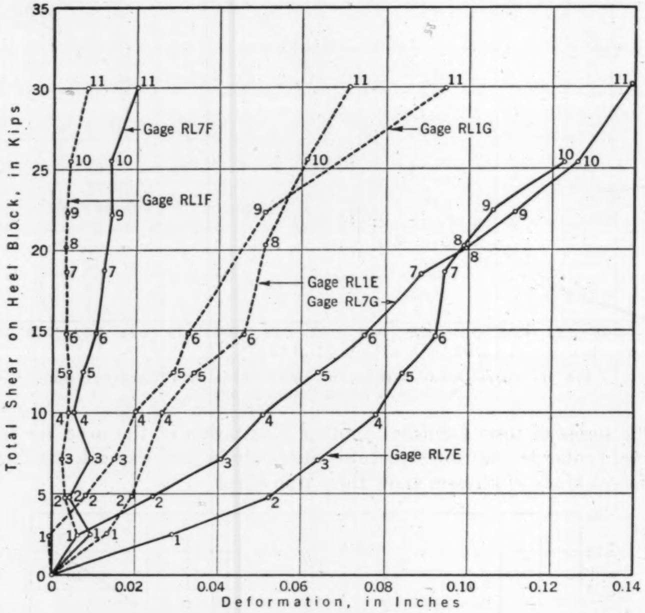


FIG. 12.—DEFORMATIONS AT VARIOUS GAGES ON THE BUTT BLOCKS OF JOINTS L1 AND L7, TRUSS TAC-3

Details of one of the frames are shown in Fig. 13, the other frame being practically identical. The connections of both frames were eccentric—that is, the center lines of connected members at a joint did not meet at a common point.

Testing.—The frame had its loads applied as shown diagrammatically in Fig. 13, in which the points of application of the loads were changed during the test. Deflections and joint deformations were measured as usual; and, in addition, gages were installed on one chord to measure the relative movement of the center plank with respect to the outside plank.

Trouble was experienced in testing both frames because of the large deformations. The loading jacks had a maximum extension of 6 in. so that considerable loading, unloading, and reblocking were necessary to obtain failure of the frames. Thus, testing of frame TB–1 consumed 3 hours, and testing of frame TB–2 required 1 hour and 10 min.

In the test frame TB–1 the first failure occurred at joint U1 and almost simultaneously at joint L10, in tension perpendicular to the grain. About the same time, failure occurred in the lower chord splices. Upon dismantling the truss, it was found that the split rings were missing in one each of the vertical web members of joints L7, L8, and L9, although grooves for the rings had been provided. As failure did not occur in these web members, these conditions apparently did not affect the load capacity of the frame.

In frame TB–2, the tension splices of the lower chord started to open at small loads. When a load of 17.2 kips on the frame had been reached, the center 3-in. by 12-in. plank of the lower chord broke near joint L5 (see Fig. 14). The increase of the load to 23.5 kips sheared the wood in the end distances in the lower chord splices as shown in Fig. 13.

Results of these tests are summarized in Table 2 and deflections for frame TB–1 for five loadings are shown in Fig. 13. Fig. 15 indicates the center deflection for the same frame.

TABLE 2.—SUMMARY OF TEST RESULTS; STRESSES ON THE TRUSSED FRAMES OF THE BRAZIL BUILDING (B)

Frame	Deflections					Ultimate load (kips)	Splice load[b] (kips)	R_{ca}	
	At a Load of 10 Kips		At Other Loads, P					1939	Splice[c]
	Inches	Ratio[a]	P	Inches	Ratio[a]				
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
TB–1.....	2.3	1/233	26.8	5.0	1/107	30.1	117.8	2.95	3.5
TB–2.....	2.5	1/214	22.0	5.8	1/ 93	17.2[d]	99.8	2.48	3.0

[a] Ratio of span. [b] Load on a splice at failure, computed by the ordinary truss theory. [c] The ratio of R_{ca} for the connection of members U1–L2 and U9–L10. [d] In frame TB–2 the center stick (3 in. by 12 in.) broke in bending at a load, on the frame, of 17.2 kips. The splices failed completely when the load on the frame was increased to 23.5 kips.

Fig. 16 shows the relative movements of the three parts of the lower chord under load. The location of the gages for these points is shown in Fig. 13.

Results of Tests, Trusses TB.—The action of these frames under load is complicated and analysis of test results, therefore, is difficult. Neglecting the bending effects of the chords and using direct stresses as computed by the usual truss theory, the values of R_{ca} calculated for the splices are 2.95 and 2.48. If the bending moments in the two chords, due to deformation of the truss as a whole, are considered, these ratios would be changed.

The importance of stresses due to deflection in shallow trusses, frames, or girders is illustrated in the failure of frame TB–2 in which the through, 3-in. by 12-in. stick of the chord member broke in bending, mainly as the result of

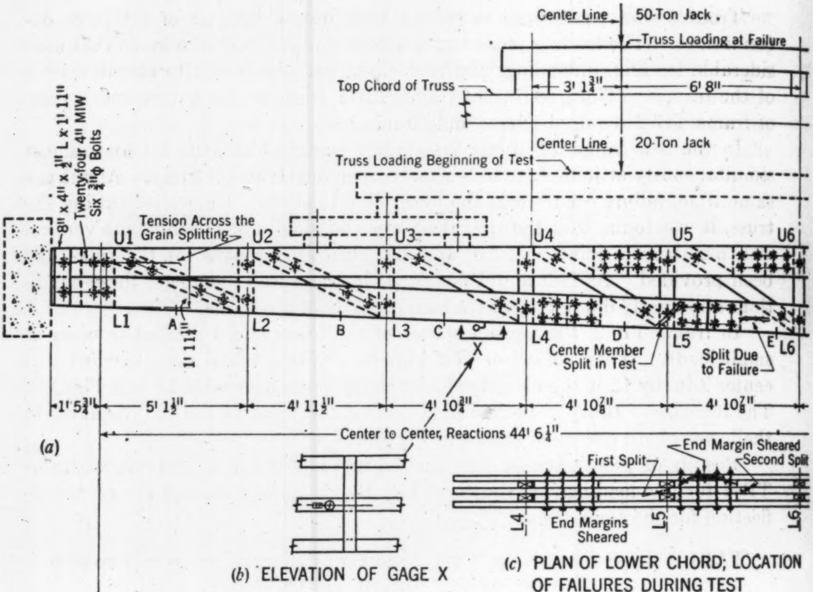
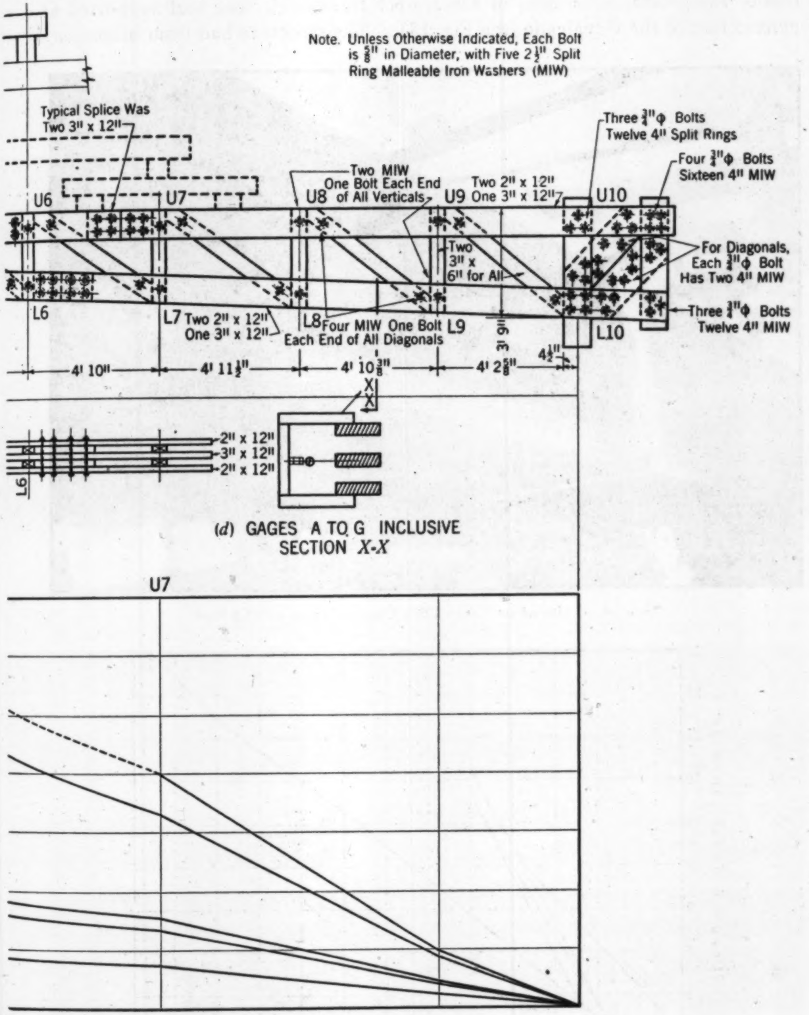

Center Line
50-Ton Jack
Truss Loading at Failure
Top Chord of Truss
3' 1¾"
6' 8"
Center Line
20-Ton Jack
Truss Loading Beginning of Test
8" x 4" x ½" L x 1' 11"
Twenty-four 4" MIW
Six ¾"φ Bolts
Tension Across the Grain Splitting
U1
U2
U3
U4
U5
U6
L1
A
1' 11¼"
L2
B
L3
C
X
L4
D
L5
E
L6
Center Member Split in Test
Split Due to Failure
1' 5¾"
5' 1½"
4' 11½"
4' 10¾"
4' 10⅞"
4' 10⅞"
Center to Center, Reactions 44' 6¼"
(a)
(b) ELEVATION OF GAGE X
First Split
End Margin Sheared
Second Split
L4
L5
L6
End Margins Sheared
(c) PLAN OF LOWER CHORD; LOCATION OF FAILURES DURING TEST
U4
U5
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
Deflection Upper Chord, in Inches
(e) DEFLECTIONS OF TRUSS TB-1
Reading No. 17 Load on Truss 24,000 Lb
Reading No. 14 Load on Truss 26,800 Lb
Reading No. 5 Load on Truss 18,820 Lb
Reading No. 4 Load on Truss 14,940 Lb
Reading No. 2 Load on Truss 7,400 Lb

Fig. 13.—Details of Trus

deformations in the adjacent splices of the 2-in. by 12-in. sticks, and of deformations of the frame as a whole.

At the two ends of frame TB–1, failures occurred simultaneously in tension perpendicular to the grain of the chord members. Failure in tension perpendicular to the grain at eccentrically connected joints was illustrated in this


Note. Unless Otherwise Indicated, Each Bolt is ⅝" in Diameter, with Five 2½" Split Ring Malleable Iron Washers (MIW)
Typical Splice Was Two 3" x 12"
Two MIW One Bolt Each End of All Verticals
Two 2" x 12" One 3" x 12"
Three ¾"φ Bolts Twelve 4" Split Rings
Four ¾"φ Bolts Sixteen 4" MIW
For Diagonals, Each ¾"φ Bolt Has Two 4" MIW
Three ¾"φ Bolts Twelve 4" MIW
Two 3" x 6" for All
U6
U7
U8
U9
U10
L6
L7
L8
L9
L10
Two 2" x 12" One 3" x 12"
Four MIW One Bolt Each End of All Diagonals
3' 9"
4½"
4' 10"
4' 11½"
4' 10⅜"
4' 2⅝"
X
X
2" x 12"
3" x 12"
2" x 12"
L6
(d) GAGES A TO G INCLUSIVE
SECTION X-X
U7

TB-1; TB-2 IS SIMILAR

test as well as in other tests of this program, in which the members of a joint were connected eccentrically. The nature of the failures emphasizes the importance of the consideration of secondary stresses in main members, splices, and connections in shallow trusses which may be expected to show relatively large deflections and deformations at working loads.

Cavalcade (C) *Cantilever Truss.*—Opportunity was afforded to test, in place, one of two cantilever trusses of the timber framework that had supported a canvas roof of the Cavalcade (see Fig. 17). The structure had been in service, exposed to the elements, for about 2 years prior to testing, and appeared to be in unusually good condition. The framing lumber was Douglas fir, appraised as "select structural" grade.

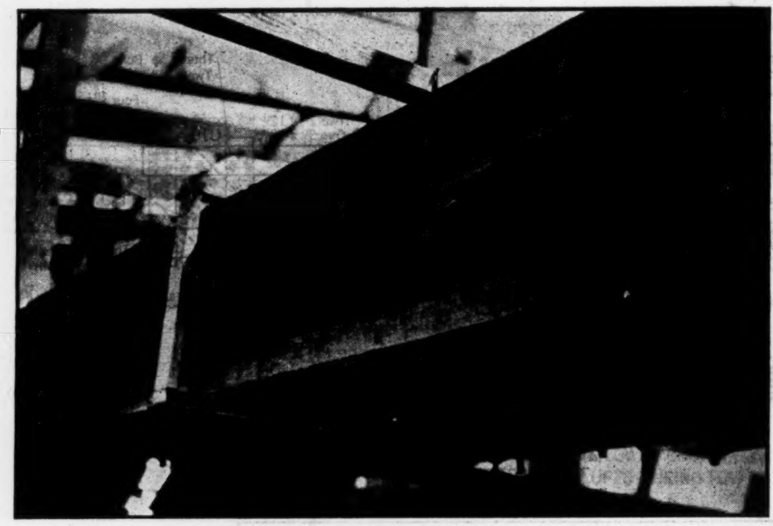


FIG. 14.—FAILURE OF TRUSS TB-2 UNDER A LOAD OF 17.2 KIPS

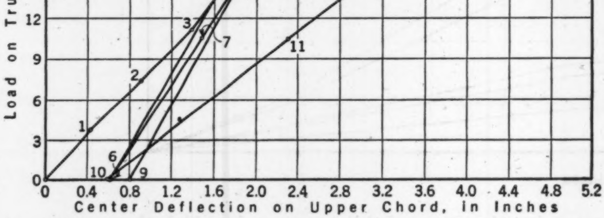


FIG. 15.—DEFLECTION OF THE CENTER OF THE UPPER CHORD, FRAME TB-1

Important joints in the framing were bolted or joined by connectors, the latter involving flush-type shear plates (Fig. 18) with steel gusset plates and split rings. Unless otherwise indicated, all bolts in Fig. 18 were $\frac{3}{4}$ in. in diameter. All bolts, including those at split-ring connectors were provided with 3-in. by 3-in. by $\frac{3}{16}$-in. washers. No bolts were tightened or

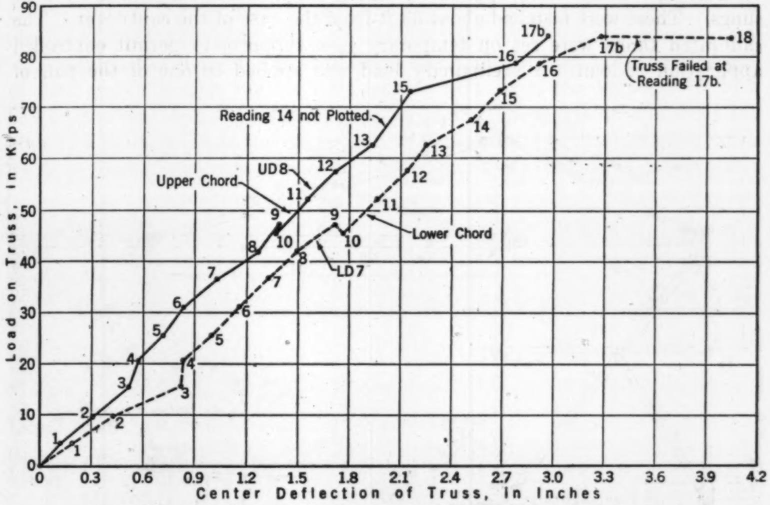


Fig. 16.—Relative Movements, Three Parts of the Lower Chord, Frame TB-1 (for Location of Gages, See Fig. 13)

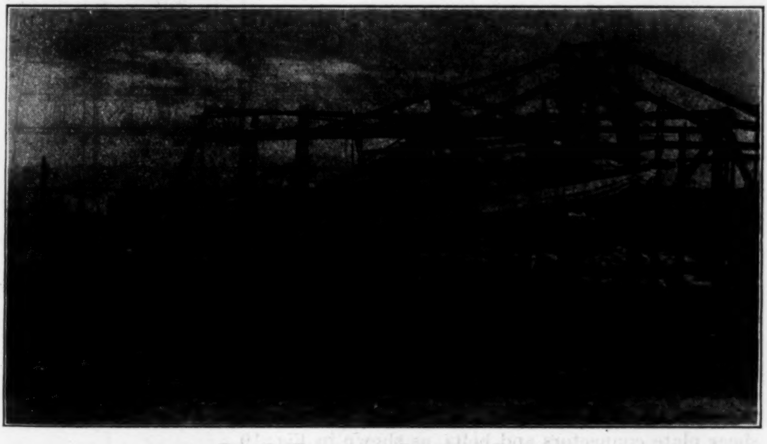
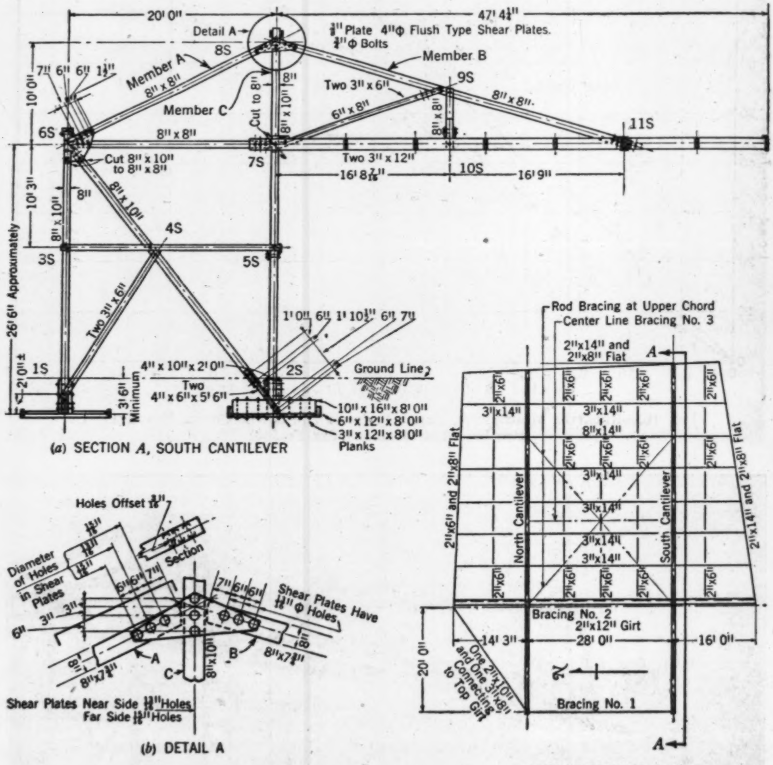


Fig. 17.—Failure of a Cantilever Frame

frames repaired prior to the application of test load. The gusset plate at joint 6S, Fig. 18, shows the typical minimum dimensions for such plates.

Testing.—Loads were applied by filling water tanks suspended by rope slings. These were fastened about 33 ft from the base of the cantilever. The calibrated tanks were set on temporary jack supports to permit controlled application of load. A preliminary load was applied to one of the pair of cantilever frames, to check the workability and safety of the loading apparatus, by slowly releasing the supporting jacks. Shortly after load transfer occurred, the anchor arm of the crown joint (8S in Fig. 18) failed by the shearing out of shear plate connectors and bolts, as shown in Fig. 19.


20' 0"
47' 4½"
Detail A
¾" Plate 4"Φ Flush Type Shear Plates.
¾" Φ Bolts
8S
Member A
8" x 8"
Member B
Member C
Cut to 8"
8" x 10"
Two 3" x 6"
6" x 8"
9S
8" x 8"
8" x 8"
11S
6S
7S
Two 3" x 12"
16' 8 7/16"
10S
16' 9"
Cut 8" x 10" to 8" x 8"
8" x 10"
10' 0"
10' 3"
26' 6" Approximately
2' 0" ±
3' 6" Minimum
4S
3S
5S
1S
2S
Two 3" x 6"
1' 0" 6" 1' 10½" 6" 7"
4" x 10" x 2' 0"
Two 4" x 6" x 5' 6"
Ground Line
10" x 16" x 8' 0"
6" x 12" x 8' 0"
3" x 12" x 8' 0" Planks
Holes Offset 3/16"
Diameter of Holes in Shear Plates
Section
Shear Plates Have 13/16" Φ Holes
Shear Plates Near Side 13/16" Holes
Far Side 13/16" Holes
A
B
C
8" x 7½"
8" x 10"
Rod Bracing at Upper Chord
Center Line Bracing No. 3
2" x 14" and 2" x 8" Flat
2" x 6"
3" x 14"
8" x 14"
North Cantilever
South Cantilever
2" x 6" and 2" x 8" Flat
2" x 14" and 2" x 8" Flat
Bracing No. 2
2" x 12" Girt
14' 3"
28' 0"
16' 0"
20' 0"
One 2" x 10" and One 3" x 8" Connecting to Top Girt
N
Bracing No. 1

(a) SECTION A, SOUTH CANTILEVER

(b) DETAIL A

(c) ROOF FRAMING PLAN

Fig. 18.—Details of Cantilever Frame TC-1

The calculated load in the failed member was 64,800 lb although, as shown on later examination of the joint, a lesser load may have caused the progressive failure that occurred. The deflection observed at the end of cantilever at time of failure was about 8 in.

Results of Tests, Trusses TC.–Joint 8S (Fig. 18) in which failure occurred consisted of three 8-in. by 8-in. timber members connected by $\frac{1}{4}$-in. steel plates drilled with $\frac{13}{16}$-in.-diameter holes and three pairs of shear plates, 4 in. in diameter, fitted with $\frac{3}{4}$-in. bolts.

Although the character of the failure was typical—that is, a shearing out of wood and pulling through of bolts causing splitting of the members—failure occurred at less than the anticipated total load. An examination of the joint after test indicated that the center pair of three pairs of shear plates, 4 in. in diameter, had holes $\frac{15}{16}$ in. in diameter. The other two pairs had $\frac{13}{16}$-in.-diameter holes, although all were fitted with $\frac{3}{4}$-in. bolts. The bolt holes in member A (which were bored halfway through the timber from each side) were found to be offset between opposite faces of the timber about $\frac{1}{16}$ in. Excessive loading of connectors and bolts was evident in member A by the $\frac{1}{16}$-in. indentations in the bolts at the contact points between shear plates and steel gusset plates, and by the distortion of bolt holes in connectors and gusset plates, which indicated that the ultimate shearing capacity of each bolt had been approached. These observations indicated that the connectors were loaded progressively, probably not more than two connectors being loaded at one time.

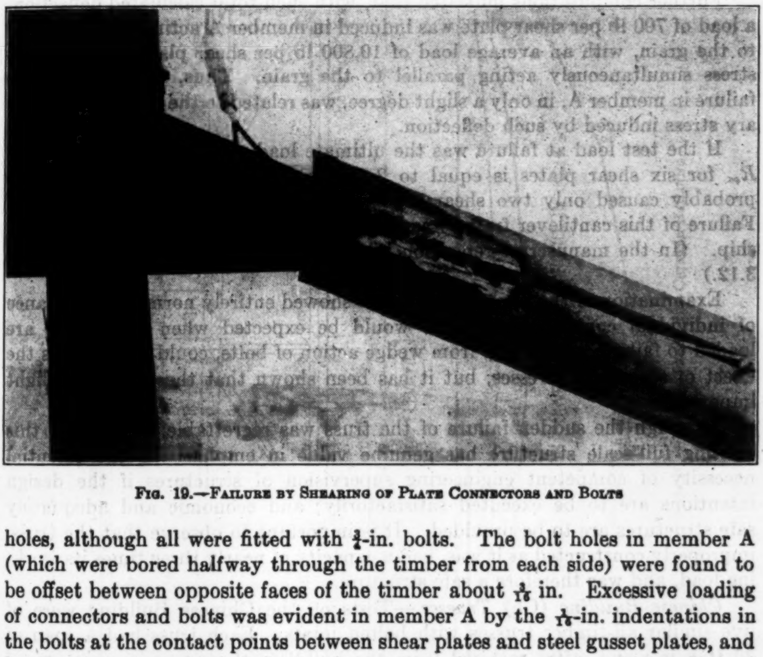


FIG. 19.—FAILURE BY SHEARING OF PLATE CONNECTORS AND BOLTS

Although the fabrication of members B and C (Fig. 18) was found to be essentially correct, the three shear plates on one side of compression member C were found to have $\frac{15}{16}$-in.-diameter holes instead of the required $\frac{13}{16}$-in.-diameter holes for $\frac{3}{4}$-in. bolts. No evidence of distress was found in tension member B or in compression member C.

On the basis of the foregoing estimated failure load, the computed tension in member A was 64,800 lb, or a load of 10,800 lb per shear plate; and an approximate deflection at the end of the cantilever of about 8 in. was observed at time of failure. The possible influence of secondary stresses due to cantilever deflection, using the foregoing load, was investigated. The computed deflections were:

Cause	Deflection (in.)
Elongation of members	3.17
Slip of joint fasteners	3.31
Total	6.48

Further computations indicated that, with such total calculated deflection, a load of 700 lb per shear plate was induced in member A acting perpendicular to the grain, with an average load of 10,800 lb per shear plate due to direct stress simultaneously acting parallel to the grain. Thus, it appeared that failure in member A, in only a slight degree, was related to the action of secondary stress induced by such deflection.

If the test load at failure was the ultimate load of the truss, the value of R_{ca} for six shear plates is equal to 2.81, although the faulty construction probably caused only two shear plates to act effectively at any one time. Failure of this cantilever frame was obviously the result of careless workmanship. (In the manuscript report on joint tests, R_{ca} for joint J11 is given as 3.12.)

Examination of the joint after the test showed entirely normal performance of individual connectors, such as would be expected when connectors are loaded to failure. Splitting, from wedge action of bolts, could be taken as the effect of secondary stresses, but it has been shown that these were of slight importance.

Although the sudden failure of the truss was regrettable, the test of this existing full-scale structure has genuine value in emphasizing the essential necessity of competent engineering supervision of structures if the design intentions are to be executed satisfactorily; and economic and adequately safe structures are to be provided. It is important to observe that the truss, improperly constructed as it was, had a capacity of nearly three times its working load, and was therefore a safe structure.

Chinese Building (Ch) *Trusses.*—Tests on the Chinese Building were of five similar all-timber trusses with bolted joints. Each truss had a span of 36 ft 8 in.; and, after removal from the building, these trusses were stored outside for about 4 months before testing. When tested, their moisture content was about 12%.

Fig. 20 shows a detail of one of the trusses. All truss members were single sticks, of 2-in. dimension stock, No. 1 common Douglas fir. The compression diagonals were framed on one side of the chords and the tension verticals on the opposite side. This detail was later found to be an important factor in the behavior of the trusses under test loads. Single bolts, with cut-steel washers under heads and nuts, were used in all joints except the four end joints. Bolts were found to be reasonably tight at the time of testing; they were not tightened. Although there were a number of knots, none seemingly reduced

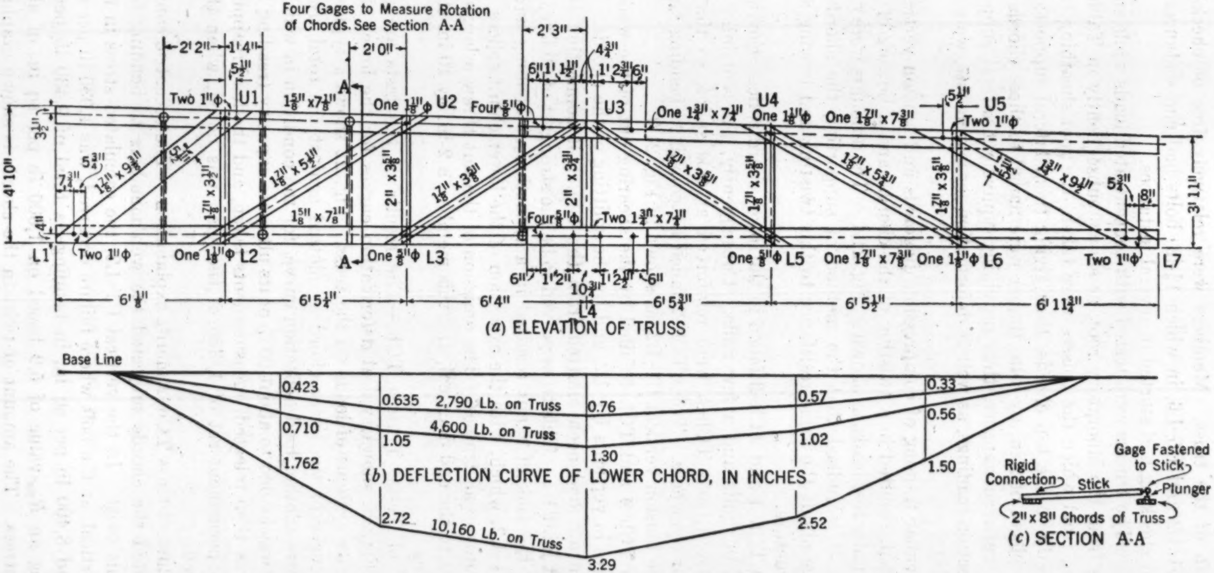

Four Gages to Measure Rotation of Chords. See Section A-A
2'2"
1'4"
5½"
2'0"
2'3"
4¾"
6" 1' 1½"
1' 2¾" 6"
5½"
Two 1"φ
U1
1⅝" x 7⅞"
A
One 1⅛"φ
U2
Four ⅝"φ
U3
One 1¾" x 7¼"
U4
One 1⅛"φ
One 1⅞" x 7⅜"
U5
Two 1"φ
4'10"
3½"
5¾"
7¾"
4"
1⅞" x 9¾"
1⅞" x 3½"
1⅞" x 5½"
1⅝" x 7⅞"
2" x 3⅝"
1⅞" x 3⅝"
2" x 3¾"
Four ⅝"φ
Two 1¾" x 7¼"
1⅞" x 3⅝"
1⅞" x 3⅝"
1⅞" x 5¾"
1⅞" x 3⅝"
1⅞" x 9½"
5¾"
8"
3'11"
L1
Two 1"φ
One 1⅛"φ
L2
A
One ⅝"φ
L3
6" 1'2"
10¾"
1'2½" 6"
L4
One ⅝"φ
L5
One 1⅞" x 7⅜"
One 1⅛"φ
L6
Two 1"φ
L7
6'⅛"
6'5¼"
6'4"
6'5¾"
6'5½"
6'11¾"
(a) ELEVATION OF TRUSS
Base Line
0.423
0.710
1.762
0.635
1.05
2.72
2,790 Lb. on Truss
4,600 Lb. on Truss
10,160 Lb. on Truss
0.76
1.30
3.29
0.57
1.02
2.52
0.33
0.56
1.50
(b) DEFLECTION CURVE OF LOWER CHORD, IN INCHES
Rigid Connection
Stick
Gage Fastened to Stick
Plunger
2" x 8" Chords of Truss
(c) SECTION A-A

FIG. 20.—TYPICAL TRUSS, THE CHINESE BUILDING

the strength of the truss. Members were relatively free of checks except verticals U1–L2 and U5–L6, in which 1⅛-in. bolts had end distances of only 3¾ in.—and checking had started in several instances.

Testing.—The trusses were tested with concentrated loads while supported by hangers from the laboratory roof, as shown subsequently in Table 9. In the building from which the trusses were taken, the roof sheathing had been nailed directly to the top chords, thus giving them lateral support. In the tests short pieces of 2-in. by 8-in. timber were nailed to these chords to form T-sections between hangers, thus providing equivalent lateral support. As usual, deflection readings, as well as deformation measurements, were made at various points.

The peculiar framing of compression diagonals and tension verticals, previously noted, resulted in a twisting of the chords, and a bowing of the web members, under test loads, as shown in Fig. 21*A*. The amounts were measured by gages. It is judged that the method of supporting the chords during testing gave about the same resistance to this twisting and bowing as in the original structure.

Trusses TCh–1 and TCh–3 failed in the lower chord splices, each of which had four ⅝-in. bolts and a few nails. Consequently, the remaining trusses, TCh–2, TCh–4, and TCh–5 were reinforced at this splice, so that failure would occur elsewhere. Truss TCh–2 failed by combined bending and direct stress in the tension vertical U1–L2, as shown in Fig. 21*B*.

Trusses TCh–4 and TCh–5 failed by the shearing out of the wood in the end distances in verticals U1–L2, and by the splitting of the chords. In truss TCh–5, the nut had been omitted from the 1-in. bolt connecting the three members at joint U1. This truss proved to be the strongest of all five, possibly because of the lack of a nut resulted in a reduction of bending moment in member U1–L2, which, with the exception of the lower chord splices, was the weakest factor in all trusses. The omission of the nut threw a larger part of the moment to the end diagonal, and this member, a 2-in. by 10-in. stick, was amply strong.

Results of Tests, Trusses TCh.—The results of the tests are given in Table 3. Fig. 20 shows typical deformation curves for the lower chord for three separate systems of loads on the truss. Fig. 22 shows a typical center deflection curve of the lower chord as plotted against the total truss load. Fig. 23 shows another load deflection curve, for the condition in which one of the trusses was loaded to about 90% of its ultimate capacity (reading No. 11); the load was then released, measurements taken, and the load reapplied. In the test, a permanent set of 1/330 of the span was noted, when the loads were released.

The failure of truss TCh–2 merits explanation. The eccentric connections which twisted the chords subjected the web members to bending moments resisting this twist. In the vertical U1–L2, the calculated stress in the 2-in. by 4-in. vertical, at the bolt where failure occurred, was 1,030 lb per sq in. in tension, and 8,400 lb per sq in. in bending—a total of 9,430 lb per sq in., representing an R_{ws}-value of 5.9 based on 1,600 lb per sq in. of allowable combined stress. The amount of twist in the chords is shown graphically in

Fig. 24. As mentioned previously, this vertical U1–L2 had very short end distances for the size of its connection. In all the trusses, splits occurred in these end distances at loads below the ultimate capacity of the truss; nevertheless, the splits did not directly cause failure of the trusses. If the allowable load in the verticals is taken as the sum of allowable loads on the bolts (2,200

FIG. 21*A*.—TWISTED CHORD AND BOWED WEB MEMBER, UNDER TEST LOADS

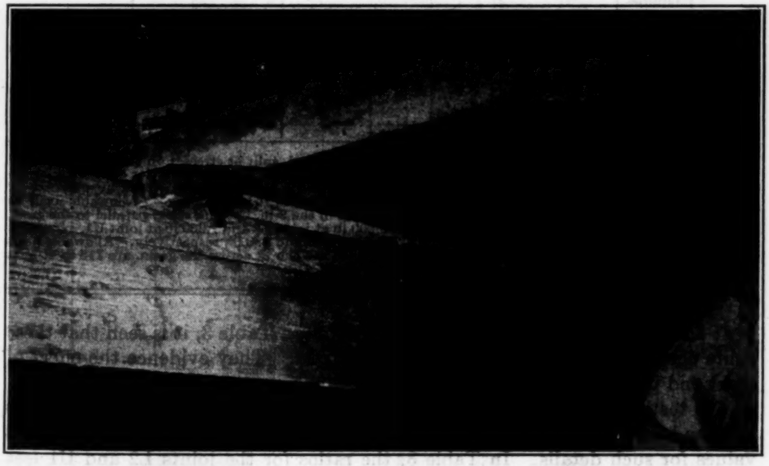


FIG. 21*B*.—TENSION AND BENDING FAILURE OF A VERTICAL MEMBER

TABLE 3.—SUMMARY OF TEST RESULTS; STRESSES IN THE TRUSSES OF THE CHINESE BUILDING (Ch)

Truss	CENTER DEFLECTIONS (FRACTIONS OF SPAN)			Location of truss failure	TOTAL MAXIMUM LOAD ON TRUSS		
	At $P=1{,}850$ lb on truss	At Other Loads			(Pounds)	(Lb per lin ft of truss)	(Lb per sq ft of roof)
		P (lb)	Deflections				
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
TCh-1...	1/690	7,390	1/323	Lower chord splice	8,980	350	175
TCh-2...	1/870[a]	10,160	1/141	U1–L2[b]	11,380	443	222
TCh-3...	1/685[a]	9,070	1/119	Lower chord splice	9,970	387	194
TCh-4...	1/783	7,450	1/169	Joint U1 in U1–L2[c]	9,420	367	184
TCh-5...	1/885	8,910	1/149	Joint U1 in U1–L2[c]	11,680	455	228

[a] These are approximately the deflections caused by design loads, which correspond with 72 lb per lin ft of truss, or 36 lb per sq ft of roof. [b] Broke in combined tension and bending at point L2. [c] Shear in the end distance of the 1-in. bolts at joint U1 in member U1–L2; then the chords split from the twisting action.

TABLE 3.—(*Continued*)

Truss	TOTAL MAXIMUM STRESS (LB)		R_{bs} of splice[f]	R_{ws} at U1–L2[h]	RATIOS R_{bs} BASED ON BOLT VALUES[j]			TOTAL ELAPSED TIME	
	On U1–L2 at failure[d]	Lower chord splice			At Point of Failure;[k] with End Distance		Design values	Hr	Min
					Neglected	Included			
	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
TCh-1..	3,628	14,319	5.3	....	....	....	....	4	20
TCh-2..	4,594	(Reinforced)	....[g]	5.9[i]	2.07	4.14	6.4	2	30
TCh-3..	4,025	15,896	5.7	....	....	....	....	3	0
TCh-4..	3,803	(Reinforced)	....[g]	....	2.0	3.0	5.9	1	50
TCh-5..	4,791[e]	(Reinforced)	....[g]	....	2.4	4.9	7.3	2	15

[d] Ends of members U1–L2 and U5–L6, that were not checked, split at stresses, in the members, equal to from $F=2{,}100$ lb to $F=3{,}600$ lb. [e] Nut omitted at U1. [f] Using a value of 90 lb for each assisting 16-penny nail. [g] Did not fail in splice because it was reinforced. [h] At the point where member U1–L2 broke in bending. [i] Based on an allowable load of 1,600 lb per sq in. in bending. [j] Allowable loads on member U1–L2 as governed by bolts, based on their values in the 2-in. by 4-in. stick, parallel to the grain, and without reducing for the short end distance, are 2,200 lb at joint L2 and 1,960 lb at joint U1. To allow for short end distances these values are reduced 50% to 1,100 lb and 980 lb, respectively. Allowable loads on U1–L2 as governed by the bolts, based on their values in the chord, across the grain, are 715 lb at joint L2 and 653 lb at joint U1. [k] In member U1–L2 not in chord.

lb at L2 and 1,960 lb at U1) as given in footnote[j], Table 3, it is seen that these splits occurred early in the progress of the tests. They evidence the effect of short end distances for bolts.

No tests appear to have been made on the complicated action of the bolts as found in these joints; hence, the question arises as to the allowable bolt values for such details. In Table 3, the ratios for the joints L2 and U1 (see Col. 14, Table 3) are based on the perpendicular to the grain value of bolts as given in Col. 9. Table 3 neglects the action of the diagonal on the far side. From this point of view, the ratios would seem ample. On the other hand, the failures in trusses TCh-2, TCh-4, and TCh-5 were in this vertical itself

(U1–L2) where the controlling value would be the parallel to grain value of the bolts, without reducing for short end distance (see footnote[f], Table 3).

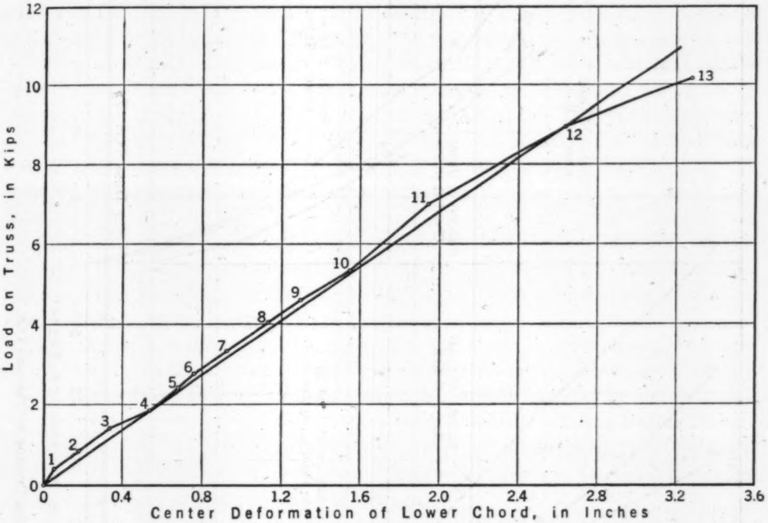


FIG. 22.—LOAD-DEFLECTION CURVE FOR TRUSSES TCH

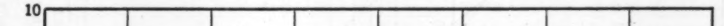


FIG. 23.—LOAD-DEFLECTION CURVE FOR TRUSS TCH-3

Based on these values (a procedure which seems more nearly correct), the ratios R are lower.

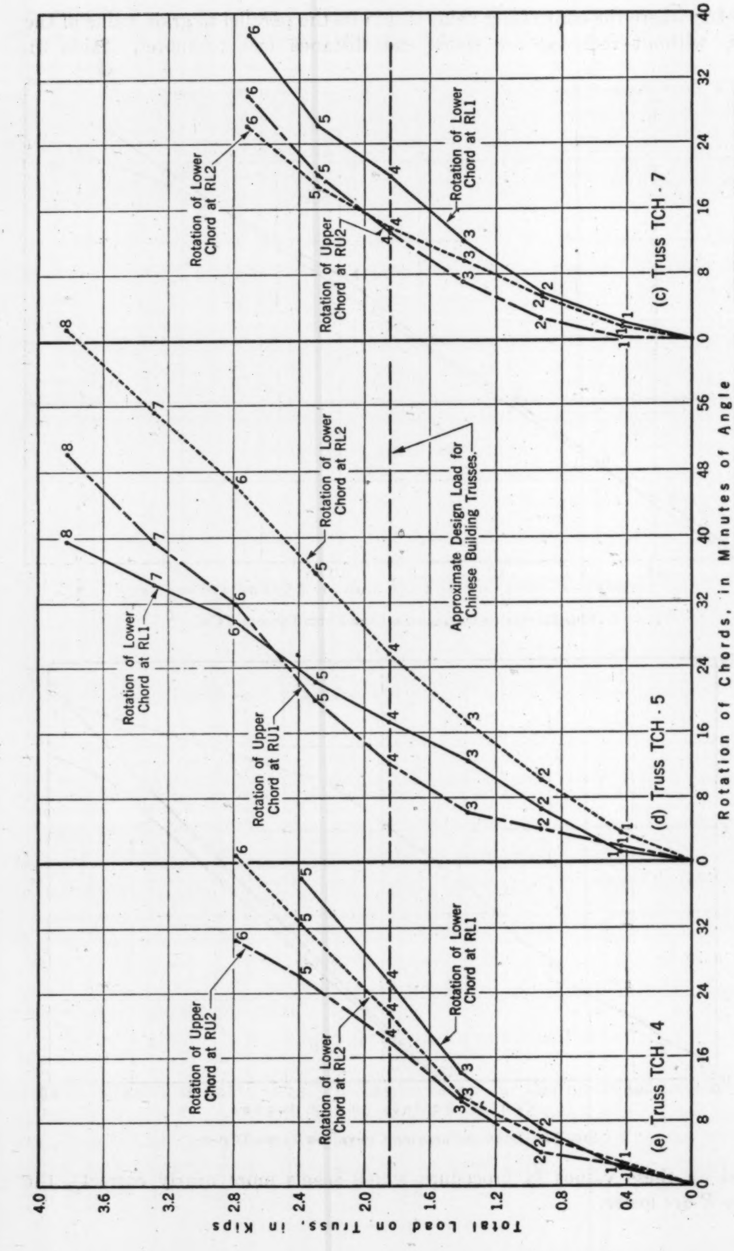


FIG. 24.—MEASURED ROTATION OF THE CHORDS OF TRUSSES TCH

In trusses TCh-4 and TCh-5, the chords developed long splits at failure—probably another result of the twisting of the bolts. These trusses were adequate for the functions they served, although the manner of failure was unexpected and not in accordance with results that might have been expected from usual or conventional investigatory computations. A change in the proportions of members would have affected the distribution and magnitude of stresses. Increasing the thickness of chords, for example, would have increased the bending moments in verticals and diagonals and would also have affected the connections. An analysis of additional stresses in truss members and their connections caused by sizes and arrangement of members would seem to be justified when such factors influence the carrying capacity of the truss.

Apparently, the use of cut washers did not decrease the strength of the trusses. When bolts are subjected to shear only, or when the washers become loose through shrinkage of the wood, the size of the washers appears to have little influence upon the strength of the connection.

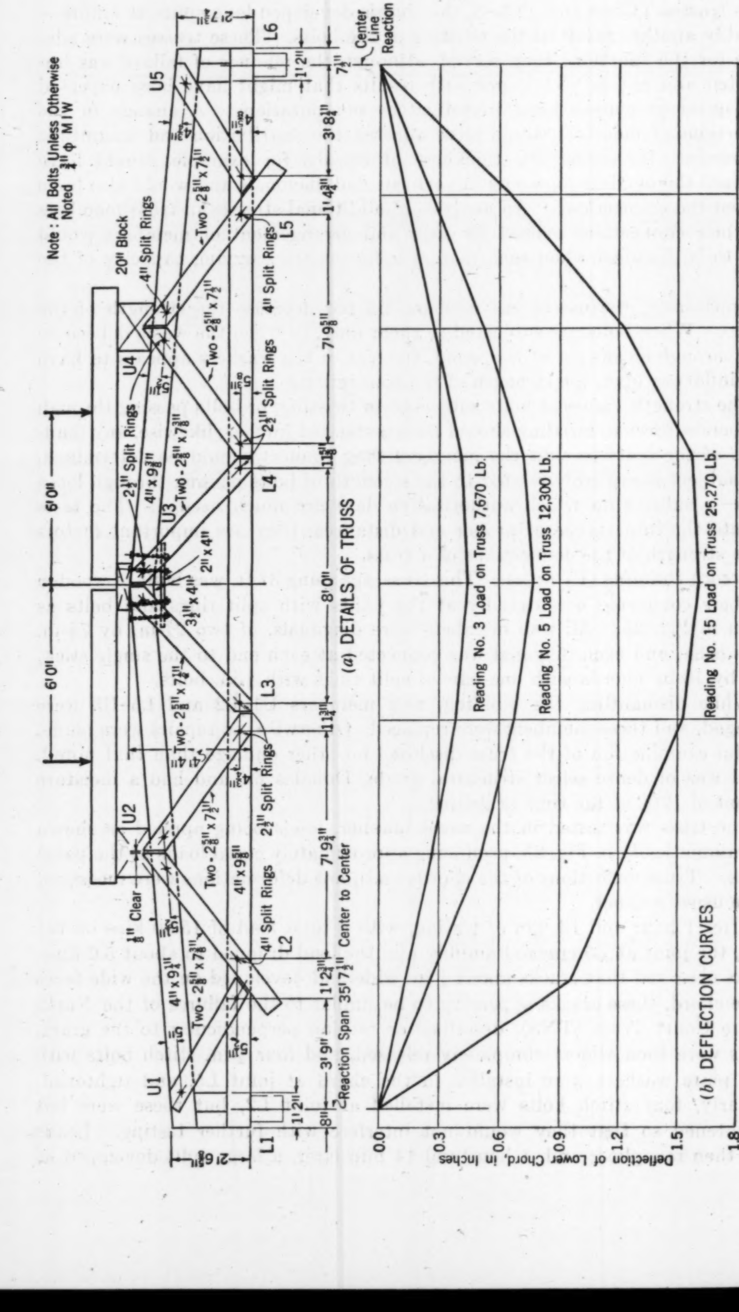
The strength values of bolts subjected to twisting, or bolts passing through members subject to twisting should be investigated further; likewise, any tendencies of the bolts to split the members they connect should be determined. This subject is probably related to the strength of bolts passing through loose fillers—a subject on which authoritative data are much needed. The tests indicate the importance of proper end distances; they are important factors in the strength of the connections of a truss.

French Building (F) *Truss.*—This truss, spanning 37 ft, was built of wooden members connected eccentrically at the joints with split rings and bolts as shown in Fig. 25. All web members were diagonals, of two 2⅝-in. by 7½-in. sticks each, and each diagonal was connected at each end to the single stick, 4-in. by 10-in. chords with one pair of split rings with ¾-in. bolts.

While dismantling the building, web members U1–L2 and L5–U5 were damaged, and these members were replaced. Otherwise, no repairs were made. Careful examination of the truss disclosed no other damage than that noted. Wood was of dense select structural grade, Douglas fir, and had a moisture content of 12% at the time of testing.

The truss was tested in the usual manner, loads being applied as shown diagrammatically in Fig. 25, producing approximately equal loads at the panel points. Truss deflections of the chords and joint deformations were measured in the usual manner.

After 1 hour and 13 min of testing, with a total load of 15.00 kips on the truss, the joint at L5 cracked audibly and the load dropped to about 5.0 kips. It was observed that cracks about ¼ in. wide had developed on the wide faces of the chord, these cracks appearing to be similar to the failures of the North Square Court Truss (TNS), described as tension perpendicular to the grain. Loads were then almost completely released, and four ½-in. stitch bolts with large plate washers were installed in the chord at joint L5, and tightened. Similarly, four stitch bolts were installed at joint L2, but these were left untightened so that they would not interfere with further testing. Loads were then reapplied; and, 1 hour and 14 min later, a large split developed at


(a) DETAILS OF TRUSS

(b) DEFLECTION CURVES

Note: All Bolts Unless Otherwise Noted 3/4" φ M I W
20" Block
4" Split Rings
2 1/2" Split Rings
Two - 2 5/8" x 7 1/2"
Two - 2 5/8" x 7 3/8"
4" x 9 5/8"
4" x 9 1/2"
2" x 4"
3 1/2" x 4"
1" Clear
1/8" Clear
U1
U2
U3
U4
U5
L1
L2
L3
L4
L5
L6
6' 0"
6' 0"
8' 0"
7' 9 3/4"
7' 9 5/8"
11' 2 3/4"
11' 3/4"
11' 3/4"
1' 4 3/4"
3' 9"
3' 8 1/4"
1' 2"
1' 2"
8"
7 1/2"
2' 6 5/8"
2' 7 3/4"
Reaction Span 35' 7 1/2" Center to Center
Center Line Reaction
Deflection of Lower Chord, in Inches
0.0
0.3
0.6
0.9
1.2
1.5
1.8
Reading No. 3 Load on Truss 7,670 Lb.
Reading No. 5 Load on Truss 12,300 Lb.
Reading No. 15 Load on Truss 25,270 Lb.

FIG. 25.—TYPICAL TRUSS, THE FRENCH BUILDING

joint L2 when the truss was loaded with 27.1 kips. This failure, shown in Fig. 26, was apparently also caused from tension perpendicular to the grain. The photograph reveals that the split is widest at the center line of the joint, tapering off to nothing farther from the joint. After this failure the loads were released and the stitch bolts at joint L2 were tightened. Loads were reapplied, when, at a maximum load of 31.2 kips, joint L2 failed again along the original split. In this second failure, because the chord member was held tight together at the joint by stitch bolts, the split grew wider as it went farther from the joint. The widening of these splits was apparently caused by the large bending stresses induced through the eccentric connection which tended to rotate the chord. Immediately after failure, the truss was sustaining a load of 21.7 kips, which was allowed to remain overnight. The following morning this load had dropped to 18.5 kips.

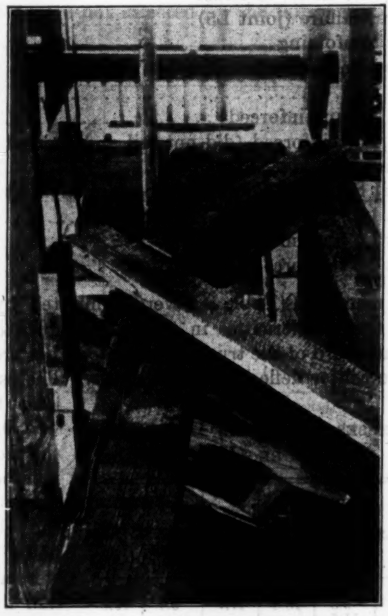


FIG. 26.—FAILURE OF TRUSS TF AT JOINT L2, BEFORE TIGHTENING THE STITCH BOLTS

Results of Tests, Truss TF.—The results of these tests can be summarized as follows:

Item	Description	Quantity
	Deflection of Truss at an Allowable Load of 9.8 Kips (Total Load Based on Dense Select Structural Grade):	
1	In inches	0.60

Item	Description	Quantity
2	Percentage of span	1/720
	Maximum Recorded Deflection (Truss Load, 25.3 Kips):	
3	In inches	1.62
4	Percentage of span	1/270
	Maximum Truss Load, in Kips:	
5	At first failure (joint L5)	15.0
6	After reinforcing	31.2
	Ratio R_{ca} as Limited by the Strength of:	
7	Joint L2	3.2
8	Joint L5 unreinforced	1.5
9	Joint L5 reinforced (did not fail and showed no sign of distress)	3.1
	Average Unit Shear, in Pounds per Square Inch, at Failure in:	
10	Joint L2	450
11	Joint L5 unreinforced	214
12	Joint L5 reinforced (did not fail)	450

Stresses in the members (Table 4) were computed by an approximate method that takes the eccentricities in the joints into account. Deflection curves for the bottom chord of the truss for three loadings are shown in Fig. 25. Fig. 27 indicates the center deflection of the truss. The estimated allowable load W_{ca} on the truss was 9.8 kips, based on the connections of the members that failed.

TABLE 4.—STRESSES FOR AXIAL LOADS IN MEMBERS AT FAILURE (KIPS)

Description	U1–L2	U5–L5		U2–L2	U4–L5	
		Unreinforced	Reinforced[a]		Unreinforced	Reinforced[a]
Stress	36.7	17.1	36.7	28.7	15.0	31.3
Vertical component	18.5	9.0	18.8	17.7	9.4	19.6
Stress, s_{ca}	11.5	11.5	....	....	....	....

[a] The reinforced joints did not fail.

Prior to reinforcing the chords with stitch bolts, the ultimate loads were about one and one-half times the estimated allowable load, giving a ratio R_{ca} of 1.7. After the stitch bolts had been provided, the capacity of the truss was approximately doubled, with $R_{ca} = 3.1$, indicating that the stitch bolts would be of considerable value if they could be kept tight. The nature of the failures of this truss emphasizes again that the stress of tension perpendicular to the grain is likely to be the determining factor in the strength of joints in which the members are eccentrically connected. If such connections are to be used, the probable amount of this stress should be determined so that the limiting capacity of the connections will be known.

Merry-Go-Round Building (M) *Trusses.*—Two all-timber trusses (TM–1 and TM–2) of similar design, spanning 54 ft 1 in. and 51 ft 4½ in., and using the double Warren type of framing, were obtained from the Merry-Go-Round Building and tested to failure. Top and bottom chords were approximately parallel, 6 ft 8 in. apart at the center, and all members were connected concentrically using split rings and ¾-in. bolts. Trusses and columns were connected to form rigid-frame bents for resistance to lateral forces. Details of the trusses are shown in Figs. 28 and 29. In dismantling the building, the trusses were pulled over and fell on the ground, injuring several of their members. Repairs, consisting of the addition of stitch bolts and pads, were made to truss TM-1, as noted in Fig. 28, and to one diagonal and at one reaction point of truss TM-2, as shown in Fig. 29. As a result of the repairing, it is believed that the test results were not affected by the fall of the trusses.

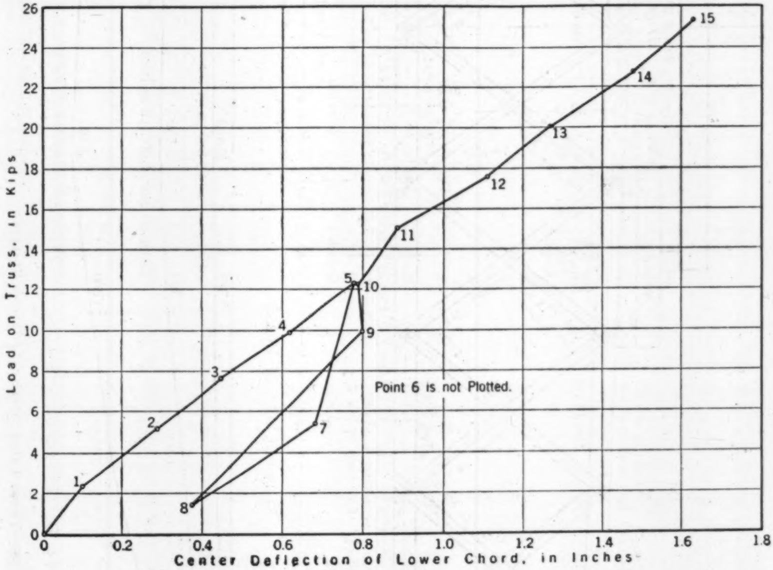


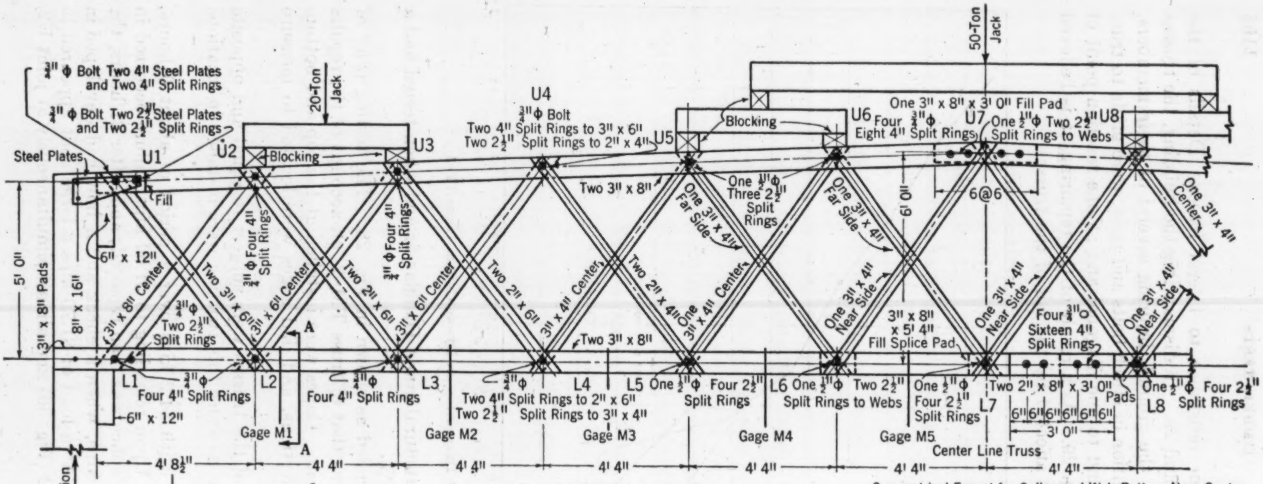
FIG. 27.—CENTER DEFLECTION ON LOWER CHORD, TRUSS TF-3

Timbers were of select structural grade Douglas fir and when tested had a moisture content of about 15%.

Testing.—Loads were applied as shown in Fig. 28. The testing span of TM-1 was 54 ft 1 in., whereas that of truss TM-2, on account of the repairs at the left end, was 51 ft 4½ in. Gages were installed at the top and bottom chords to measure truss deflections, and dial gages were located to measure local joint deformations.

The chords of each half of the trusses were subject to equal and opposite twisting moments. Gages to measure this twisting were installed in locations as shown in Fig. 28.

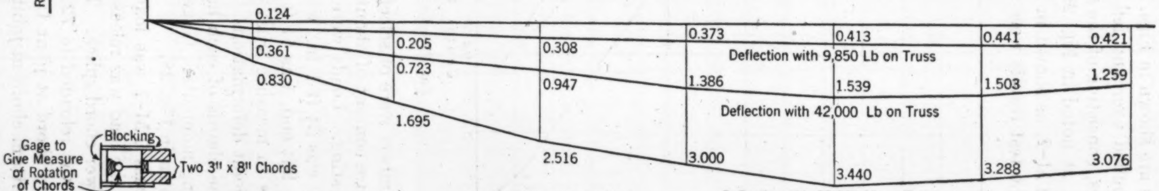
Truss TM-1 was loaded with 54.2 kips over a period of about 6 hours, when the load was released. Upon reloading to 51.60 kips, failure occurred in the lower chord splice. The splice was then repaired; and, after 21 hours, the truss was reloaded to 72.90 kips, when a check and a bow had developed in the top chord at joint U7. At a load of 81.70 kips a large split appeared in the bottom chord at joint L2 (Fig. 30) and almost simultaneously at joint U2.



3/4" ϕ Bolt Two 4" Steel Plates and Two 4" Split Rings
3/4" ϕ Bolt Two 2 1/2" Steel Plates and Two 2 1/2" Split Rings
Steel Plates
Fill
6" x 12"
5' 0"
8" x 16"
3" x 8" Pads
U1
U2
U3
U4
U5
U6
U7
U8
L1
L2
L3
L4
L5
L6
L7
L8
20-Ton Jack
50-Ton Jack
Blocking
3/4" ϕ Bolt
Two 4" Split Rings to 3" x 6"
Two 2 1/2" Split Rings to 2" x 4"
One 3" x 8" x 3' 0" Fill Pad
Four 3/4" ϕ Eight 4" Split Rings
One 1/2" ϕ Two 2 1/2" Split Rings to Webs
3" x 8" Center
Two 3" x 6"
3/4" ϕ Two 2 1/2" Split Rings
3/4" ϕ Four 4" Split Rings
3" x 6" Center
Two 2" x 6"
3" x 4" Center
Two 2" x 4"
One 3" x 4" Center
Two 3" x 8"
One 1/2" ϕ Three 2 1/2" Split Rings
One 3" x 4" Far Side
One 3" x 4" Near Side
6' 0"
6@6
3" x 8" x 5' 4" Fill Splice Pad
Four 3/4" ϕ Sixteen 4" Split Rings
A
A
3/4" ϕ Four 4" Split Rings
Two 4" Split Rings to 2" x 6"
Two 2 1/2" Split Rings to 3" x 4"
One 1/2" ϕ Four 2 1/2" Split Rings
One 1/2" ϕ Two 2 1/2" Split Rings to Webs
Two 2" x 8" x 3' 0" Pads
6" 6" 6" 6" 6" 6"
3' 0"
Gage M1
Gage M2
Gage M3
Gage M4
Gage M5
Center Line Truss
Reaction
4' 8 1/2"
4' 4"

(*a*) DETAILS OF TRUSS

Symmetrical Except for Splice and Web Pattern Near Center,



(*b*) SECTION A-A SHOWING M-GAGES

(*c*) DEFLECTION CURVES

FIG. 28.—TRUSS TM-1, MERRY-GO-ROUND BUILDING

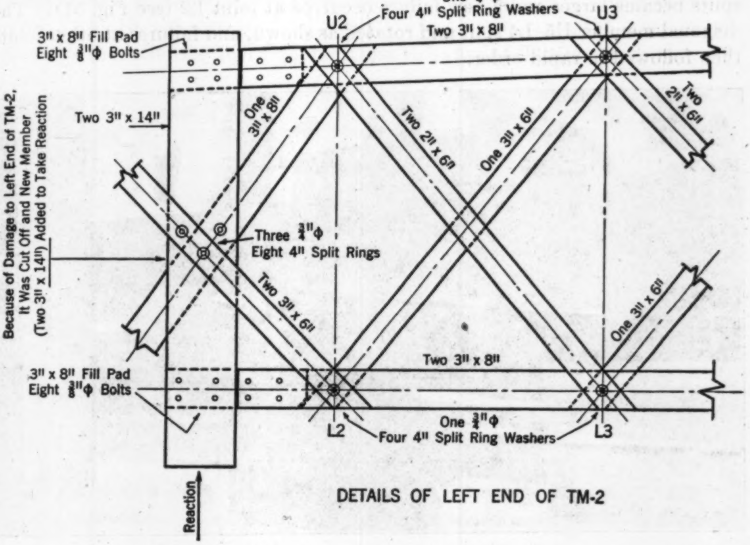

One ¾"φ
Four 4" Split Ring Washers
U2
U3
Two 3" x 8"
3" x 8" Fill Pad
Eight ⅜"φ Bolts
Two 3" x 14"
One 3" x 8"
Two 2" x 6"
One 3" x 6"
Two 2" x 6"
Because of Damage to Left End of TM-2,
It Was Cut Off and New Member
(Two 3" x 14") Added to Take Reaction
Three ¾"φ
Eight 4" Split Rings
Two 3" x 6"
One 3" x 6"
Two 3" x 8"
3" x 8" Fill Pad
Eight ⅜"φ Bolts
One ¾"φ
Four 4" Split Ring Washers
L2
L3
Reaction
DETAILS OF LEFT END OF TM-2

FIG. 29.—DETAILS OF LEFT END OF TRUSS TM-2 (REMAINDER OF THIS TRUSS THE SAME AS TM-1, FIG. 28)

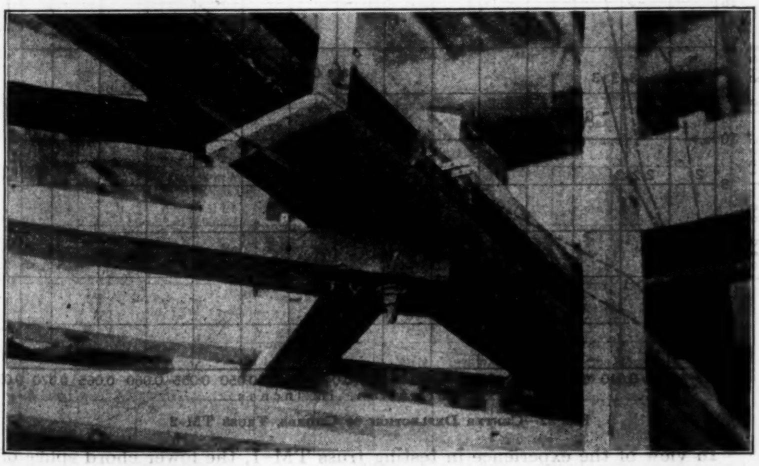


FIG. 30.—FAILURE OF BOTTOM CHORD, TRUSS TM-1, AT JOINT L2

After releasing the load, a total of 83.30 kips was then applied, when these splits became larger and a new failure occurred at joint L2 (see Fig. 31). The diagonal member U5–L4 split and rotated as shown, and failures at other joints then followed in rapid order.

Fig. 31.—Rotating and Splitting of Diagonal Member at the Failure of Truss TM-1

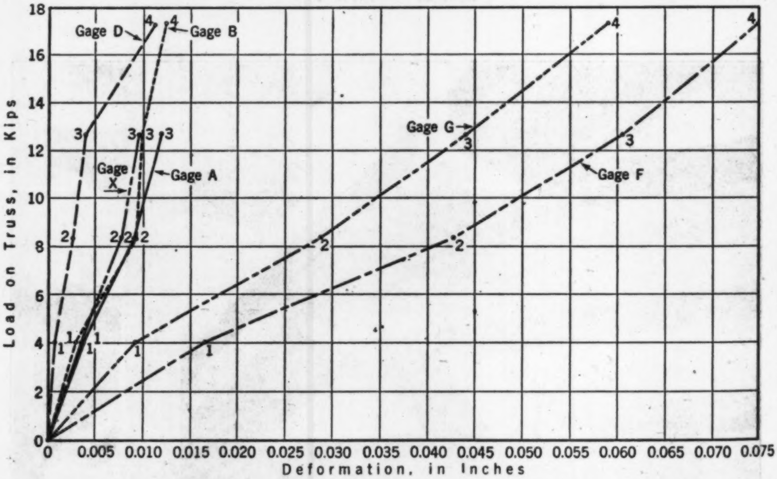


Fig. 32—Center Deflection of Chords, Truss TM-2

In view of the experience in testing truss TM–1, the lower chord splice of truss TM–2 was reinforced before testing. When the load had reached 83.30 kips, a crack about 24 in. long opened on one side of the lower chord of joint

L2, and failure occurred at a maximum load of 88.20 kips. The load then dropped to 80.70 kips—which load was allowed to remain on the truss overnight. On the following morning, the truss still carried 62.10 kips.

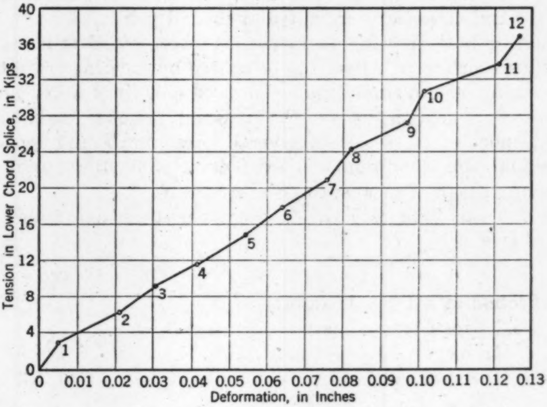


FIG. 33.—DEFORMATION AT THE LOWER CHORD SPLICE, TRUSS TM-1

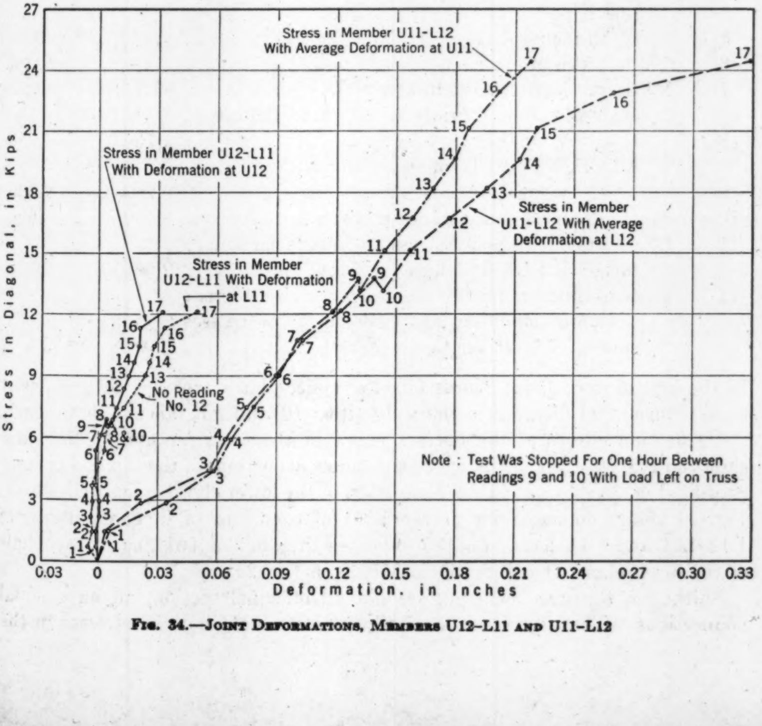


FIG. 34.—JOINT DEFORMATIONS, MEMBERS U12-L11 AND U11-L12

Failures in this truss were quite similar to those in truss TM–1—that is, by a lengthwise splitting of the chords. This truss also showed distress simultaneously at various points before the maximum loads were reached. Like truss TM–1, the top chord bowed and split at joint U7. In addition, shear failures in the end distances were noted at several points.

When testing both trusses, cracking noises were heard at loads of from 30 kips to 40 kips which were believed to be caused by shearing of split-ring cores. Upon disassembly of the trusses, some cores of split rings were found sheared. Most of these were near the ends of the trusses in the chord members. At the lower chord splice of truss TM–1, several cores were found broken off with indications that this condition had existed for some time prior to testing; possibly having occurred in wrecking the structure.

Results of Tests, Trusses TM.—The significant test data to be recorded for these trusses are:

Item	Description	TM–1	TM–2
	Deflection at a Truss Load of:		
	25.3 Kips (Approximate Design Load)—		
1	In inches	1.1	0.99
2	Ratio of span	1/567	1/630
	81.7 Kips—		
3	In inches	3.6	....
4	Ratio of span	1/173	....
	88.2 Kips—		
5	In inches	....	3.0
6	Ratio of span	....	1/187
7	Maximum load in truss in kips	83.3	88.2
	Allowable Load on Critical Members, in Pounds:		
8	Diagonal U5–L4	4,750	4,750
	All Other End Diagonals, Based on—		
9	49° value in chords	9,200	9,200
10	Reduced end distance in webs	8,600	8,600
11	Chord unit compressive stress, in pounds per square inch, joint U7, at failure	2,530	2,320
12	Ratio of ultimate load to the designer's requirement that roof loads be 30 lb per sq ft for trusses spaced 17.75 ft, center to center	3.3	3.5

In the case of item 9, the chords failed at splice plates, whereas designs based on a reduced end distance in the webs (item 10) did not fail in this manner.

Deflection curves of the bottom chord of the truss for five loadings are shown in Fig. 29. Fig. 32 indicates the center deflections of the top and bottom chord. Fig. 33 shows the total elongation of the lower chord splice under load. Fig. 34 shows deformations at the joint of each end of diagonal members U12–L11 and U11–L12. Fig. 35 indicates the angular twisting of each half of the lower chord at points M1 to M5, as in Fig. 29.

Although the web members are not all of equal section, all have equal connections. The trusses are statically indeterminate but the stresses in the web members were calculated on the assumption that each web system took 50% of the shears.

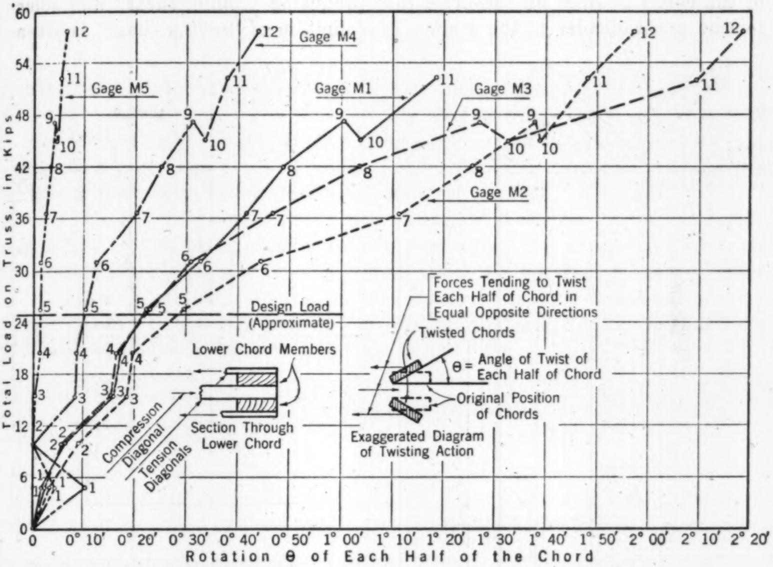


FIG. 35.—ANGULAR TWISTING OF EACH HALF OF THE LOWER CHORD, TRUSS TM-2 (SEE FIG. 29)

In truss TM-1 the first failure occurred in the lower chord splice—possibly the result of damage during wrecking. Therefore, no recommendations or conclusions are presented relative to this failure.

TABLE 5.—RATIOS R_{ca} BASED ON THE WORKING CAPACITY OF TIMBER CONNECTORS

Member	Stress,[a] F (kips)	RATIO R_{ca}	
		Chords[b]	Web[c]
Truss TM-1:			
U1-L2	27.95	3.05	3.25
U2-L1	28.48	3.09	3.30
U3-L4	12.00	2.53	2.53
U5-L4	12.00	2.53	2.53
Truss TM-2:			
I0-L2	31.00	3.38	3.60

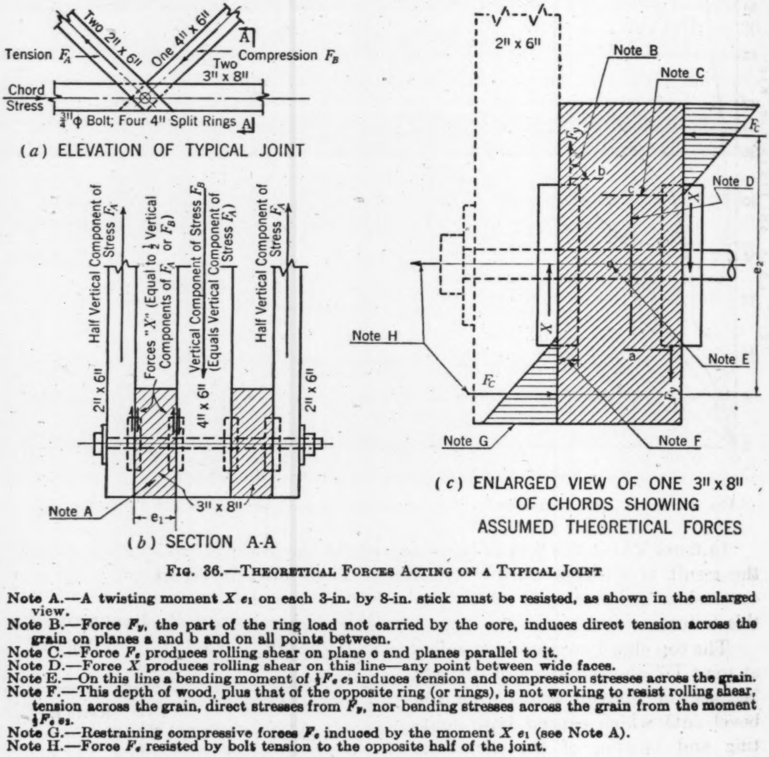
[a] Stresses in diagonals at failure of truss. [b] Ratios based on connector values in chords. [c] Ratios based on connector values in the web. The truss did not fail in this manner because the wood in the end distances did not shear out.

The top chord compression splice at joint U7 showed signs of distress because of uneven bearing on the bevel cuts which caused both splitting and bowing of the members. This bow was accentuated because of shrinkage and also because, in the test, the center panel point was unloaded.

The ratios (R_{ca}) given in Table 5 were based on the working capacity of timber connectors P_e although failure of the trusses came about through twisting and consequent splitting of the chords, and was not due to the connectors.

Fig. 36 illustrates one conception of theoretical forces acting on a typical joint. The vertical components of the stresses on diagonal members are

assumed to create bending moments and shears perpendicular to the grain of the chord members. In addition, loads are transferred from rings on one face of the chord to rings on the other face producing "rolling shear" and some tension perpendicular to the grain. (A definition of "rolling shear" is given


Two 2" x 6"
One 4" x 6"
Tension F_A
Two 3" x 8"
Compression F_B
Chord Stress
A
A
¾"φ Bolt; Four 4" Split Rings
Half Vertical Component of Stress F_A
Forces "X" (Equal to ½ Vertical Components of F_A or F_B)
Vertical Component of Stress F_B (Equals Vertical Component of Stress F_A)
Half Vertical Component of Stress F_A
2" x 6"
4" x 6"
2" x 6"
3" x 8"
Note A
e_1
2" x 6"
Note B
Note C
Note D
Note E
Note F
Note G
Note H
F_c
F_y
X
a
b
c
e_2

(*a*) ELEVATION OF TYPICAL JOINT

(*b*) SECTION A-A

(*c*) ENLARGED VIEW OF ONE 3" x 8" OF CHORDS SHOWING ASSUMED THEORETICAL FORCES

FIG. 36.—THEORETICAL FORCES ACTING ON A TYPICAL JOINT

Note A.—A twisting moment $X e_1$ on each 3-in. by 8-in. stick must be resisted, as shown in the enlarged view.
Note B.—Force F_y, the part of the ring load not carried by the core, induces direct tension across the grain on planes a and b and on all points between.
Note C.—Force F_c produces rolling shear on plane c and planes parallel to it.
Note D.—Force X produces rolling shear on this line—any point between wide faces.
Note E.—On this line a bending moment of $\frac{1}{2}F_c e_2$ induces tension and compression stresses across the grain.
Note F.—This depth of wood, plus that of the opposite ring (or rings), is not working to resist rolling shear, tension across the grain, direct stresses from F_y, nor bending stresses across the grain from the moment $\frac{1}{2}F_c e_2$.
Note G.—Restraining compressive forces F_c induced by the moment $X e_1$ (see Note A).
Note H.—Force F_c resisted by bolt tension to the opposite half of the joint.

subsequently under the heading, "Part II. Joint Tests: Compression or Tension Splice Joints with Plywood Splice Plates and Split Rings." The complexity of the problem makes it pertinent to cite a statement by John A. Scholten, as follows:[3]

> **"Considerable progress has been made in the theoretical stress analysis of connector joints and in correlating the results with basic data on the mechanical properties of the wood and metal. The fact remains, however, that the stress distribution is so complicated, and the assumptions involved are so often invalid, that actual tests must be relied on to provide the necessary design data."**

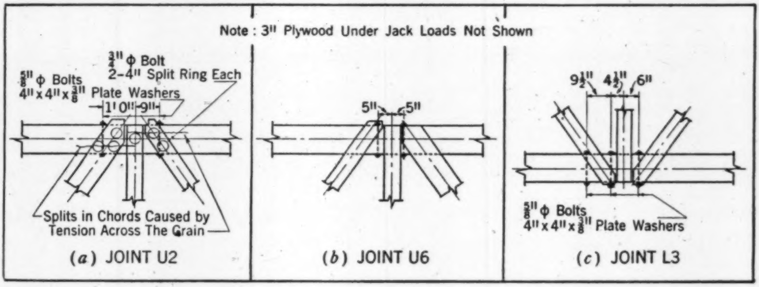
---

[3] "Timber-Connector Joints: Their Strength and Design," by John A. Scholten, *Technical Bulletin No. 865*, Forest Products Laboratory, Forest Service, U.S.D.A., Madison, Wis., March, 1944, p. 14.

In thin members, such as these chords, grooves of split rings take out a relatively large area of the chord section, thereby reducing resistance of the chords to stresses caused by the forces shown in Fig. 36.

The tests show that the present methods of design analysis usually employed in trusses of this type (using connector values but neglecting the stress effect on the connected members) may not, and probably will not, reveal the critical stress factors. For example, Fig. 36 illustrates the many forces that may be acting in an apparently simple connection. Because of the wide application of these types of joints, further full-sized tests and study should be continued in an endeavor to establish a rational basis of design.

North Square Court (NS) *Truss.*—Perhaps the most important results obtained in this test program were from the test of a truss taken from the North Square Court area. This truss, with all members of wood, about 10.5 ft deep, spanning about 48 ft, had double stick web members connected eccentrically to the 6-in. by 12-in. chords by 4-in. split rings with $\frac{3}{4}$-in. bolts. Details of the truss are shown in Figs. 37 and 38(*b*).


Note : 3" Plywood Under Jack Loads Not Shown
$\frac{3}{4}$" ϕ Bolt
2-4" Split Ring Each
$\frac{5}{8}$" ϕ Bolts
4" x 4" x $\frac{3}{8}$" Plate Washers
1' 0"
9"
Splits in Chords Caused by Tension Across The Grain
5"
5"
9$\frac{1}{2}$"
4$\frac{1}{2}$"
6"
$\frac{5}{8}$" ϕ Bolts
4" x 4" x $\frac{3}{8}$" Plate Washers

(*a*) JOINT U2 (*b*) JOINT U6 (*c*) JOINT L3

FIG. 37.—TYPICAL JOINTS TNS AND DETAILS OF STITCH BOLTS ADDED DURING TESTS (SEE FIG. 38(*b*))

The workmanship of the truss was excellent, as was evidenced by good fits, indicated by inspection before testing and verified by examination of the disassembled members and their connections after testing. The wood was of structural grade Douglas fir in good condition with the exception of the 6-in. by 12-in. chords. These were of boxed-heart material and had some checks, those in the top chord near joint U6 being about $\frac{1}{4}$ in. wide and extending toward the heart from both faces. For reasons to be explained subsequently, these checks apparently did not affect the test results.

The truss was stored outdoors for about two and one-half months after its removal from the structure and was tested at an average moisture content varying from about 10% to 14%. Because of shrinkage of the timbers, bolts were generally loose from about $\frac{1}{4}$ in. to $\frac{5}{16}$ in.

Testing.—The truss was tested in the typical manner. About one hundred gage points were placed as indicated in Fig. 39 to measure deformations. Loads were applied as shown subsequently in Table 6. Gages LD–1 to LD–11 and UD–1 to UD–13 (see Fig. 39) measured deflections of the truss and indicated bending deformations of the chords caused by eccentric joint connections.

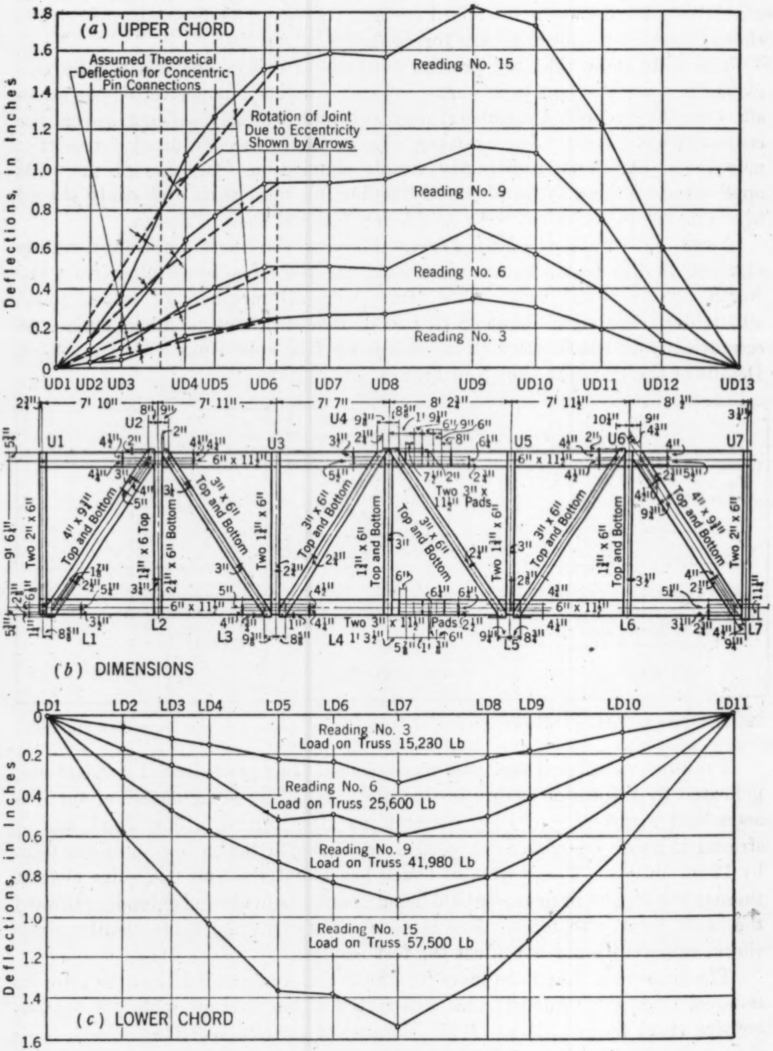
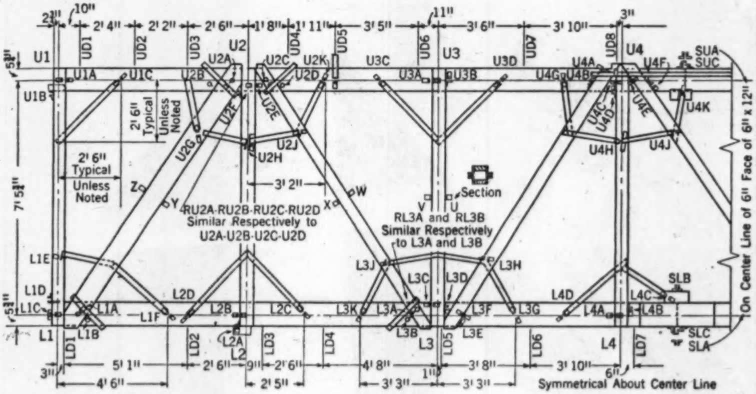


FIG. 38.—LOAD-DEFLECTION CURVES, TRUSS TNS

Angular changes between members as well as slips of connectors were also measured. Additional sets of gages were placed on three of the web members to measure the bending effects caused by direct loads, since these loads were applied at the faces of the members rather than at their centers of gravity.


2½"
10"
2' 4"
2' 2"
2' 6"
1' 8"
1' 11"
3' 5"
11"
3' 6"
3' 10"
3"
UD1
UD2
UD3
UD4
UD5
UD6
UD7
UD8
U1
U2
U3
U4
5½"
U1A
U1B
U1C
U2A
U2B
U2C
U2D
U2K
U2E
U2F
U2G
U2H
U2J
U3A
U3B
U3C
U3D
U4A
U4B
U4C
U4D
U4E
U4F
U4G
U4H
U4J
U4K
SUA
SUC
2' 6"
Typical
Unless
Noted
2' 6"
Typical
Unless
Noted
7' 5½"
3' 2"
Z
Y
X
W
V
U
Section
L1E
L1D
L1C
L1A
L1B
L1F
L2D
L2B
L2A
L2C
L3K
L3J
L3A
L3B
L3C
L3D
L3E
L3F
L3G
L3H
L4D
L4A
L4B
L4C
SLB
SLC
SLA
5½"
L1
L2
L3
L4
LD1
LD2
LD3
LD4
LD5
LD6
LD7
3"
5' 1"
2' 6"
9"
2' 6"
4' 8"
1"
3' 8"
3' 10"
6"
4' 6"
2' 5"
3' 3"
3' 3"
On Center Line of 6" Face of 6" x 12"

RU2A-RU2B-RU2C-RU2D Similar Respectively to U2A-U2B-U2C-U2D

RL3A and RL3B Similar Respectively to L3A and L3B

Symmetrical About Center Line

FIG. 39.—HALF-SECTION, TRUSS TNS, SHOWING LOCATION OF GAGE POINTS

A total load on the truss of 42,000 lb was applied gradually over a period of 8 hours. Loads were then released and testing was resumed about 48 hours later. After reaching a load of 62,600 lb a cracking noise was heard and a split opened in the lower chord at joint L3. This split was ½ in. wide, and was increasing in width as the pressure on the loading jacks decreased. Loads were then released and stitch bolts were installed in the chord at joints L3 and L5 as shown in Fig. 37, although the nuts of the bolts were not tightened until the truss refused to take further load.

When the test was resumed, failure developed again at joint L3 at the same load of 62,600 lb. When it was apparent that the truss would not resist greater loads, the stitch bolts were released and the nuts were tightened. Testing was resumed; and, at a load of 65,020 lb on the truss, joint U2 failed, as shown in Fig. 40 (note the uncovered ring groove on the left-hand diagonal). Further testing after reinforcing the joint with stitch bolts brought about deformation in the top chord, as shown in Fig. 41. After further reinforcing joint U2, the next failure occurred at joint U6 in a similar manner at a load of 67,400 lb. It is important to record that, in spite of existing deep checks at joint U6, splitting followed new lines, although at some points they were only ½ in. apart. Fig. 37(*a*) shows the position of splits in the chords which was typical for all three joint failures. The location of the new splits was seemingly independent of existing checks in all cases.

Results of Tests, Truss TNS.—The results of this test are summarized in Table 6.

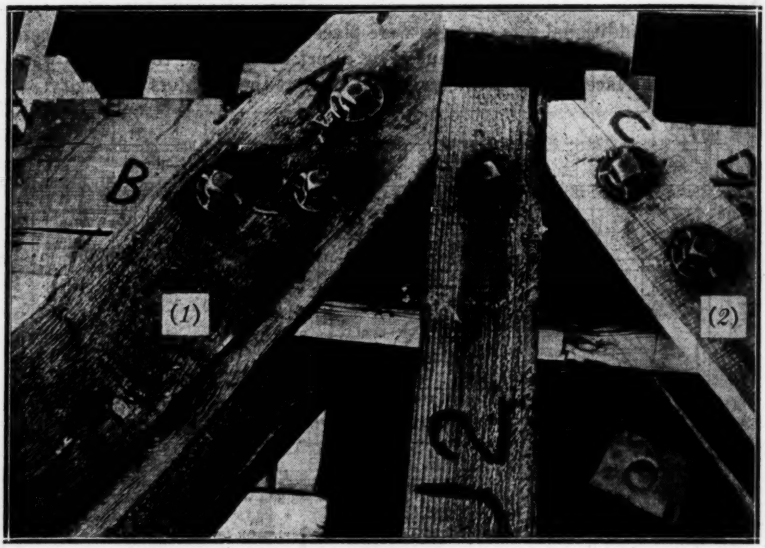


FIG. 40.—FAILURE AT JOINT U2 OF THE NORTH SQUARE COURT TRUSS

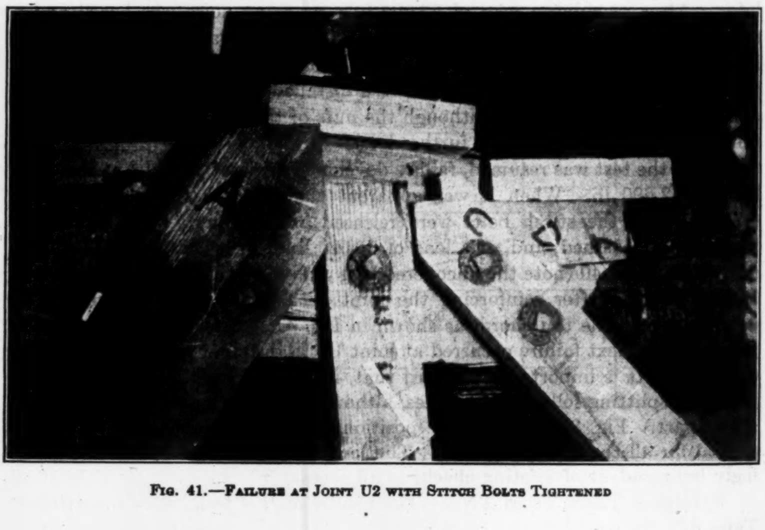


FIG. 41.—FAILURE AT JOINT U2 WITH STITCH BOLTS TIGHTENED

TABLE 6.—STRESSES, ALLOWABLE LOADS, AND RATIOS FOR THREE FAILURES, TRUSS TNS

Description	JOINT L3[a]			JOINT U2[a]		JOINT U6[a]	
	U4–L3	U3–L3	U2–L3	U2–L3	U2–L1	U6–L5	U6–L7
Maximum truss load at failure (kips)	64.6			65.0		67.4	
Stress, F, in the member at failure (kips)	0.9	15.9	24.6	25.5	48.4	22.1	49.6
Allowable load, P (kips)	17.9	7.7	17.9	17.9	27.2	17.9	27.2
Ratio, R_{ss}, at failure	....[b]	2.07	1.37	1.42	1.78	1.23	1.83

[a] After failures shown in table, each individual joint was reinforced to permit further testing and failure at other joints. [b] Not a factor in failure.

The following information concerning deflections was observed:

Item	Description	Quantity
	Deflection at the Allowable Truss Load of 36.5 Kips, Based on an Allowable Stress in Diagonal U2–L3 Corresponding to a Load of 930 Lb per Lin Ft of Truss:	
1	In inches	0.84
2	Ratio of span	1/685
	Maximum Recorded Deflection, at a Truss Load of 57.5 Kips:	
3	In inches	1.56
4	Ratio of span	1/370

Fig. 38 shows the deflections of the bottom and top chords under various loadings. The effect of bending moments in chords, caused by the eccentric connections of the web members, is illustrated by the dotted lines in Fig. 38(*a*). Fig. 42 indicates the center deflections of the bottom chord of the truss at various stages of loading. Fig. 43 gives data on the bowing of the two sticks of diagonal L1–U2, because the loads were applied at their faces instead of at the centers of gravity. This condition occurs quite commonly in timber design, and is generally neglected. As the timbers in such a connection shrink and bolts become loose, the end restraint of the members thus connected becomes less effective with consequent increase in bending stresses. Members composed of two sticks, as were these web members of this truss, would have their load capacity increased at only slight additional expense if their sticks were tied together to form an integral member similar to the "battened" steel column.

Fig. 44 shows the slips of connectors at either end of diagonal member L1–U2. The average slip of a connector (split ring) was about 0.06 in. for a load of 4,000 lb applied at an angle of 50° with the grain of the wood. Fig. 45 shows the deformations at four gage points at joint U2. Using these data, the approximate angular change between the various members may be calculated.

The most important feature of this test was the demonstration that failure in tension perpendicular to the grain in the chord members at the eccentric joints limited ultimate load-carrying capacity. In the history of timber testing, there seems to be no previously published record of this type of failure. It is known that, when designing this truss, shearing and bending stresses created in the chords by the eccentric connections were carefully investigated; yet, in spite of the unusual care exercised in the design and in spite of unusually good workmanship in construction, the truss failed at a R_{wa}-ratio or a R_{ca}-ratio of 1.78—an unexpectedly low value. This weakness would probably not have been developed in tests on small specimens. Again the vital importance of tests on actual trusses is demonstrated.

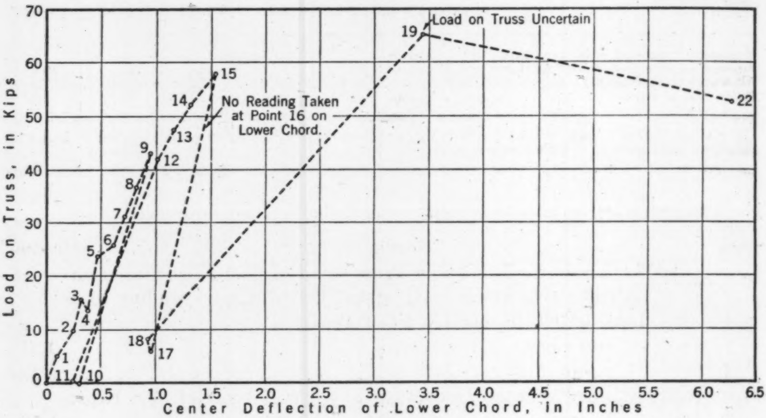


FIG. 42.—CENTER DEFLECTION OF LOWER CHORD, TRUSS TNS

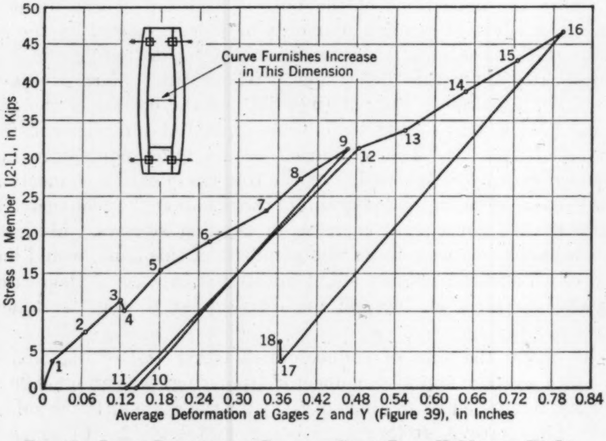


FIG. 43.—STRESS-DEFORMATION CURVE AT GAGES Z AND Y, MEMBER U2-L1

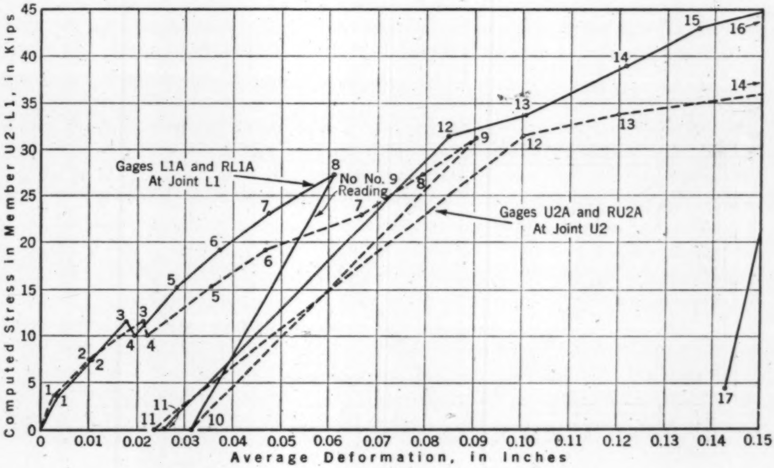


Fig. 44.—Slip of the Connectors on Diagonal L1-U2

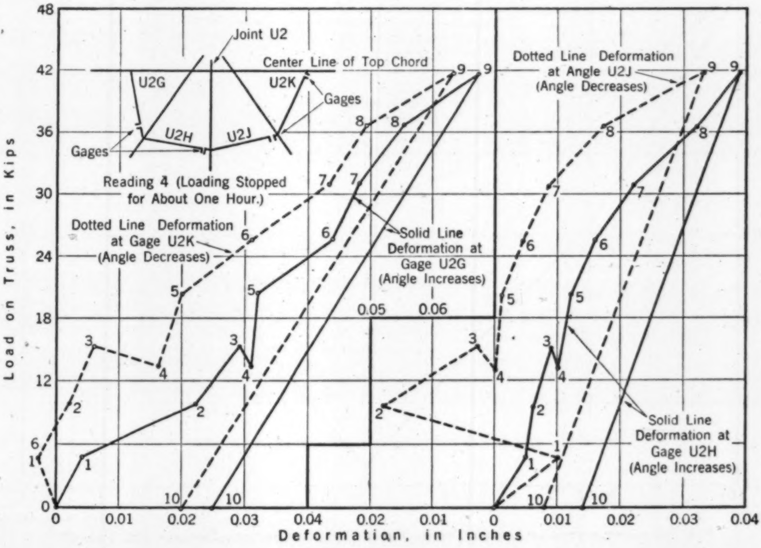
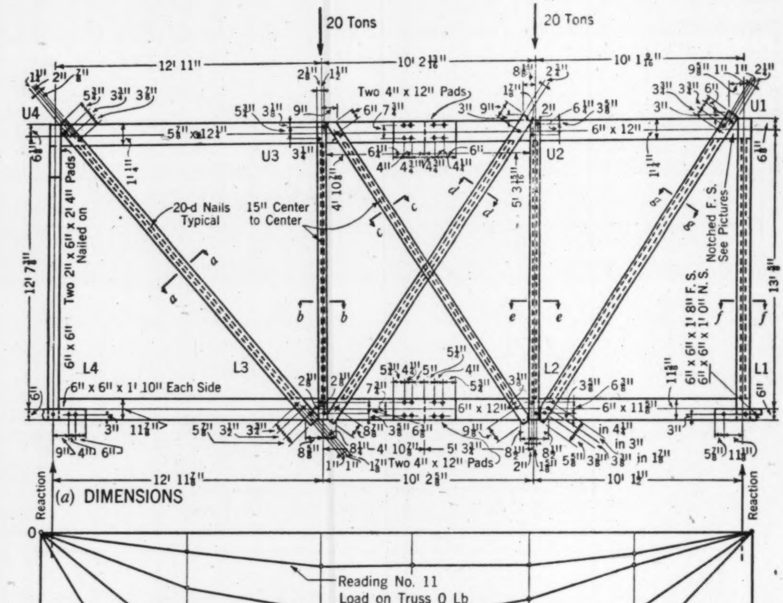


Fig. 45.—Deformations at Joint U2



(a) DIMENSIONS

(b) DEFLECTIONS

FIG. 46.—DEFLECTION OF LOWER CHORD, TRUSS TSF (FOR SECTION DETAILS, SEE FIG. 47)

San Francisco Building (SF) *Truss.*—This test was of an all-timber truss with bolted connections taken from the San Francisco Building. It had a span of 33 ft 4 in. and a depth of 13 ft 10 in., and its chords were substantially parallel. When tested it had a moisture content of about 14%.

The details of this truss are shown in Figs. 46(*a*) and 47. Web members were I-shaped, made of three 2-in. by 6-in. sticks nailed together. The 2-in. by 6-in. webs were butted against the 6-in. by 12-in. chords of the truss, being angle cut in the base of the diagonals, and the flanges were carried over the sides of the chords and bolted through with ⅞-in. round bolts with malleable iron washers. Timbers were of select structural grade Douglas fir, and were in good condition except at joint U1, where the end of one flange of the diagonal U1–L2 had been notched to provide clearance for a roof rafter. This flange had a check which had started at the point of the notch. Shrinkage of the wood had caused all bolts to become loose from about $\frac{1}{16}$ in. to ⅛ in.

Attention is called to the pattern of the bolt arrangement at the end connections of U1–L2 and U4–L3 (see Fig. 46(*a*)). At joint U1, for example, the longitudinal center lines of bolts were staggered by about 1 in. only; and this feature, combined with the notching and checking, proved to be the cause of an early failure of this joint.

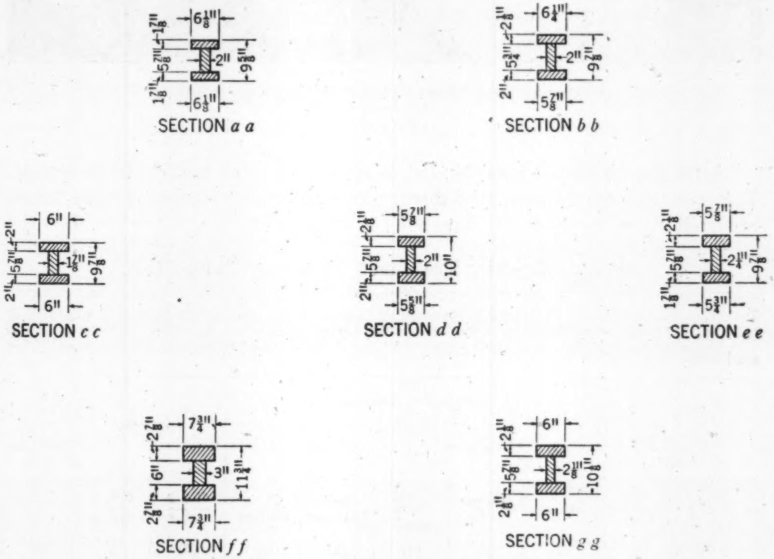


Fig. 47.—Section Dimensions (See Fig. 46(*a*))

Testing.—The truss was installed horizontally in the testing structure, and tested in the usual manner. Testing was started late one afternoon, and loads of 1,840 lb and 1,650 lb were applied to panel points U2 and U3, respectively. Deformations and deflections were measured, loads were released, and the recovery was measured. The next day, the very slight recovery overnight was measured and loads were again applied, The first failure occurred at joint U1 in the notched end of U1–L2, at a total truss load of 18,700 lb. The damaged 2-in. by 6-in. flange of member U1–L2 was removed (see Fig. 48(*a*)). The block had sheared out by the bolt, and its width is the same as the diameter of bolt. Fig. 48(*a*) reveals no evidence of wedge action.

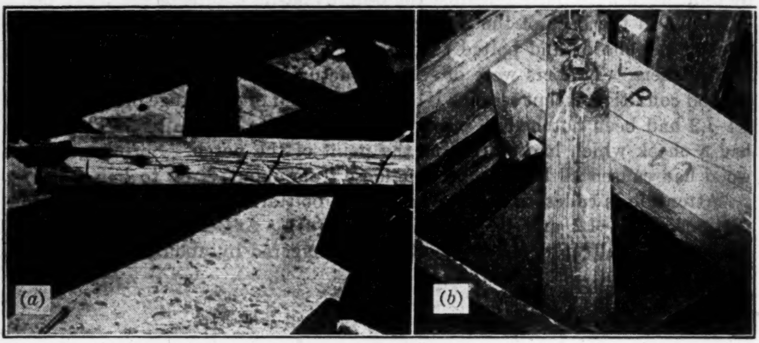


FIG. 48.—FAILURES OF UPPER CHORD JOINTS, TRUSS TSF
(*a*) JOINT U1
(*b*) JOINT U4

After this failure, it was decided to repair the joint and continue testing. This was done on the next (the third) day, and testing continued until failure occurred in one flange of the diagonal L3–U4 at joint U4 at a total load on the

TABLE 7.—SAN FRANCISCO BUILDING TRUSS TSF

Deflection.—At a total allowable load (corresponding to about 375 lb per lin ft of truss, or a total load of 8.2 kips), the deflection at the center of the bottom chord was 0.25 in.—that is, 1/1,600 of the span length. The maximum recorded deflection (with a load of 40 kips on the truss) was 1.70 in.—that is, 1/235 of the span length.

Joint	STRESS IN END DIAGONALS					RATIO R_{bs}		Residual stress (lb)
	At Failure[a]			Allowable (Lb)[b]				
	(Kips)	(Lb per lin ft)	Total load (kips)	End bearings	Cross bearing	$\frac{\text{Col. 2}}{\text{Col. 6}}$	$\frac{\text{Col. 2}}{\text{Col. 5}}$	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Joint U1	18.7	1,100	24.0	9,468	6,390	2.94	1.98	15,511
Joint U4								
Part failure[c]	31.9	1,750	38.4	9,000	6,840[d]	4.65	3.54	34,800[e]
Total failure[c]	34.8	1,900	41.6	9,000	6,840[d]	5.09	3.87	19,800

[a] Stress *F* in the member (Col. 2) corresponding to applied loads of *w* (Col. 3) or *W* (Col. 4). [b] Based on the bolts of the given joints being subjected to end bearing (Col. 5, at 0° to the grain) or cross bearing (Col. 6) for stress on the end diagonal. In Joint U1 (6-in. by 12-in. chord) the cross grain was at about 57° and in Joint U4 at about 45°. [c] Only the top member of the I-section failed at 31.9 kips. The loading was increased until at 34.8 kips the entire connection failed—top and bottom. [d] Not critical. [e] The load was increased to a maximum of 34,800 lb.

truss of 31,900 lb. Without repairing the joint, the load was increased until, at 34,800 lb, the other flange of the diagonal at joint U4 also failed. Fig. 48(*b*) illustrates these last failures.

Results of Tests, Truss TSF.—All failures of the truss occurred in the wood at the bolted connections. Fig. 46 shows the deflection curves of the lower chord. Fig. 49 shows the center deflection of the lower chord plotted against the total load on the truss, and Fig. 50 shows the axial deformation of the diagonal member U1–L2 with respect to the 6-in. by 12-in. chord at joint U1.

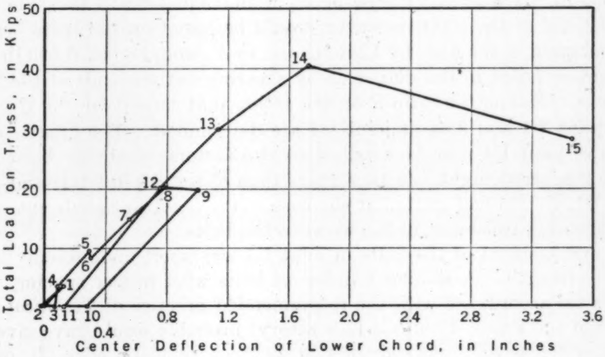


FIG. 49.—CENTER DEFLECTION, LOWER CHORD, TRUSS TSF

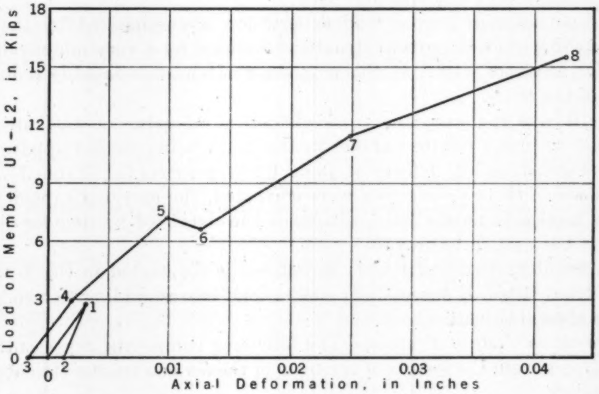


FIG. 50.—DEFORMATION OF U1–L2 WITH RESPECT TO UPPER CHORD TRUSS TSF (CONNECTION CONSISTS OF THREE ¾-IN.-ROUND BOLTS)

The allowable load on this truss is that determined by the safe capacity of the joint detail (U1) that failed—in this case determined by the allowable load on the bolts, P_b. The computed load (8,200 lb) corresponds to about 375 lb per ft of truss. If the failure of the defective notched joint U1 is excluded as subnormal, the truss may be stated to have failed at a load of 31,900 lb, corresponding to about 1,750 lb per lin ft of truss.

It is important to note that the joints did not fail in the manner indicated by calculations of their strength. To illustrate, Table 7 indicates that, in joint U1, the allowable load for the diagonal U1–L2 is the allowable load for the bolts in the end of the diagonal based on bearing parallel to the grain of the two 2-in. by 6-in. diagonals, or P_b would equal 9,648 lb. The critical allowable load of the joint, however, would be based on the bolts bearing at 57° to the grain in the 6-in. by 12-in. chord, or P_b would equal 6,390 lb. Joint U1, however, failed in the diagonals by shearing out the ends of the 2-in. by 6-in. sticks. Basing the ratio R on the action that caused failure, R-values of 1.98 at joint U1 and 3.54 at joint U4 are determined. The greatly superior strength of joint U4 may be ascribed to two factors—first, the bolts at joint U4 were staggered slightly farther apart than those at joint U1; and, second, at joint U1, checks had started at the base of the reentrant angle cut, weakening the shearing resistance of the wood to the bolts.

The arrangement of the bolts in joint U1 was poor, undoubtedly lowering the joint strength. A smaller number of bolts with proper spacing and end distances (in accordance with the recommended practice of *Technical Bulletin No. 332*[4] of the Forest Products Laboratory) probably would have given equal strength, whereas only one pair of 1-in. bolts, with 1½-in. edge distances and 8-in. end distances would have yielded a higher strength than the three bolts with the arrangement and spacings used.

The importance of proper field supervision is emphasized by the unfortunate notching of an important structural member for a very minor purpose—the location of a roof joist. In this case the notch unquestionably reduced the strength of the truss.

Before it was repaired, the truss yielded a R_{ba}-value of 2.94, in bearing at an angle to grain, failure parallel to the grain being caused apparently by design and construction defects at joint U1 (see footnote[b], Table 7). After repairs in which these deficiencies were corrected, the carrying capacity of the truss was increased nearly 75%, although the detail of bolting for joint U4 could have been much improved.

An inherent property of timber structures is illustrated in this test: Even after the truss failed, it successfully held a load corresponding to two or three times the allowable load.

Comments on Tests of Trusses.—The following comments, representing conclusions drawn from the results of the tests of the sixteen trusses (twenty tests), seem justified:

1. Of the twenty tests made, on sixteen trusses, the results show that—five failed because the wood members ruptured at joint details; six failed at bolted joints; and nine failed at connector joints, due to types or combination of stresses not anticipated.

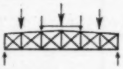
2. The truss with the highest R-value of 7.3 was truss TCh–5 (see Table 8) which had its compression diagonals framed on one side of the chords and its

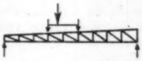
---

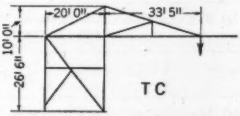
[4] "The Bearing Strength of Wood Under Bolts," by George W. Trayer, *Technical Bulletin No. 332*, Forest Products Laboratory, Forest Service, U.S.D.A., Madison, Wis., October, 1932.

tension diagonals on the other—a type discussed in conclusion 6. Except for the end diagonals, connections of web members to chords were made with single bolts.

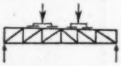
TABLE 8.—Summary of Tests, Douglas Fir Timber Trusses

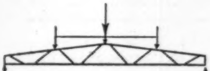

TAC

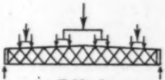

TB

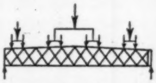


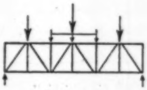
TC

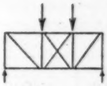

TCH


TF


TM-1


TM-2


TNS


TSF

Test No.	R	Deflection		Type and location of failure
		Design load	Ultimate load	
(a) Alameda-Contra Costa Building Trusses, TAC; Span, 39 Ft 4¾ In.; Height 6 Ft 0¼ In.; Select Grade of Timber; Rod-and-Block Wood Diagonals, and Steel Rod Verticals				
TAC-1..	3.15	1/790	1/255	Failed by shear in the butt blocks. Low shear values were the result of the shape of butt blocks and the combined stresses.[a]
TAC-2..	2.95	1/665	1/257	
TAC-3..	2.55	1/775	1/378	Failed by shear in the butt blocks. The surfaces of the butt blocks were paraffined thus reducing friction and yielding lower test results.
TAC-4..	2.0	1/775	1/468	
(b) Brazil Building Trusses TB; Span 44 Ft, 6¼ In.; Height 2 Ft, 10 In.; Select Grade of Timber; Split Rings on Three Stick Chords and Two Stick Webs				
TB-1....	2.95[b]	1/233	1/107	The chords failed in tension across the grain, TB-2 failing at the center chord splice.
TB-2....	2.48[b]	1/214	1/191	
(c) Cavalcade Cantilever-Frame, TC; Span 33 Ft, 5 In.; Height 10 Ft; No. 1 Timber, with Shear Plates and Split Rings				
TC......	....	....	....	The crown joint failed under preliminary loading; careless workmanship.
(d) Chinese Building Trusses. TCh; Span 38 Ft, 8¾ In.; Height 4 Ft 4¼ In.; No. 1 Dimension Timber; All Single Stick Members with Bolted and Nailed Joints				
TCh-1...	5.3	1/690	1/323	Failure was by tension in the lower chord splice. There was considerable distortion under load because of unsymmetrical framing.
TCh-2...	6.4	1/870	1/141	
TCh-3...	5.7	1/685	1/119	Failure was by bending and direct stress in the end diagonal.[a]
TCh-4...	5.9	1/783	1/169	Bolts sheared out of end diagonal.[a]
TCh-5...	7.3	1/885	1/149	

TABLE 8.—(*Continued*)

Test No.	R	Deflection		Type and location of failure
		Design load	Ultimate load	
(*e*) French Building Trusses, TF; Span 35 Ft, 7½ In.; Height 3 Ft, 6 11/16 In.; Select Grade of Timber; Split Rings, with One Stick Chords and Two Stick Webs				
TF......	1.5[d]	1/720	1/270	Cross-grain failure by tension in the chord at joint L2.[e]
(*f*) Merry-Go-Round Building Trusses, TM; Height[f] 6 Ft; Select Grade of Timber; Split Rings in Two Stick Chords (One System); Diagonals Are One Stick and Two Stick Members				
TM-1...	3.3[g]	1/567	1/173	Failed by chords splitting at several points.[h]
TM-2...	3.60[g]	1/630	1/187	Bottom chord split at joint L2.[h]
(*g*) North Square Court Building Trusses, TNS; Span 47 Ft, 6½ In.; Height 10 Ft, 5¾ In.; Structural Grade of Timber, with Split Rings in the One Stick Chords and the Two Stick Web				
TNS....	1.78	1/685	1/370	Cross-grain failure by tension[i] in the chord at joint L3.
(*h*) San Francisco Building Trusses, TSF; Span 33 Ft, 4 In.; Height 12 Ft, 10 In.; Select Grade of Timber with Bolted Joints and Nailed Members				
TSF.....	2.0[j]	1/1,600	1/235	Bolts sheared at joint U2, emphasizing importance of proper bolt spacing and arrangement.

[a] The results of these tests show a friction value of 0.315 in the butt blocks. [b] In this shallow truss the chord bending effect was ignored in computing the R-values. [c] The bottom chord splice was reinforced. There was considerable distortion under load. [d] R varied between 1.5 and 3.6. [e] Stitch bolts held excellently after first failure, adequately reinforcing joint. [f] Span lengths: TM-1, 54 ft, 1 in.; and TM-2, 51 ft, 4½ in. [g] These values apply after the chord splices had been reinforced. [h] Initial failure at the bottom chord splice which, apparently, had been injured in dismantling. [i] Eccentric joints require check for cross-grain tension. [j] R varied between 2.0 and 5.1.

3. The deflection of trusses was apparently independent of type, the all-timber truss with modern connectors and the truss with butt-block connections for compression members and tension web members of steel having about equal stiffness.

4. The average deflection for parallel-chord trusses with depth-to-span ratios of 1/5 to 1/8 was, for allowable loads, about 1/700 of the span (varying from 1/567 to 1/885).

5. A number of trusses failed at unexpected loads, such failures emphasizing the lack of information on, or knowledge of, the local stresses to be expected in addition to the direct stresses. Even had check computations been made on the design, using conventional methods, some of these weaknesses probably would not have been disclosed, because in many cases customary methods of truss analysis would not have taken into account the stresses that actually caused failure.

6. The necessity of considering tension perpendicular to the grain for correct design of timber joints was one of the outstanding results of these tests. Examples of this need were disclosed in the cases of eccentric connections of truss members, as (*a*) where chord and web members did not intersect at common points, and (*b*) where two web members connecting at a joint or

panel point were framed on the opposite sides of the chords, causing rotation of the chords, bowing of the webs, or both, with resultant splitting of the chord members.

7. The importance of adequate end distances for bolts, proper spacing of bolts, and especially proper arrangement or pattern of the bolts in bolt clusters,[5] was signally illustrated by some of the truss failures.

8. The possibility and effect of combining important stresses should be investigated in some designs. Suitable reductions of allowable stresses should be made if necessary. For example, where shear and flexural stresses occur simultaneously in a timber detail, as in the end butt block of the Alameda-Contra Costa Counties Building trusses (TAC), an assumed working stress in shear of 150 lb per sq in. resulted in a low R_{wa}-value although this unit shear, and even a value of 180 lb per sq in. is allowed in a joint by authoritative recommendations without qualifying limitations for type of detail or presence of other stresses.

9. The high values of R_{ba} were noticeable for bolted connections where the size, spacing, and arrangement of the bolts were carefully selected, although it must be remembered that the design loads recommended for bolts by the Forest Products Laboratory are based on proportional limit loads rather than on ultimate loads.

10. The indentation of the end diagonals of the Alameda-Contra Costa Counties Building trusses (TAC) into the beveled surfaces of their butt blocks indicates the wisdom, in this type of truss, of paying particular attention to the compression areas afforded the ends of diagonal web members, if a balanced design of truss is considered a desideratum.

II. JOINT TESTS

Joints with Steel Gusset Plates and Shear Plates.—This part of the test program consisted of tests on seventeen joints in which from two to four (but generally three) members were connected together with steel gusset plates and flush-type shear plates. Fourteen of the joints were taken from the stage framing of the Cavalcade, two from the cantilever root trusses of the same structure, and one from the Triumphal Arch. All the joints except one (joint J1), were concentrically connected, or substantially so—that is, the center lines of the members met at a common point.

All joints had been carefully handled during removal and in transportation, so that their strength factors would not be altered by such operations. The wood was in good condition except in a few instances recorded in the summarized results of Table 9 and noted in the following descriptions of the individual tests:

Joint J2.—Although there was some slight checking, the pretest condition of the members of this joint was excellent. Failure was characterized by splitting and shearing.

Joints J3–J7.—The pretest condition of these joints was excellent. Failure was characterized by splitting and shearing.

---

[5] "Wood Handbook," Forest Products Laboratory, Forest Service, U.S.D.A., Madison, Wis., 1940, p. 136.

TABLE 9.—SUMMARY OF TESTS, JOINTS WITH STEEL GUSSET PLATES

Joint	DIMENSIONS (IN.)		End distance, connectors[c] (in.)	P_s (kips)	MAXIMUM LOADS (KIPS)			RATIOS R		Deformation in failed member (in.)	Stress, F, in member after failure	Testing time (min)
	Size of timber[a] (in.)	No. of bolts[b]			One shear plate	Failed member	P_l	R_{us}	R_{sl}			
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
J2....	6×14	3	7	25.1	19.1	114.4	60.0 ±	4.56	2.38	0.062	88.4	137[h]
J3....	6×10	2	4½	15.1	13.3	53.24	31.0 ±	3.53	2.05	0.033	10.7	31
J4....	6×10	2	5½	16.4	14.3	57.1	40.9 ±	3.48	2.49	0.085	6.2	44
J5....	6×6	2	8½	18.1	21.9	87.7	46.0 ±	4.84	2.54	0.0735	0	25
J6....	6×6	2	8½	18.1	21.8	87.0	59.5 ±	4.81	3.28	0.030	0	20
J7....	6×6	2	7¾	18.1	22.8	91.2	45.6 ±	5.04	2.53	0.06	17.2	30
J8....	6×14	3	6	18.1	....	86.83	32.9 ±	4.77	1.82	0.11	11.5	39
J9....	6×6	2	8⅜	18.1	21.7	86.9	44.4 ±	4.79	2.45	0.036	6.1	25
J10...	6×10	2	5	17.1	....	77.7[d]	52.5 ±	4.54	3.06	0.085	50.6	61
J11...	8×10	2	6	15.5	12.1	48.4	38.1 ±	3.12	2.46	0.094	5.9	23
J12...	8×10	2	6	15.5	15.8	63.1	39.1 ±	4.08	2.52	0.094	18.0[f]	30
J13...	6×14	3	7½	25.1	....	126.4[d]	50.0 ±	5.06	1.98	0.22	....[g]	57[i]
J14...	8×12	2	8¾	18.1	11.3	45.2	0[e]	2.5	....	0.201	10.9	34
J15...	3×10[a]	4	9¾	18.1	13.1	52.5	0[e]	2.91	....	0.056	10.2	54
J16...	3×10[a]	4	10	18.4	13.4	53.8	<37.5[e]	2.93	....	0.095	7.1	37
J17...	2⅝×5⅝[a]	4	10½	18.1	....	33.1	0[e]	1.83	....	....	....	8

[a] Single stick members at each joint except J15, J16, and J17, which were double sticks. [b] Bolts were all ⅞ in. in diameter except J11 and J12, which were ¾ in. Except joints J15, J16, and J17 each bolt included two 4-in. shear plates. The three joints mentioned were designed with one 4-in. shear plate on each bolt. After disassembling of joint J8 it was discovered that three shear plates had been omitted from one side. [c] Connectors were spaced uniformly 6 in., center to center, except joint J16 which was 6½ in. [d] In joints J10 and J13, the compression member failed. In the other joints failure occurred in a tension member. [e] No "proportional limit load" evident. [f] Held by the side; not sheared. [g] Did not fail. The joint was holding but deforming. [h] 1 hr 9 min on one day and 1 hr 8 min the succeeding day. [i] Tested for 24 min; then permitted to stand at the design load over one week end and tested for 33 min on Monday.

Joint J8.—The members of this joint were grooved for six shear plates. After disassembly it was discovered that the three plates on one side had been omitted. Failure was by tension, characterized by splitting and shearing on one side.

Joint J9.—The pretest condition of this joint is reported as "good."

Joint J10.—The pretest condition was excellent except for a large natural check in the compression member that failed. This member was a boxed-heart stick, and failure was characterized by shearing, splitting, and bearing. The shear plate slid into the large check.

Joint J11.—This joint came from a cantilever which failed at a relatively low load because of careless workmanship. The pretest condition of the member that failed was excellent except that one shear plate had a hole $\frac{15}{16}$ in. in diameter, whereas the remaining shear plates had $\frac{13}{16}$-in. holes. The threads of the bolts were in bearing on the steel gusset plates and the shear plates. Failure was characterized by shearing and splitting, the bolts being almost sheared through and badly bent.

Joint J12.—The pretest condition of the tension member that failed was excellent. As in joint J11, the threads of the bolts were in bearing on the steel gusset plates and shear plates, and the bolts sheared through at the threaded sections.

Joint J13.—Despite some checking, the pretest condition of the compression member that failed was good. Members 1 and 3 of this joint did not fail. The joint came together and the members butted. The wood adjacent to the shear plates had failed and the bolts were badly bent.

Joint J14.—The tension member that failed in this joint had appeared in excellent condition until it was disassembled. Evidently the workmen had grooved the holes and knocked away the wood between the shear plates before assembling the joint. Failure was characterized by splitting and shearing. Bolts were badly bent; and failure occurred by shearing along old lines where split rings had been installed badly and where chips of wood had been removed.

Joints J15 and J16.—The members of this joint were in good condition, with some checking at the ends of tension members.

Joint J17.—The members of this joint were in excellent condition before testing. There were no checks in the tension members, but there were knots and a slope of 1/12 in the grain that caused bending failure in the wood. The tension member failed by tension and bending of wood. The shear plate did not fail.

Except for $\frac{3}{8}$-in. gusset plates of joints J11 and J12, all gusset plates were $\frac{5}{16}$ in. thick; $\frac{3}{4}$-in. bolts were used in joints J11 and J12, otherwise all bolts were $\frac{7}{8}$ in. in diameter. The bolt holes, in both steel plates and wood members, in accordance with recommendation of the Forest Products Laboratory, were $\frac{1}{16}$ in. oversize.

The members that failed in joints J10 and J17 were structurally defective; but, aside from these joints, the timber was select structural grade of Douglas fir. At the time of testing all members had an average moisture content of from 12% to 15%. Shrinkage had caused the wood members to lose their once snug fit with the steel gusset plates, but no remedial measures, such as tightening the nuts of bolts passing through the connectors, were attempted.

In all joints the holes in the wood members and in the shear plates matched those of the steel gusset plates very well. In all the joints removed from the Cavalcade stage framing (joints J1 to J10, inclusive, and J13 and J15 to J17, inclusive), at least one of each pair of gusset plates had been drilled in the field. In joints J11, J12, and J14, the members had been bored from both sides, but good fits had been obtained.

Testing Procedure.—The manner in which the joints were installed in the testing machine is shown in Fig. 1, and Fig. 51 shows, diagrammatically, the make-up of joints and method of loading. In detail, the load was applied in each test by an hydraulic jack which was placed at the end of one member, usually the chord member and termed member No. 1. A new tension connection was built at one end of one diagonal and the third member was blocked up to resist the induced compression. One exception to this procedure is: In joint J13, all three members were placed in compression; all connectors were 4-in. flush-shear plates; all bolts were $\frac{7}{8}$ in. in diameter except as noted in Fig. 51; and all gusset plates were $\frac{5}{16}$ in. thick except as noted.

Gages were installed to measure deformations of the various members with respect to the steel plates—that is, the slip of gusset and shear plates with respect to the wood members. Other gages were installed to indicate the angular rotation between members. The purpose of these last measurements was to obtain a basis from which the secondary stresses induced in a member when its connection deformed under load might be estimated. None of the

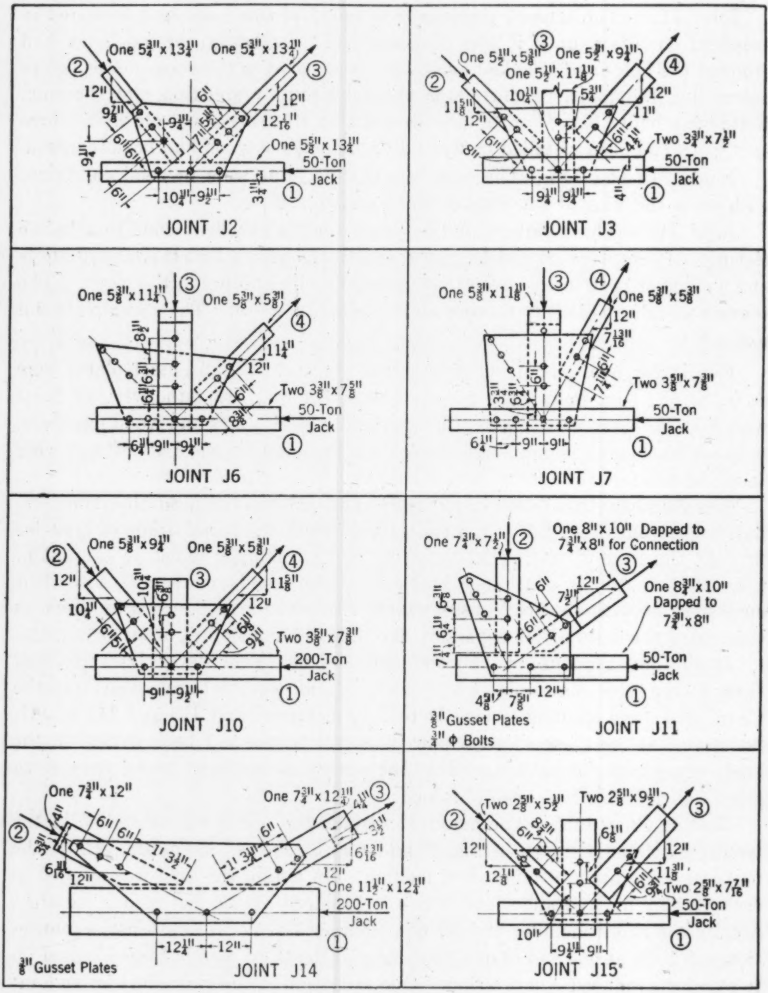
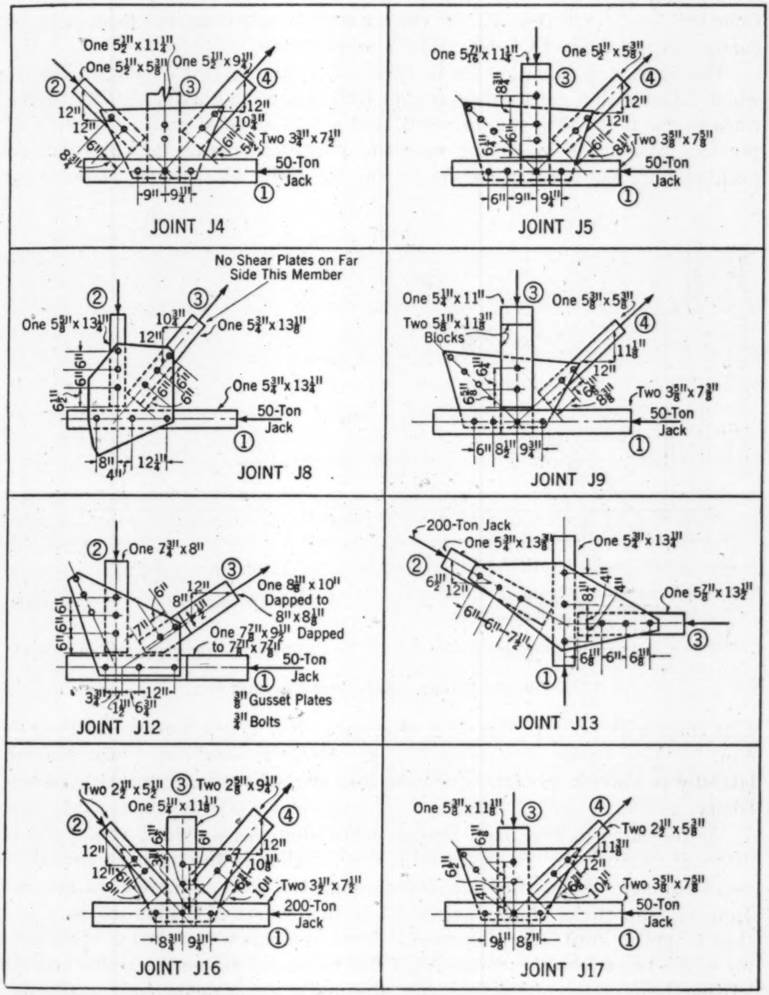

One 5¾"x13¼"
One 5¾"x13¼"
12"
9⅞"
9¼"
6"
12"
12 1/16"
One 5⅝"x13¾"
50-Ton Jack
10¼"
9½"
3¼"
JOINT J2
One 5½"x5⅝"
One 5½"x9¼"
One 5½"x11⅛"
10¼"
5¾"
12"
11"
Two 3¾"x7½"
50-Ton Jack
9¼"
9¼"
4"
JOINT J3
One 5⅝"x11¼"
One 5⅝"x5⅝"
8½"
11¼"
12"
Two 3⅜"x7⅝"
50-Ton Jack
6¼"
9"
9¼"
JOINT J6
One 5⅜"x11⅛"
One 5⅝"x5⅝"
12"
7 13/16"
Two 3⅝"x7⅞"
50-Ton Jack
6¼"
9"
9"
JOINT J7
One 5⅝"x9¼"
One 5⅜"x5⅝"
12"
10¼"
11⅝"
12"
Two 3⅝"x7⅜"
200-Ton Jack
9"
9½"
JOINT J10
One 7¾"x7½"
One 8"x10" Dapped to 7¾"x8" for Connection
12"
One 8¾"x10" Dapped to 7¾"x8"
50-Ton Jack
12"
4⅝"
7⅝"
⅜" Gusset Plates
¾" ϕ Bolts
JOINT J11
One 7¾"x12"
One 7¾"x12¼"
6"
1' 3½"
1' 3½"
6 13/16"
6 1/16"
12"
One 11½"x12¼"
200-Ton Jack
12¼"
12"
⅜" Gusset Plates
JOINT J14
Two 2⅝"x5½"
Two 2⅝"x9½"
6⅛"
12"
12⅛"
11⅜"
Two 2⅝"x7 7/16"
50-Ton Jack
10"
9¼"
9"
JOINT J15
①
②
③
④

Fig. 51.—Make-Up of Joints

bolts in the joints were tightened before testing, and the members of all joints were usually quite loose in their enclosing gusset plates. Because of this condition, all secondary effects and stresses are believed to be negligible and no attempt has been made to evaluate them. Loads were applied in increments of from about 5 kips to 10 kips to approximately the estimated design load; they were then released, the recovery was measured, and the loads were reapplied until failure had occurred in some member or its connection. The

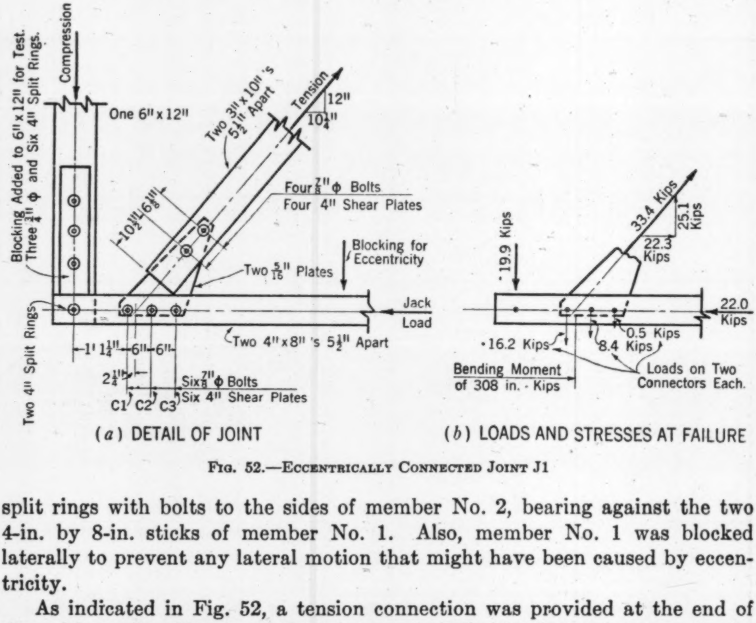

One 5½"x11¼"
One 5½"x5⅜"
One 5½"x9¼"
12"
10¾"
Two 3¾"x7½"
50-Ton Jack
JOINT J4
One 5⁷⁄₁₆"x11¼"
One 5½"x5⅜"
Two 3⅝"x7⅝"
50-Ton Jack
JOINT J5
No Shear Plates on Far Side This Member
One 5⅝"x13¼"
One 5¾"x13⅜"
One 5¾"x13¼"
50-Ton Jack
JOINT J8
One 5¼"x11"
Two 5⅛"x11⅜" Blocks
One 5⅜"x5⅜"
Two 3⅝"x7⅜"
50-Ton Jack
JOINT J9
One 7¾"x8"
One 8⅝"x10" Dapped to 8"x8⅛"
One 7⅞"x9½" Dapped to 7⅞"x7⅞"
50-Ton Jack
⅜" Gusset Plates
¾" Bolts
JOINT J12
200-Ton Jack
One 5¾"x13⅜"
One 5¾"x13¼"
One 5⅞"x13½"
JOINT J13
Two 2½"x5½"
Two 2⅝"x9¼"
One 5¼"x11⅜"
Two 3½"x7½"
200-Ton Jack
JOINT J16
One 5⅜"x11⅜"
Two 2½"x5⅜"
Two 3⅝"x7⅝"
50-Ton Jack
JOINT J17

WITH STEEL GUSSET PLATES

gages were generally removed before the ultimate loads were reached. After failure the joints were dismantled, while the results of the test, together with any pertinent observations, were noted and photographs taken. Fig. 1 shows a typical failure in the tension diagonal.

Eccentrically Connected Joint J1.—Fig. 52 shows the detail of this joint. It will be noted that the center line of the diagonal is eccentric by 3¾ in. with the centroid of the three pairs of shear plates with its connection to the chord (member No. 1). In Fig. 52, for convenience of reference, the three pairs of connectors are marked C1, C2, and C3, respectively.

The joint was found to be in excellent condition, with member No. 1, which failed with a moisture content of 13%, a specific gravity of 0.53, twelve rings to the inch, 30% summerwood, and a 1.18 slope of grain. Because of previous experience in testing eccentric joints of trusses in which failure resulted from the stress of tension across the grain, blocks were fastened by split rings with bolts to the sides of member No. 2, bearing against the two 4-in. by 8-in. sticks of member No. 1. Also, member No. 1 was blocked laterally to prevent any lateral motion that might have been caused by eccentricity.


Blocking Added to 6" x 12" for Test.
Three 3/4" ϕ and Six 4" Split Rings.
Compression
One 6" x 12"
Two 3" x 10" 's 5½" Apart
Tension
12"
10¼"
Four 7/8" ϕ Bolts
Four 4" Shear Plates
10½"
6⅜"
Blocking for Eccentricity
Two 5/16" Plates
Jack Load
Two 4" x 8" 's 5½" Apart
Two 4" Split Rings
1" 1¼" 6" 6"
2¼"
Six 7/8" ϕ Bolts
Six 4" Shear Plates
C1 C2 C3
19.9 Kips
33.4 Kips
25.1 Kips
22.3 Kips
22.0 Kips
0.5 Kips
16.2 Kips
8.4 Kips
Bending Moment of 308 in. - Kips
Loads on Two Connectors Each.

(*a*) DETAIL OF JOINT (*b*) LOADS AND STRESSES AT FAILURE

FIG. 52.—ECCENTRICALLY CONNECTED JOINT J1

As indicated in Fig. 52, a tension connection was provided at the end of diagonal member No. 3 and the joint was installed in the testing truss in the usual manner. Loads were applied through the jack at the end of the two 4-in. by 8-in. sticks of member No. 1 in increments of 5 kips released, and then reapplied until failure occurred. Measurements were taken of the widening of the two sticks of member No. 1 due to tension across the grain. At the left-hand connector of this member, the maximum widening of the member was registered, being 0.01 in., whereas, at the right-hand connector, the widening of the member was only 0.001 in.

Failure occurred through splitting of the two pieces of member No. 1 at a total compression in this member of 22 kips and a simultaneous tension in member No. 3 of 33.4 kips. The following excerpts from the laboratory notes are pertinent:

> "At a load of 16 kips in Member #1 a check in one of the two sticks of this member was opening and a new crack started, while at 20.8 kips the other stick also gave, failing in bending. The first stick went gradually

up to 16 kips and then opened up in tension across the grain following an old check that was about ½″ deep. This check kept getting larger until at 20.8 kips the other stick failed in flexure. The first stick failed by bending in the portion that had been split off due to tension across the grain. Both of the sticks also failed by twisting due to the connection of the gusset plates to only one side of each member.

"It was noted that the sticks of Member #3 bent due to the connections of the gusset plates on their sides. This caused the ends of the members to lift off about ⅛″ or more from the gusset plates. Both sticks showed the twisting, or rotational, action of the pull on the inside of members, causing a twist or rotation at failure of an amount sufficient to lift or lower the inside edge of the members about ⅝″ to ¾″."

Results of Tests, Joint J1.—The tension in member No. 3 at failure produced an axial stress of 22.3 kips in member No. 1 (due to the eccentricity of the connection), and a transverse shear of 25.1 kips. The axial stress may be assumed to have been resisted equally by the three pairs of shear plates in the member, or by a shear of 7.4 kips per pair of connectors. Because of the eccentricity of the connection (which was really doubly eccentric, since the chord was of two pieces, each connected by one face only), the total transverse shear was not carried equally by the three pairs of connectors. Computations indicate that the left-hand pair of connectors, C1, took a transverse shear of 16.2 kips; and the center pair of connectors, C2, took a transverse shear of 8.4 kips, whereas the transverse shear in the right-hand pair of connectors, C3, was 0.5 kips. Consequently, the maximum resultant shear was found in connectors C1, equal to 17.8 kips, and at a slope of 66° with the grain of the member. The stress P_c on a pair of shear plates, for nondense structural grade of Douglas fir, at an angle of 66° to the grain, is 8,250 lb. Therefore, R_{ea} was $\frac{17.8}{8.25}$, or only 2.16, instead of a value ranging from 3.5 to 5 as was found in the concentric joints of the series. This illustrates the loss of efficiency occasioned by the eccentric connection.

The test emphasizes the fact that the resistance of a joint of this type is lowered by an eccentric connection: First, because of the increased load and increased transverse shear on one or more connectors (in this case 1.55 times the average) resulting from the eccentricity; and, second, because of the increased intensity of stress in tension perpendicular to the grain.

At failure in the two 4-in. by 8-in. pieces of member No. 1, bending stress of 4.90 kips per sq in. occurred. Based on this bending stress, R_{wa} would be 3.04, this value of R having little meaning because failure was not caused by bending. The twisting, or rotation, of the two sticks has been mentioned in the excerpt from the laboratory's notes. This twisting, or rotation, was caused by the steel gusset plates pulling on the sides of the members rather than through their centers of gravity. This phenomenon may be seen in Fig. 53 where the near face of the 4-in. by 8-in. sticks has spread about ⅝ in. The bending effects caused by plate connections applied on one side of a member is further illustrated in the same photograph where, at the left, the ends of the 3-in. by 10-in. sticks of diagonal member No. 3 are shown separated from the steel gusset plates, which the sticks had caused to bend.

Concentrically Connected Joints.—The details of the concentrically connected joints are illustrated in Fig. 51. Stress ratios, R, are given in Table 9, Cols. 9 and 10.

Joint J2.—Shrinkage of the wood caused the members entering this joint to separate from the gusset plates a distance of $\frac{3}{16}$ in. to $\frac{1}{4}$ in.; and, during this test, this effort resulted in a buckling of gusset plates. At a load of 42.5 kips in member No. 3, the bolts were bending and the gusset plates at member No. 1 were noticed to be buckling at a load, in the member, of 88.5 kips. Failure occurred in member No. 3 by shearing and splitting of the wood at a total stress of 114.4 kips.

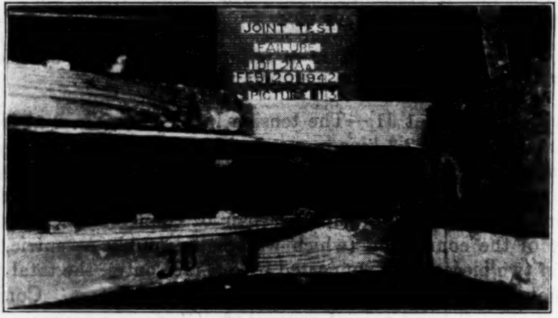


Fig. 53.—Effect of Connecting Steel Plates on Only One Side of Each Member, Joint J1

Member No. 1, with six shear plates at 10-in. spacing, resisted without failure or apparent distress 24.6 kips per shear plate, giving a R_{ca}-value of 4.45. This high value of R may be regarded as substantiating the standard spacing of 9 in. recommended for shear plates, although it is to be noted that the gusset plate on one side of member No. 1 had moved laterally 0.11 in. with respect to the member whereas the relative movement on the other side was 0.40 in.—an average of 0.26 in. The large difference in the relative movements of the two faces is difficult to explain, for there was no evident lateral eccentricity of the joint. The following values based on ultimate loads taken by member No. 3 are of interest:

Ultimate load......................	114.4 kips
Proportional limit load..............	60.0 kips
Ratio R_{ca}.........................	4.56

Joint J3.—At a total axial stress of 53.1 kips in the tension diagonal member No. 4, this member failed by shearing and splitting, a large split occurring through the bolts and shear plates. Failure went through a deep check. Previous to this failure, and at a stress of 46 kips in this member, a large cracking had been heard and the load had fallen off. The end distance in member No. 4 was only 4½ in.

Joint J4.—Although, in this joint, the tension diagonal, member No. 4, failed at a total axial stress of 57.1 kips, with typical shearing and splitting, a new split, about 3½ in. deep, had occurred in the end of the member at a stress of 24.2 kips. This split was located in the center of the member, parallel to the center line of the shear plate.

Joint J5.—Unlike the most of the other joints this joint failed suddenly. Member No. 4, the tension diagonal, failed by typical shearing and splitting at a total stress of 87.7 kips. On dismantling the joint, it was noted that member No. 3, in compression, showed no distress and that it was well fitted. At the time that member No. 4 failed, each shear plate in member No. 3 had resisted 10.2 kips. The end distance was 8½ in. (Figs. 1 and 54(a)).

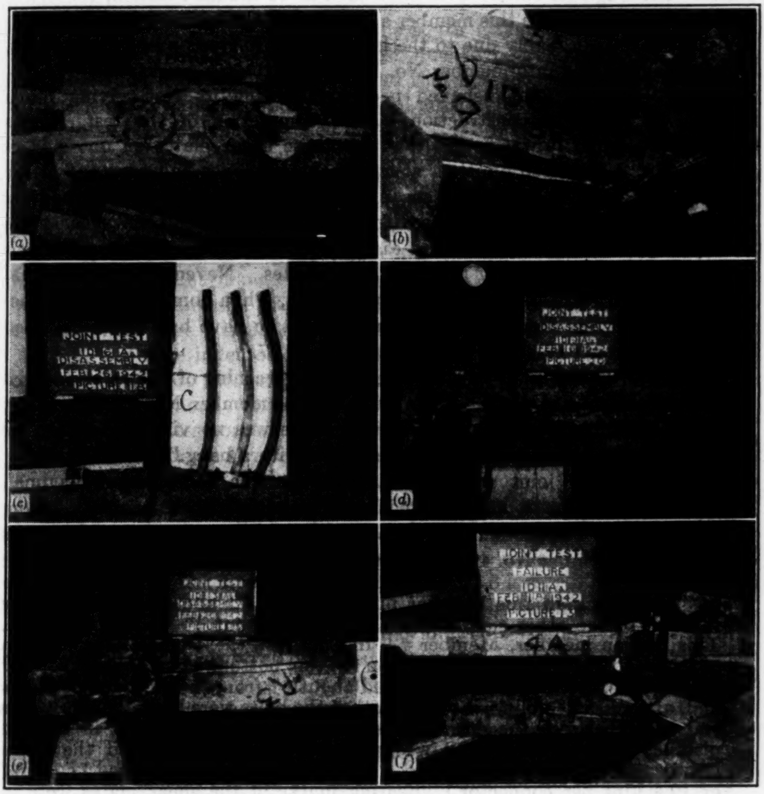


Fig. 54.—The Failure of Concentrically Connected Joints

(a) Joint J5
(b) Joint J8
(c) Joint J8
(d) Joint J10
(e) Joint J11
(f) Joint J17

Joint J6.—The test behavior of this joint was similar to that of joint J5. The end distance was 8½ in.

Joint J7.—Member No. 4, the tension diagonal that failed, did not fit tightly between the gusset plates, because of shrinkage which was estimated to be about $\frac{1}{16}$ in. This member failed, with typical shearing and splitting, at 91.2 kips. Its end distance was 7¾ in.

Chord member No. 1 consisted of two 3⅝ in. by 7⅝ in. sticks, each connected to its gusset plate on one side only, but each stick was fastened with four shear plates. Although its connections were such as to induce rotation, there was no evidence of distress in this member at the maximum load of 44 kips (5.5 kips per shear plate), probably due to the lateral support given by member No. 3, which extended well into member No. 1, between the two sticks.

Joint J8.—This joint behaved peculiarly during its test, giving much evidence of defective construction. After completion of the test, it was taken apart, when it was found that the three shear plates on one side of member No. 3 had been omitted, although their grooves had been cut in the wood. This omission caused the shear plates on the other side of the member to carry almost the entire load. Fig. 54(*b*) shows one of the shear plate grooves without its shear plate, and Fig. 54(*c*) shows the bent bolts with their evidence of excessive bearing from the shear plates. Nevertheless, the actual capacity of the joint, as determined by the test, when compared to the theoretical capacity as designed, assuming the joint to have had all its designed shear plates, would give $R_{ca} = 3.61$, and $R_{ct} = 1.37$.

Joint J9.—This joint gave visual and audible warning of failure at an axial load in member No. 4 of 82.5 kips, the failure of member No. 4 occurring at 87 kips. At this ultimate load, however, there was no visual crack. The laboratory notes record that the failure "came with a bang."

Joint J10.—This joint was defective, in that the compression diagonal, diagonal member No. 2, had a boxed heart, although, when examined before testing, the joint was pronounced to be in excellent condition; however, it was observed that member No. 4 had a large check. The laboratory notes record:

> "The check in this member opened about ½" and split all the way through to the other side, where a new split about ½" wide was opened. This tension member, Member #4, did not fail, there being no indication of any timber distress. No cracks on the top, bottom, or end of the member were visible, but the stick had moved about ⅝" from the steel gusset plates and, after disassembling, it was seen that the bolts were badly bent. Cores and wood between the shear plates had sheared, although not in the end distance area. Member #2 was as expected; the cores and the wood between shear plates had sheared. (This member had withstood 77.7 kips, or 19.4 kips per connector.) Bearing marks of the shear plates were in evidence on all bolts but the shear plates were not damaged (see Fig. 61 [54(*d*)])."

In this joint, the 6-in. by 6-in. member which was in tension, resisted without failure a load of 111.4 kips on two pairs of 4-in. shear plates, spaced at 6 in., with an end distance of 9 in. Thus, each shear plate resisted a load of 28 kips, equal to six and one half times its allowable load.

This member was a single stick, with its connection in double, instead of single, shear; and, although the spacing of connectors was only 6 in., the end distance was 9 in.—the two important factors leading to the high strength. The test of this joint has led to the suggestions, given subsequently under the heading, "Comments on Tests of Joints with Steel Gusset Plates," that: (1) Perhaps the reduced values of allowable connector loads for 6-in. spacings are not justified; and (2) the allowable loads for connectors in single shear should be less than one half of the allowable loads when the connectors are in double shear.

Joint J11.—This joint, taken from the south Cavalcade cantilever truss (which had failed under preliminary test due to poor workmanship), was defective. Upon dismantling it was found that one shear plate had a $\frac{13}{16}$-in. hole for a $\frac{3}{4}$-in. bolt and that both shear and gusset plates bore on the threads of the bolts. Fig. 54(*e*) shows failed member No. 3 after disassembly.

Joint J12.—Fig. 51 illustrates that joint J12 had $\frac{3}{8}$-in. instead of $\frac{5}{16}$-in. gusset plates. The joint was obtained from the north Cavalcade cantilever truss (not the truss that had failed) in which it formed the top detail on the rear end of the anchor arm of the cantilever. To all appearances the joint was in excellent condition, having no important checks or cracks and having undergone but small shrinkage—probably the maximum shrinkage was $\frac{1}{32}$ in.

At a total axial stress of 63.1 kips, member No. 3 failed in its connection. The bolts, which were $\frac{3}{4}$ in. in diameter, sheared off at the nuts, at the root of the thread, and were indented at the ends near the head. The member did not split; neither were there any checks or cracks in the wood of the member. The wood did not shear. After dismantling the joint, it was found that the gusset and shear plates bore on the bolt threads, that a number of shear plates with $\frac{13}{16}$-in. holes had been used with $\frac{3}{4}$-in. bolts, and that these bolts had been bent about $\frac{3}{8}$ in.

Joint J13.—In this joint all members were tested in compression. The joint was in excellent condition with the bolts giving evidence of about $\frac{1}{4}$-in. shrinkage of the wood. The test continued over two days. On the first day, the load on member No. 1 reached a maximum of 15.8 kips. Then, because of mechanical difficulties, the test was stopped. The load automatically dropped to 10.0 kips, and the joint with this load was allowed to stand overnight. On the following day, the load was reapplied, reaching a maximum of 59.2 kips in member No. 1 and 112.1 kips in member No. 3. At a load of 80.4 kips in member No. 2, this member was splitting and (see Fig. 51) the gap between the end of member No. 3 and the side of member No. 1 had closed. With further increase in load, member No. 2 continued to split. The gusset plates were seen to be bowing slightly at the maximum load on the joint (126.4 kips in member No. 2). The shear plates in this member had sheared the wood, and its bolts were badly bent. Because of the bearing of the ends of member No. 2 and member No. 3 against the sides of member No. 1, no direct failure had occurred—that is, the load did not diminish, although it could not be increased (see Table 9).

Joint J14.—As in joint J12, the gusset plates were $\frac{3}{8}$ in. instead of $\frac{5}{16}$ in. The joint was seemingly in excellent condition, the bolts indicating a shrinkage of the wood from $\frac{1}{8}$ in. to $\frac{1}{4}$ in. Although the structure of the Triumphal Arch from which the joint was obtained had been wrecked by falling, the joint gave no visible evidence of having been damaged.

As the load was applied, deformations of the tension diagonal member No. 2 seemed to be quite large; and, at an axial stress of 29.75 kips in member No. 2, the shear plates in this member appeared to have moved about 0.20 in. Distress of the joint was audible. At a stress of 47.8 kips in member No. 2, the split in the member became wider, but the failure of the joint occurred in member No. 3 at a stress of 45.2 kips, by splitting and shearing of the wood by the shear plates.

On disassembling the joint it was found that the wood cores of the shear plates had been broken and sheared off, stains indicating that this damage had occurred prior to the test. Further evidence of prior damage was the fact that the wood between the shear plates on one side of the connection of member No. 3 and part of the wood between the shear plates was missing. The other part was loose and its edges stained, as if the joint had weathered after the wood had sheared off.

Joint J15.—The joint was in good condition except for a check in one stick of member No. 2 and a small check in one stick of member No. 3. Failure occurred in the tensional diagonal (member No. 3) at an axial stress of 52.5 kips by typical shearing of the wood in the end distance and between the shear plates, combined with the usual splitting. This member was not of solid section, but was composed of two sticks, each connected by one side only; hence, $R_{ca} = 2.91$.

Joint J16.—The joint was in excellent condition except for a check in one of the sticks of tension diagonal member No. 4, the check extending from the last shear plate to the end of the timber. The joint was opened $\frac{1}{8}$ in. before testing. Failure occurred suddenly in member No. 4 by the usual shearing and splitting. As in joint J15, note the lower strength of the two stick member, in that $R_{ca} = 2.93$.

Joint J17.—Before testing, the joint appeared to be in excellent condition, shrinkage of the wood being small. Failure occurred in member No. 4 at a stress of 33.1 kips, one of the two sticks of member No. 4 failing in combined bending and tension (see Fig. 54(*f*)). This member was found to have a 1/12 slope of grain, and a knot at the point where the split at failure intersected at the edge of the stick. The gusset plate bent badly as the member could not take the bending moment induced by its eccentric connection.

Comments on Tests of Joints with Steel Gusset Plates.—The foregoing results lead to ten broad conclusions, as follows:

(1) The necessity of careful and continuous inspection during construction, if material, workmanship, and resultant strength commensurate with the design are to be expected, is particularly illustrated by the tests of these joints. For example, joints J8 and J14, before testing, gave the appearance of being entirely normal. Indeed, an extract from the laboratory notes of the test of joint J8 reads—

> "The test of this joint was to have been one of the most important of the Test-Program because of the very excellent condition of the joint and its careful handling prior to the test. However, during the test the behavior of the joint was so peculiar that it was suspected of being deficient internally, and after the test it was taken apart. Then only was it found that the joint had but one-half of its indicated number of shear-plates, those on one side having been completely omitted."

Joint J14 was also found to have been damaged in fabrication; joint J10 had a member with boxed heart with resultant checking; in joints J10, J11, and J12, short bolts had been used, causing gusset and shear plates to bear on the bolt threads; and joint J12 actually failed by shearing of the bolts. Also, in joints J11 and J12 shear plates were used with the wrong size of bolts.

(2) Even with the defects described in item (1), joints J8 and J14 did not fail until they had sustained twice the allowable connector loads P_a for 9-in. spacings and end distances and two and one-half times the allowable loads for 6-in. spacings and end distances.

(3) Effects that can occur due to the presence of boxed heart, knots, or sloping grain are illustrated by joints J10 and J17, in which one or more of these characteristics influenced or contributed to failure.

(4) In a two-piece member each piece is connected on one face only, or with the connectors of each piece in single shear, whereas a one-piece member has connectors in both faces—that is, a double-shear connection. The decreased capacity of the connection of a two-piece member is illustrated by comparison of the average maximum loads at failure of joints J15 and J16 (13.20 kips per shear plate) with the average of joints J2, J5, J6, J7, and J9 (21.6 kips per shear plate). These joints all had end distances well in excess of 6 in. The relative efficiency of the one-piece member with double-shear connection as compared with the two-piece member with single-shear connection raised the question as to whether joints in single shear may properly be considered as having one half of the strength of joints in double shear.

(5) The use of a ¾-in. bolt with a 4-in. shear plate, having a 15/16-in. hole intended for use with a ⅞-in. bolt, would seem to be a comparatively easy oversight on the part of the carpenter. Some method of avoiding this condition should be developed.

(6) In these tests no ⅞-in. bolt used with a 4-in. shear plate failed in shear.

(7) The increased capacity of one-piece tension members in double shear, with end distances of 8 in. or 9 in. over one-piece tension members similarly connected but with end distances of only 6 in. or less, was in the ratio of from 1.4 to 1.0—an increase in capacity of 40%.

(8) The slip of shear plates at the allowable or working load was found to be from about 0.05 in. to 0.06 in., confirming published values.

(9) In view of the comment in items (4) and (7), it will be enlightening to group eight of the joints, J3, J4, J5, J6, J7, J9, J15, and J16 (all having 6-in. center-to-center spacings of connectors, but with varying end distances) into three divisions governed by end distance and type of shear and then to note the average ratio value of R_{ca} of each group. Such a grouping is given in Table 10. The joints with connections in single shear, even though they had

ample end distances (group 1), had low values of R. The joints with connections in double shear, even with substandard end distances (group 2), showed increased R-values; and the joints with connections in double shear and with greater than standard end distances (group 3) had the highest values of R. If the nine tests reported in Table 10 are judged sufficient in number to warrant drawing conclusions it would appear that, in this type of joint—

TABLE 10.—EFFECT OF END DISTANCE AND TYPE OF SHEAR

Group	Joints	End distance (in.)[a]	Average R_{ss}
1[b]........	J15 and J16	10	2.92
2[c]........	J3 and J4	4.5 to 5.5	3.50
3[c]........	J5, J6, J7, and J9	7.75 to 9	4.87

[a] Connector spacing 6 in., center to center. [b] Single shear. [c] Double shear.

a. Shear plates at connections in single shear seem to be considerably less effective than such connections in double shear, at least for relatively thin members (3 in.). This condition may be due to stress caused by eccentricity, shrinkage, or other factors. In some cases, the allowable load on a connector in a double-shear connection should be materially greater than that on a connector in a single-shear connection.

b. The reduction in allowable loads for connectors with a spacing of 6 in. center to center, but with a 9-in. end distance, is too severe.

c. The 7-in. end distance for full allowable load for connectors now recommended as standard, as against the former recommended standard of 6 in., is amply justified. Further increase of end-distance requirements might result in even higher allowable loads.

(10) A comparison of the ratios obtained from the joint tests which failed in the shear plates, joints J2 to J9, J12, J15, and J16, is afforded by the data in Table 11(*a*). Joints J15 and J16 were those in single shear which failed at lower ratios than the others. Without these two joints the comparison would be as shown in Table 11(*b*). The joints were fabricated from unseasoned material and tested after seasoning in service. For this condition the recommended working stresses in the TECO Manual of 1939[1] require a reduction in stress of 20%. Calculated on this basis the test results would give comparisons 25% higher than those shown in Table 11.

TABLE 11.—COMPARISON OF EXPECTED AND R-VALUES FROM TESTS

Source	Expected values	(*a*) JOINTS J2 TO J9, J12, J15, AND J16		(*b*) JOINTS J2 TO J9, AND J12	
		Average	Range	Average	Range
Manual, TECO[a].....	3.5	4.16	2.91 to 5.04	4.43	3.45 to 5.04
Basic values, FPL[b]...	4.0	4.78	3.55 to 5.80	5.09	4.00 to 5.80

[a] "Timber Connector Construction," Manual of the Timber Eng. Co., Inc., Washington, D.C., 1939.
[b] Forest Products Laboratory, Madison, Wis.

However, it should be noted that in these tests the load was always parallel to grain, under which condition the effect of fabricating with green timber and testing after seasoning in place is much less severe than when loads are applied at an angle to grain.

Joints with Plywood Gusset Plates, Bolted or Split-Ring Connections.—Joints J18 to J22, inclusive, consisted of one bolted joint connected with a plywood gusset plate and four other joints in which a member was spliced with plywood splice plates fastened with split rings.

The first joint (J18) was removed from the San Francisco Building where it had served as part of the lateral bracing system of the lower chord of the roof trusses. The details of the joint before testing are shown in Fig. 55. The joint consisted of a $2\frac{1}{4}$-in.-thick seventeen-ply Douglas fir plywood gusset plate to which were connected two diagonal members, each a 2-in. by 8-in. stick, with $\frac{3}{4}$-in. bolts in single shear. (This plywood was of "exterior" type, made with phenol formaldehyde resin adhesive.) The other members of the joint (not concerned in the test) were parts of a 4-in. by 10-in. strut and the lower chords of one of the original trusses, consisting of 2-in. by 8-in. and 3-in. by 12-in. members also connected with $\frac{3}{4}$-in. bolts. The joint was in excellent condition and was tested at a moisture content, in the 2-in. by 8-in. diagonal, of about 13.5%.

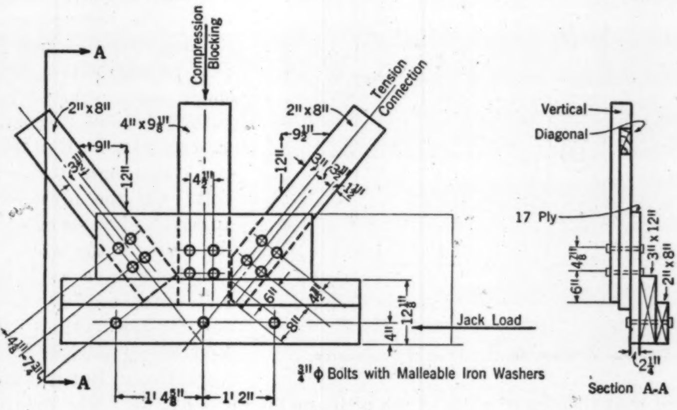


FIG. 55.—DETAILS OF JOINT J18, SAN FRANCISCO BUILDING

The other four joints, J19, J20, J21, and J22, were constructed in the laboratory, of materials from the Exposition buildings. Their details are shown in Fig. 56. Each joint consisted of a center 6-in. by 8-in. block with two splice plates of twenty-nine-ply plywood $3\frac{5}{16}$ in. thick and 8 in. wide, connected to the 6-in. by 8-in. block with eight 4-in. split rings and four $\frac{3}{4}$-in. bolts and with a 9-in. spacing and a 7-in. end distance (see Fig. 56). The grain of the face ply ran parallel to the direction of the load in two of the tests, J19 and J20, and perpendicular in the other two, J21 and J22. The plywood was obtained from the Federal Building where it had been exposed to the weather for about $2\frac{1}{2}$ years.

Bolted Joint (J18) with Plywood Gusset Plates.—Joint J18 failed in the 2-in. by 8-in. tension member by shearing out the end, the end distance being 6 in. The washers were pulled into the face of the plywood causing some splitting of the top plies and creating a corrugated effect. Meanwhile, the bolts had been bent into an S-shaped form.

The total stress in the diagonal at failure was 42 kips. The connection of the diagonal, in which four $\frac{3}{4}$-in.-round bolts were in single shear, had a computed total safe load, based on the 2-in. by 8-in. stick, of 4.7 kips, or 1.16 kips per bolt. On the plywood at an assumed value of 45° to the grain of the wood, the four bolts have a value of 3.11 kips, or 780 lb per bolt. These values yield *R*-values of 8.95 and 13.5, respectively; but it is to be noticed that the slip of bolts at the ultimate load was 0.35 in. The deformation at a working load of 4.7 kips on the four bolts was 0.05 in.

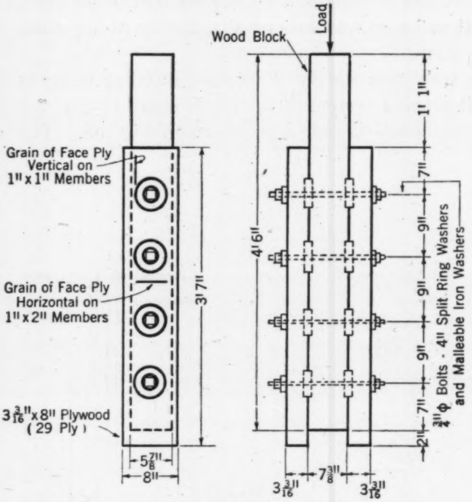

Wood Block
Load
Grain of Face Ply Vertical on 1" x 1" Members
Grain of Face Ply Horizontal on 1" x 2" Members
3 3/16" x 8" Plywood (29 Ply)
3' 7"
4' 6"
5 7/8"
8"
7 3/8"
3 3/16"
3 3/16"
1' 1"
7"
9"
9"
9"
7"
2"
3/4" φ Bolts · 4" Split Ring Washers and Malleable Iron Washers

FIG. 56.—DETAILS, JOINTS J19 TO J22, INCLUSIVE

The proportional limit of the bolts is estimated to have been about 8.0 kips, giving a ratio of proportional limit load to allowable, or working load, R_{bl} of 1.7, based on the bolt value of the 2-in. by 8-in. member. The corresponding slip of bolts was 0.10 in. Summarizing these results (in inches):

Based on the 2-in. by 8-in. stick—
$R_{ba} = 8.95$
$R_{bl} = 1.7$
Based on the plywood—
$R_{ba} = 13.5$
Slip of the bolts—
At allowable load $P_b = 0.05$ in.
At proportional limit
load P_l = 0.10 in.

Compression or Tension Splice Joints with Plywood Splice Plates and Split Rings.—Joints J19 to J22, inclusive, with plywood splice plates failed in two types of failure, as illustrated in Figs. 57. Fig. 57(*a*) shows the crushing and buckling of the individual plies of the plywood when subjected to compression loads—the first type. The other type of failure is shown in Fig. 57(*b*) in which the fibers of the plies running perpendicular to the direction of the load are seen to have sheared and "rolled" in the plywood, causing a shear failure at the base of the split rings.

After dismantling the four joints, it was found that all the ring cores in the plywood had failed and that the cores in the rings of the 6-in. by 8-in. center block remained intact. In the ring cores, about 25% of the core area failed in the glue; the remainder failed in the wood. The failures always occurred at a ply located at approximately the bottom of the ring and in a ply whose grain was perpendicular to direction of the load. The cores evidently failed by "rolling shear" action. (The term "rolling shear" has been applied when the shearing force tends to deform the hollow wood cells diagonally in a direction perpendicular to their fibers.)

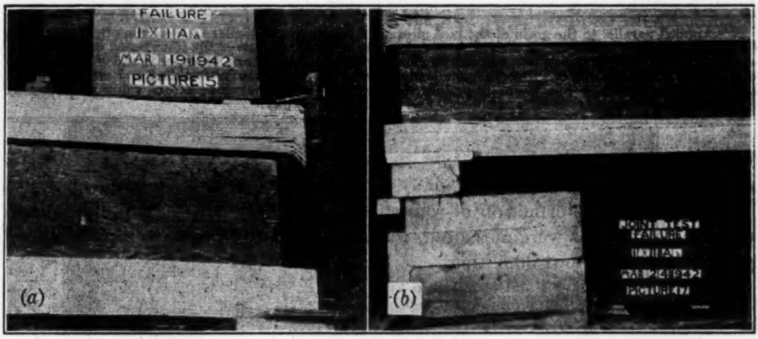


Fig. 57.—Two Types of Failure Joints with Plywood Splice Plates
(a) Crushing of Plywood
(b) Rolling Shear Failure in Lower Plywood Pad

The results of these four tests are summarized in Table 12, which shows maximum loads, manner of failure, and other pertinent data. The working values of split rings in plywood have been taken as the TECO allowable loads for group A woods with direction of load at 60° to the grain. The working capacity of plywood in compression was assumed at 1,735 lb per sq in., and the allowable rolling shear value at 60 lb per sq in.—both of these stresses being those recommended by the Douglas Fir Plywood Association based on tests by the Forest Products Laboratory.

Results of Tests.—The ratio R_{ba} of 8.95, as found in the test of joint J18, is characteristic of bolts; also a value of R_{bt} of 1.7, based on the 2-in. by 8-in. member, seems proper. The tests indicate that the joint had been proportioned correctly. The tests also indicate that an increase in the edge distances in plywood would reduce any effects of rolling shear at a critical plane, and that the plies running parallel to the loads are the effective ones in resistance to compression. It follows that the shearing and crushing strength of the plies needs particular attention in the design of joints of this type. Additional test data in connection with both bolt and connector fastenings when used

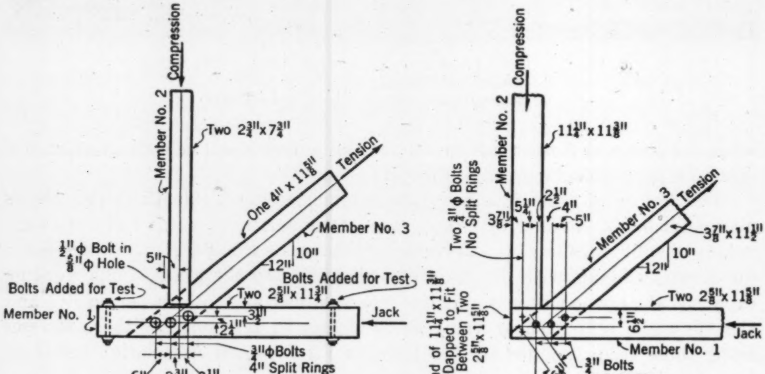
TABLE 12.—JOINTS WITH PLYWOOD SPLICE PLATES AND 4-IN. SPLIT-RING CONNECTORS

Joint	Load on Joint, in Kips				R_{ca}	Deformation at allowable load (in.)	P_l	R_{sl}	Residual load[f] (kips)	Testing time (min)
	Maximum	Allowable, Based on:								
		Split rings[c]	Plywood (compression[d])	Plywood (shear[e])						
J19....	130.0[a]	40.96	45.6	41.5	3.17	0.045	61	1.49	98.8[g]	35
J20....	145.5[a]	40.96	45.6	41.5	3.5	0.048	84	2.06	83.0[h]	5
J21....	142.3[b]	40.96	42.9	41.5	3.47	0.074	82	2.00	...[h]	10.5
J22....	148.3[b]	40.96	42.9	41.5	3.62	0.041	70	1.71	73.5[h]	35[i]

[a] Applied parallel to the grain of the face plies. [b] Applied perpendicular to the grain of the face plies. [c] Based on group A wood; 60° to the grain. [d] Allowable or working loads of the joint based on plywood in compression at 1,735 lb per sq in. [e] Based on an allowable rolling shear of 60 lb per sq in. [f] After failure the joint was still capable of supporting the loads shown. [g] Failure was by buckling and crushing of the splice plate, as shown in Fig. 57(*a*). [h] Failure was by buckling and crushing of one splice plate and by rolling shear on the other. [i] 29 min one day and 6 min the next day.

with plywood, such as information on plywood splice plates and gusset plates, would be helpful in design, especially when large loads are involved.

The average deformation of joints J19 to J22, at their maximum loads at which deformations were measured, was about 0.24 in., corresponding to a load per split ring of about 30.5 kips.


Compression
Member No. 2
Two 2¾" x 7¾"
Tension
One 4" x 11⅝"
Member No. 3
½" ϕ Bolt in
⅝" ϕ Hole
5"
10"
12"
Bolts Added for Test
Bolts Added for Test
Two 2⅝" x 11¾"
Member No. 1
Jack
3½"
2¼"
¾" ϕ Bolts
4" Split Rings
6"
3¾"
3½"
Compression
Member No. 2
11¼" x 11⅜"
Two ¾" ϕ Bolts
No Split Rings
3⅞"
5¼"
2½"
4"
5"
Member No. 3
Tension
3⅞" x 11½"
10"
12"
End of 11¼" x 11⅜"
Dapped to Fit
Between Two
2⅝" x 11⅝"
Two 2⅝" x 11⅝"
6½"
Jack
Member No. 1
¾" Bolts
4" Split Rings
6½"

(*a*) FROM PACIFIC BUILDING TRUSSES; J23

(*b*) FROM PACIFIC BUILDING TRUSSES; J24

FIG. 58.—DETAILS OF JOINTS WITH SPLIT-RING CONNECTORS

Joints with Split-Ring Connections.—This series of tests comprised six joints, J23 to J28, inclusive, in which the members were connected by 4-in. split rings and bolts. Two of the joints, J23 and J24, were from an 80-ft roof truss of the Pacific Building. The other four, J25 to J28, inclusive, were compression or tension splices from the stage framing of the Cavalcade of America.

Joints J23 and J24.—One of the truss joints, J23, was taken from an interior panel of the truss, and the other, J24, formed the end joint of the same truss at its supporting column. Fig. 58 shows the details of these two joints. Diagonal member No. 3, Fig. 58(*a*), connects to chord member No. 1 by three pairs of split rings, and vertical member No. 2 was intended to bear on the top side of member No. 1.

The trusses from this building were dropped and damaged during dismantling. The following excerpt from the laboratory notes is pertinent:

> "Joint J-23 was obtained from the Pacific Building where it was an interior joint of one of the 80′ span roof trusses. It was only in fair condition for the two sticks of the chord, Member #1, were badly split. Also, Member #3 had a large check. The truss had been dropped in wrecking the building but the joint had suffered no visible damage. Because of splitting of the chord member, stitch bolts were placed as shown in Fig. 73 [58(*a*)]."

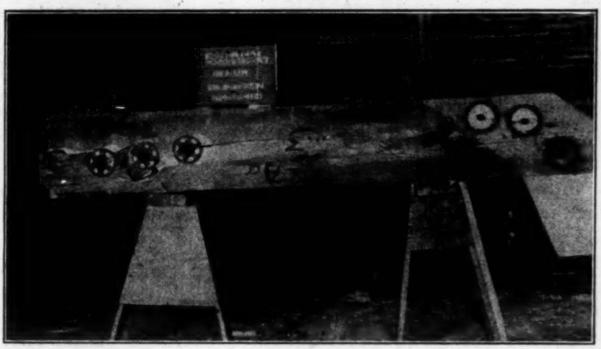


FIG. 59.—FAILURE OF JOINT J24

Although the second of these truss joints, joint J24, was in a somewhat damaged condition prior to the test, the results apparently were not affected. The members of both joints J23 and J24 were of select structural grade Douglas fir, and the two joints were tested at a moisture content of about 15%. No shrinkage was noted that would cause looseness of the bolts; but, in joint J23, there was a space of $\frac{5}{16}$ in. between vertical member No. 2 and chord member No. 1.

Joint J23 failed by splitting of diagonal member No. 3, at a total stress of 44.3 kips, or only 7.4 kips per ring, corresponding to one and one-half times the allowable load. The existing large check had opened into the wide crack. After disassembling the joint, it was noticed that all the ring cores had been sheared off before testing, except one core in member No. 1 and one core in member No. 3. These two particular cores were not even sheared at the completion of the test. The bolts were not bent. The wood in contact with the two end rings appeared to be in good condition, but the inside ring had "slid" into the check.

In the test of joint J24 (see Fig. 58(*b*)), failure did not take place in the original joint but in the tension connection installed for testing purposes at one end of diagonal member No. 3. When the connection had failed, several difficulties prevented its reconstruction, so the test was discontinued. Fig. 59 shows the member after disassembly, and the shearing out of shear plates in the tension connection may be noted. There were several small knots in this particular location. Attention is called to the grooves and cores of the split rings which remained in good condition, although the connectors had been loaded to three times their allowable loads. The maximum load on the member was 87.2 kips.

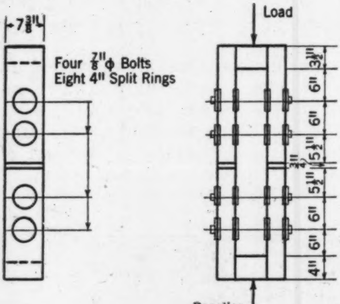


FIG. 60.—TYPICAL DETAILS OF JOINTS J25 TO J28, INCLUSIVE

Joints J25 to J28, Inclusive.—Each splice joint consisted of a 6-in. by $7\frac{1}{2}$-in. center timber, spliced with two $3\frac{1}{2}$-in. by $7\frac{1}{2}$-in. splice pads, and it had a single row of four pairs of 4-in. split rings with $\frac{7}{8}$-in. bolts—a total of eight split rings (see Fig. 60). The joints were in excellent condition. Shrinkage of the wood had resulted in a gap between the center stick and each splice pad of a measured amount of from $\frac{1}{16}$ in. to $\frac{5}{16}$ in.—an average of $\frac{3}{16}$ in. For convenience of testing, in which the joints were placed in compression, a $\frac{3}{4}$-in. to $\frac{7}{8}$-in. gap was provided at the center of the length of splice pads. The movement of splice members was measured. Because of the limitation of slip of joints as determined by the $\frac{3}{4}$-in. to $\frac{7}{8}$-in. gap between the splice pads, the maximum resistance of the split rings was not realized. Table 13 and Table 14 summarize the results, and Fig. 61 shows the stress deformation curves of the four joints.

TABLE 13.—TRUSS JOINTS WITH SPLIT RINGS

Joint	Location	Ultimate load (kips)[a]	P (kips)[b]	R_{ea}	Slip (in.)[c]
J23..........	Intermediate Truss Joint: Tension member..........	44.3	28.9	1.55	....[d]
J24..........	End Reaction Truss Joint: Tension connection........ Joint....................	 87.23 87.23[e]	 23.8 28.9	 3.65 3.00[e]	 .... 0.059

[a] In tension member. [b] Six split rings. [c] Slip of split rings longitudinally in member No. 3 at the design allowable load P. [d] Not evident. [e] In tension member, observed proportional limit was at a tension of 42.3 kips giving an R_{et}-value of $\frac{42.3}{28.9} = 1.46$. This tension member did not fail in the joint, failure occurring in the tension connection provided for testing.

Results of Tests.—Reference to the stress deformation curves of Fig. 61, all plotted to the same scales, and also to Table 14, indicates the erratic test behavior of these joints. The slopes of the first sections of the stress-deformation curves of joints J25, J26, and J27 are very flat as compared with the

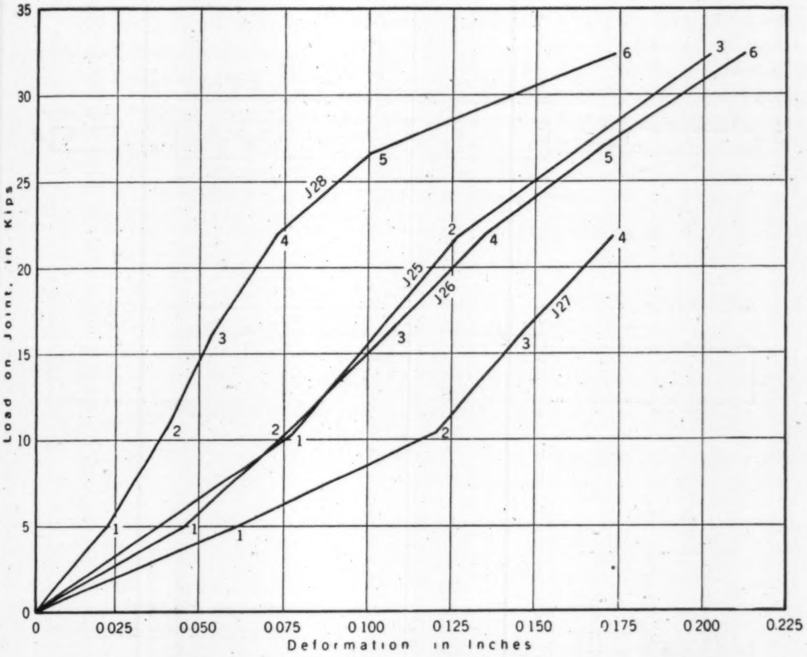


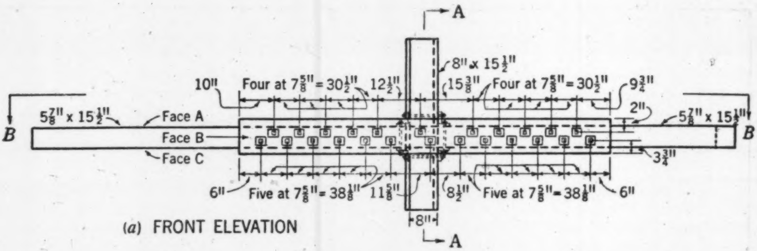
FIG. 61.—LOAD-DEFORMATION CURVE, JOINTS J25 TO J28, INCLUSIVE (SPLICE WITH FOUR, 4-IN. SPLIT RINGS AND TWO ¾-IN. BOLTS ON EACH SIDE)

similar part of the curve of joint J28, indicating that a relatively large initial deformation took place in joints J25, J26, and J27. Up to a load of 10 kips the curves of joints J25 and J26 coincide, having a slope of about 4 on 7, whereas the curve of joint J27 has a relatively very flat slope. Joint J28, which

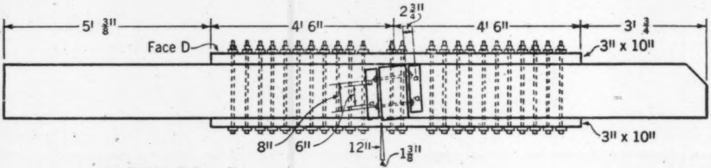
TABLE 14.—SPLICE JOINTS (J25 TO J28) WITH SPLIT RINGS

Joint	Gap between spliced members (in.)	Maximum Loads on Joint[a]		Proportional limit load[b] (kips)	Allowable load[c] P_e	Ratio[a] R_{ee}	Deformation[d]	Opening caused by shrinkage[e]
		Four rings (kips)	Kips per ring					
J25........	¾	52.5	13.1	22.0	20.2	2.6	0.119	⅛
J26........	¾	51.5	12.9	22.0	20.2	2.55	0.126	1/16
J27........	¾	51.5	12.9	....[f]	20.2	2.55	0.166	1/16
J28........	¾	55.5	13.8	22.0	20.2	2.75	0.067	⅛

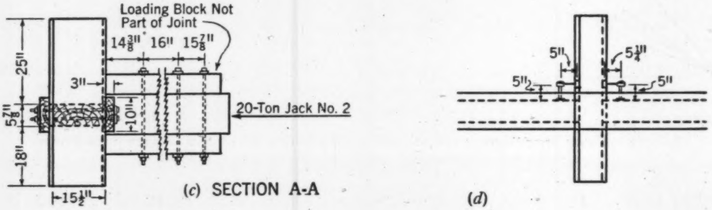
[a] The joints did not fail completely, but the test was stopped when the opening between the spliced members was closed. For the maximum load per split ring there was a slip of ¾ in. [b] Values of the proportional limit load are very uncertain because of the large increments of test load. The ratio R_{ee} is equal to the proportional limit load (22.0 kips) divided by P_e, or 1.09. [c] The basic allowable load on four 4-in. split rings is 22.0 kips; but this value was first increased 12% to 24.6 kips because of the ¾-in. bolts, and then decreased to 20.2 kips (or 5.05 kips per split ring) because of the spacing of 6 in., center to center. [d] Deformation, in inches, at a load of 20.2 kips. [e] Total width of the two spaces between the splice pads and the center block, caused by shrinkage. [f] Not measured.


A
8" x 15½"
10"
Four at 7⅝" = 30½"
12½"
15⅜"
Four at 7⅝" = 30½"
9¾"
2"
5⅞" x 15½"
Face A
Face B
Face C
B
5⅞" x 15½"
B
3¾"
6"
Five at 7⅝" = 38⅛"
11⅝"
8½"
Five at 7⅝" = 38⅛"
6"
8"
A

(a) FRONT ELEVATION


5' ⅜"
4' 6"
2¾"
4' 6"
3' ¾"
Face D
3" x 10"
3" x 10"
8"
6"
12"
1⅜"

(b) PLAN B-B


Loading Block Not Part of Joint
14⅜"
16"
15⅞"
25"
3"
5⅞"
10"
20-Ton Jack No. 2
18"
15½"
5"
5¼"
5"
5"

(c) SECTION A-A

(d)

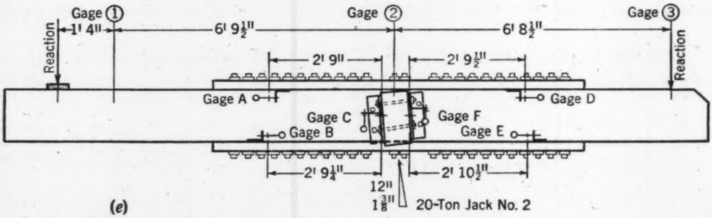

Gage ①
Gage ②
Gage ③
Reaction
1' 4"
6' 9½"
6' 8½"
Reaction
2' 9"
2' 9½"
Gage A
Gage D
Gage C
Gage F
Gage B
Gage E
2' 9¼"
2' 10½"
12"
1⅜"
20-Ton Jack No. 2
(e)

FIG. 62.—MOMENT JOINT, TYPE I (J29, J30, AND J31) (BOLTS ¾ IN.; PLATE WASHERS, $\frac{5}{16}$ × 3 × 3; AND ANGLES, $\frac{5}{16}$ × 4 × 4)

shows a slip of 0.06 in. at the allowable load of its connectors, $P_e = 20.2$ kips, is the only one of the four joints that exhibited normal behavior. These joints were taken apart after testing, and no defective workmanship was evident, so that the lack of consistency in their behavior cannot be explained.

Moment Joints.—The series of bolted joints J29 to J34 consisted of six splices of 6-in. by 16-in. and 8-in. by 16-in. wooden beams which had been part of the framing supporting a canvas roof over the seats of the Cavalcade. All splices had been calculated to resist moment as well as shear, and the framing was designed to support a live load of 5 lb per sq ft plus dead load, with a 50% increase in the usual working stresses. Some of the beams had been continuous, with spans of 40 ft and 42 ft, whereas others had been 21-ft cantilevers extending beyond their supporting girders.

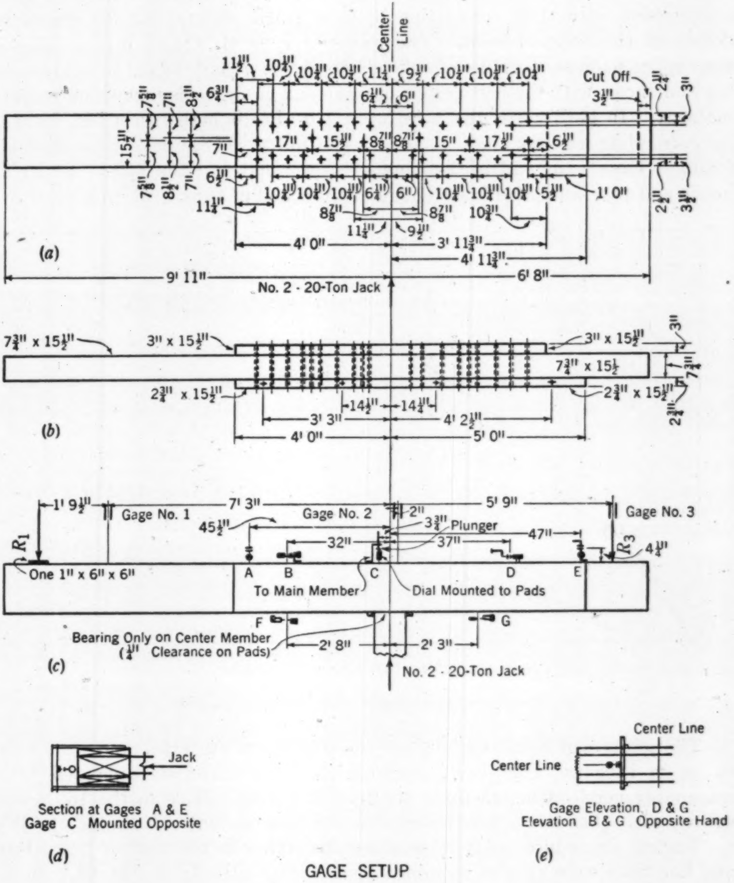

Center Line
Cut Off
(a)
9' 11"
6' 8"
4' 0"
3' 11¾"
4' 11¾"
No. 2 - 20-Ton Jack
(b)
7¾" x 15½"
3" x 15½"
2¾" x 15½"
3' 3"
4' 2½"
14½"
4' 0"
5' 0"
(c)
R1
R3
1' 9½"
7' 3"
5' 9"
2"
Gage No. 1
Gage No. 2
Gage No. 3
45½"
32"
3¾" Plunger
37"
47"
4¼"
One 1" x 6" x 6"
A
B
C
D
E
F
G
To Main Member
Dial Mounted to Pads
Bearing Only on Center Member
(¼" Clearance on Pads)
2' 8"
2' 3"
No. 2 - 20-Ton Jack
Jack
Section at Gages A & E
Gage C Mounted Opposite
(d)
Center Line
Center Line
Gage Elevation D & G
Elevation B & G Opposite Hand
(e)
GAGE SETUP

FIG. 63.—MOMENT JOINT, TYPE II (J32, J33, AND J34) (STITCH BOLTS, ⅝ IN.; BOLTS, ¾ IN.; PLATE WASHERS, $\frac{5}{16} \times 3 \times 3$)

The spliced joints were of two types. Type I (joints J29 to J31, inclusive) had its splice pads on the tension and compression faces of the beams, the beam being cut in two to allow its supporting girder to pass through within its depth. It had steel angle clips bolted to its sides and to its supporting girder to resist shear. The splice pads were designed to act in direct tension or compression with their connecting bolts in single shear. Fig. 62 shows the details of this type of joint. The joints of type II, shown in Fig. 63, had their splice pads on the sides of spliced beams, the pads designed to act in bending and shear, with the bolt groups on each end of the pads furnishing resisting moment. In both types the bolts were $\frac{3}{4}$ in. in diameter with 3-in. by 3-in. by $\frac{5}{16}$-in. plate washers under the heads and nuts. These bolts were generally found to have a tight fit in their holes in the wood, but the "play" under the heads and nuts indicated a shrinkage of the wood of from $\frac{1}{8}$ in. to $\frac{1}{4}$ in.

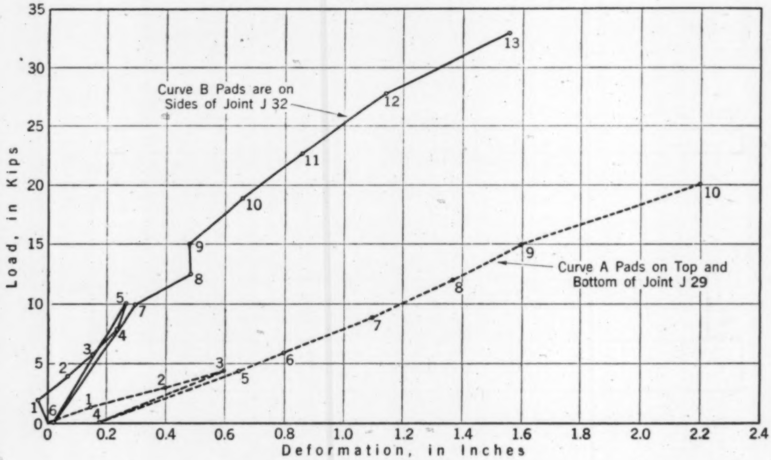


FIG. 64.—CENTER DEFLECTION OF MOMENT JOINTS

The joints had been carefully removed from the structure and were found to be in excellent condition; workmanship and materials appeared to be reasonably good, although some checking had taken place in the 16-in.-wide splice pads. The joints were tested at a moisture content of from 15% to 25%.

Testing Procedure.—After installing the joints in the testing truss, loads and reactions were applied as indicated diagrammatically in Fig. 63. In five of the joints, one concentrated load was applied directly over the splice, and in the sixth joint two equal loads were applied about 3 ft apart, near the center of the splice.

Deflections under the loads were measured on all joints; but, since these measurements were taken on one side of the beams only, some erratic results were obtained, and consequently the actual values of deflections are to be taken with caution from the deflection curves A and B of Fig. 64.

Loads were applied with an hydraulic jack and through blocking to the splice points. When loads that produced bending moments of about 20 ft-kips for type I joints and 30 ft-kips for type II joints had been applied, the loads were released, and deflections and deformations were measured. Joints were then reloaded until failure occurred.

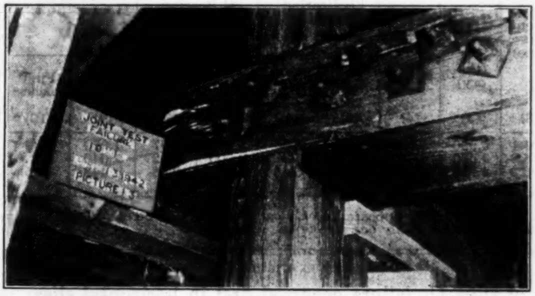


FIG. 65.—FAILURE OF JOINT J31 IN BENDING AND TENSION OPPOSITE THE SPLICE POINT

Results of Tests.—Failure in joints of type I occurred in the tension pads, caused by direct tension and secondary bending stresses. Fig. 65 shows a sharp bend in the pads which created large secondary bending stresses. No attempt was made to calculate these secondary stresses.

TABLE 15.—STRESS RATIOS, TESTS OF TOP AND BOTTOM SPLICE PADS, MOMENT JOINTS, TYPE I, J29 TO J31

Joint	Maximum supported load (kips)	Maximum moment (ft-kips)	Ultimate shear (lb per bolt)	Ratio, R_{ts}	Stress in tension pad[a] (lb per sq in.)	Estimated load P_l (lb per bolt)	Ratio, R_{tl}
J29.........	28.8	105.9	7,450	6.31	2,730	3,862[b]	3.27
J30.........	30.5	111.3	7,767	6.57	2,970	4,601	3.90
J31.........	29.8	105.3	7,471	6.31	2,740	3,515	2.98

[a] Computed stress in the tension pad neglecting bending effects. [b] Was at least 3,862 lb per bolt.

TABLE 16.—DEFORMATIONS OF TOP AND BOTTOM SPLICE PLATES, MOMENT JOINTS, TYPE 1, J29 TO J31
(For Loads and Stress Ratios, see Table 15)

Joint	BOLT DEFORMATIONS, IN INCHES, UNDER THE FOLLOWING LOADS PER BOLT:					JOINT DEFLECTIONS[a]			Span length[b] (ft)
	984 lb	1,180 lb	3,862 lb	4,000 lb	5,394 lb	984 lb per bolt (in.)	Maximum at Center		
							(Kips)	(In.)	
J29......	0.057	0.070	0.1670	....	....	0.52	20.1	2.21	14.75
J30......	0.056	0.068	0.182	0.185	0.2563	0.420	30.5	3.86	14.77
J31......	0.0477	0.060	0.164	0.1707	....	0.395	29.8	3.54	14.54

[a] At the point of application of the load. These values are to be accepted as relative only, since measurements were taken only on one side of the splice. [b] Beam section, 6 in. by 16 in.; splice pads, one each top and bottom, 3 in. by 10 in. by 9 ft; and twenty-two ¾-in. bolts.

TABLE 17.—LOADS AND DEFORMATIONS, SIDE SPLICE PADS, MOMENT JOINTS, TYPE II, J32 TO J34

Joint	Maximum load (kips)	Maximum moment (ft-kips)	BENDING STRESS (LB PER SQ IN.)		THEORETICAL BOLT STRESS AT ULTIMATE LOAD				HORIZONTAL THEORETICAL SHEAR IN PADS (LB PER SQ IN.)		MAXIMUM RECORDED DEFLECTIONS[b]	
			Main member (theory)	Splice pads (computed)	If Bearing Parallel to Grain (0°)		If Bearing Across the Grain (90°)					
					(Lb)	R_{bs}	(Lb)	R_{bs}	Ultimate	First cracks[a]	(Kips)	(In.)
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
J32..	27.9	101.1	4,900	5,880	13,000	5.54	5,800	4.97	1,020	330	27.9	1.249
J33..	32.8	115.3	4,460	6,010	17,300	7.3	9,000	5.92	1,040	....	27.7	1.146
J34..	33.4	96.9	4,850	5,280	12,600	5.36	4,370	2.88	443	415	26.8	1.772

[a] Computed shear at the first appearance of shear cracks. [b] To be considered relative only, because the measurements were taken on only one side of the splice.

In joints of type II, failure occurred first in horizontal shear at the pads, as evidenced by offsets at the ends. Usually this first shear failure followed existing checks, but new cracks were also opened connecting to the existing checks—in a manner quite typical of horizontal shear in beams. In a few cases the shearing cracks developed in entirely new and unchecked material. After the splice pads were split into two or more pieces by shearing, each part then broke in bending, or by subsequent splitting.

TABLE 18.—DIMENSIONS IN INCHES, MOMENT JOINTS, TYPE II (J32 TO J34), WITH SIDE SPLICE PLATES
(See Table 17)

Joint	Beam sections	Splice pads	Number of bolts (¾-in. ϕ)
J32....	6×15¾	2×15¾ 3×15¾	40
J33....	7¾×15½	3×15½ 2¾×15½	38
J34....	6×15½	2¾×15½[a] 2¾×15½[a]	32

[a] Length 14 ft.

The results of the tests are summarized in Tables 15, 16, 17, and 18, from which it may be observed that the joints with side pads (type II) are nearly two and one-half times stiffer than those of type I with top and bottom pads. Fig. 66 shows typical bolt deformation in the joints with top and bottom pads (type I). Comparing the results of Tables 15 and 16 with the bolt slips of Fig. 66, it is noted that considerable deflection of the beam took place with only small bolt slips because of the large ratio of span length to depth of beam—that is, a beam deflection of 0.4 in. corresponds to a bolt slip of 0.075 in.

In joints of type II the determination of amount and direction of shear in each bolt is a complex and indeterminate problem, unlike similar problems in steel construction, because of the difference in the elastic properties of the wood parallel and perpendicular to the grain. It may be conceived that the total moment in the splices is resisted by a combination of two types of bolt action. In the one type, assuming that the bolts act parallel to the grain of the wood only in resistance to the bending moment, each bolt would act in double shear and the pads could probably be split along the center line of the beam with no loss in strength or rigidity. This type of action would produce tension and compression in the pads, but no horizontal shear. Col. 6, Table 17, shows the bolt stresses under such action. In the second type of bolt action, the bolts are assumed to bear perpendicular to the grain of the wood only; and, hence, their pressures would induce large shearing as well as bending stresses in the pads. Bolt stresses for this type of action are shown in Col. 8, Table 17, and the calculated horizontal shear is indicated in Col. 10.

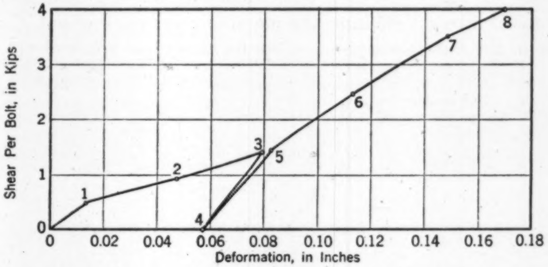


Fig. 66.—Typical Bolt Deformation Curve, Joints of Type I, with Top and Bottom Pads (Bolts, ¾ In. Round in Single Shear)

The total resistance of such a splice must result from a combination of the two types of action. The distribution between the two types of bolt action will depend on the dimension of the joint and the relative elastic properties of the bolts and the wood in the two directions, and is highly indeterminate. Even if only a small proportion of the total bending moment is resisted by bolt action perpendicular to the grain of the wood, high shearing stresses must occur in the pads—as was evidenced in the tests of the joints of type II by the fact that all the first failures were caused by horizontal shear. Further tests and study are needed on splices of this type if a definite and rational method of design is to be determined.

The strength ratios obtained in these tests were adequate—even large. Both types of joints supported approximately the same ultimate loads, but the joints with the side pads were much stiffer. In all these tests, failure occurred in the splice pads, and not in the bolts. Probably the same loads could have been supported by using fewer bolts. In type II joints, it is probable that even larger loads could have been obtained by increasing the thickness of the pads, thus reducing the horizontal unit shearing stresses. In type I joints, secondary bending stresses, usually neglected in design, were the primary causes of failure.

Butt-Block End Joints of Trusses.—Two butt-block end, or heel, joints of identical design (J35 and J36), removed from the 80-ft span roof trusses of the Mission Trails Building, were tested to failure. They formed the connection of the lower chords of the trusses to the building columns. In each joint a heavy butt block supported the end diagonal and was connected to the lower chord with six 4-in. split rings. The members were of select structural Douglas fir, and the joints were tested at a moisture content of 17%. A detail of one of the joints is shown in Fig. 67. Unfortunately for the test results, the butt blocks were of boxed-heart material; and, although aluminum paint had been applied to their ends, considerable checking had occurred.

Joint J35.—In the test of joint J35, when the end diagonal, member No. 1, carried a total of stress of 95 kips, the member was forced about $\frac{3}{16}$ in. into the beveled face of the butt block; and, when its stress had reached 111.5 kips, the indentation had increased to $1\frac{5}{8}$ in. The block itself sheared and split along check planes that followed the rays of the boxed heart. The maximum slip of the split rings was 0.05 in.

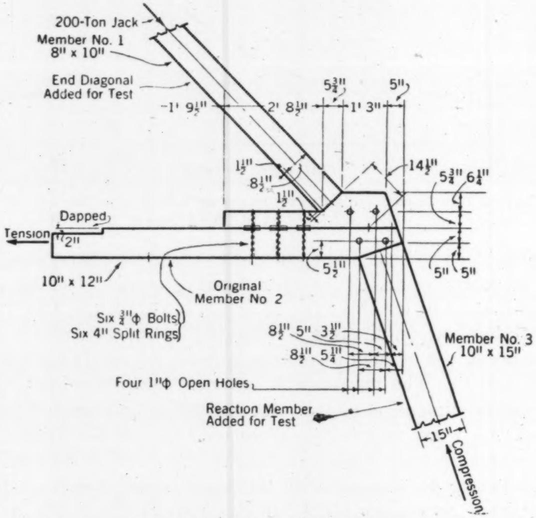

200-Ton Jack
Member No. 1
8" x 10"
End Diagonal
Added for Test
1' 9½"
2' 8½"
5¾"
1' 3"
5"
1½"
8½"
1½"
14½"
5¾"
6¼"
Dapped
Tension
2"
10" x 12"
Original
Member No 2
5½"
5"
5"
Six ¾"ϕ Bolts
Six 4" Split Rings
8½"
5"
3½"
8½"
5¼"
Member No. 3
10" x 15"
Four 1"ϕ Open Holes
Reaction Member
Added for Test
15"
Compression

FIG. 67.—BUTT-BLOCK END JOINT J35

Joint J36.—In the test of joint J36, to prevent failure by crushing the beveled face of the butt block, a steel plate was installed at the end of diagonal member No. 1. During the test this steel plate was pressed into the block about ½ in., and the block was sheared on a horizontal plane about halfway up the beveled cut. In spite of these failures, the loads did not decrease appreciably, and the maximum load was recorded after the shearing occurred. This condition was undoubtedly caused by the action of the steel plate in reducing the shear on the plane of failure.

When the maximum load on the joint had been reached, it was noted that the reaction bearing of the supporting post (member No. 3), which had replaced the original column for testing purposes, had been pushed into the lower chord member from about 1¼ in. to 1½ in. At this time no further increase in load could be obtained, but the joint continued to hold the full load for about 1 hour, at which time the test was discontinued.

TABLE 19.—BUTT-BLOCK HEEL JOINTS

Allowable, or Working Loads and Stresses:

Longitudinal shear, in pounds per square inch	150
Bearing stress at 45° with the grain, in pounds per square inch	563
Bearing stress at 70° with the grain, in pounds per square inch	380
One split ring, in pounds per split ring	5,500

Joint	MAXIMUM LOAD STRESSES					REACTION BEARING (70° CUT)		RATIO R_{wa} BASED ON		
	Shear			Beveled Face of Block (45° Cut)				Shear in the block	Bearing 45° on the block	Bearing 70° on the chord
	Six split rings[a] (kips)	One split ring[a] (kips)	Unit shear at base of block (lb per sq in.)	Total load (kips)	Stress (lb per sq in.)	Total load (kips)	Stress (lb per sq in.)			
J35...	75.8[b]	12.6	277	111.5	1,640	86.0	574	1.85	2.93[e]	1.51[e, f]
J36...	96.2[b]	16.0	296[c]	139.5[d]	1,453[d]	105.7	729	1.97	2.57[e]	1.91[e]

[a] Neglecting friction effects. [b] Split rings did not fail. [c] The base block sheared at this load but the load did not decrease. [d] Steel plate had been added. [e] These factors are of doubtful value since the working stress in bearing, on inclined planes, should be based on the indentation in the wood. [f] Did not fail.

Results of Tests.—The results of the tests are summarized in Table 19 in which the various maximum and working loads and stresses, corresponding unit stresses, and ratios R are shown. The working stresses of 563 lb per sq in. and 380 lb per sq in. (Table 19) are based on the formula developed by R. L. Hankinson:[6]

$$N = \frac{PQ}{P \sin^2 \theta + Q^2 \cos^2 \theta} \qquad (1)$$

in which P is the allowable unit compressive stress parallel to the grain; Q is the allowable unit compressive stress perpendicular to the grain; N is the allowable unit compressive stress at angle θ to the grain; and θ is the angle to the grain of applied load.

Joint J35 failed by shear in the butt blocks after the diagonal had been impressed about 1⅝ in. into the 45° cut of the block. The split rings were intact whereas the block was completely split.

As in joint J35, the block in joint J36 was split, and failed in shear; but the load did not decrease. The reaction bearing (the post) sank 1¼ in. into the 70° bevel cut of the chord. Despite the steel plate, the 45° bevel dug an impression ½ in. deep.

If the working load for which the end diagonal was designed is taken as the inclined component of the working capacity of the six 4-in. split rings, which fastened the butt block to the chord, such axial and diagonal stress would have been 46.5 kips. The corresponding unit stress on the beveled face of the butt block would have been 580 lb per sq in.; and the corresponding unit shearing stress in the block, 150 lb per sq in. Judged by the failure of the butt block, these working stresses were too high for the quality of butt block used. A bearing stress of 450 lb per sq in. and a shearing stress of 90 lb per sq in. would have been more in order. These values would have given R-values of about 3, based on the ultimate shearing stress and on the butt-block bearing stress at which an indentation of $\frac{3}{16}$ in. was observed. A boxed-heart stick should not have been used for so important a member as the end butt block, for such a stick is liable to serious checking.

---

[6] "Investigation of Crushing Strength of Spruce at Varying Angles of Grain," *Circular No. 259*, U. S. Air Service Information, Washington, D.C., 1921.

Even had the block been of sound material, failure at such comparatively low unit stresses leads one to question whether the generally accepted working stresses for longitudinal shear in joints and bearing on inclined surfaces are not too high. Probably the detail of block could have been modified to provide a lower shearing stress; but, to reduce the bearing stress, a larger end post would have been necessary.

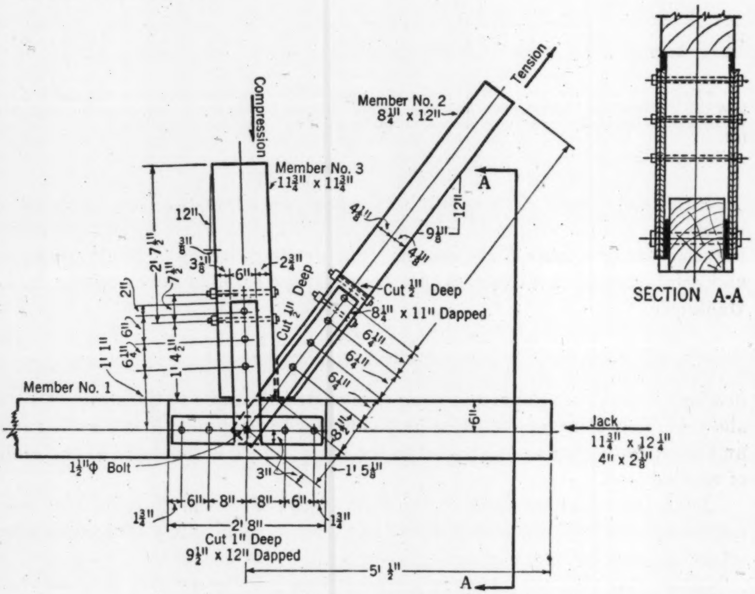

Compression
Tension
Member No. 2
8¼" x 12"
Member No. 3
11¾" x 11¾"
A
12"
4⅛"
9⅛"
4⅛"
12"
⅜"
3⅛"
6"
2¾"
2' 7½"
7½"
2"
Cut ½" Deep
Cut ½" Deep
8¼" x 11" Dapped
6¼"
6¼"
6¼"
1' 1"
6¼"
6"
1' 4½"
Member No. 1
6"
8½"
Jack
11¾" x 12¼"
4" x 2⅛"
1½"Φ Bolt
3"
1' 5⅛"
6"
8"
8"
6"
1¾"
1¾"
2' 8"
Cut 1" Deep
9½" x 12" Dapped
5' ½"
A
SECTION A-A

Fig. 68.—Strap and Pin Joints—Joint J37

Strap and Pin Joints.—Tests were made of three joints of similar design (J37, J38, and J39) in each of which a pair of steel pin plates were bolted to each member entering the joint; and each pair of pin plates was then connected to a single pin. One of the joints is shown in Figs. 68 and 69. The pin plates were 6 in. by ⅜ in. Each pair was fastened to each side of each wood member with 1-in.-round bolts, and interconnected with a single 1½-in.-round pin. The joints were in excellent condition; and the wood was of select grade Douglas fir and was tested at a moisture content of about 15%. Joints J37 and J38 had some checks at the bolts in their diagonal tension member—member No. 3. Shrinkage was not apparent in joint J37; but shrinkage in joint J38 was estimated at about ⅜ in., whereas that in joint J39 was estimated at ⅛ in. All bolts were tight in their holes in the wood. The joints were provided with ½-in. stitch bolts as shown in Fig. 68.

The results of the tests are summarized in Table 20. Fig. 70 shows the slip of steel plates with respect to the wood in tension member No. 3, joint J39, which is the member that failed. Fig. 71(*a*) shows the typical splitting of this tension member at the ultimate load, and Fig. 71(*b*) shows the same failure after the steel plates had been removed. From Table 20, it can be noted that the *R*-values of the failed members were 3.03 in joint J37, 3.31 in joint J38, and 5 and 5.9 in joint J39. The values of the proportional limit loads were not determined. Joint J39 took a greatly increased ultimate load as compared to the loads taken by joints J37 and J38.

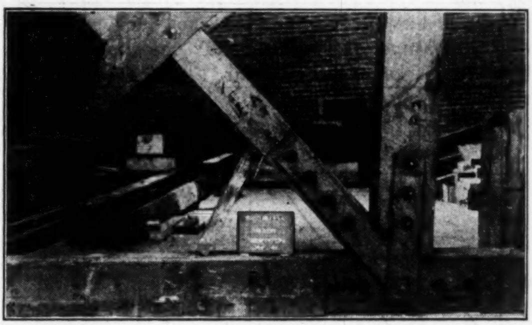


FIG. 69.—STRAP AND PIN JOINT BEFORE TESTING

TABLE 20.—BOLTED STRAP AND PIN JOINTS

Joint	LOADS ON MEMBERS, IN KIPS						Ratio,[a] R_{ba}	DAP OF BOLTS[b] (IN INCHES)		End distance[c] (in.)	Residual load (kips)	Elapsed time[d] (min)
	Maximum, at Failure			Allowable, P_b				At P_b	At end of test			
	No. 1	No. 2	No. 3	No. 1	No. 2	No. 3						
J37..	40.2	50.6	63.7	21.0	15.7	21.0	3.03	0.068	3¼	8½	61.8	41
J38..	50.2	46.5	69.3	21.0	15.7	21.0	3.31	0.070	1⅛	9⅛	29.6	23
J39..	95.7	84.1	123.4[e]	21.0	15.7	21.0	5.9	0.012	0⅛	9¼	71.0	128

[a] Ratio R_{ba} for member No. 3, which failed. [b] Measured on one side of joint only. [c] End distance on the first bolt at member No. 3. [d] Elapsed time of testing. [e] The tension connection used consisted of six 4-in. shear plates with ¾-in. bolts, 12-in. end distance, and 6-in. spacing. In joint J39, this connection successfully withstood 123.4 kips, the largest maximum load of the three tests. *P* for six shear plates equaled 30.5 kips without reducing for substandard spacing and 23.2 kips if reduced for 6-in. spacing. Thus, $R_{ss} = \frac{123.4}{23.2} = 5.32$.

As stated, there were checks in the tension members of these last two joints which may have lowered the ultimate loads; but it scarcely seems possible that these slight difficulties were responsible for the great difference in load-carrying capacity. After failure, in spite of the splitting shown in the photographs, the joints continued to carry considerable load, as shown in Table 20.

After dismantling, it was found that the bolts nearest the pins were bent the least, whereas the bolts farthest from the pins had larger deformations, possibly due to the action of the stitch bolts. During the tests the members began splitting at the ends; therefore, it is believed that, if the stitch bolts had been placed at the ends of the members, their ultimate loads would have been greater. The excess of the ultimate loads of joints J37 and J38 is unexplainable, except perhaps that it furnishes an example of the old adage that "the exception proves the rule." The increased capacity is apparently not the result of a better quality of wood. Member No. 3 of joint J39 had a boxed heart; it had 25% summerwood as against from 30% to 50% summerwood, respectively, in the

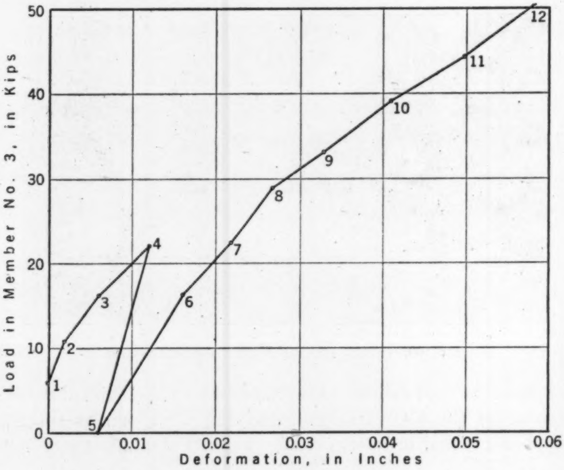


FIG. 70.—SLIP OF STEEL PLATES WITH RESPECT TO WOOD TENSION MEMBER NO. 3

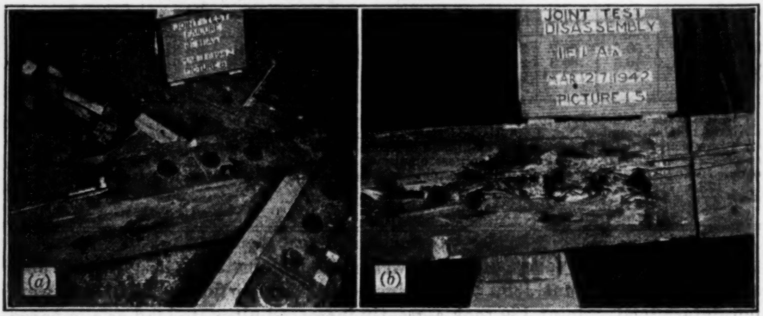


FIG. 71.—TEST OF TENSION MEMBER NO. 3
(a) View of Split in Tension Member at Failure
(b) Same as (a) with Steel Plates Removed

same members of joints J37 and J38; and it had twenty rings to the inch as compared to fourteen and thirty eight, respectively, in joints J37 and J38.

Bolted Joints with Steel Straps.—Tests were made of two joints of identical design taken from the East Cambodian Towers in which the members were interconnected by steel straps and bolts as shown in Fig. 72.

Diagonal member No. 3 consisted of two 3-in. by 14-in. sticks with an 8-in. by 14-in. filler block between. The member was connected to two steel straps, each $\frac{1}{2}$ in. by 12 in., with three $1\frac{1}{8}$-in. bolts. The horizontal member (member No. 1) was a 14-in. by 16-in. stick with four $1\frac{1}{8}$-in. bolts connecting to the same steel straps, the bolts bearing at an angle of $34\frac{1}{2}°$ to the grain of the member.

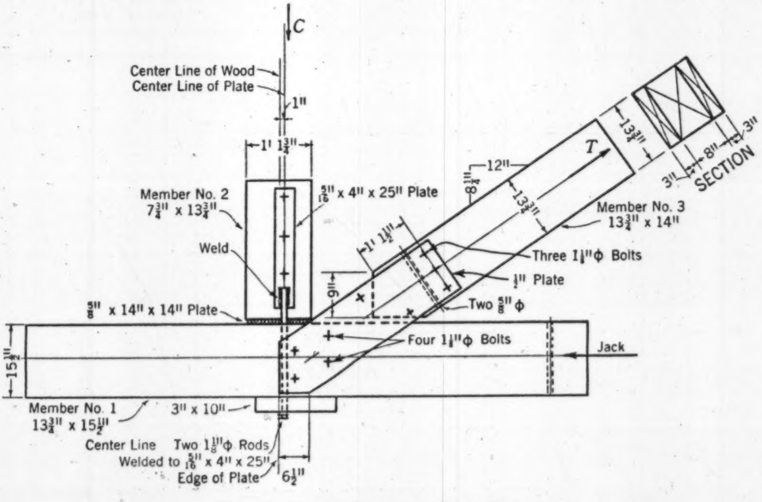

C
Center Line of Wood
Center Line of Plate
1"
1' 1¾"
Member No. 2
7¾" x 13¾"
5/16" x 4" x 25" Plate
Weld
⅝" x 14" x 14" Plate
12"
8¼"
13¾"
13¾"
T
3"
8"
3"
3"
SECTION
Member No. 3
13¾" x 14"
1' 1½"
Three 1⅛"ϕ Bolts
½" Plate
9"
Two ⅝" ϕ
Four 1⅛"ϕ Bolts
Jack
15½"
Member No. 1
13¾" x 15½"
3" x 10"
Center Line
Two 1⅛"ϕ Rods
Welded to 5/16" x 4" x 25"
Edge of Plate
6½"

FIG. 72.—BOLTED JOINT WITH STEEL STRAPS

Vertical member No. 2, an 8-in. by 14-in. stick, bore on a steel plate on the top of member No. 1 and in addition had a steel tension connection to the same member. Joint J40 is detailed in Fig. 72. The wood of both joints was of select structural grade of Douglas fir in good condition. The measured shrinkage amounted to about $\frac{1}{16}$ in. in 14 in. of width. At the time of testing the wood had a moisture content of about 15%.

In the test of joint J40, the tension connection was made to the two 3-in. by 14-in. diagonals using the 8-in. by 14-in. block between them as a filler. In the test of joint J41, the pull was applied directly to the 8-in. by 14-in. stick, so that the two 3-in. by 14-in. sticks acted as loose fills in the transfer of loads to the steel plates. In joint J40, failure took place not in the original joint, but in the new tension connection of member No. 3.

In the test of joint J41, loads were applied until the deformation became so large that the joint merely yielded without further increase in load. Consequently, the test was discontinued because of these large deformations, with the joint still holding its maximum load. The joint was then taken apart and

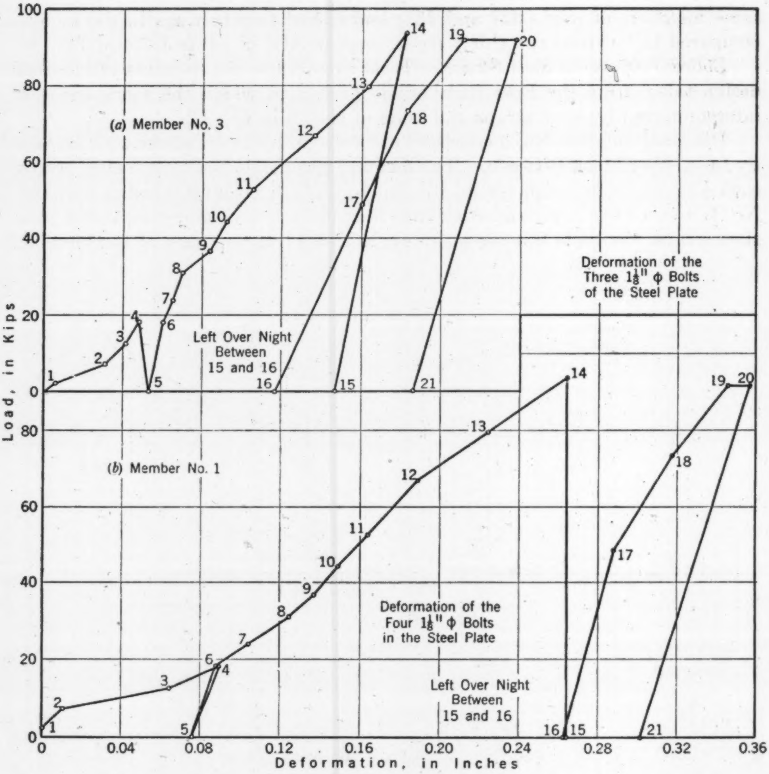


FIG. 73.—STRESS-DEFORMATION CURVE OF BOLTS

TABLE 21.—BOLT AND STEEL-STRAP JOINTS

Joint	Maximum stress[a] member No. 3 (kips)	Allowable Loads P_b on Total Number of 1⅛-In. Bolts		Stress Ratios			Residual load, member No. 3 (kips)	Load P_l in member No. 3	Bolt Deformation (Inches)				Elapsed time (days)
				R_{bs}		R_{ss}			At Load P_b		End of Test		
		Member No. 3[b]	Member No. 1[c]	Member No. 1	Member No. 3	Testing connection			Member No. 1	Member No. 3	Member No. 1	Member No. 3	
J40..	93.9	19.9	18.4	5.1	4.7	4.3	85.4	66.7[b,d]	0.089	0.048	$\frac{1}{4}$	$\frac{3}{8}$	2
J41..	129.2	19.9[e]	18.4	7.0	6.5	....	129.2	...[d]	0.051	0.050[f]	$\frac{3}{8}$	$1\frac{1}{4}$	2

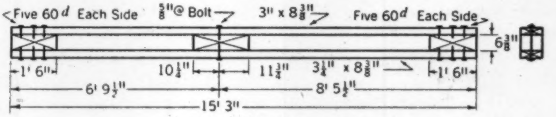
[a] The tension member. [b] Three bolts, loads parallel to the grain. [c] Four bolts, loads at an angle of 34½° to the grain. [d] Not evident. [e] Through two loose filler plates. [f] Includes loose filler plates.

the blocks split open to observe the deformations of the bolts, which amounted to 1½ in. at points of maximum bending. Fig. 73 shows the stress-deformation curves of the joints and Table 21 summarizes the results. The slips of bolts in both joints at their allowable loads were the normal slips (about 0.05 in.), but the influence of the loose fillers in joint J41 is evidenced in the much larger slips of the bolts in this joint at their maximum load.

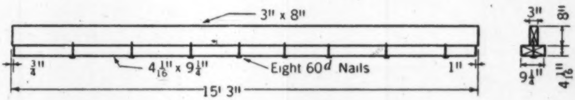
III. Column Tests

Built-Up Columns.—

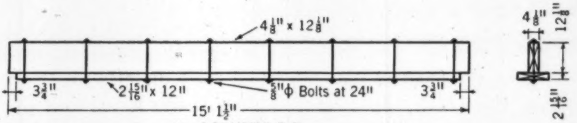
Spaced Column and T-Section Columns.—Because of financial limitations, the somewhat extensive original program of tests of built-up columns and struts had to be curtailed so that only three built-up columns were tested; however, they represented three separate types.



(a) COLUMN C 1



(b) COLUMN C 2



(c) COLUMN C 3

Fig. 74.—Details of Built-Up Timber Columns

Column C1, taken from the Mission Trails Building, was made of two 3-in. by 8-in. sticks spaced 6 in. apart, held together at midheight by a block with a single ⅝-in. bolt and at the ends by end blocks fastened only by nails. Column C2 was from the San Francisco Building, and column C3 was from the Mission Trails Building, where it had formed the top chord of the bracing truss for the main framing of this building. Each of the two columns consisted of two sticks forming T-sections, the two pieces of column C2 held together by nails and those of column C3, by bolts. Details of the three columns are shown in Fig. 74. The columns, about 15 ft long, were of select structural grade of Douglas fir and were tested at a moisture content of about 15%. The shrinkage of column C1 was estimated to be about ⅜ in., and that of column C3 was from about ⅛ in. to ¼ in.

The ends of all columns were carefully squared. In columns C2 and C3 of the T-section, the flanges of the T-section were cut back about 2 in. so that the loads would bear on the vertical web only. All three columns were installed horizontally in the testing machine and the loads applied through steel plates bearing on the ends, as spherical or rolling bearing blocks were not available. Gages were installed to measure deflections caused by buckling, as well as compression deformations. Loads were applied in increments of from 15 kips to 20 kips, about 20 min being consumed for testing each column.

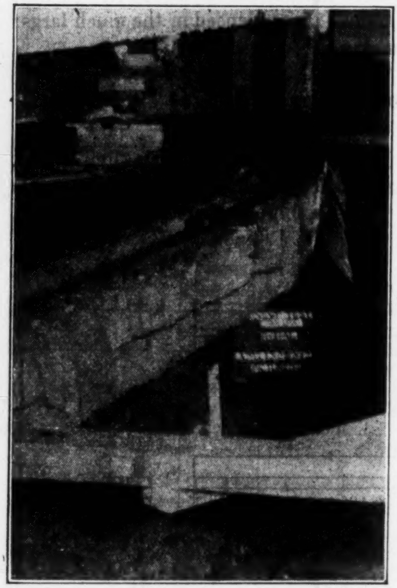


Fig. 75.—Bowing Failure of Column C1

Column C1, the spaced column, started buckling at a load of about 50 kips, but supported a maximum load of 60 kips. Then the resistance to load decreased, and the buckling increased. The manner of failure is illustrated in Fig. 75.

In testing column C2, of the nailed **T**-section, at a load of 70 kips, there was a visual indication of a sine curve with about $\frac{1}{16}$-in. amplitude. At a load of 106.3 kips, the web of the **T**-section suddenly buckled and tore loose from the flange, the nails ripping chunks of wood out of the web section.

Column C3, with a bolted **T**-section, did not fail in buckling. At a load of 182.7 kips, the head of the jack which bore directly on the web of the column crushed the wood. Just before failure, it was noted that the free edge of the web was buckling very slightly.

Results of Tests.—The results of tests are summarized in Table 22. Fig. 76 shows the relationship between the center buckling and the supported load for column C1. The column had a definite proportional limit load of 50 kips. At this load the computed value of E was 1,600,000 lb per sq in. The individual members of this column had an L/d-ratio of 61 corresponding to an L/r of 210 (L = height of column; d = least depth of section; and r = least radius of gyration). The maximum unit stress at failure was 1,146 lb per sq in.

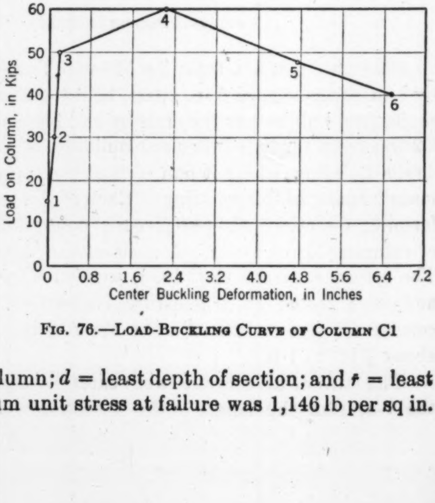


Fig. 76.—Load-Buckling Curve of Column C1

TABLE 22.—TESTS OF BUILT-UP COLUMNS (MOISTURE CONTENT, 14%, AND PARALLEL GRAIN)

Column	Specific gravity	Rings per inch	Percentage summerwood	Length (ft)	Buckling began at: Load (kips)	Buckling began at: (Lb per sq in.)	Maximum supported load: Load (kips)	Maximum supported load: (Lb per sq in.)	Ratios L/d: Web only	Ratios L/d: T-section	Allowable load (lb per sq in.)[a]	Ratio R_{wa}
C1..	0.46	14.5	25	15.25	50.0	955	60.0	1,146	..[b]	...[b]	295[c]	1,146 ÷ 295 = 3.9[d]
C2[e].	0.55	20	28	15.25	...[f]	...[f]	106.3	4,431[g]	61	19.7	1,178[g]	4,431 ÷ 1,178 = 3.75
C3[e].	0.49	28	42	15.12	...[h]	...[h]	182.7	4,677[g]	44	15.1	1,193[g]	4,677 ÷ 1,193 = 3.92

[a] Modulus of elasticity $E = 1{,}600{,}000$ lb per sq in.; allowable $s = 1{,}200$ lb per sq in; and $L/d = 61$. [b] For the entire length of column C1, $L/d = 61$, the maximum between the block and the end being 30.7. [c] For a spaced column, using split-ring connectors. [d] $R_{wa} = 955/295 = 3.3$. [e] Loads were applied only to the web (vertical) of the T-section. [f] Buckling began at failure, web tearing loose from the flange. [g] Unit stress in the web section. [h] This specimen did not fail as a column and did not buckle. It failed by local crushing under the jack, with a bearing pressure of 6,172 lb per sq in.

The nailing of column C2 proved adequate to support the web, and the bolting of column C3 was even more effective. These three tests are too few in number to warrant drawing any definite conclusions. However, the loads that they carried are impressive, particularly when it is considered that: (1) They had seen continuous service for about two and one-half years; (2) they were fastened only with nails and bolts; (3) the shrinkage of the wood had rendered the bolts ineffective in tension; and (4) the bolts were not tightened for the tests. The tests tend to lend confidence in these types of columns, particularly for use as bracing struts, for which they are usually employed.

The characteristics of the wood of these columns as shown in Table 22 are interesting.

IV. PLYWOOD TESTS

This division of the test program comprised the investigation of plywood sheathing when used as a structural element resisting lateral forces, like winds and earthquakes. Plywood-sheathed walls were so utilized in a number of the Exposition buildings.

When the test program was first organized, it had been expected to test large plywood panels from the various buildings, but this procedure was later found to be impracticable. As an alternative, it was decided to investigate three basic requirements in the design of plywood diaphragms. Therefore, test specimens were built from plywood panels that had been exposed to the elements at the Exposition site for a period of about 2 years, and three groups of tests were made as follows:

Group 1.—Strength of nail fastenings under lateral loading.

Group 2.—On the shearing strength of plywood.

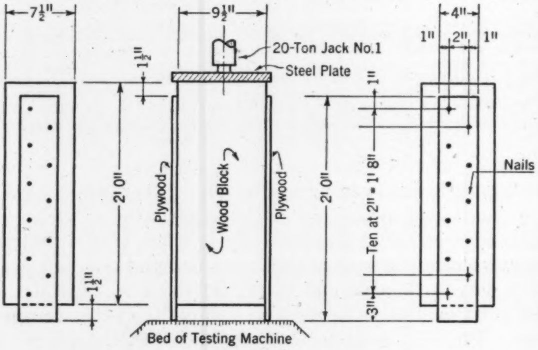
Group 3.—On the buckling of the plywood under shear loads with various spacings of studs.

All three groups of tests were run on all three of the following types of plywood:

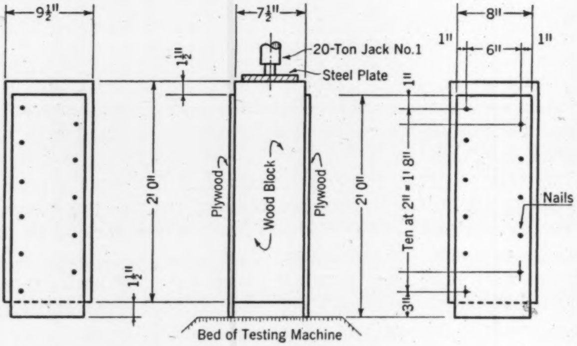
Type 1.—Three-ply $\frac{3}{8}$-in.-thick Douglas fir plywood unpainted panels that had been used in the interior of buildings and were in fair condition. Glue was a moisture resistant adhesive of the soya bean or casein type.

Type 2.—Three-ply $\frac{7}{16}$-in.-thick Douglas fir plywood specimens removed from exterior walls. They had one coat of paint and were generally in good condition.

Type 3.—Five-ply $\frac{7}{16}$-in.-thick Douglas fir plywood specimens removed from exterior walls. They had one coat of paint and were generally in good condition.


7½"
9½"
4"
1"
2"
1"
20-Ton Jack No.1
Steel Plate
1½"
1"
Plywood
Wood Block
Plywood
2' 0"
2' 0"
Ten at 2" = 1' 8"
Nails
1½"
3"
Bed of Testing Machine

(*a*) TYPE A


9½"
7½"
8"
1"
6"
1"
20-Ton Jack No.1
Steel Plate
1½"
1"
Plywood
Wood Block
Plywood
2' 0"
2' 0"
Ten at 2" = 1' 8"
Nails
1½"
3"
Bed of Testing Machine

(*b*) TYPE B

FIG. 77.—DETAILS OF PLYWOOD NAIL SPECIMENS

Glues for types 2 and 3 were phenolic resin adhesives or phenol-formaldehyde synthetic resin adhesives.

The preceding determinations were made in the Tacoma laboratory of the Douglas Fir Plywood Association, from samples submitted by the committee.

Group One; Lateral Strength of Nails in Plywood.—An important criterion of the strength of the average plywood diaphragm is the lateral strength of the nail fastenings to the studs. Before designing the Exposition structures, preliminary tests had been made which indicated that six-penny nails used in plywood $\frac{7}{16}$ in. thick failed first by bending and then by the heads pulling through the plywood. Consequently, when a design called for heavy shears (up to about 1,000 lb per lin ft) to be carried on plywood diaphragms, special roofing nails were specified, with a heavy No. 10 gage shank $1\frac{3}{4}$ in. long, to resist in bending and bearing, and a large head, $\frac{7}{16}$ in. in diameter, to resist pulling through the plywood. For heavy loads, these nails proved to be economical since their number was reduced. These roofing nails, together with more usual types, were tested in this series.

Specimens.—Eighty-two specimens were made up as shown in Fig. 77, including the three types of plywood mentioned previously. Some were to be tested with the loads parallel, and some with the loads perpendicular to the grain of the face plies. Nails were four-penny, six-penny, and eight-penny common wire nails, and two types of special roofing nails—all galvanized. One roofing nail was $1\frac{3}{4}$ in. long, No. 10 gage, with $\frac{7}{16}$-in.-round head, and the other was identical except that its head was $\frac{5}{16}$ in. in diameter. Nails were driven "home." On specimens of type B that had less than twenty-two nails, the top nails shown in Fig. 77(*b*) were omitted. There was no change in the number of bottom nails. The interior block which received the point of the nail was cut from an 8-in. by 10-in. piece that had a moisture content of 14% and a specific gravity of 0.476. All center blocks for the specimens were cut from the same 8-in. by 10-in. piece, so that little variation in results should be expected from this source. Special care was taken in cutting and assembling the specimens to have square cuts and even bearings for distributing the loads equally among the nails. Each specimen had from 14 to 22 nails, so that each test represents the average action of several nails.

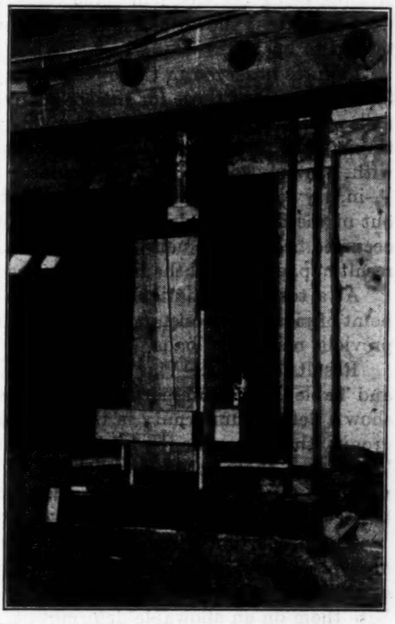


Fig. 78.—View of Plywood Nail Test Specimen in the Testing Frame

Testing.—Specimens were tested in a special part of the large testing frame, using a 20-ton hydraulic jack for applying the load. A view of a specimen in the testing frame is shown in Fig. 78. Two Federal dial gages were used, with the wood mountings as shown in the photograph. These gages were attached to the specimen to measure the deformation between the center block and each piece of plywood.

Because hand pumping was used with the hydraulic apparatus, the speed of testing was indeterminate. Pumping was done at an approximately uniform rate. It required from about 2 min to 2½ min to reach the limit of the dial gages—indicating a deformation of about 0.25 in. Dials were then removed and about 2 min more were required to reach a total deformation of from ½ in. to ⅞ in., at which point the tests were stopped.

Type A specimen (see Fig. 77) was used for all the first tests. Trouble was experienced with some of these specimens, in that the plywood failed in column action, before the nails failed. To remedy this condition, type B specimens were used because of their increased compression area, the number of nails being decreased in some cases. Some specimens with heavier nails were braced, as shown in Fig. 78, to prevent a side movement if the plywood tended to buckle.

Failures observed in these tests were of three types. The majority of the failures—most of those for the six-penny, eight-penny, and the special roofing nails with $\frac{5}{16}$-in. heads—occurred because the nails bent and then the head pulled through the plywood. This also happened in the case of the four-penny nails with ⅜-in. plywood. In the remainder of the four-penny nail tests, on the $\frac{7}{16}$-in. plywood, failure generally occurred because the nail bent and then pulled out of the block to which the plywood was nailed. The third type of failure occurred because the nails bent and then the heads broke and pulled off. This result happened in all the tests of the special roofing nails with the $\frac{7}{16}$-in. heads.

At a total deformation of from ½ in. to ⅞ in., all the nails had passed the point of maximum resistance and were yielding while supporting less than their previous maximum loads.

Results of Tests.—Fig. 79 contains the load-deformation curves of the tests and Table 23 is a summary of the results. The individual tests of each group show a definite uniformity in the great majority of cases. In ten of the fifteen groups, the values for the loads acting at 90° to the grain of the face plies are larger than those acting parallel. This condition is most noticeable for light nails and for the three-ply plywoods. In the case of the five-ply plywood and larger nails, these values are closer together.

The proportional limits of the nail strength values, as given in Table 23, are very approximate since nail tests invariably yield irregular curves. In establishing safe working nail values for practical uses, it appears desirable to base them on an allowable deformation, such as 0.01 in.

Further economies in plywood construction might be realized if nails with heavier shafts and larger heads were to be manufactured. Economies in labor as well as in materials would also result since large nail spacings could be used.

Group Two; Plywood Shear Tests.—The shear resisting ability of the plywood in plywood diaphragms is of importance; yet few data on this subject are available. For this reason panels were made and tested in which the plywood had been nailed to wood studs in a most substantial manner for the purpose of

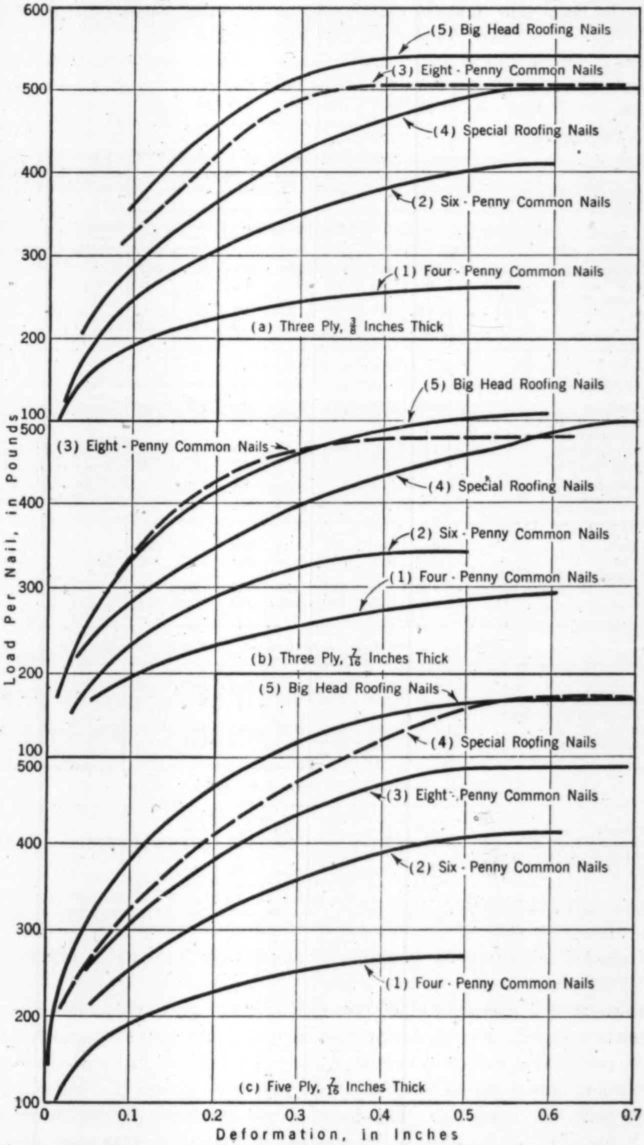


FIG. 79.—LATERAL STRENGTH OF NAILS IN PLYWOOD

TABLE 23.—OBSERVED LOADS, IN POUNDS, SUMMARY OF NAIL TESTS ON PLYWOOD SPECIMENS

Description	THREE PLY, $\frac{3}{8}$ IN.		THREE PLY, $\frac{7}{16}$ IN.		FIVE PLY, $\frac{7}{16}$ IN.		Average value	LOAD BASED ON FACTOR OF:			
	Parallel to the grain	Cross grain	Parallel to the grain	Cross grain	Parallel to the grain	Cross grain		1.6 on P_l	In 0.01 slip	1.33 on P_l	In 0.02 slip
(*a*) FOUR-PENNY NAILS											
Ultimate load	234	305	278	326	258	291	282	...	...	...	...
Proportional limit, P_l	105	113	105	105	92	123	108	68	...	81	...
Observed Load at Deformation of:											
0.01 in.	105	117	96	113	90	122	108	...	68	...	...
0.02 in.	125	144	119	135	112	145	130	...	...	...	81
(*b*) SIX-PENNY NAILS											
Ultimate load	363	424	374	317	450	385	385	...	...	...	...
Proportional limit, P_l	110	155	123	137	147	148	137	86	...	103	...
Observed Load at Deformation of:											
0.01 in.	110	140	108	143	157	138	133	...	83	...	...
0.02 in.	130	172	132	174	184	171	160	...	...	...	100
(*c*) EIGHT-PENNY NAILS											
Ultimate load	520	489	474	495	501	484	492	...	...	...	...
Proportional limit, P_l	195	203	157	209	203	185	191	120	...	144	...
Observed Load at Deformation of:											
0.01 in.	172	194	154	197	187	175	179	...	112	...	...
0.02 in.	210	228	190	232	220	205	213	...	...	...	135
(*d*) SPECIAL ROOFING NAILS WITH $\frac{5}{16}$-IN.-ROUND HEADS											
Ultimate load	440	500	486	469	554	582	505	...	...	...	...
Proportional limit, P_l	167	151	172	155	187	202	172	107	...	129	...
Observed Load at Deformation of:											
0.01 in.	149	145	166	127	188	174	158	...	99	...	...
0.02 in.	185	175	201	167	216	213	193	...	...	...	119
(*e*) SPECIAL ROOFING NAILS WITH $\frac{7}{16}$-IN.-ROUND HEADS											
Ultimate load	520	565	494	525	561	506	541	...	...	...	...
Proportional limit, P_l	220	220	180	189	250	212	212	133	...	159	...
Observed Load at Deformation of:											
0.01 in.	205	194	163	214	190	195	193	...	121	...	...
0.02 in.	249	240	219	251	248	258	244	...	...	...	145

preventing either failure by buckling of the plywood sheets or failure in the nails.

Specimens.—There were three test specimens of plywood panels—a typical one is shown in Fig. 80. Exterior load-carrying members were made of wide timbers, permitting sufficient roofing nails to be applied at each edge so that the nails themselves would not fail. The panels were also blocked to prevent the plywood from buckling. Fig. 81(*a*) shows the top. At the two corners, where loads were not applied, a gap was left so that the frame itself would not carry any load. Loads were applied directly to the frame on two of the diagonal corners. To prevent bearing of the loads on the plywood itself, edges were clipped on a diagonal thereby providing bearing for plates on the frame only.

The general setup is shown in Fig. 81(*b*). The hydraulic jack is shown at the left, and the reaction into the testing truss at the right is taken through a short heavy beam. Gages were installed as indicated in Fig. 80(*b*). One dial gage A measured the lengthening of the tension diagonal, and another gage B measured the shortening of the entire frame from timber to timber as close to the corners as was practicable. Four gages were installed to measure the nail slip. As these slips were small and correspond to the series of tests on nails, the results are not discussed. Fig. 81(*b*) shows the dial gage setups as seen from the back of the machine. Testing procedures for each panel were as follows:

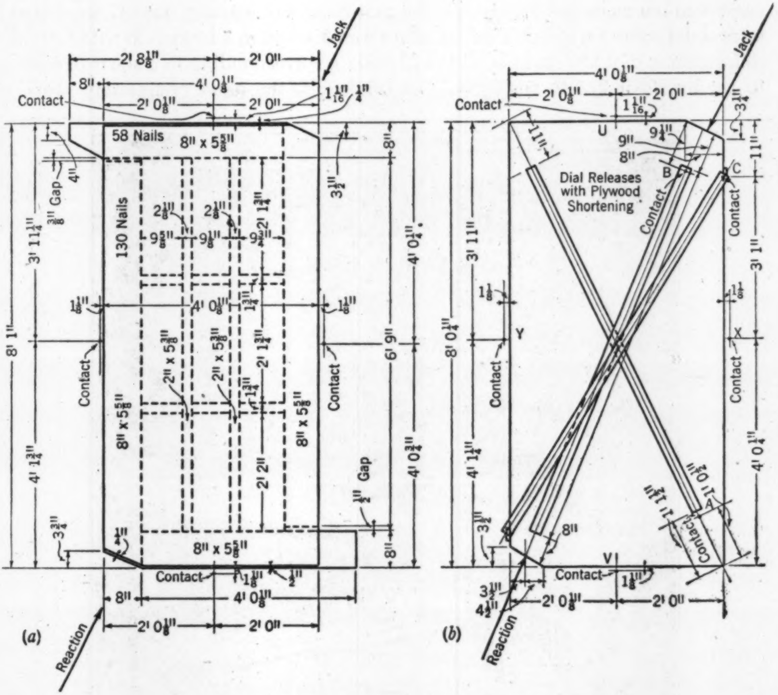

Jack
2' 8⅛"
2' 0"
8"
4' 0⅛"
Contact
2' 0⅛"
2' 0"
1 1/16"
¼"
58 Nails
8" x 5⅝"
4"
⅜" Gap
3' 11¼"
130 Nails
2⅛"
2⅛"
9⅝"
9⅛"
9¾"
2' 1¾"
3½"
8"
4' 0¼"
1⅛"
4' 0⅛"
1¾"
1⅛"
8' 1"
Contact
2" x 5⅜"
2" x 5⅜"
2' 1¾"
1¾"
6' 9"
Contact
8" x 5⅝"
8" x 5⅝"
4' 1¾"
2' 2"
½" Gap
4' 0¾"
3¾"
¼"
8" x 5⅝"
8"
Contact
1⅛"
½"
8"
4' 0⅛"
2' 0⅛"
2' 0"
(a)
Reaction
Jack
4' 0⅛"
2' 0⅛"
2' 0"
Contact
1 1/16"
3¼"
11"
U
9¼"
9"
8"
B
C
Dial Releases with Plywood Shortening
Contact
Contact
11"
3' 11"
3' 1"
1⅛
1⅛
Y
X
8' 0¼"
Contact
Contact
4' 1¼"
4' 0¼"
1' 0¾"
1' 1¾"
A
Contact
3½"
8"
V
Contact
1⅛"
3½"
4½"
2' 0⅛"
2' 0"
(b)
Reaction

FIG. 80.—TEST SETUP, PLYWOOD SHEAR TESTS
(*a*) Construction of Specimen
(*b*) Location of Gages

Test No. 1; Three-Ply Panel, ⅜ In. Thick.—This panel was placed in the machine and loaded with a 50-ton hydraulic jack. In the first 21 min of testing, the deformation at gage C, measuring the total frame and panel deformation, was 0.331 in. At this point, with 19,800 lb on the panel, the dial gages were removed and the loading continued until a deformation of about 1 in. was obtained after an interval of about 5 min. At this time the panel failed in shear at a load of 23,000 lb, corresponding to a shear of 2,571 lb per ft, or 575 lb per sq in. At failure, the panel split lengthwise in many fine lines. After failing, the panel still held a load of about 19,800 lb.

Test No. 2; Three-Ply Panel, $\frac{7}{16}$ In. Thick.—This panel was loaded in the same manner as in test No. 1. In the first 20 min of loading, gage C registered a total deformation of 0.241 in. Failure was reached in 6 more min with a total deformation of 0.380 in. at a load of 37,500 lb, corresponding to a shear of 4,068 lb per lin ft, or 775 lb per sq in. At this point, the panel completely "tore" at the third points. Loading was continued until a deformation of 2 in. was noted, the load dropping gradually to about 20,000 lb, which was maintained until the end of the test. After the load was removed, the panel looked as shown in Fig. 82, with offsets of about $\frac{3}{4}$ in.

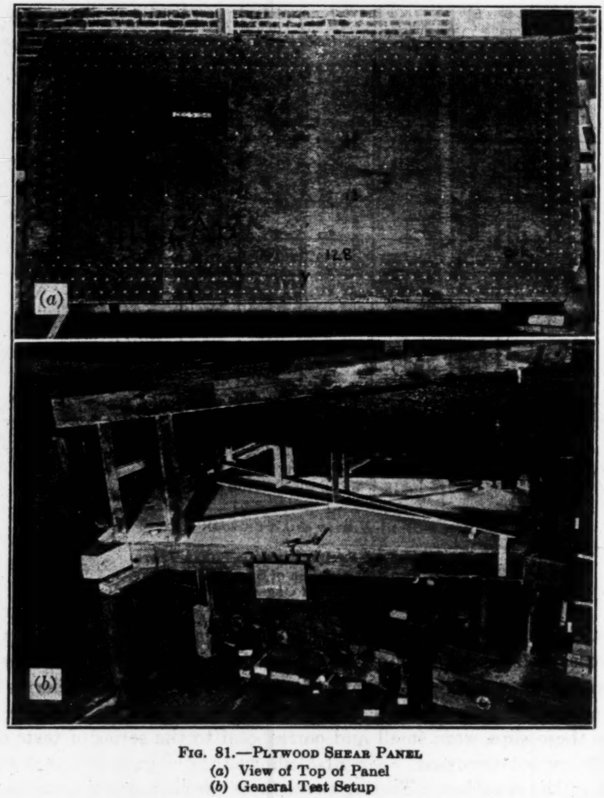


Fig. 81.—Plywood Shear Panel
(a) View of Top of Panel
(b) General Test Setup

Test No. 3; Five-Ply Panel, $\frac{7}{16}$ In. Thick.—The panel was installed in the testing frame and testing was started with a 20-ton jack. Loading was begun and 24 min were required to reach a maximum load of 32,170 lb, with a deformation of 0.239 in. at gage C and a corresponding shear of 3,597 lb per lin ft. Since it was found impossible to increase the foregoing load because of external conditions, it was released, noting a permanent deformation of 0.043 in. at gage C. No detrimental effects were noted. The following day, testing continued with a 50-ton jack which required 20 min to show a total deformation of 0.278 in. at a load at failure of 51,300 lb—corresponding to a shear of 5,736 lb per lin ft or 1,090 lb per sq in. Failure consisted of a single tear down the middle of the panel.

Fig. 82.—Shear Failure of Three-Ply Panel, $\frac{7}{16}$ In. Thick

Results of Tests.—For all three panels, Fig. 83 shows curves relative to the lengthening and shortening of the diagonals under load in terms of a 12-in. gage length and shears per linear foot acting on the plywood. The horizontal displacement of the wall (Δ) caused by plywood deformation can be computed with the aid of these curves and the formula:

$$\Delta = 2.5\, K\, L \qquad (2)$$

in which (see Fig. 83(*b*)) L is the height of the panel; and K is a factor derived from the curves (deformation at design shear). All the times given in Fig. 83(*c*) are for deformation on the compression diagonal. It may be noted that the tension diagonal deformed slightly more than the compression diagonal, which seems to agree with other tests on timber and plywood. Deformations, caused by shear in the plywood, are relatively small, especially compared to nail slips. In most designs they may be neglected.

Table 24 presents a summary of ultimate loads and deformations. According to these tests, five-ply plywood, as compared with three-ply plywood of the same thickness, has about 40% greater ultimate shearing strength. The results of these tests would indicate that the capacity of plywood in shear has

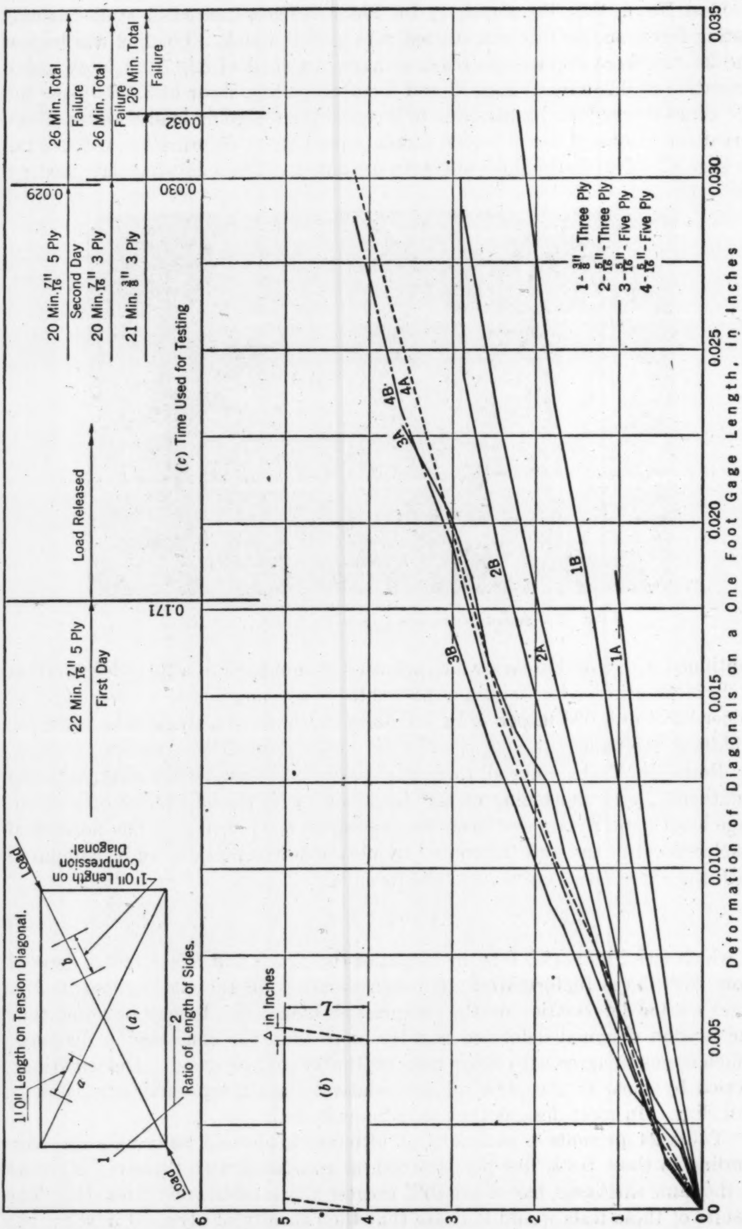


FIG. 83.—SHEAR-DEFORMATION CURVES, PLYWOOD SPECIMENS, GROUP TWO TESTS

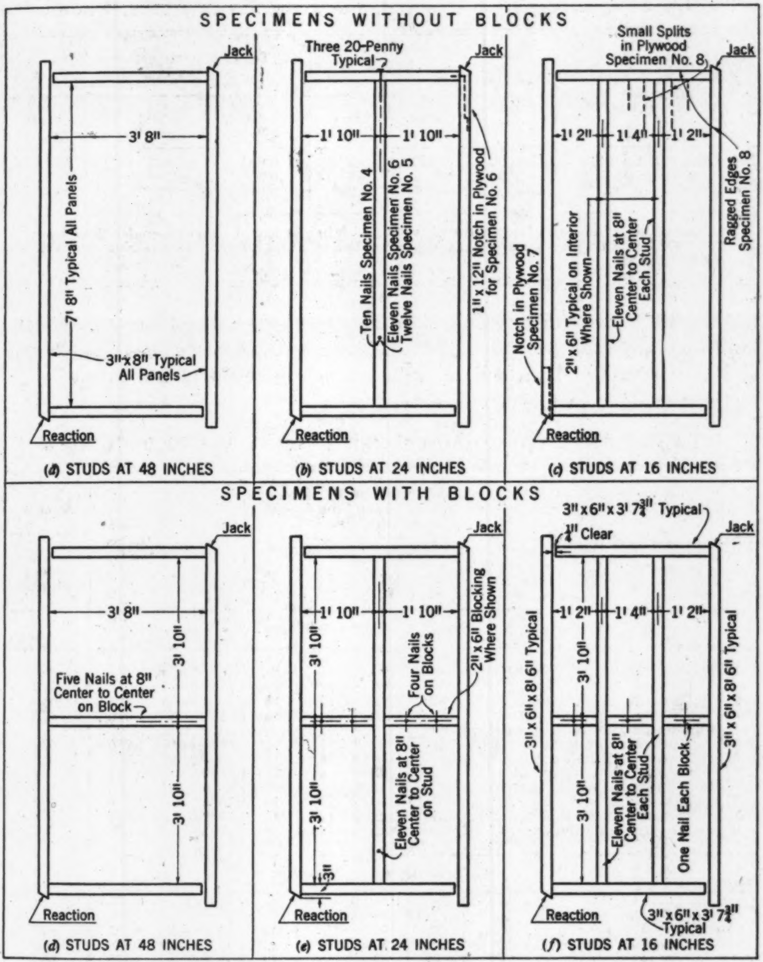

SPECIMENS WITHOUT BLOCKS
Jack
3' 8"
7' 8" Typical All Panels
3"x8" Typical All Panels
Reaction
(a) STUDS AT 48 INCHES
Three 20-Penny Typical
Jack
1' 10"
1' 10"
Ten Nails Specimen No. 4
Eleven Nails Specimen No. 6
Twelve Nails Specimen No. 5
1" x 12" Notch in Plywood for Specimen No. 6
Reaction
(b) STUDS AT 24 INCHES
Small Splits in Plywood Specimen No. 8
Jack
1' 2"
1' 4"
1' 2"
Ragged Edges Specimen No. 8
Notch in Plywood Specimen No. 7
2" x 6" Typical on Interior Where Shown
Eleven Nails at 8" Center to Center Each Stud
Reaction
(c) STUDS AT 16 INCHES
SPECIMENS WITH BLOCKS
Jack
3' 8"
3' 10"
Five Nails at 8" Center to Center on Block
3' 10"
Reaction
(d) STUDS AT 48 INCHES
Jack
1' 10"
1' 10"
3' 10"
Four Nails on Blocks
2" x 6" Blocking Where Shown
3' 10"
Eleven Nails at 8" Center to Center on Stud
3"
Reaction
(e) STUDS AT 24 INCHES
3" x 6" x 3' 7¾" Typical
¼" Clear
Jack
1' 2"
1' 4"
1' 2"
3" x 6" x 8' 6" Typical
3' 10"
3' 10"
Eleven Nails at 8" Center to Center Each Stud
One Nail Each Block
3" x 6" x 8' 6" Typical
Reaction
3" x 6" x 3' 7¾" Typical
(f) STUDS AT 16 INCHES

FIG. 84.—DETAILS OF PANELS TESTED FOR BUCKLING (SEE TABLE 25)

not been fully utilized because of the difficulty of making adequate connections. This deficiency may be remedied by the use of specially manufactured nails, as previously mentioned, by gluing, or by some other equally efficient method. Because of the increase in the use of plywood, these problems appear to be worthy of further study so that its potential strength values can be utilized more economically.

Group Three; Plywood Buckling Tests.—This group of tests included the investigation of plywood panels in which the buckling of the plywood itself

TABLE 24.—SUMMARY OF RESULTS, PLYWOOD SHEAR TESTS, GROUP 2

Plywood	Load on panel (lb)	Shear		Total Deformation,[a] Gage C (in Inches)		
		(Lb per lin ft)	(Lb per sq in.)	1,000 lb	2,000 lb	Ultimate load
Three ply, $\frac{3}{8}$ in. thick	23,000	2,571	575	0.085	0.28	0.469
Three ply, $\frac{7}{16}$ in. thick[b]	37,500	4,068	775	0.053	0.123	0.380
Five ply, $\frac{7}{16}$ in. thick	51,300	5,736	1,090	0.019	0.106	0.278

[a] Total deformation (in inches) of the plywood and frame at gage C, Fig. 80, under three loads: A shear of 1,000 lb per lin ft; a shear of 2,000 lb per lin ft; and, at an instant before reaching the ultimate load. [b] See Fig. 82.

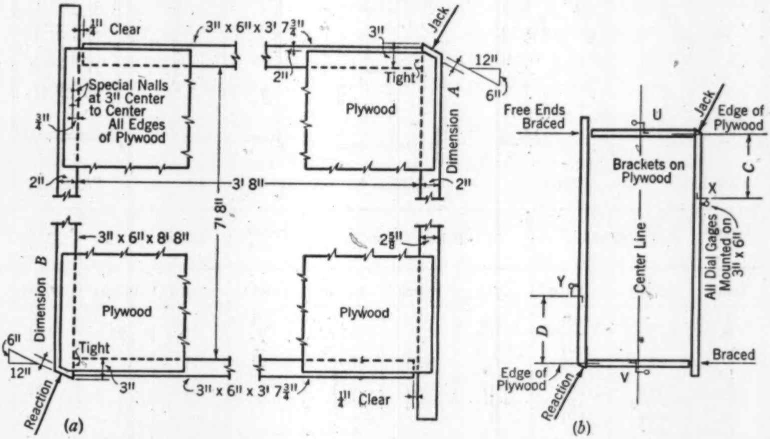
might be the strength criterion. For this purpose, eighteen specimens were built up of plywood $\frac{3}{8}$ in. and $\frac{7}{16}$ in. thick, using stud spacings of 16 in., 24 in., and 48 in., with and without blocking at the midheight of the panels. One test was made of each type of specimen.

TABLE 25.—REPORT OF BUCKLING TESTS IN PLYWOOD PANELS

No.	Plywood (see Fig. 84)	Number of Nails on Each Side (Fig. 85(*b*))				Ultimate load[a] (lb)	Shear (lb per lin ft)	Nail Value[b]
		X	Y	U	V			Pounds per nail at ultimate
	(*a*) SPECIMENS WITHOUT BLOCKS							
1	Stud Spacing, 48 In. (Fig. 84(*a*)):							
	Three ply, $\frac{3}{8}$ in. thick	32	32	15	15	14,940[a,c]	1,670	445
2	Three ply, $\frac{7}{16}$ in. thick	32	32	15	15	15,420[a,d]	1,724	460
3	Five ply, $\frac{7}{16}$ in. thick	32	32	14	14	14,940	1,670	477
	Stud Spacing, 24 In. (Fig. 84(*b*)):							
4	Three ply, $\frac{3}{8}$ in. thick	32	32	14	14	15,920[a,d]	1,780	509
5	Three ply, $\frac{7}{16}$ in. thick	32	32	14	14	20,640	2,308	659
6	Five ply, $\frac{7}{16}$ in. thick	32	32	14	14	18,580	2,077	594
	Stud Spacing, 16 In. (Fig. 84(*c*)):							
7	Three ply, $\frac{3}{8}$ in. thick	31	31	14	14	15,920	1,780	509
8	Three ply, $\frac{7}{16}$ in. thick[e]	32	32	14	14	15,920	1,780	509
9	Five ply, $\frac{7}{16}$ in. thick	32	32	14	14	20,640	2,308	659
	(*b*) SPECIMENS WITH BLOCKING							
	Stud Spacing, 48 In. (Fig. 84(*d*)):							
10	Three ply, $\frac{3}{8}$ in. thick	31	31	14	14	19,100[a,f]	2,135	617
11	Three ply, $\frac{7}{16}$ in. thick	32	32	14	14	17,700[a,g]	1,979	565
12	Five ply, $\frac{7}{16}$ in. thick	32	32	14	14	19,220	2,149	614
	Stud Spacing, 24 In. (Fig. 84(*e*)):							
13	Three ply, $\frac{3}{8}$ in. thick	32	32	15	15	13,500[h]	1,509	402
14	Three ply, $\frac{7}{16}$ in. thick	32	32	14	14	15,920	1,780	509
15	Five ply, $\frac{7}{16}$ in. thick	32	32	15	15	14,000[i]	1,565	417
	Stud Spacing, 16 In. (Fig. 84(*f*)):							
16	Three ply, $\frac{3}{8}$ in. thick	33	33	15	14	20,100	2,247	620
17	Three ply, $\frac{7}{16}$ in. thick	32	32	14	14	16,430	1,839	525
18	Five ply, $\frac{7}{16}$ in. thick	32	32	15	14	19,600	2,192	605

[a] In only five cases (Nos. 1, 2, 4, 10, and 11) was there any visible tendency to buckle. The nails failed in each case (except as noted by footnotes [b] to [g]). [b] Ultimate nail value (pounds per nail). In specimens Nos. 1, 4, 10, and 11 the nails were not the cause of failure (see footnote [a]). [c] Failure by buckling only. [d] The specimen buckled first and then the nails failed. [e] Ragged edges and ragged spots on top and left sides (see Fig. 84(*c*)). [f] The buckling test was stopped when the load on the specimen would increase no further. [g] Failure was by buckling but the nails were well bent and pulling through. [h] Failure was by nails pulling through the plywood. [i] Test was stopped when the clipped edges touched each other. [j] Nails did not fail completely.

Specimens.—The eighteen panels, measuring 4 ft by 8 ft, were constructed as shown in Table 25 with Figs. 84 and 85. Greater than usual edge distances of the nails were used because all plywood sheets already had a series of nail holes at the edges. In addition, some of the edges were slightly ragged. As in the shear tests, a space was left between framing members to minimize the amount of load that the frame itself would carry. Edges of the plywood were clipped so that no part of the jack or reaction forces would bear directly on the plywood, routing forces thereby through the frame and the nails. The plywood was fastened to the frame with special roofing nails (No. 10 gage 1¾ in. long, with $\frac{5}{16}$-in. heads) spaced 3 in. center to center at the edges and 8 in. center to to center on interior contacts. All nails were driven "home."



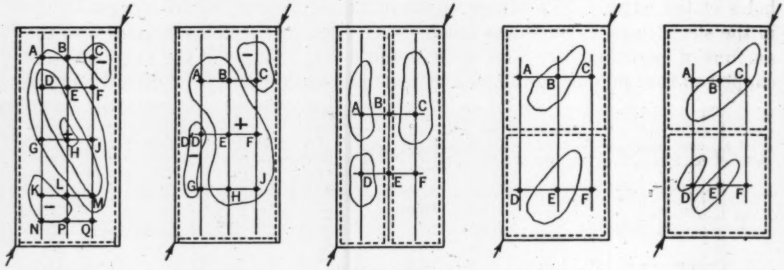
¼'' Clear
3'' x 6'' x 3' 7¾''
3''
Jack
12''
6''
2''
Tight
Special Nails at 3'' Center to Center All Edges of Plywood
¾''
Plywood
Dimension A
2''
3' 8''
2''
7' 8''
3'' x 6'' x 8' 8''
2⅝''
Dimension B
Plywood
Plywood
6''
12''
Tight
3''
3'' x 6'' x 3' 7¾''
¼'' Clear
Reaction
(a)
Free Ends Braced
U
Jack
Edge of Plywood
Brackets on Plywood
C
X
Center Line
All Dial Gages Mounted on 3'' x 6''
Y
D
Braced
Edge of Plywood
V
Reaction
(b)

FIG. 85.—CORNER DETAILS AND KEY TO TYPICAL DIMENSION IN TABLE 25, FOR ALL BUCKLING TESTS

Testing.—The panels installed in the testing frame were like those used in the shear tests. Loading was applied with a 20-ton hydraulic jack, in a manner similar to that described for the shear tests (Fig. 81(*b*)). Gages were placed at the four edges of the plywood to measure the nail slips. Braces were required to hold the frame in position. Since the web is off center from the chords, the near corner of the panel (Fig. 81(*b*)) tends to move up whereas the corners under the jack and reactions tend to move down. Loads were applied in increments, interrupted long enough to take readings of the dial gages. Panels were tested to failure, or until the deformations were large enough that the edges of the plywood bore on the bearing blocks at the reaction of the jack.

Thirteen of the eighteen panels failed by yielding of the nails (Table 25), most of them pulling through the plywood. In a few cases, the nails pulled out of the studs. In these thirteen tests, several panels failed after holding loads at large deformations for some time. Final failure sometimes occurred from a secondary effect, the block nearest the jack rotating and splitting at large nail deformations. In five specimens, failure is attributed to buckling, as shown by the contours and deflection readings in Table 26.

TABLE 26.—DEFLECTIONS (INCHES) OF PLYWOOD PANELS AT FAILURE BY BUCKLING; GROUP THREE (RAISED SURFACES ARE DENOTED +)



PANEL 1 PANEL 2 PANEL 4 PANEL 10 PANEL 11

Point[a]	Panel No. 1		Panel No. 2			Panel No. 4	Panel No. 10		Panel No. 11
Load[b]	12,960	14,940	8,950	11,920	14,940	15,920	13,000	18,000	17,700
A....	....	+0⅝	+0⅜	....	+1⅜	+0⅝	....	....	0
B....	....	+0 5/16	+0 5/16	....	+1 1/16	0	0⅜	0¼	+0¼
C....	....	−0⅛	−0 7/16	....	−0 1/16	+0⅝	....	....	+0¼
D....	+1	+1⅝	+0¼	+0⅜	+0⅜	−0¼	....	....	−0⅜
D′...	....	....	....	....	−0⅜	....	....	....	....
E....	+0⅞	+0¾	+0⅝	+0⅜	+1⅝	0	0⅜	1	+0¼
F....	+0 1/16	0	+0¾	+0⅜	+0¼	0	....	....	0
G....	+0⅝	+0⅝	0	....	+0⅜	....	....	....	....
H....	+1¼	+2	+0¾	....	+0 5/16	....	....	....	....
J....	+0 13/16	+0 13/16	+0⅝	....	+1⅜	....	....	....	....
K....	−0⅛	−0⅛	....	....	....	....	....	....	....
L....	+0⅜	+0⅜	....	....	....	....	....	....	....
M...	+0⅜	+1⅛	....	....	....	....	....	....	....
N....	....	−0⅛	....	....	....	....	....	....	....
P....	....	−0⅛	....	....	....	....	....	....	....
Q	....	+0⅜	....	....	....	....	....	....	....

[a] Deflections are given for the points shown in the accompanying diagrams. [b] Loads on the panel, in pounds, at the time of observation.

Results of Tests.—The data on each test panel together with ultimate loads, manner of failures, etc., are shown in Figs. 84 and 85, and in Tables 25 and 26.

Average test values for all panels at various stud spacings are as follows:

Stud spacing (in.)	Ultimate stress (lb per lin ft)
48	1,889
24	1,875
16	2,025
Average	1,928

From these tests, it appears that a working load of 500 lb per lin ft for plywood ⅜ in. thick is allowable since it would provide a ratio R_{wa} of 3—the lowest ultimate load recorded being 1,509 lb per lin ft. Even in the panel failing at that load, failure occurred in the nails and not in the panel.

The addition of intermediate blocking seems to have had little, if any, effect upon the strength within the limited range of stud spacings and plywood thicknesses investigated.

These tests were not sufficient in number or scope to be conclusive in determining critical buckling proportions even in stud spacings as large as 4 ft.

The ultimate nail values obtained in the buckling tests are comparable with previous nail tests. For example, the average ultimate strength of a special roofing nail with $\frac{5}{16}$-in. head as found in the nail tests of group one was 505 lb per nail; and the average for the buckle tests of group three was 508 lb per nail in those tests in which the nails caused failure.

Conclusions on Plywood Tests.—The following two conclusions on the plywood tests seem justified: (*a*) The value of plywood as a diaphragm to carry large loads, even where panels are as thin as $\frac{3}{8}$ in., appears to be established beyond reasonable doubt; and (*b*) the use of stud spacings larger than the customary 16 in. or 24 in. appears to be possible.

Because of the financial limitations of this program, other contemplated features of plywood panel construction could not be investigated. These further studies are recommended to the consideration of others. Among them are the following:

I. The buckling resistance of plywood beyond the scope of the present paper;
II. The strength of plywood panels in which the edge distances of nails are $\frac{1}{2}$ in. or less, as is the case where 2-in.-wide studs are used;
III. The capacity of nails when plywood panels are nailed to each other rather than to studs or posts; and
IV. The characteristics of large plywood panels, with and without openings, using details of construction as found in current best practice, particularly where bottom sills are bolted to foundations.

DISCUSSION

James H. Carr, Jr.,[7] Assoc. M. ASCE.—Much has been added to the knowledge of timber design and construction by this interesting study, which has already resulted in further studies being made by those who had advance information on the results. The Timber Engineering Company, one of the sponsors of the San Francisco (Calif.) study, for three years has been carrying on a series of tests on full-scale trusses, has completed a series of tests on eccentric joints, and has developed, therefrom, an empirical formula. In addition, it has sponsored research at the Massachusetts Institute of Technology at Cambridge on the distribution of stress around timber connectors.

There are three treatments in the paper with which the writer did not agree at the time of preparation. The first of these is that in many places in the paper the factors relating test load to design load are given when the failure was not directly connected to the design load used. In such cases, the factor given represents only the minimum amount of test load to which the section of the truss was submitted and does not indicate the maximum load it might sustain. The factor therefore is meaningless.

The second is that the comparisons in regard to allowable connector values are based on data published in 1939, although the original designs were not prepared on the basis of these data, but on data published in 1935. It is believed that it would have been more correct to compare either with the 1935 data or with the recommendations published in 1943, which were in effect, and current practice, at the time the report was in preparation, and which continue to be recommended and used in 1947.

The third is that much of the paper centers around a so-called factor of "tension perpendicular to grain," occurring particularly in eccentric joints. The writer disagreed with this theory at the time the report was in process and stated that he felt the tests indicated and closely followed another theory, advanced by Everett Lank, that the shear theory for notched beams more closely fitted the data. The authors, while stressing the "tension perpendicular to grain" theory, offer no design approach. Mr. Lank has subsequently been proved correct by a series of tests[8] substantiating the shear theory, and an empirical formula for design application based on this theory has been developed, as follows:

$$\frac{3}{2}\frac{V}{b\,d_e} = 1.5\,H \qquad (3)$$

in which V is the total vertical shear; b is the width of the section; and d_e is the effective depth of the section, or the distance from the inside edge of the connector (nearest the unloaded edge of the member) to the loaded edge of the member.

Application of these three comments to the data presented brings forth the results given hereafter. In order to simplify the presentation, all values of R_{ca} as developed in the paper for the 1939 data and those based on the 1943 data, published by the National Lumber Manufacturers' Association,[9] are given in Table 27.

---

[7] Secy., Timber Eng. Co., Washington, D. C.

[8] "Tests of Eccentric Split-Ring Joints Yield Rational Design Formula," by R. H. Glass, *Engineering News-Record*, May 29, 1947, pp. 69–70.

The report states the first failure for truss TB–1 is at joints U1 and L10. It is not clear as to just what the initial failure was. However, a comparison on current data gives the result shown in Table 27(*a*).

Applying the shear formula to the French Building truss at failure, the 1943 factor becomes 2.56 (Table 27(*b*)) as against the 1939 factor reported, based on connector values (which did not fail) of 1.7, an appreciable increase.

In the Merry-Go-Round Building trusses the lumber failed and not the connector joint, hence the *R*-values are not true results. However, based on current recommendations the factors of Table 5 become those shown in Table 27(*c*).

As to the discussion of theoretical stress around a connector (under the heading, "Merry-Go-Round Building (M) Trusses"), a paper to be presented at the 1947 Fall Meeting of the Society on tests conducted at the Massachusetts Institute of Technology, under the direction of Prof. Albert Dietz, Assoc. M. ASCE, discusses data developed by use of stress coats and electrical strain gages. Current recommendations include factors developed in these tests; connectors of improved design have also been developed and are now available.

In the North Square Court truss tests the failure was due to eccentric joints and hence the shear formula applies. Factors secured by this method show that for joint L3 the 1943 factor is 2.43. Although they do not reflect the true failure, the factors on the 1943 data would correct the factors in Table 6 to those given in Table 27(*d*).

Also, in Table 8, the changes in factors, where applicable, based on the 1943 data would be as shown in Tables 27(*a*), (*b*), (*c*), and (*d*). The writer disagrees entirely with conclusion 6 (see heading, "Comments on Tests of Trusses") as

TABLE 27.—COMPARISON BETWEEN RATIO OF TEST LOAD STRESS AND DESIGN CAPACITY AS BASED ON 1939 DATA AND ON 1943 RECOMMENDATIONS

Truss (1)	Joint (2)	R_{ca}	
		1939	1943
(*a*) BRAZIL BUILDING TRUSSES			
TB–1	....	2.95	3.86
TB–2	....	2.48	3.28
(*b*) FRENCH BUILDING TRUSS			
TF	....	1.7	2.56[a]
(*c*) MERRY-GO-ROUND BUILDING TRUSSES			
TM–1	L2	3.25	4.5
	L1	3.30	4.6
	L4	2.53	3.1
TM–1	....	3.3	4.6
TM–2	....	3.6	4.9
(*d*) NORTH SQUARE COURT BUILDING TRUSS			
TNS	L3	2.07	2.43
	U2	1.78	2.99
	U6	1.83	3.06
TNS	....	1.78	2.99
(*e*) JOINT TESTS (STEEL GUSSETS)			
....	J2	4.56	5.61
....	J3	3.53	4.30
....	J4	3.48	4.17
....	J5	4.84	6.44
....	J6	4.81	6.40
....	J7	5.04	6.70
....	J8	4.77	5.78
....	J9	4.79	6.38
....	J10	4.54	5.73
....	J11	3.12	3.58
....	J12	4.08	4.55
....	J13	5.06	....[b]
....	J14	2.5	3.33
....	J15	2.91	3.87
....	J16	2.93	3.96
....	J17	1.83	....[b]
(*f*) JOINT TESTS (ECCENTRIC)			
....	J1	2.16	3.12
(*g*) JOINT TESTS (SPLIT RINGS)			
....	J23	1.55[c]	1.82
....	J24	3.00[c]	3.69
....	J25	2.6[d]	3.55
....	J26	2.55[d]	3.48
....	J27	2.55[d]	3.48
....	J28	2.75[d]	3.75

[a] Based on shear formula. [b] Connectors did not fail. [c] Not true factors, as grooves sheared and test setup failed. [d] Tests not carried to failure of connector joints.

---

[9] "National Design Specification for Stress-Grade Lumber and Its Fastenings," National Lumber Mfrs. Assn., Washington, D. C., 1944.

it is believed that eccentric joints may be properly and capably handled with the shear formula.

On the basis of the 1943 data, the factors in Table 9 should be corrected as shown in Table 27(*e*). Based on the shear formula application, the 1943 factor for test joint J1 becomes 3.3. However, if connector factors must be considered, the factors are changed as shown in Table 27(*f*). As regards the tests of joints with ring connectors, the factors given in Tables 13 and 14 should be corrected to the 1943 values, as shown in Table 27(*g*).

After eleven years' experience, the writer feels that the comments or conclusions on the tests of joints with steel gusset plates are broader than the data available.

However, in view of the results of this program, it is this writer's opinion that the designer who uses the 1943 recommendations, including the shear formula, is assured of factors which are more than safe when based on these tests inasmuch as the smallest corrected true factor is 2.43, whereas a factor in the neighborhood of 2 would be considered sufficient by experienced timber designers and research men.

Howard J. Hansen,[10] Assoc. M. ASCE.—The descriptions of the equipment used and the methods of testing are quite complete, and will be valuable guides in future tests of full-scale trusses and joints involving eccentricity and loads at an angle to the grain. In most cases, the tests of any one type were too few in number and the variables too involved to provide an exact design procedure for similar types. The writer is, however, much pleased that this extensive test program has been published and is particularly interested to note that the complete report including test data, curves, field notes, photographs, and calculations is filed at several strategic points throughout the United States.

The sudden failure under test loads of the Cavalcade cantilever truss (TC) and the detailed examination of joints in other trusses again emphasizes the need for competent engineering supervision of timber structures if the factor of safety expected in design is to be provided in actual construction. This same fact was brought out during World War II when timber structures were built with little or no competent engineering inspection and investigations of several failures showed that failure was due to improper construction.

The R_{ca}-values calculated for the Brazil Building (TB) trusses did not include the bending effects of the chords. This fact was noted in the report. However, since the stresses due to bending could be rather high, they should have been included in the computations of the R_{ca}-values. The fact that the nature of the failures in the joints of these and other trusses emphasizes the importance of secondary stresses is significant. It is believed that many typical designs of shallow timber trusses have not been checked for secondary stresses in the past. The authors mention the need for additional tests on joints in which a combination of shearing stress, direct stress, and bending stress is involved. This is one of the greatest needs in the field of timber engineering and additional experimental data on combined stresses will aid materially in the development of more efficient and economical design in timber.

---

[10] Prof. of Mechanics and Acting Head, Industrial Eng. Dept., Univ. of Florida, Gainesville, Fla.

Although the data on the lateral strength of nails in plywood are worthwhile and may be used as a guide for further experimental investigations, considerably more data on nailed construction are definitely needed. The tests covered in the report do not include a wide enough range in nail sizes, spacing, and end and edge distances to predict the behavior of nailed joints other than those tested.

TABLE 28.—COMPARISON OF R_{ca}-VALUES FOR JOINTS AS BASED ON 1939 AND 1943 ALLOWABLE LOADS

Joint	R_{ca}		Col. 2 − Col. 3[a]
	1943	1939	
(1)	(2)	(3)	(4)
1.....	2.5	2.16	+0.34
2.....	4.62	4.56	+0.06
3.....	3.44	3.53	−0.09
4.....	3.33	3.48	−0.15
5.....	5.1	4.84	+0.26
6.....	5.08	4.81	+0.27
7.....	5.32	5.04	+0.28
8.....	4.47	4.77	−0.30
9.....	5.06	4.79	+0.27
10.....	4.52	4.54	−0.02
11.....	2.82	3.12	−0.20
12.....	3.68	4.08	−0.40
13.....	4.91	5.06	−0.15
14.....	2.3	2.5	−0.20
15.....	3.05	2.91	+0.14
16.....	3.12	2.93	+0.19
17.....	1.94	1.83	+0.11

[a] Excess or deficiency of 1943 values over 1939 values.

As noted in the report, the buckling tests of plywood panels were not exhaustive enough to provide design details. The conclusion that stud spacing may be increased beyond the customary 16 in. or 24 in. seems feasible, but additional tests on Douglas fir structural plywood should be conducted along the lines investigated by the United States Forest Products Laboratory at Madison, Wis., on aircraft plywoods.

TABLE 29.—VALUES OF *R* FOR SHEAR TESTS OF DOUGLAS FIR PLYWOOD

Grade[a]	Stress (Lb per Sq In.)		*R*
	Ultimate	Allowable	
(1)	(2)	(3)	(4)
(a) Three Ply, ⅜ In. Thick			
G-2-S....	575	480	1.22
G-1-S....	575	456	1.28
So-2-S...	575	420	1.39
So-1-S...	575	384	1.52
Sheathing	575	360	1.62
(b) Three Ply, $\frac{7}{16}$ In. Thick			
G-2-S....	775	480	1.61
G-1-S....	775	456	1.70
So-2-S...	775	420	1.84
So-1-S...	775	384	2.02
Sheathing	775	360	2.15
(c) Five Ply, $\frac{7}{16}$ In. Thick			
G-2-S....	1,090	480	2.27
G-1-S....	1,090	456	2.39
So-2-S...	1,090	420	2.60
So-1-S...	1,090	384	2.84
Sheathing	1,090	360	3.03

[a] "G" denotes "Good"; "S" denotes "Side"; and "So" denotes "Sound."

In the joint tests the factor of safety of each joint was calculated on the basis of recommended allowable loads from the TECO manual.[1] A comparison of the R_{ca}-values based on a later TECO manual[11] is given in Table 28.

There are several questions that have arisen from the study of this report and it is believed that the value of the report will be enhanced if the answers to these questions are given. The questions are:

1. What approximate method was used in computing the stresses in the members of an eccentric joint?
2. How was the lower chord splice of truss TM-2 reinforced before testing?
3. Why were blocks added to joint J1 under test loads? Would this give a true picture of the type of failure for this joint?
4. Cannot some explanation for the large difference in movement of the gusset plates in joint J2 be given?

---

[11] "Design Manual for Teco Timber Connector Construction," Manual of the Timber Eng. Co., Inc., Washington, D. C., 1943.

5. Is the statement under the heading, "II. Joint Tests: Comments on Tests of Joints with Steel Gusset Plates: (10)," that "* * * in these tests the load was always parallel to grain" correct?

The values of *R* based on the ultimate shear loads for various thicknesses of Douglas fir plywood, as obtained by the test results given in the report and the allowable stresses for various grades of Douglas fir plywood, are given in Table 29. The report does not mention the grade of plywood used and the values in Table 29 may prove valuable as a comparison of the *R*-values for the different grades.

In conclusion, the report is a valuable contribution to the literature and design data pertaining to timber structures, as well as an indication of the need for more full-scale tests of timber trusses, joints with combined stresses, nail tests, and structural plywood tests of all types. It is true that no two identical timber trusses or joints will behave exactly the same under load; but, as more tests of this nature are conducted, the results can be correlated with those contained in this report.

E. G. STERN,[12] Assoc. M. ASCE.—The report of the committee contains much valuable information for timber designers and fabricators of timber structures. It is of interest to learn what comments designers and fabricators of the tested timber structures may make; whether their assumptions, theories, and computations were verified by the test data, and which theories were contradicted. It is believed that additional valuable information could be obtained from the designers and fabricators if a review of their designs and construction logs could be made available on the basis of the test data obtained.

The fabricators probably can give the answer to this question: Were split rings and shear plates, respectively, omitted purposely in joints L7, L8, and L9 of truss TB-1, and in joint J8 because they were considered unnecessary?

Several of the findings seem to confirm the previously made assumption that, during initial loading and failures of a minor nature, adjustments take place in the local and secondary stresses—often anticipated but not included in detailed computations—which result in reduction and more uniform distribution of such internal stresses and, thus, make the structure safer than it was before adjustments took place. Although such adjustments may increase the load-carrying capacity of the structure, the deformation and total deflection may also be increased beyond those computed. Rolling shear of the core within split rings may often be accounted for as a result of such adjustments.

It is hoped that the appropriate industry will sponsor performance of some of the recommended, additional tests in order to secure the needed information. It cannot be expected that engineers or architects will employ timber structures as long as some of the basic design data are lacking and not available to them. Industry may be reminded that several United States technical institutions and universities are willing to assist industry in obtaining such vital data by assigning specific problems to graduate students interested in timber structures or by assigning their experienced staff to industry-sponsored projects.

---

[12] Associate Prof. and Head, Wood Constr., Virginia Polytechnic Inst., Blacksburg, Va.

Melvin W. Jackson,[13] **Jun. ASCE.**—The San Francisco Timber Test Program Committee calls attention to the necessity of considering tension perpendicular to grain for the correct design of timber joints, but it does not suggest how this may be taken into consideration. The problem raised is apparently an unsolved one. In testing approximately 250 ring-connectored timber joints of the type shown in Fig. 86, the writer obtained some information pertinent to this case.[14] The test specimens were intended to simulate conditions in a truss joint of the type shown in Fig. 36. The investigation was partly an outgrowth of a previously published report of this type of failure.[15]

In the test specimens (Fig. 86) member Nos. 1, 3, and 5 were end-matched 2 in. by 8 in. by 18½ in. long, and the cross members, A and B, were similarly end-matched, but varied in dimensions. The lower bolt was provided to prevent excessive spread of member Nos. 1 and 5 under load. Standard procedure was followed in the laboratory in installing 4-in. split-ring connectors in the faces between each member. The test specimens, to which the data given hereafter apply, were of Douglas fir, structural grade (allowable working stress = 1,200 lb per sq in.). Two general types of failures occurred in members A and B; that is, either cores sheared, or the horizontal members split longitudinally. The tests were stopped at the first indication of failure to determine the type of initial failure. It was necessary to take the joint apart to determine whether a core had sheared.

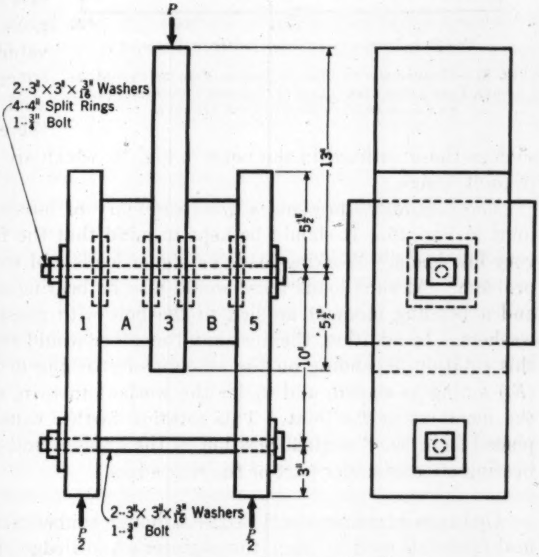


Fig. 86.—Assembly of Test Joint

Tension perpendicular to grain tests following the recommended procedure of the American Society for Testing Materials were made from members A and B, and these values and the load at initial failure in the five-member specimens were compared. Where the horizontal members were 2 in. thick (nominal dimensions) and 20 in. or more long, and where the initial failure occurred by splitting, the correlation shown in Fig. 87 was obtained. Fig. 87 indicates that there is some correlation between the strength of a timber in tension perpendicular to grain, and the load which will cause it to split in a joint of the type tested. It is apparent, however, that the problem in this case cannot be solved simply by dividing the total stress in tension across the member by an allowable working stress in tension perpendicular to grain, because doubling the strength of the material in tension perpendicular to grain does not double the load required to produce initial failure. Possible explanations of this fact may be that the law of stress distribution within the member is not known or that the member and the control tests for establishing the tension perpendicular to grain values are subject to stress concentrations not evaluated. On the other hand, there may be other factors, such as those outlined in the notes in Fig. 36, which are not usually considered in joint design.

---

[13] Champaign, Ill.

[14] "An Experimental Investigation of Some Properties of Five-Member Ring-Connectored Timber Joints," by Melvin W. Jackson, thesis presented to the Univ. of Illinois at Urbana, in 1945, in partial fulfilment of the requirements for the degree of Master of Science in Civil Engineering.

[15] "Preventing Failure of Timber Joints," by Charles Mackintosh, *Civil Engineering*, December, 1943, p. 573.

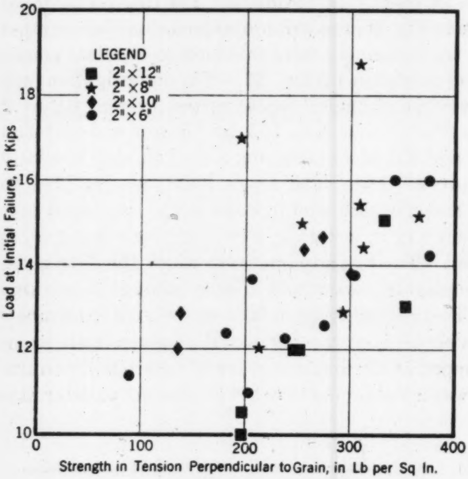


Fig. 87.—Comparison of Load at Initial Failure in 5-Member Test Joint with Tests of Tension Perpendicular to Grain in 20-in. and Longer Members

The committee presents a good summary of forces acting on the typical joint in Fig. 36. It should be kept in mind that the forces shown are under very low loads. Working loads or higher loads add to the complexity of the problem. At such loads there would also be bearing stresses under the bolt, and a bending moment applied to the bolt with resisting stresses under the washers. In addition, the horizontal member would rotate, the magnitude of this rotation depending on the amount of crushing of fibers under the forces (F_s) acting as shown, and under the washers in turn, depending on the fit of the members in the joint. This rotation further causes the rings to be displaced from their original position in the grooves and adds additional lateral bearing stresses under part of the ring edges.

Charles Mackintosh,[16] Esq.—Although timber is one of the oldest structural materials used by man, the engineer's knowledge of its proper use in large frame structures is appallingly meager. This paper makes a signal contribution to the understanding of the actions of timber joints. It makes unmistakably clear the fact that the strength of the bolt, and split-ring (or other) connection in a given joint is dependent not upon the tables of values of such connecting devices, but rather upon the nature of the connection itself and the timber in the connection.

---

[16] Cons. Engr., Mackintosh and Mackintosh, Los Angeles, Calif.

The tables of safe loads for bolts, split rings, and other connections are, in general, based upon the expected safe load when the failure is at the maximum value obtained from the most advantageous position of the connector. These tables do not generally anticipate the failure of the timbers in the connection. It is stated in the "Foreword," that nearly one half of the failures occurred "* * * because of types or combination of stresses not anticipated"—which is a most revealing evaluation of the present quality of engineering design.

The importance of secondary stress due to bending in the shallow trusses of the Brazil Building is well emphasized. This writer has found that similar importance should be attached to any condition where concentrated long-time loads occur on the chords between panel points of normally proportioned trusses. A heavily loaded beam supported between panel points on the lower chord of a timber truss may cause such a deflection over a long period of time that enough additional stress is caused by eccentricity to produce failure.

In discussing the failure of the cantilever frame in the Cavalcade Building, the paper emphasizes that oversized bolt holes may have contributed to the failure of the joint by causing some successive loading. If the effect of the oversized hole is so great when shear plate connectors are used, should not the manufacturer's recommended value be questioned? These values are based upon ultimate loads having greater movement than the tolerance caused by enlarged bolt holes. It is clear that final failure will occur if the loads which caused an initial movement of ⅛-in. are maintained for a sufficient length of time. Such failure is attributed by some timber research men to "creep" or "progressive failure."[17] In the data published relative to split rings, "* * * the loads listed as maximum are the highest loads obtained within or at a slip in the joint of 0.60 inch, beyond which the tests were continued."[18] Is not 0.60 in. much too great a slip to give any worthwhile data for the design of connector joints subject to long continued load?

In discussing the heel failures of the trusses in the Alameda-Contra Costa Counties Building, the paper states (under the heading, "Detailed Report of Tests,") that: "Failures in the blocks at these relatively low unit [shearing] stresses were probably due to the abrupt change in sections * * *." Reentrant grooves in timber for various connectors cause abrupt changes in section and greatly influence the relative safe values of long-time and short-time loads. Unfortunately, few data are available on any type of connector stressed by long-time loads. Such tests have been initiated in both Australia and the United States, but are of such recent origin that final reports are not available. Oversized holes have long been prohibited by skilled timber truss manufacturers because of the large deflection in trusses with oversized holes, rather than because of the reduced value of the connection.

The necessity of considering tension perpendicular to grain and proper end distance and spacing has been well presented by the committee, and these comments point to more complete analysis than most engineers have been accustomed to making. This analysis must be supplemented by additional series of tests such as those sponsored by most universities.[14] The fullest use of timber cannot be made until such test data are available. A practical solution may lie in a type of connector which obtains more uniform attachment to the to the wood and does not require large end and edge distances.

---

[17] "Green Timbers in Australian Structures," by Ian Langlands, *Engineering News-Record*, September 6, 1945, p. 94.

[18] "Timber-Connector Joints: Their Strength and Design," by John A. Scholten, *Technical Bulletin No. 865*, Forest Products Laboratory, Forest Service, U. S. D. A., Madison. Wis., March, 1944, p. 39.

Because of timber truss failures in Los Angeles, Calif., and particularly the failure of a splice in a truss using shear plates, the Superintendent of Building and Safety of the City of Los Angeles requested recommendations from the Structural Engineers' Association of Southern California. The subsequent report of this association and its Committee on Timber Research puts into usable form some of the conclusions to which the paper points.

The report recommends wider use of grade marking, straight grain tension members, the rejection of previously recommended methods [19] for computing spaced column values, greater attention to the moisture contents of lumber, calculation of tension across grain stresses and stresses due to eccentricity, and a reduction of connector values below those recommended by the manufacturers.

The authors of this paper are to be commended for the material aid that they have given to the development of a better understanding of structural design in wood.

SIDNEY NOVICK,[20] JUN. ASCE.—It is of interest to compare the results of the plywood shear buckling tests (under the heading, "IV Plywood Tests; Group 3") with the values at which failure in buckling might have been predicted on the basis of previously published information concerning shear buckling of rectangular plywood plates. The method and results of this comparison are described below.

All calculations require basic assumptions regarding the specific plywood constructions. From the descriptions in the text, it was inferred that plywood type 1 was of the sanded face ply variety, and that types 2 and 3 were unsanded; the thicknesses of the individual plies could then be found from data issued by the Douglas Fir Plywood Association.[21] Assuming the modulus of elasticity along the grain for a single ply to be 1,600,000 lb per sq in. and that across the grain to be 0.045 times the value along the grain,[22] all necessary plywood stiffness properties may be found.

A convenient method for calculating the theoretical buckling stress of a simply supported rectangular plywood plate under shear has been presented elsewhere.[23] Later empirical investigation[24] (with plates acting as webs of box beams) indicated that the theoretical stresses require correction within a given range of plate dimensions. From the empirical data the writer has derived a curve[25] from which the required correction factors can be obtained. Specific values necessary to derive factors needed for the ultimate buckling stress calculation were also obtained from other published data.[26] Values given in Table 24 were used as the maximum plywood shear stresses in these calculations.

---

[19] "Wood Structural Design Data, " *Supplement No. 4*, National Lumber Manufacturers' Association, Washington, D. C., 2nd Ed., 1939.

[20] Structural Dept., John B. Pierce Foundation, Housing Research Div., Raritan, N. J.

[21] "Technical Data on Plywood," Douglas Fir Plywood Assn., Tacoma, Wash., p. 9, Section 2.

[22] "Design of Plywood I-Beams," by Howard J. Hansen, *Transactions*, ASCE, Vol. 112, 1947, p. 955.

[23] "Buckling of Flat Plywood Plates in Compression, Shear or Combined Compression and Shear," *Bulletin No. 1316* and Supplements, Forest Products Laboratory, Forest Service, U.S.D.A., Madison, Wis., 1942–1945.

[24] "Design of Plywood Webs in Box Beams," *Bulletin No. 1318* and Supplements, Forest Products Laboratory, Forest Service, U.S.D.A., Madison, Wis., 1943–1944, Fig. 16.

The comparison of results of the foregoing calculations with the actual shear stresses at failure, the latter derived from Table 25, is given in Table 30. This tabulation shows that the scatter in the percentage variation values is large for the specimens that failed by buckling. That this behavior is typical is evident from a consideration of the Forest Products Laboratory (FPL) results:[23] For the range of plate dimensions in Table 30(*a*), the FPL data show a variation between 50% and 150% greater than the values on the empirical buckling curve (the latter, of course, taken along the lower limits of the range of values). It appears that the average fixity of plywood plates when glued or nailed sufficiently to insure shear buckling is well above the simply supported condition, but that the variations in the extent of fixity lead to large variations, not only between the actual stresses and those predicted, but also in the extent of the disagreement, so that the actual values are not in constant relation to the theoretical.

TABLE 30.—COMPARISON OF PREDICTED SHEAR BUCKLING STRESS AND ACTUAL SHEAR FAILURE STRESS IN PLYWOOD

Specimen No. (1)	STRESS (LB PER SQ IN.)		Variation of actual from predicted stress (%) (4)
	Predicted shear buckling (2)	Actual shear failure (3)	
(*a*) FAILURE DUE TO BUCKLING			
1	135	370	+174
2	170	330	+ 94
4	265	395	+ 49
10	220	475	+116
11	340	375	+ 11
Average	...	...	+ 89
(*b*) FAILURE DUE TO NAILS			
3	240	320	+ 33
5	405	440	+ 9
6	580	395	− 32
7	370	395	+ 7
8	580	340	− 41
9	850	440	− 48
12	395	410	+ 4
13	295	335	+ 11
14	445	340	− 24
15	615	300	− 51
16	370	500	+ 35
17	605	350	− 42
18	865	420	− 51
Average	...	...	− 15

From the results for the specimens in Table 30(*a*), the specimens in Table 30(*b*) might have been expected to fail in shear buckling at values proportionately above the predicted buckling stresses. It is evident that the actual failure of the nails at much lower values forestalled development of the requisite buckling stresses. Thus, the results for the specimens in Table 30(*b*) are in qualitative agreement with the behavior expected on the basis of the previously published information.

Results of the shear buckling tests reported in the paper, therefore, tend to confirm previously published data and the general method for design calculation derived therefrom. The general method, in turn, is applicable within the entire practicable range of design variables, the latter including the spacing of studs and intermediate blocking, and the thickness of plywood sheathing.

---

[25] *Transactions*, ASCE, Vol. 112, 1947, p. 968, Fig. 5.

[26] "Design of Wood Aircraft Structures," *Bulletin No. 18*, Army-Navy-Commerce Committee on Aircraft Requirements, U. S. Govt. Printing Office, Washington, D. C., 1942.

It should be particularly noted that by this method the value of the intermediate blocking can be found in explicit quantitative terms, making it necessary to alter the implied conclusion in the text that the value of such blocking is open to question.

FRANK J. HANRAHAN,[27] M. ASCE.—Those closely associated with this enormous, complicated test program are to be congratulated, both for their unceasing efforts in the face of great obstacles and for the important contribution made to the knowledge of timber engineering. These investigations, together with extensive research by the United States Forest Products Laboratory (Madison, Wis.), the Timber Engineering Company (Washington, D. C.), and others have resulted in substantial changes and technical improvements in engineering design recommendations. Such modifications are reflected in a design specification[9] which has been widely adopted as a design standard by government agencies and many other organizations. Among the items which were more accurately evaluated and are incorporated in this specification are the effect of end distances, spacings, edge distances, thickness of member, angle of load to grain, moisture condition of lumber at time of fabrication and in service, and loads for multiple fastenings—all in relation to the basic design capacity of connectors.

Among other things, the analysis of these 1941 tests, and the report on them, focused attention on the need for further study of eccentrically loaded joints and the cross-grain stresses resulting from such eccentric loading. In spite of studies conducted subsequent to analysis of these tests, no satisfactory criteria for designing eccentric joints and taking care of the resulting stresses had been developed prior to the issuance of the "National Design Specification for Stress-Grade Lumber and Its Fastenings" in 1944. There appeared to be little correlation between values obtained from standard tension perpendicular to grain tests and values obtained from tests of eccentric joints which presumably were loaded in cross-grain tension. Consequently, the aforementioned specification recommended avoidance of eccentric joints.

Subsequently extensive research studies developed an empirical approach which when applied to a wide range of tests showed a proper relationship between allowable design loads and test loads at failure. This recommendation, which currently is officially incorporated as an amendment in the national design specification and which assumes the mathematical form of the horizontal shear formula but with allowable shear stresses 50% greater than otherwise permitted, may be stated as follows:

> "In computing the horizontal shear in eccentric joints the depth of the member shall be assumed as reduced by the distance between the unloaded edge of the member and the nearest edge of the connector. Where bolts

---

[27] Chf. Engr., National Lumber Mfrs. Assn., Washington, D. C.

> alone are used, the depth shall be reduced by the distance between the unloaded edge and the center of the bolt. Shearing stress for joint details may be taken as 50% greater than the horizontal shear values otherwise permitted."

In addition, the stresses induced by axial loading, if any, and bending moment should be calculated and taken care of in the design in the customary manner.

The effect of these technical advances through the incorporation of the additional technical findings of recent years in current recommendations[28] is indicated by the values in Table 27.

When the structures tested were designed for the 1939 Golden Gate International Exposition, in San Francisco, Calif., the designer did not have access to design recommendations later than those of 1935, which were considerably less refined than those of 1939. Design stresses for lumber and design loads for fastenings higher than those commonly used in 1935 were used in the Exposition buildings. John J. Gould,[29] M. ASCE, chief structural engineer for the Exposition, has stated:

> "Split rings gave satisfactory service despite working stresses 30 to 50 per cent more than usual. * * *
>
> "The wholly satisfactory performance of exposition structures, designed with 50 to 100 per cent increases of the usual timber unit stresses, suggests that where competent engineers make the design and supervise construction and maintenance, building codes should be revised to provide for increased unit stresses—perhaps 30 to 50 per cent more than now permitted."

It should be noted that the ratios given in the report and in discussion approach true safety factors for those instances where failure occurred. The so-called factor of safety as commonly used is not a true factor of safety for any material in the sense that the load required to produce failure of the completed structure is the design load times the factor. This so-called factor of safety contains allowances for variability of material, inaccuracies of design and fabrication, and other factors. When the structure is tested to failure, many of these items are incorporated physically in the structure itself, instead of as allowances in the so-called factor of safety. Consequently, the true factor of safety as determined by test of the structure logically should be lower than the factor of safety as commonly stated.

It is interesting to note from Table 27 that, for those joints in which failure is approached under test load, the lowest ratio of test load to design load is approximately 3 when based on current design recommendations[28] and is approximately 1¾ when based on those for 1939. Hence, these particular data indicate that present design recommendations actually result in structures approximately 70% stronger than those based on 1939 recommendations, in spite of the fact that the current basic allowable design values for lumber and fastenings are 20% higher than in the 1939 recommendations.

---

[28] "National Design Specification for Stress-Grade Lumber and Its Fastenings (1944) Including 1947 Amendments," National Lumber Mfrs. Assn., Washington, D. C., 1947.

[29] "Experience in Maintenance of Large Timber Structures," by John J. Gould, *Engineering News-Record*, August 15, 1940, pp. 72–74.

Although the writer does not intend to imply that such a percentage increase in strength can be expected generally, the data do indicate that improvements in design recommendations do offset increases in allowable design stresses in so far as total strength of structure is concerned. The data also illustrate the greater technical accuracy and reliability of the design recommendations incorporated in the aforementioned design specification[28] and the increased safety of structures conforming to this current specification.

Walter J. Ryan,[30] M. ASCE.—The Timber Test Committee of the Society's San Francisco Section deserves the thanks of the profession for publishing this report of its project, hindered as it was by many obstacles that threatened to make its completion impossible. We are fortunate to have this record, which marks a big step in the progress of timber construction from an art to a science.

Builders of the ninteenth century carried forward the knowledge of timber construction that had been accumulated since the beginning of its development. They were artisans, and the structures they built were erected by men trained and interested in their work. Following the introduction of newer materials of construction, engineering methods became an essential part of the development. Now that timber engineering is becoming a science instead of an art, it is suffering from a lack of these artisans of former generations and from the lack of understanding of a common material. The very name "lumber," so frequently used in the United States, is an indication of the lack of respect for a familiar product. This report illustrates clearly how intricate a material wood really is, and the designer of timber structures should keep the ten notations, made in the "Foreword," always in mind. These findings[31] support the "ten commandments" of a west coast association.

Most of the grief in design and construction in timber is caused by a lack of uniformity of strength within the piece due to "grain," changes through "seasoning," and variation within and between "grades." These tests will supplement laboratory studies and leave no excuse for the designer to notch a timber or neglect an analysis of the various stresses acting on a member. Lumber grading rules are nearly as perfect a guide as human interpretation makes possible. However, some pieces may be unsuited for the specific application. The framer must have the knowledge and the authority to cull such items and to select material suitable for the particular purpose. The report illustrates instances where this was not done. The principal difficulty came from shrinkage and checking. This problem is inherent with the use of green timber and seems to be without a solution. The answer is in the use of seasoned material.

Air seasoning of large timbers is not practical. Drying in a kiln is difficult and expensive. Modern practice can secure the most satisfactory results by treatment in a retort. This can be accomplished with or without the addition of preservative. Such trusses as those used in the Treasure Island Buildings will be built of seasoned or treated material in the future.

---

[30] Weyerhaeuser Timber Co., Tacoma, Wash.

[31] "Ten Essential Standards for Fabricated Timber Structures," West Coast Lumbermen's Assn., Portland, Ore., 1946.

The design of the first heavy trusses built in the United States with split rings was based on the tables published by the manufacturer, Carl Tuchscherer,[32] of Breslau, Germany, supported by a limited series of tests made at the University of British Columbia (Canada) in Vancouver. Later tests proved that the values assigned, especially to the large 9-in. and 11-in. rings, were excessive, but that the bridges served adequately for railroad loading until the timber was destroyed by decay. The bolts in these trusses were kept tight.

In the San Francisco tests, the committee saw fit to test the joints in their condition in the structure. This condition reflected the result of shrinkage and little or no maintenance. There was lack of uniformity and, in the case of loose bolts, there was the loss of the frictional resistance which is an important element in the strength of a timber joint. Perhaps these results will help to emphasize the importance of continuous inspection and maintenance. A more valuable result will be attained, however, when adequate tests are made with perfect examples and engineers, builders, and owners are taught to treat the design and the structures with the care they deserve.

WALTER M. PRATT,[33] Assoc. M. ASCE.—In the "Foreword" of this paper reference is made to "The need of keeping adjacent faces of members in contact * * *." The moisture content of the wood is apt to vary even after long periods of service in a structure; there may possibly be no moisture equilibrium condition. Use of moisture control might avoid many problems in wood construction. In contrast to various treatments of woods to resist moisture, so-called "fog-outlets" are available from some of the automatic sprinkler companies. These units are designed to control the humidity in a building automatically.

The Committee refers favorably to the use of plywood. Bonding strength of the glue used does not seem to be mentioned, but it might reasonably be as good as the wood in longitudinal shear, or from 100 lb per sq in. to 150 lb per sq in. The wider use of glue at surfaces of contact might be helpful, just as is the use of phenolic resin glues in laminated wood truss work, particularly with recent developments of the high-frequency electric current method of application. Casein glues might be practicable for use without special heat facilities for temperatures as low as 40° F. Surfaces of contact referred to would include splice plates, heel pads, and assemblies of web members in contact with chords.

In Fig. 28 the bottom chord joint pad has a contact of 7½ in. by 18 in., or 135 sq in. Using a unit stress of 100 lb per sq in. on this area, the total transmitted load is 13,500 lb. Because of a possible lack of uniformity in sizing the wood, a reduction factor could be applied, but it seems likely that the stress in the split rings shown could be materially reduced, at the same time keeping the adjacent surfaces in contact. Without adhesives, the friction between wood surfaces in contact can be a factor in carrying loads, but it is often lost through wood shrinkage, and its amount is doubtful.

---

[32] "Der Holzbau," by Theodor Gesteschi, Springer, Berlin, 1926.

[33] Mgr., Pratt Roof Truss Co., Detroit, Mich.

The background of these tests is such that all the truss work shown is of "ordinary" construction and not "slow burning." A slow burning truss can be defined roughly as one in which no single member has an area of less than 56 sq in. Use of the authors' data on plywood for gussets suggests possibilities for future truss designs where greater resistance to fire is desired through slow burning construction.

Several joints tested involved the use of "split rings," and in several cases the wood cores sheared. The results indicate the value of tests on full-size trusses, compared with laboratory tests often made. The tension across grain phase is very properly emphasized. A hollow ring dowel, and any round dowel, must necessarily produce a splitting action. There is no constant angle to the grain in a round dowel. The force in the dowel may be at an angle to the grain, but the bearing of the ring dowel on the wood is never at a single angle—rather it is at many angles.

Referring to Fig. 36, it is also possible that the core within a ring may receive load before the exterior bearing of the ring, producing an extra "punching shear." The wood core may or may not take up one half the ring load. It is customary, but not necessarily correct, to assume that each half of the ring takes one half of the load. With a ring 4 in. in inside diameter having a center bolt $\frac{3}{4}$ in. in diameter, the core shear area is 12.12 (say, 12) sq in. At 150 lb per sq in. its load transmitting value would be 1,800 lb. Let the value for this ring be 6,400 lb at 0° to the wood grain and let the value at 90° be 4,500 lb. If the core is working independently, and is limited in capacity by horizontal shear, the opposing halves of the ring must carry 4,600 lb and 2,700 lb, respectively. Dividing by the projected area of $4 \times 1 \times \frac{1}{2}$ or 2 sq in., the resultant unit stresses would be 2,300 lb per sq in. and 1,350 lb per sq in., respectively—even if the bearing were truly at 0° and 90°, respectively. The allowable stresses for these rings, given in "III. Nomenclature: Explanation of Terms and Abbreviations," paragraph 3, can be compared with the design stresses described under the same heading (paragraph 1).

If a hollow square dowel or a hollow rectangular dowel were available, the core could be analyzed on the basis of the allowable stresses mentioned in the "Nomenclature." Thus, a hollow square dowel, 4 in. by 4 in. by 1 in., at 90° to the grain would develop a resistance of $2 \times 4 \times \frac{1}{2} \times 350 = 1{,}400$ lb, compared with the 4,500 lb previously stated for the 4-in. split ring.

The round dowel is used partly because a convenient turning tool is available; a hollow square dowel could be tooled with a routing machine. As the authors state, workmanship is a very important factor. There is also the possibility that round wood cores shrink to something approaching an ellipse. From an economic standpoint, structural use from the core is desirable, but the dowel design should have balance, and the same factor of safety could well be used for ring cores as is demanded for the remainder of the structure.

To approach the phase of tension across the grain from a different viewpoint, a round dowel can be compared with a square one. If a hollow square dowel, 4 in by 4 in. by 1 in., has a value of only 1,400 lb at right angles to the grain, and if a 4-in. standard split ring has a value of 4,500 lb under similar conditions, it can be assumed that a ring-type dowel never acts at any one angle to the grain. Consider the wedging and splitting action of such a 4-in. square dowel set at 45° to the grain. It would have a theoretical bearing area of $2 \times 4 \times \frac{1}{2} \times 1.41 = 5.6$ sq in. At an allowable stress of 540 lb per sq in. (determined by the Hankinson formula), this area will transmit a stress of 3,050 lb, which compares with 5,500 lb—the value of a 4-in. split ring at a 45° bearing angle. It seems likely then that tension across the grain is inherent in the ring-type dowel, and that forces that tend to separate the fibers of the wood are set up.

Regarding reentrant cuts in wood—some may be eliminated. If a joint assembly needs a cluster of ring dowels, the web members could at times be balanced around resultant force lines as shown in Fig. 88, using only one row of dowels, as is shown in the bottom chord.

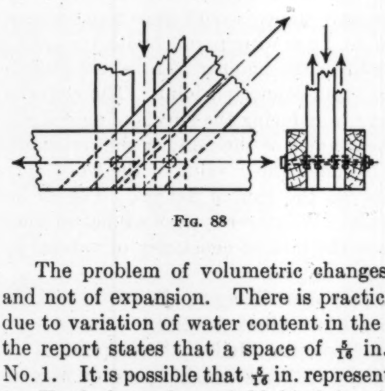


FIG. 88

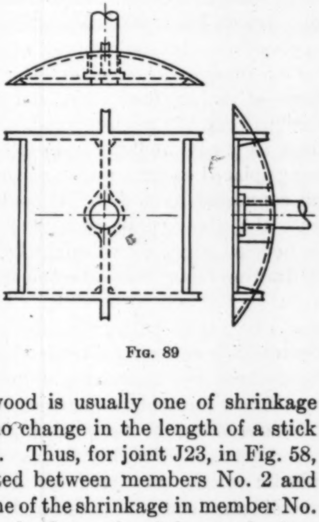


FIG. 89

The problem of volumetric changes in wood is usually one of shrinkage and not of expansion. There is practically no change in the length of a stick due to variation of water content in the wood. Thus, for joint J23, in Fig. 58, the report states that a space of $\frac{5}{16}$ in. existed between members No. 2 and No. 1. It is possible that $\frac{5}{16}$ in. represents some of the shrinkage in member No. 1. Vertical members of this kind, bearing on the flat grain of the wood, often show such a separation, and the authors rightly recommend that such details be avoided. The use of free body diagrams will often assist the designer to avoid such bearings. Thus, if the vertical members in Fig. 88 were fastened on the outside of the chord, the angle to the grain on chord would be 90°. Placed as shown, the diagonal is the only member working at an angle to the grain.

However, there is the possibility in a joint assembly that different members shrink at different rates and at different angles. Splice pads may have different water contents than the member joined and they may shrink at a different rate than the member, thereby causing possible secondary stresses, particularly if more than one row of bolts or connectors is used. Each member may shrink in both width and thickness, the shrinkage usually being about one third less radial to the annual growth rings than tangential to them. Depending on how the lumber is cut from the log, shrinkage may be unbalanced and cause warping and twisting. Lumber is certainly an intriguing material for the designer. It would seem best, as often as possible, to keep wood connectors on the center lines of joined pieces, and to avoid clusters of off-set connectors.

The report stresses the need for new thought and improvements on timber connectors. It indicates the desirability of a type of connector that can be analyzed readily and also the desirability of reducing flat grain bearings, of providing for wood shrinkage, and of preventing the overstepping tolerances by manual labor. In order to avoid the complications in stress analysis of ring dowels, the use of plane surface contacts instead of rounded contacts would greatly simplify analysis. The report definitely uses, in making comparisons, "connector allowable load." Such loads were apparently developed from tests by others. Undoubtedly, in a great deal of construction, hollow ring dowels have been used with satisfactory results, but their continued wide application, even at reduced values, would still leave many engineers hoping for an improved connector after considering all the factors—safety, rational basis of design, feasibility, and economy. Perhaps one of the writer's own designs (Fig. 89) might furnish some constructive thought along these lines, at least as a new angle of approach. The unit shown would have two arcuate wings placed at right angles to the grain, and one wing parallel with the grain, all connected by a web. It could be used in the manner of a "shear plate," with the whole connection computed as a pin-connected joint. The concave or beveled edges shown might be effective in reducing the effects of shirnkage. A limiting factor might be the single shear value of the bolt, or the resistance value of the bolt to bending. The longitudinal shear value of the wood core could be regulated by the distance between the pair of flanges. The single center fin is entirely subtended by the bolt. Whatever type of connector may be devised, the underlying difficulties are the lack of constancy of volume in wood, and its nonhomogeneous character.

Concerning the workmanship—some play must likely be allowed in making wood fittings. Possibly the use of cementitious fillers (similar in character to plastic wood), powdered metal with an asphaltic binder, thermoplastics, or metal shims could be considered. Tolerances in workmanship possibly may be overcome by means other than providing flexibility in a dowel.

COMMITTEE OF THE SAN FRANCISCO (CALIF.) SECTION, ASCE, ON TIMBER TEST PROGRAM: HAROLD B. HAMMILL,[34] M. ASCE, CHAIRMAN, JOHN J. GOULD,[35] HENRY D. DEWELL,[36] AND LEON H. NISHKIAN,[37] MEMBERS, ASCE, AND HENRY J. DEGENKOLB,[38] ASSOC. M. ASCE.—Although the number of discussions of the report of tests of timber structures from the Golden Gate International Exposition at San Francisco, Calif., has not been great, the committee feels gratified with those presented. The committee realizes that the proper study of a report of this character requires great effort before an engineer is prepared to present a discussion.

---

[34] Cons. Civ. Engr., San Francisco, Calif.

[35] Cons. Engr., San Francisco, Calif.

[36] Mr. Dewell died on March 20, 1946.

[37] Mr. Nishkian died on June 2, 1947.

[38] Structural Engr., San Francisco, Calif.

Since the completion of testing for this report, many more data on plywood have been published by the United States Forest Products Laboratory in Madison, Wis., and the Douglas Fir Plywood Association. It is gratifying to find that the tests are in general agreement with later, more comprehensive tests and data as reported by Mr. Novick. Regarding stud spacings and the need for intermediate blocking as questioned by Professor Hansen and Mr. Novick, the committee's conclusions referred to the use of plywood in stud walls for resistance to lateral forces and not to glued plywood web girders. Obviously, these conclusions would be open to question when applied to glued plywood webs, which are usually stressed more highly in shear.

In reference to the questions of fabrication raised by Professor Stern, the split rings and shear plates omitted in certain joints and trusses were omitted by error and not by design. Grooves had been made in the timbers in all cases and carelessness on the part of the erection crews was responsible for the omissions.

The thought contained in Mr. Mackintosh's question—"If the effect of the oversized hole is so great where shear plate connectors are used, should not the manufacturer's recommended value be questioned?"—has no connection with the failure of the cantilever frame in the Cavalcade Building. The report states that this frame was very carelessly fabricated and the relation of bolt size to bolt hole was not in accordance with manufacturer's recommended values. In fact, the report shows that, in tests of other specimens, when manufacturer's values were used with proper workmanship, excellent *R*-values were obtained.

Regarding various questions raised by Professor Hansen, the committee comments as follows:

1. In computing the stresses in a truss with large eccentricities, preliminary stresses were calculated assuming concentric joints. Then diagonal member stresses were corrected for the change in slope and the additional leverage caused by the eccentricities, assuming the points of inflection to be at the midpoints. The forces were then checked for balance and further corrections were applied as necessary.

2. The lower chord of truss TM-2 was reinforced with four 3 by 8 pads, boxing the section, twenty ½-in.-round bolts, and eight ⅜-in.-round stitch bolts.

3. In joint J1, it seemed obvious to the investigators, from previous types of tests, that failure would come at a very low load and that the test would not yield any new information. Therefore, the blocking at the sides of the vertical member was added to investigate a new condition. It is felt that this reinforcement greatly increased the capacity of the joint but manifestly did not give a true picture of the failure of the joint as originally constructed.

4. Regarding the testing of joint J2, an inaccurate statement (see heading, "II. Joint Tests: Concentrically Connected Joints—Joint J2") was made. At the next to the last reading for the first series of tests the top plate had moved 0.132 in., and the bottom plate had moved 0.137 in. When retested the next day, the plates moved approximately the same amount—up to the last reading. Then the top plate moved a total of 0.11 in., and the bottom plate took a sudden jump to 0.40 in. On disassembly it was found that the shear plate on the bottom side had cracked and broken, thus explaining the large deformation.

5. For joints J2 to J17, inclusive, it is correct to state that "* * * in these tests the load was always parallel to grain."

Mr. Carr disagrees with several statements upon which the committee comments as follows:

The *R*-Values Used in the Report.—In many cases where an *R*-value is given for a member that did not fail, data were furnished to show that the element in question successfully held at least the loads shown. Where *R*-values were so given, the fact was clearly stated. In other cases, for example, truss TNS (see Table 6), all factors were given, as there is unquestionably a doubt as to which was the critical one. In such cases, the highest factor was recorded and used in the tables and discussions. In any case, the comment that the "* * * factor given represents only the minimum amount of test load * * *" is not justified by the facts.

The Use of 1939 Connector Values.—In reference to Mr. Carr's comment on the use of 1935 or 1943 connector values versus 1939 connector values, the committee may be subject to some criticism for using the 1939 values instead of the 1935 values, but certainly not for failing to use the 1943 values. This subject was discussed at great length and the committee decided the report would be more valuable if based on currently used connector values. The committee still considers the decision sound although it was known that new connector values were to be published in 1943. With the tests based on a definite table of connector values, any interested designer can determine, from the information given in the report, the factors found by use of any different table of connector values. Great difficulties were encountered in the preparation of this report and the major part of the computations was completed before publication of the 1943 connector values. The committee believes that the report could never have been published if new computations had been required for results based on the 1943 table of connector values.

It should be mentioned that the major difference in *R*-values between 1943 and 1939 (see Table 27) is found in the 20% stress reduction for members fabricated green and used after seasoning. A more realistic comparison of the basic revisions is given by Professor Hansen in Table 28, which the committee, in general, checks. Although, in Table 28, the effects of moisture conditions when fabricated were neglected, it should be noted that few, if any, improvements of original inconsistencies were made. In fact, when adding all the pluses and minuses, the improvement is +0.41 for seventeen joints, or an average of +0.02 per joint. Furthermore, this improvement is indiscriminate and does not increase low factors nor decrease high ones. Any other improvement is due to the blanket 20% seasoning factor.

If other parts of the test results were compared with the latest design specifications,[9] all tests on bolts would show a 150% increase in R-values to account for the presently recommended 67% reduction for fabrication when green, as illustrated in Table 31.

Both Messrs. Carr and Hanrahan have mentioned the new "National Design Specification"[9] and advocate its use, although Table 31 lists large R-values. The use of a specification requiring such large R-values does not result in sound economic designs. With the exception of joint UI of truss TSF (design and fabrication mistakes are detailed in the report), these bolted joints gave some of the highest R-values of the whole test program. Nevertheless, in accordance with new design standards, the factors for these bolted joints fabricated green and tested dry are increased 150%, whereas similar connector joints are increased 25%, although some connector joints loaded at certain angles to the grain proved to be deficient. Certainly, with such a large variation in R-values, further revision should be investigated in the new "National Design Specification" for bolts, and similar revisions should be considered for connectors with parallel to grain loading.

TABLE 31.—R-VALUES OF CERTAIN BOLTED CONNECTIONS

Designation	RATIO, R_{ba}		Designation	RATIO, R_{ba}	
	Test[a] report	National design specifications[b]		Test[a] report	National design specifications[b]
(1)	(2)	(3)	(1)	(2)	(3)
(a) TRUSSES			(b) JOINTS		
TCh-1	5.3	13.2	J18	8.95	22.4
TCh-2	6.4[c]	16.0	J29	6.31	15.8
TCh-3	5.7	14.2	J30	6.57	16.4
TCh-4	5.9	14.8	J31	6.31	15.8
TCh-5	7.3	18.2	J32	5.54	13.8
TSF	2.94	7.38	J33	7.3	18.2
	2.0[d]	5.0	J34	5.36	13.4
	5.1	12.9	J37	3.03	7.60
	4.65	11.6	J38	3.31	8.3
			J39	5.9	14.7
			J40	5.1	12.7
				4.3	10.7
			J41	7.0	17.5

[a] Based on the recommendations of the Forest Products Laboratory, Madison, Wis. [b] Based on "National Design Specification for Stress-Grade Lumber and Its Fastenings," National Lumber Mfrs. Assn., Washington, D. C., 1944. [c] Bolts. [d] At joint U1.

Tension Across Grain.—Mr. Carr received the preliminary report in progressive stages and therefore knows that considerable space was given to discussing the tension across grain features of certain failures and that a design method was tentatively advanced. This method was supported by certain tests,[39] which were made by the University of California at Berkeley on specimens furnished by the Timber Test Committee. The final edition of the report had to be drastically shortened and as a result the "tension across grain" method was deleted. The following is quoted from Mr. Carr's discussion:

---

[39] "Tests and Analysis of Crossgrain Tension in Eccentric Wooden Truss Joints," by O. H. Degenkolb and J. J. Holstein, a thesis presented to the University of California at Berkeley, Calif., in 1942, in partial fulfilment of the requirements for the degree of Bachelor of Science.

> "The authors, while stressing the 'tension perpendicular to grain' theory, offer no design approach. Mr. Lank has subsequently been proved correct by a series of tests[8] substantiating the shear theory, and an empirical formula for design application based on this theory has been developed * * *."

Everett Lank's formula has been developed from tests of fifty-four small-size, dry, selected laboratory specimens with results varying more than 100%. From experience on the timber test program the committee is convinced of the soundness of the conclusion (see heading, "A. Foreword"), which calls attention to "1. The inadequacy of small individual tests as a premise for determining the behavior and strength of full-sized structures." The committee believes that conclusions from such small-size tests should not be considered in preparing specifications for universal use until the results of such tests have been checked by testing full-sized specimens.

With regard to factors of safety, it is realized by the committee that, in testing complete structures or trusses, smaller factors are acceptable than for individual joints. The committee does not disagree with a factor of 2 as noted in the last paragraph of Mr. Carr's discussion, provided all factors of design are considered, including long-time loading.

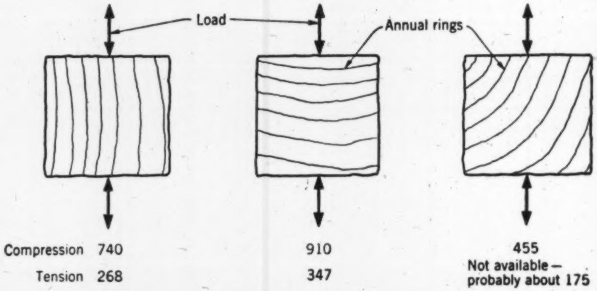


FIG. 90.—ULTIMATE STRENGTHS, IN POUNDS PER SQUARE INCH, OF DOUGLAS FIR AT 12% MOISTURE CONTENT LOADED PERPENDICULAR TO THE GRAIN AND AT VARIOUS ANGLES TO THE DIRECTION OF ANNUAL RINGS

The committee is very much interested in the results found by Messrs. Jackson and Mackintosh on their joint tests. Fig. 87 confirms the committee's opinion that the stresses in a typical timber joint using split rings are definitely connected with the tension perpendicular to the grain strength of wood. Particular notice has been paid to the variation of the tension strength from 90 lb per sq in. to 380 lb per sq in., as this confirms the committee's observations. It is believed that the worst condition for tension across the grain is not usually reported (see Fig. 90). For example, the "Wood Handbook"[2] gives average values of 740 lb per sq in., 910 lb per sq in., and 455 lb per sq in. for the compressive strength perpendicular to grain of wood when load is applied parallel, perpendicular, and at 45°, respectively, to the annual rings. The Forest Products Laboratory in unpublished data also gives 268 lb per sq in. and 347 lb per sq in. as the average tensile strengths when load is applied parallel and perpendicular to the annual rings, but neither knowledge of tests nor test data are available for conditions at 45°. However, by analogy to the compression tests, the 45° condition would possibly be the minimum, with an average value of about 175 lb per sq in. Considering variability and other factors, this could help explain many erratic results. In the University of California[39] tests, it was inferred that this tension strength at 45° might be related to the number of annual rings per inch; the closer the annual rings, the higher the tension strength. In the joint shown in Fig. 86, it is felt that it would have been better if the horizontal members A and B had been longer, since the University of California tests confirm the theoretical distribution of tension stress as extending each side of the point of application of the force about two and one-half times the depth of the member. Thus, for example, for a 2 by 6 member the minimum length is 5 × 6 in., or about 30 in., and for a 2 by 12 member, it is about 5 ft. The effective length of tension surface, from both tests and analyses, seems to be about 3.1 times the depth of member.

The Douglas Fir Plywood Association reports that three of the four subjects recommended by the committee for further consideration have already been considered. The general case of plywood buckling, together with specific applications, has been solved both theoretically and experimentally by the Forest Products Laboratory.[26] The Treasure Island tests confirm the formulas of the Forest Products Laboratory.

The lateral bearing strength of nails having reduced edge distances in plywood is discussed, and test data are presented in Section 6 of "Technical Data on Plywood."[40]

Some information on the behavior of large plywood sheathed walls, both with and without openings, under wracking loads is contained in an article by Alan D. Freas,[41] Assoc. M. ASCE. Further test data on the shear strengths of specific prefabricated plywood panels may be found in various reports of the National Bureau of Standards.

In Table 29, Professor Hansen gives values of R on the ultimate shear loads of Douglas Fir Plywood shear tests. It is unlikely that the grades tested were higher than sound two sides, according to officials of the Douglas Fir Plywood Association who examined these panels for the committee. The allowable stresses used are evidently based on stresses recommended for loads at 45° to the grain, which are twice those recommended for parallel and perpendicular directions. N. S. Perkins and D. Countryman of the Douglas Fir Plywood Association, in a communication to the committee, state that in their opinion these R-values should be based upon the parallel condition, which would give double the values. The tests were made with plywood in "pure" shear. Neglecting edge conditions, shear is equal in all directions. The committee presents this information without comment.

---

[40] "Technical Data on Plywood," Douglas Fir Plywood Assn., Tacoma, Wash.

[41] "Guides to Improved Framed Walls for Houses," by Alan D. Freas, *Engineering News-Record*, October 17, 1946, p. 117.

The committee appreciates the favorable comments contained in the discussions of Messrs. Pratt and Ryan, and the helpful suggestions contained in correspondence from Messrs. Perkins and Countryman of the Douglas Fir Plywood Association.

The committee deeply regrets that the following engineers who contributed so much to the development of this report did not live to see it published: Henry D. Dewell and Leon H. Nishkian, Members, ASCE, members of the committee; Chester J. Hogue, M. ASCE, of the West Coast Lumberman's Association; and J. E. Mackie, Assoc. M. ASCE, of the National Lumber Manufacturers' Association.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2356

INVESTIGATION OF DRAINAGE RATES AFFECTING STABILITY OF EARTH DAMS

By F. H. Kellogg,[1] M. ASCE

With Discussion by Messrs. Harry R. Cedergren, Ross M. Riegel, T. W. Lambe, W. L. Shannon, and F. H. Kellogg

Synopsis

The investigation described in this paper was undertaken for the purpose of studying the effects of increased rates of reservoir drawdown on the stability of an earth dam. Field observations of the rates of drainage of several dams were compared with rates computed from a method of analysis similar to procedures used in the theory of heat. Computed and observed rates agreed fairly well. To obtain a wider range of comparison, laboratory observations were made of models and sand samples under a variety of conditions. Both analytical and observed results indicated that the possibility of mass slides resulting from reservoir drawdown has been somewhat exaggerated, but that "sloughing" due to a process of flotation or "piping" induced by drawdown may deserve more attention than it has received. Limitations for the method of analysis are suggested.

Introduction

Anyone who has watched a spring during the dry part of the summer must have been impressed with the efficiency of nature's system of rationing, which continuously distributes water stored in some long-forgotten wet season at rates that instantly adjust themselves to the supply on hand. Engineers in many fields have endeavored to learn the mechanics of this system and to apply it to problems of water supply, land reclamation, and stabilization of foundations and subgrades, as well as to petroleum production and agriculture. They have been successful as long as water is constantly supplied to the ground from an external source. When underground pore spaces constitute the only source of outflowing water, engineers must either use empirical rules, with their numerous qualifications and local exceptions, or rely on flights into the realm of theory

---

Note.—Published in September, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Prof. of Civ. Eng., Univ. of Mississippi, University, Miss.

that often require a certain amount of "nature faking" to facilitate mathematical treatment.

It was necessary to choose between these alternatives in searching for a solution to a specific problem, relative to the effect of rate of reservoir drawdown on the stability of an old earth dam. Both were tried. The best results were obtained from an analytical method similar to that used in problems involving heat and consolidation. A description of that method, together with supporting data from field and laboratory observations and a discussion of the application of the results to estimates of slope stability, constitutes the principal part of the subject matter presented in this paper. Some suggestions as to the limitations of the methods used are offered in the hope that other engineers can supply a wider variety of data indicating the applicability of such analyses to drainage problems in general.

The term "steady state" is used to describe a condition of constant rate of seepage and a distribution of heads that does not vary with time. Seepage through a dam when the reservoir is held at constant level, so that all water escaping is replaced from the pool, is an example of this condition. The term "unsteady state" is used to represent a condition of variable rate of seepage and head distribution changing with time. Seepage from a dam during a drawdown, when all escaping water is not replaced, is an example of unsteady state flow. The scope of discussion is limited to those conditions under which Darcy's law is valid. No direct consideration is given to the related subject of consolidation.

INVESTIGATION

Specific Problem Involved.—The particular problem that led to the investigations was one of judging the maximum safe rate of drawdown of the reservoir impounded by the Blue Ridge Dam, a semihydraulic fill dam built about 1930, in northwest Georgia. Since 1939, Blue Ridge Dam has been operated as a part of a regional system of power plants.

An operating rule, inherited from the original owner, limited the rate of drawdown of the reservoir to 1 ft per day or 5 ft per week, to prevent "sloughing" of the dam. In 1943, during a period of unprecedented demand for power, this rule caused a bottleneck in the operation of the entire reservoir system. The question arose as to whether the rate of drawdown could safely be increased, and if so, how much.

The problem was attacked by trying to determine the rate of decrease of the hydrostatic pressures exerted by the water in the pores of the embankment, during a given drawdown, so that (*a*) stability against mass sliding could be estimated by "slip circle" analyses,[2] and (*b*) stability against flotation or "piping" could be estimated by analysis of the stresses at the upstream face.[3]

Published Data.—An analysis of seepage in the unsteady state similar to that used in problems of unsteady state heat flow has been applied to ground waters involving very low hydraulic gradients. The method and its fallacies were discussed by Morris Muskat,[4] in 1937. Charles V. Theis[5] has used a thermodynamic analogy in computing the rates of drawdown of a pumped well. His computed data show a remarkably close agreement with observations under certain conditions. Mr. Muskat[6] has presented an analysis of steady state flow in which time enters as a boundary condition. This theory specifically applies to the rate of advancement of an oil-water interface. Karl Terzaghi,[7] M. ASCE, discusses the time rate of drainage of an ideal sand through its base. J. H. A. Brahtz[8] considers unsteady state drainage of an earth dam which involves three factors that must be determined in the laboratory—namely, permeability, percentage of air voids, and modulus of compressibility. Lacking the required facilities, the last of these factors could not be determined during this investigation, so no checks on this method could be made.

---

[2] "Pendleton Levee Failure," by Kenneth E. Fields and William L. Wells, *Transactions*, ASCE, Vol. 109, 1944, pp. 1400–1429.

[3] "Uplift and Seepage Under Dams on Sand," by L. F. Harza, *ibid.*, Vol. 100, 1935, pp. 1354–1358.

The various empirical rules generally applied to drainage involve grain size and plasticity. Their application would lead to the conclusion that clays cannot be treated by drainage. This conclusion is contrary to field experience, since many cuts in flocculent clays have been stabilized by drainage.

Since references to such literature as was available gave no procedure readily applicable to the problem at hand, it was decided to study observations of the rate of change of pore-water pressures in various earth dams in the hope of obtaining some empirical rule or of developing an analytical treatment with the aid of factual information not immediately available.

Notation.—The letter symbols in the paper are defined where they first appear, in the text or by illustration, and are assembled for convenience of reference in Appendix I.

Method of Investigation.—For the first stage of the investigation, the pore-water pressures were observed within several earth dams that had been in operation for two years or more. The initial observations were made while the elevations of the impounded reservoirs were held constant at high levels for prolonged periods. Thus, for each observation point, a pressure could eventually be read which would not change with time, indicating a steady state. Later, as the reservoir elevations were lowered, the rates of change of pore-water pressure at each point were compared with the rates of drawdown.

Obviously, any change in hydrostatic pressure indicated by sounding an open well involves a flow of water either into or out of the well. Unless this flow occurs in an infinitely short time, a sounding made during periods of changing water levels will merely represent the apex of a cone of depression or the top of a water cone. The height of such a cone will become progressively greater as the change continues.

Therefore, in this investigation, the same type of measuring devices were used that are employed in reading pore-water pressures in silts and clays.[9] Special pressure cells, reading to 0.1 in. of hydrostatic head, were developed for use in studying model embankments. The purpose of these cells was to disturb the natural flow of pore water during drainage as little as possible by the pressure measurement.

---

[4] "The Flow of Homogeneous Fluids through Porous Media," by Morris Muskat, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1937, pp. 363–365.

[5] "The Relation Between the Lowering of the Piezometric Surface and the Rate and Duration of a Well Using Ground-Water Storage," by Charles V. Theis, *Transactions*, Am. Geophysical Union, Vol. 16, 1935, pp. 519–524.

[6] "The Flow of Homogeneous Fluids Through Porous Media," by Morris Muskat, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1937, p. 453 et seq.

[7] "Theoretical Soil Mechanics," by Karl Terzaghi, John Wiley & Sons, Inc., New York, N. Y., 1943.

[8] "Notes on Analytic Soil Mechanics," by J. R. Bruggeman, C. N. Zangar, and J. H. A. Brahtz, *Technical Memorandum No. 592*, Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo., 1939, pp. 122–135.

Since pore pressures in an earth dam sometimes rise as the reservoir falls (for example, curve C, Fig. 1(*b*)), they have been converted to values that will be more sensitive indicators of response to reservoir fluctuations. These values are expressed in terms of "potential head." Potential head at a given point is defined as the sum of the pressure head at that point, u_w/γ, plus the additional head between that point and the drain, $z_p - z_d$, or

$$h = z_p - z_d + \frac{u_w}{\gamma} \qquad \text{.....................(1)}$$

in which h is potential head, at a given pore pressure cell; z_p is the elevation of that cell; z_d is the elevation of the drain or outlet; u_w is the pore pressure reading; and γ is the unit weight of water. Fig. 2(*a*) illustrates this conception. The dotted curves in Fig. 2(*a*) represent contours of potential head at zero time (initial steady state), and the curves marked by arrows represent lines of flow.

In Fig. 2(*b*), hydrographs have been plotted showing the variation in potential heads at various points within a sand model embankment during a drawdown of the pool impounded by that embankment. The potential heads (ordinates) were computed, from readings of pore pressure cells, by Eq. 1. Where observed values of head for zero time differed from those taken from the theoretical flow net of Fig. 2(*a*), the former were used in Fig. 2(*b*). The topmost curve of Fig. 2(*b*), marked "Pool," shows the time rate of drawdown of the pool. Comparing the hydrographs of the cells with the hydrograph of the reservoir, a functional relationship between the rate of change of reservoir level and the rate of change of potential head at each cell location is noted at once, whereas such a relationship is less readily apparent if only the rate of change of pore pressure is studied. (Compare Fig. 2(*b*) with Fig. 2(*c*).) Fig. 2(*b*) shows clearly how the rate of change of potential head during an unsteady state increases as the downstream toe or drain of the dam is approached.

A most important approximation revealed by such hydrographs was that, over most of the dam at least, all cells with a given potential head during periods of constant pool level would show the same hydrographs during periods of changing reservoir level, regardless of the locations of the cells. The hydrographs of cells 0 and 5, or of cells 2, 3, and 4 in Fig. 2(*b*), illustrate this. Such an observation does not hold in regions which, at the time of observation, are well above the level of the pool at that same time.

These properties of the hydrographs of pore pressure cells strongly invite the development of an empirical rule. In fact, one was formulated which expressed, quite closely, the rates of drainage for two particular dams. However, it indicated that rate of drainage is independent of permeability. Since these two dams involved extremely slow rates of drawdown, the effects of permeability

---

[9] "Measurement of Pore-Water Pressure in Silt and Clay," by K. Terzaghi, *Civil Engineering*, January, 1943 pp. 33–36.

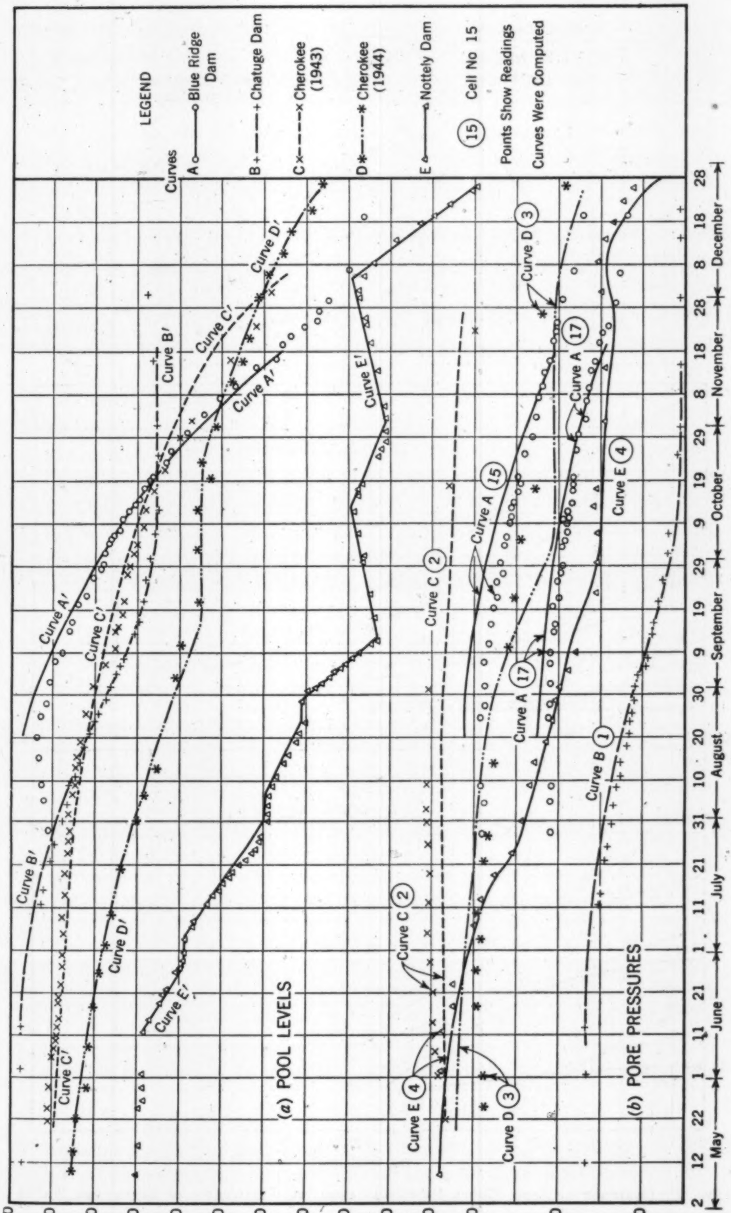


FIG. 1.—COMPARISON OF COMPUTED AND OBSERVED PORE PRESSURES

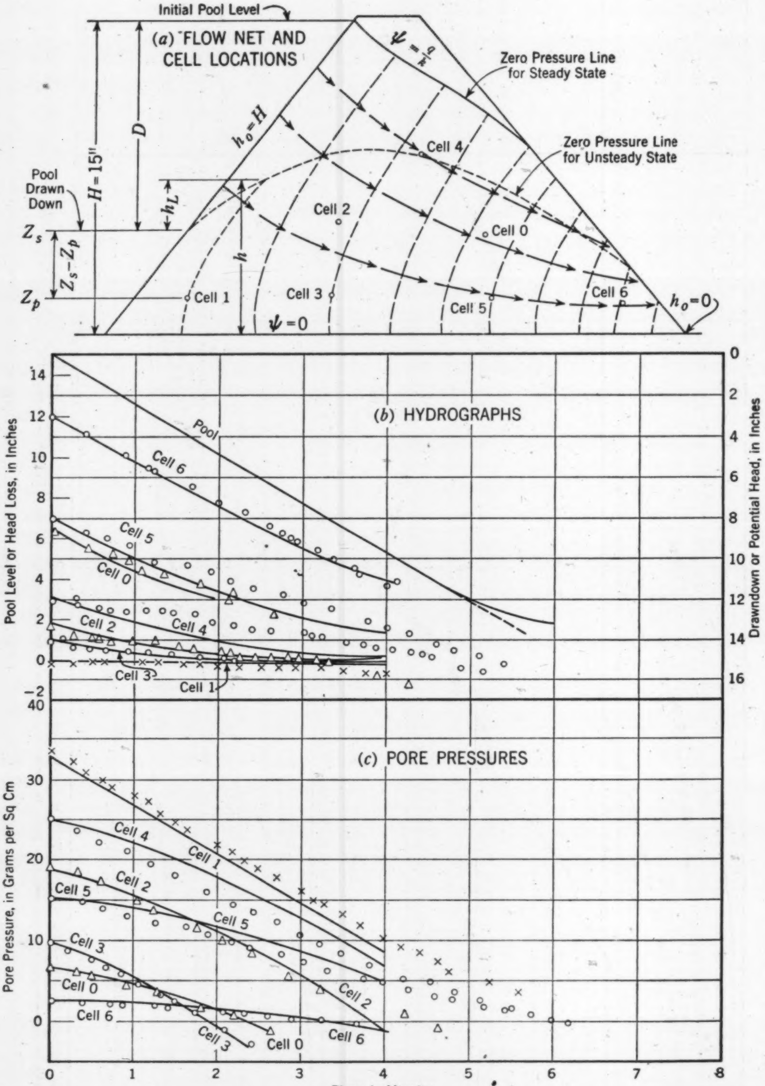


FIG. 2.—DRAINAGE OF SAND MODEL RESULTING FROM THE DRAWDOWN OF IMPOUNDED WATER

were so little as to be negligible, but the empirical rule holding for them could not be extended very far. After following various statistical studies "around a circle," such methods were abandoned in favor of a physical analysis of the problem.

Analogy Between Seepage and Heat Flow.—An analogy between unsteady state heat flow and flow of pore water squeezed from the pores of a clay during consolidation has been described in some detail by Professor Terzaghi and O. K. Froelich.[10]

In the structures involved in the present investigation, however, no definite evidences of consolidation were found, either from settlement measurements or from pore pressure observations. This section of the paper therefore refers to the problem of drainage from pore spaces, the volume of which is assumed to remain essentially constant. As one working hypothesis, the assumption has been made that under this condition analyses of drainage similar to those used in problems involving heat flow and consolidation can still be used, at least within limits.

In seepage, Darcy's law states that the average velocity is proportional to the hydraulic gradient, from which it follows, as shown in Appendix II, that the net quantity of water flowing in time dt out of an element of porous material having a volume $dx\ dy\ dz$ is $k\nabla^2 h\ dx\ dy\ dz\ dt$, in which h is the drop in head across the element and k is the permeability coefficient. In heat flow, the Biot-Fourier law states that quantity of heat per unit area is proportional to the temperature gradient, so that the net quantity of heat flowing out of an element in time dt is $k'\ \nabla^2\ T'\ dx\ dy\ dz\ dt$, in which k' is the coefficient of heat transmission and T' is the difference in temperature across the element. This quantity can also be expressed in another way: Specific heat is defined as the ratio $q_s : q_w$, in which q_s is the quantity of heat required to raise 1 g of transmitting substance 1° C and q_w is the quantity required to raise 1 g of water 1° C. Since this latter quantity may be considered as one calory, specific heat multiplied by unity gives the number of calories required to raise 1 g of conducting medium 1° C. In a volume $dx\ dy\ dz$, of this medium, there are $\rho\ dx\ dy\ dz$ grams, in which ρ is density in grams per cubic centimeter. Therefore, calling specific heat c', the quantity of heat required to raise an element of conducting medium dT' degrees is $c'\ \rho\ dx\ dy\ dz\ dT'$. If this quantity flows in time dt, then:

$$k'\ \nabla^2\ T'\ dx\ dy\ dz\ dt = c'\ \rho\ dx\ dy\ dz\ dT' \quad \text{..............} (2)$$

or

$$\frac{k'}{c'\ \rho}\ \nabla^2\ T' = \frac{\delta T'}{\delta t} \quad \text{.........................} (3)$$

Eq. 3 is the basis of the classical analyses of heat flow in the unsteady state. The factor $\frac{k'}{c'\ \rho}$ is called the "diffusivity" of the conducting medium. If a relationship similar to Eq. 3 could be demonstrated for seepage, a wealth of analytical technique would be available for problems of drainage. For such an analogy to exist, there must be a constant w_d such that

---

[10] "Theorie der Setzung von Tonschichten," by K. Terzaghi and O. K. Fröhlich, F. Deuticke, Leipzig 1936.

$$\frac{k}{w_d} \nabla^2 h = \frac{\delta h}{\delta t} \qquad (4)$$

or

$$k \nabla^2 h \, dx \, dy \, dz \, dt = w_d \, dx \, dy \, dz \, dh \qquad (5)$$

The quantity $k/w_d = c_d$ would then be a constant characteristic of the porous medium in which drainage takes place, expressing potential speed of drainage. It will be designated as the "drainability" of the material, to emphasize the analogy with diffusivity in heat transmission.

Transposing Eq. 5,

$$\frac{k \nabla^2 h \, dx \, dy \, dz \, dt}{dx \, dy \, dz} = w_d \, dh \qquad (6)$$

If "yield" is defined as the volume of water drained without replacement from a unit volume of porous material, then the left-hand side of Eq. 6 represents the yield in time dt. Designating yield w_y, Eq. 6 may be written

$$\frac{dw_y}{dh} = w_d \qquad (7)$$

The constant w_d must therefore be defined as the volume of water that can be drained from a unit volume of porous material by a unit change in head. It will be designated as "specific drainage."

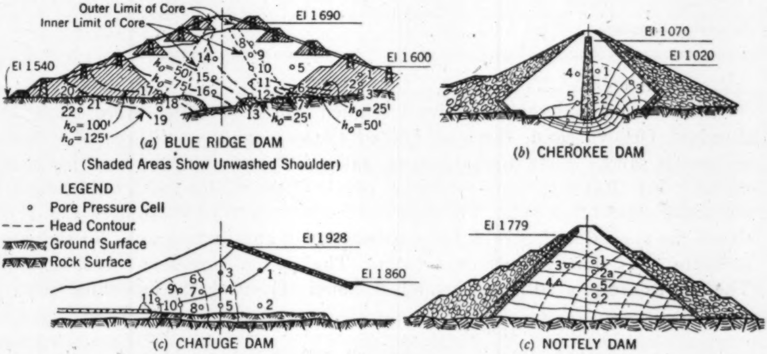
In Appendix II, various solutions of Eq. 4 are given. Values of pore pressures and yields computed from these solutions have been compared with field and laboratory observations. After these comparisons are summarized, the significance of the factors designated as drainability and specific drainage will be discussed further.

Procedure for Checking.—Observations of pore pressures were first made on earth dams that had been in operation for some time. Then, to check the conditions found and to extend them under controlled rates of drawdown, studies were made of model embankments in which a steady state was allowed to develop before applying drawdowns at uniform rates. Finally, to check the methods of computation against some factor other than pore-water pressure, sand samples were saturated and permitted to drain, the yield at various times being measured. In these samples, horizontal drainage and vertical drainage were observed separately. Also, both drainage under gravity alone and drainage under constant air pressure held at one end of the sample were studied.

For dams and model embankments, pore pressures were computed from Eq. 1, using Eqs. 30 in Appendix II to approximate values of potential head. Eqs. 30 are the result of a very crude analysis, but a rigorous solution of the problem could not be made, and the method used serves to give a fair estimate as long as points in the region of zero or negative hydrostatic pressure are not considered and as long as the drawdown is not so rapid as to leave the largest part of the region of positive hydrostatic pressure above the pool level at the time of computation.

The first steps in the computations were to take the observed pore pressure for one cell and one time and to compute the drainability from Eqs. 30. Next, this value was inserted in Eqs. 30, and the pore pressure for this cell at other times was computed, for comparison with observed values. Finally, using the same value of drainability, the pore pressures at other cells at various times were computed. However, when laboratory tests indicated a considerable variation in materials at different cells, a new value of drainability was determined for each cell.

For sand samples, a drainage index was considered, defined as the yield at any given time divided by the total yield under the conditions considered. Eq. 24 in Appendix II was used for horizontal drainage; and Eqs. 25 and 26 were used for vertical drainage. The drainability was first computed from the drainage index observed at one time, and this value was used to compute the drainage indexes at other times for comparison with observations.


Outer Limit of Core
Inner Limit of Core
El 1690
El 1540
El 1600
h_o=50'
h_o=75'
h_o=100'
h_o=125'
h_o=25'
h_o=25'
h_o=50'
1
2
3
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
El 1070
El 1020
1
2
3
4
5
LEGEND
Pore Pressure Cell
Head Contour
Ground Surface
Rock Surface
El 1928
El 1860
1
2
3
4
5
6
7
8
9
10
11
El 1779
1
2a
5
2
3
4

(*a*) BLUE RIDGE DAM

(Shaded Areas Show Unwashed Shoulder)

(*b*) CHEROKEE DAM

(*c*) CHATUGE DAM

(*c*) NOTTELY DAM

FIG. 3.—TYPICAL SECTIONS OF DAMS

Comparison of Computed and Observed Data.—The operation of the reservoirs impounded by four earth dams was such as to permit the attainment of a steady state of seepage, followed by an unsteady state induced by drawdown. Fig. 3 shows typical cross sections of these dams and the head contours during the initial steady state as determined from readings of pore pressure cells. Fig. 1(*a*) shows reservoir hydrographs for drawdown periods during which pore pressure readings were taken. Fig. 1(*b*) shows a few typical pore pressure-time curves. The curves were computed, whereas the points represent actual observations.

In general, computed pore pressures did not vary from those observed by more than 2 lb per sq in. In one case, where the cell reading was only 2 lb per sq in., and the cell was above pool level at the time of reading, a variation of 5 lb per sq in. was noted. However, since the analysis used becomes invalid at points within the embankment higher than pool level, a check could not be expected.

For Chatuge Dam, in southwestern North Carolina, and Cherokee Dam, in eastern Tennessee, a single value of drainability served to check all cells.

For Blue Ridge and Nottely dams both in northwestern Georgia, a value which would check observations at a given cell for one time would agree for other times, but not for other cells. A considerable variation in permeability was found in these two dams. Table 1 gives test data which serve to describe the materials in the various dams.

TABLE 1.—TEST VALUES FOR MATERIALS IN OBSERVED DAMS

Structure	Sand[a] (%)	Clay[a] (%)	Liquid limit	Plasticity index	Permeability coefficient[b]
Blue Ridge Dam (Georgia):					
Shoulder	73	5	....[c]	....[c]	0.3 to 3.0
Foundation	....	....	....[c]	....[c]	2.0 to 6.0
Core	52 to 73	5 to 17	30 to 42	2 to 9	0.003 to 0.007
Chatuge Dam (North Carolina)	64 to 76	4 to 8	45 to 47	4 to 15	0.0003 to 0.005
Cherokee Dam (Tennessee)	15 to 32	28 to 51	41 to 66	20 to 41	0.000008 to 0.00008
Nottely Dam (Georgia)	55 to 94	5 to 39	35 to 43	4 to 23	0.0003 to 0.2

[a] Percentage of the total sample, by dry weight. Particles with equivalent diameter greater than 0.05 mm are classified as sand and those with equivalent diameters less than 0.005 mm are classified as clay. [b] Feet per day. [c] Nonplastic.

Nine model embankments were studied, three of which were made of standard Ottawa sand, three of graded Ottawa sand, and three of a fine, micaceous sand. Four different linear rates of drawdown were used for each model. Fig. 2(*a*) shows a cross section of one of the models, with cell locations and a theoretical flow net. The material is assumed to be isotropic. Fig. 2(*b*) shows the typical hydrograph for one drawdown, and hydrographs for six pore pressure cells during that drawdown. The points represent observations. The curves are plotted from Eqs. 30, Appendix II, using the following data:

Specific drainage w_d	0.23
Permeability k, in inches per minute	0.8
Initial head H, in inches	15
Ratio of head loss to arc length $\Delta h_o/\Delta s_f$	0.6
Drainability, c_d	41.7
B	0.66

In the foregoing—

$$B = \frac{\pi^2 c_d}{H} \left(\frac{\Delta h_o}{\Delta s_f} \right)^2 \qquad \text{(8a)}$$

and

$$c_d = \frac{k}{w_d} \qquad \text{(8b)}$$

In Fig. 2(*b*) w_d is the specific drainage.

The value of the hydrograph of each cell at zero time represents the observed potential head during the initial steady state. Cells 2 and 3 show appreciably different initial values from those indicated by the theoretical flow net of Fig. 2(*a*). In Fig. 2(*c*), the hydrographs of the cells have been converted into pore pressure-time curves by Eq. 1. Results for other observations of model embankments are quite similar to those shown in Fig. 2, and to the results obtained from field observations on dams during drawdown.

The computed values shown in Figs. 1 and 2 have been obtained on the basis of steady state flow nets derived from readings of pore pressure cells. The question naturally arises as to how these computations would have checked had theoretical flow nets been used as a basis. The head distribution shown in Fig. 3 for Blue Ridge Dam is obviously unlike any that could be obtained theoretically. Since the dam is a semihydraulic fill with a fairly pervious foundation and considerable variation in the permeability of the flow system, this condition is not surprising. The flow nets for Chatuge, Cherokee, and Nottely dams are somewhat different from the corresponding theoretical flow nets for isotropic embankments. By assuming various values of anisotropy for each dam (calling "anisotropy" the ratio of permeability in a horizontal direction to that in a vertical direction), a fair agreement between observed and theoretical heads could be obtained for one particular value. The highest value of anisotropy that had to be used to obtain agreement between theoretical and observed flow nets was 50 at Guntersville Dam (in northern Alabama), but, usually, the value was between 1 and 4. Ordinarily, a theoretical flow net is made assuming an anisotropy of 1 or some other arbitrarily chosen value. Under such conditions, pore pressures computed for the unsteady state could be considerably in error.

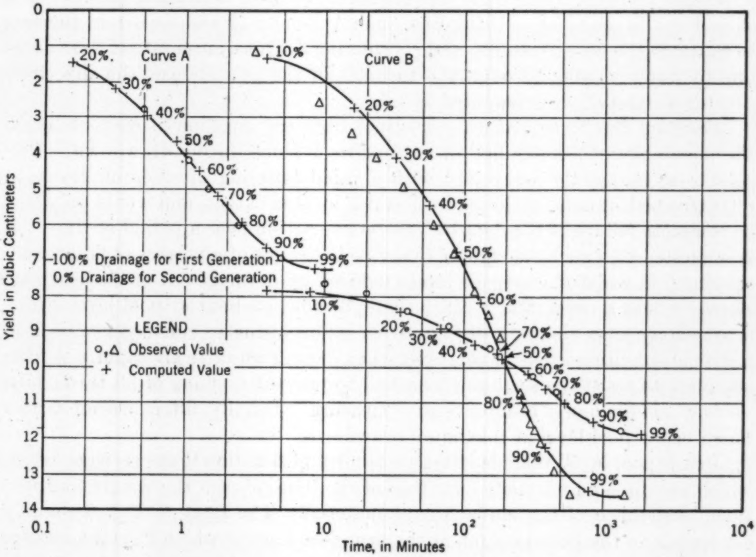


FIG. 4.—COMPARISON OF OBSERVED AND COMPUTED YIELDS

Fig. 4 shows the volume of water drained from a sand sample by gravity alone, plotted against time. Drainage occurred through vertical planes at each end of the sample. Curve A represents results for graded Ottawa sand, and curve B those for a micaceous sand, all of which passes United States standard sieve No. 30, and is retained on a No. 200 sieve. The curves were computed by Eq. 24 in Appendix II, and the points represent actual observations. The two generations of drainage indicated by curve A seem to be characteristic of medium-grained to coarse-grained sands, and indicate that there may be several independent flow systems in a sand, of which the most permeable is the only one of practical significance.

Similar comparisons were obtained when all water flowed vertically, draining from the base of the sample. Drainage was induced in this case by gravity plus a constant external stress, maintained by applying a constant air pressure at the top of the sample. The curves were computed from Eqs. 24 and 25 applied to Eq. 26 in Appendix II.

APPLICATIONS

Slope Stability.—The stability of a slope against mass sliding may be estimated by the "slip circle" method. Field evidence indicates this method is satisfactory for analyzing slope stability if the pore pressures are known.[2] In the case of an earth dam subjected to drawdown of its impounded reservoir, the pore-water pressures are frequently considered as those that would exist if no drainage whatsoever were to occur. Under such an assumption, the "factor of safety" for Blue Ridge Dam was computed as 0.95, indicating that the dam should not be standing. Using Eqs. 30 in Appendix II and Eqs. 1 in the text to estimate the pore pressures, and considering the maximum rate of drawdown due to water passing through the turbines in the powerhouse, the slip circle "factor of safety" was computed as 1.5.

Flotation Due to Drawdown.—Distinct from the question of mass sliding is the possibility of "sloughing" of the upstream slope during drawdown. This condition, frequently associated with a rapid lowering of reservoir levels, is distinguished from mass sliding in that it is local in extent, and would only lead to complete failure of the dam by progressive action over a period of time. It is suggested that such "sloughing" may be the result of a process of flotation or "piping," in which the seepage forces induced by drainage out of the upstream slope produce a quicksand. Since the hydraulic gradient at pool level during a drawdown may theoretically approach infinity, due to convergence of flow, a critical condition exists at this level when drawdown rates are rapid. A filter adequate to resist the seepage forces and to prevent washing of fill through its voids would obviously be a preventive measure. Usually, filters are not carried much below pool level at maximum drawdown.

Below pool level, there is still a possibility of flotation if the seepage forces, which are directed normal to the face of the dam, exceed the weight and frictional resistance of the embankment material. The inset in Fig. 5 shows a conception of the force polygon at the upstream face, in which F_i is the seepage force; F_ϕ is the frictional force; W is the buoyed weight; and ϕ is the angle of internal friction. The vector F_i represents the seepage force required for equilibrium. The actual seepage force at the end of a given period of drawdown may be expressed as $\gamma\, i$, in which γ is the unit weight of water and i is the hydraulic gradient at the upstream face at the time considered. Eq. 36 in Appendix II has been derived to express the approximate value of i for a constant rate of drawdown. Knowing this value, a factor of safety against flotation can be expressed as

$$S = \frac{F_i}{\gamma\, i} \qquad (9)$$

For Blue Ridge Dam, such a factor of safety was computed for a drawdown of 90 ft at 2 ft per day as 2.23. Similar computations indicated that a quick condition would be imminent at the upstream face when the rate of drawdown increased to 5.3 ft per day. Actually, the rate has been increased from the old limit of 5 ft per week to 2 ft per day without harmful effects. The additional water supply made available by such an increase represents an appreciable monetary value.

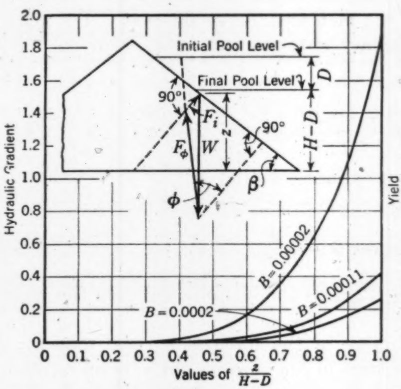


FIG. 5.—FACTORS THAT AFFECT THE PIPING RESULTING FROM DRAINAGE

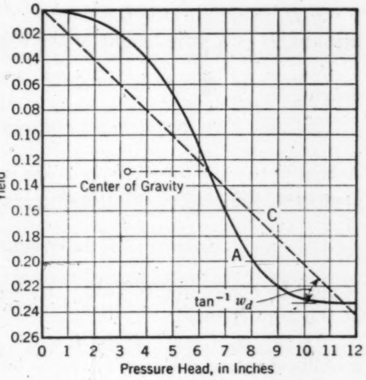


FIG. 6.—YIELD-PRESSURE CURVE (GRADED OTTAWA SAND)

The distribution of hydraulic gradient along the upstream face at points below minimum pool level is shown by the curves in Fig. 5 for materials of three different drainabilities. These curves suggest the desirability of filter protection below minimum pool level when the embankment material is not obviously free draining and the drawdown rate is appreciable.

SIGNIFICANCE OF DRAINAGE CONSTANTS

Pore Pressures in Media of Low Permeability.—Field observations of pore-water pressures in dams composed of relatively impervious materials indicate pressure drops quite similar to those that would be expected for nearly free draining media—contrary to the usual assumption that a relatively impervious material develops high excess hydrostatic pressures which dissipate very slowly. In this section, an effort will be made to explain such behavior.

Consider a porous body, such as an embankment, submerged in a pool of water so that its top is level with the water surface and imagine this body to be completely saturated. The body can be visualized as a plexus of intercommunicating capillary channels of varying cross section. If the level of the pool is now lowered slightly, the water will remain in these capillary channels, and a meniscus will form at the top of each channel. The pore water is held above the surface of the pool by capillarity.[7] To idealize for a moment, if the channel were a circular cylinder, the upward force holding the water above pool level would be $2\,\pi\,r\,\sigma_w \sin\beta$, in which r is the radius of the channel; σ_w is the surface tension of water; and β is the angle made by the meniscus with the horizontal. Resisting this force would be the weight of water above the pool, $\pi\,r^2\,\gamma\,D$, in which D is the height of the meniscus above the pool. For equilibrium, $D = h_c$, the height of the capillary water column, and

$$2\,\pi\,r\,\sigma_w \sin\beta = \pi\,r^2\,\gamma\,h_c \qquad (10)$$

Solving for h_c:

$$h_c = \frac{2\,\sigma_w \sin\beta}{r\,\gamma} \qquad (11)$$

Actually, the capillary channels are not cylindrical; they are not even uniform. However, the general relationship—

$$h_c = \text{constant} \times \frac{\sin\beta}{L} \qquad (12)$$

—may be considered to hold. In Eq. 12, L is the distance between the walls of the pore channel.

As the pool level drops further, more of the force of surface tension is mobilized to resist a drop of the water level in the pore channels by an increase in the curvature of the meniscus which increases the value of the angle β in Eqs. 11 and 12. The maximum value of $\sin\beta$ is unity, of course, and at this value the full force of surface tension is brought to resist a drop in pore-water level. In a relatively fine material, the pore width, L of Eq. 12, may be so small that a quite appreciable drawdown may occur with no drop in pore-water level ($D = h_c$). For example, in Cherokee Dam, at a distance of 70 ft above the free surface of the water, negative pore pressure and complete saturation (within the limits of accuracy of the tests used) were observed. Under such conditions (barring consolidation), the pore pressures vary from $-D$ at the top of the embankment, to zero at pool level, to $+h$ at a distance h below pool level (expressing pressure as equivalent head), and decrease as rapidly as the pool is drawn down. Thus, a relatively impervious embankment could show rates of decrease of hydrostatic pressure during drawdown similar to those exhibited by a free draining structure. In terms of Eq. 4, the permeability k is low and the yield per unit head w_d is zero; therefore, the drainability k/w_d is infinite.

Suggested Limitation of Analysis.—Eq. 12 represents a condition of equilibrium in which the pressure head created by the weight of water in the capillary channel h_c balances an equivalent force, expressed as head, which is created by surface tension. For equilibrium, the drawdown D is equal to the capillary head h_c, and the maximum value of D or h_c for this equilibrium in a channel of given width occurs when $\sin\beta = 1$. When $D > h_c$, Eq. 12 may be expressed as

$$D - \frac{\text{constant}}{L} = D - h_c = h \qquad (13)$$

in which h is the head available to produce flow in the capillary channel. The value of L is obviously not constant, but varies over a wide range, possibly similar to the variation in sizes of grains. Therefore, during a drainage due to drawdown, the largest pore channels will drain first whereas, in the smaller channels, there will simply be increases in curvatures of the menisci without appreciable loss of water. Thus, each increment of drawdown will drain water from successively smaller channels.

To illustrate these relationships, a cylinder of sand, 4 in. in diameter and 5 in. high, open at the top and base, was saturated and its base submerged. The air pressure at the top was adjusted so that the sand would remain saturated without flow. Then the pressure at the top was increased by an equivalent of 2 in. of hydrostatic head and the sand was allowed to drain until no additional water was added to the yield during a 24-hour period. This procedure was repeated at 2-in. head increments until no more water could be forced out of the sand. The total yield at the end of each pressure increment was plotted against the pressure head as shown in Fig. 6.

Although the equilibrium conditions in this experiment are different from those expressed in Eqs. 10 to 13, the principle that there is a capillary head, varying inversely as the width (or "diameter") of the capillary channel at the air-water interface (which head must be overcome by a greater force to produce drainage) holds in either case. In either case, when the pressure head is zero, there is no yield. Furthermore, the yield approaches a maximum value, beyond which there is no additional drainage. This value is called "specific yield" in ground-water hydrology;[11] and the test for specific yield is similar to that for the centrifuge moisture equivalent of a soil. Moreover, since pressure is a function of pore size, and yield, of pore diameter, the yield-pressure curve can be considered as a rough example of a pore-size distribution curve. Then, if there is any truth in the assumption that grain size is an index of pore size (as implied in the "effective size" hypothesis), the yield-pressure curve should have the same general characteristic shape as does a grain-size distribution curve—that is, it should suggest a probability curve, as does the curve in Fig. 6.

To apply the concept of the yield-pressure curve to the analyses in this paper, it is necessary to consider the draining mass as homogeneous, or better, as uniformly heterogeneous. In any element, the grain-size distribution and the pore-size distribution must be considered the same as in any other element. Consider an element at the top of the draining medium, containing pore spaces of every size in the range of the material. In any one of these pore channels, if there is a force F_d promoting drainage, and an equal capillary force, h_c, resisting drainage, there is no flow. If the force F_d is increased by an infinitesimal amount, dF_d, there is an increase in capillary resistance to $h_c + dh$ in those channels too small to permit flow, and seepage under a head not greater than $dh = dF_d$ in the larger channels. The yield increases by an amount dw_y. This increase eventually becomes smaller as successively smaller pore spaces are drained by additional increments of drawdown. Eventually, as the total yield approaches the value of specific yield, dw_y approaches zero. Therefore, the relationship in Eq. 7 ($dw_y/dh = w_d$) can at best be only an approximation. At the value of the specific yield, $dw_y/dh = w_d = 0$, and Eq. 4 becomes

---

[11] "The Occurrence of Ground Water in the United States," by O. E. Meinzer, *Water-Supply Paper No. 489*, U.S.G.S., U. S. Govt. Printing Office, Washington, D. C., 1923.

$$\nabla^2 h = 0 \quad (14)$$

which is the differential equation for steady state flow.

Since, as a material becomes finer, the specific yield becomes less and the pressure at which the value of specific yield is approached becomes greater, the curve is better approximated by a straight line, $dw_y/dh = w_d$, as indicated by the dashed line in Fig. 6. As the value of specific yield is exceeded, the straight line approximation becomes untenable. It is concluded, therefore, that analysis of drainage problems by methods analogous to those used for determining the unsteady state of heat flow is an approximation that can hold only as long as drainage water can be taken from the capillary zone. After all drainable water has been removed from the porous body, air must enter the entire surface of the porous body; and, from Eq. 14, the ensuing drainage follows the laws of the steady state. It is suggested that, in such a steady state, the source is the surface of the draining mass, the inflowing material is air, and the process is similar to one analyzed by Mr. Muskat for the rate of advancement of an oil-water interface.[6]

Since this study was made in connection with a specific problem relating to production rather than as a research project, sufficient data to support or refute the suggestions given herein as to the limitations of the analysis were not obtained. This treatment has been included only in the hope that additional information may be available which will better define the range of validity of the analysis.

CONCLUSIONS

Ordinarily, when an earth dam is subjected to drawdown of its impounded reservoir, it is assumed that the internal hydrostatic pressures will decrease very slowly, if at all, unless the slopes of the dam are composed of relatively permeable material. Such a slow decrease would materially reduce the stability of the dam. Therefore, substantial sums of money have been spent in securing large quantities of pervious materials for the upstream shoulders of earth dams, or in constructing dams with very flat slopes.

Field observations summarized in this paper indicate that, in materials of relatively low permeability, the hydrostatic pressures may decrease about as rapidly as in pervious materials, with little or no reduction in stability. These observations were in fairly good agreement with hydrostatic pressures computed from analyses similar to those used in problems of heat transmission and consolidation of clays. Such computations also agreed fairly well with observations made in the laboratory on model embankments and on volumes of water drained from sand samples under both gravity and induced pressures. Materials observed included clays, silts, and sands.

The analyses indicate that speed of drainage depends not only on permeability, but also on the volume of water that can be drained. For example, standard Ottawa sand is more permeable than graded Ottawa sand, but requires more time to drain because more water can flow out of the same volume of sand. The potential rate at which a given material can drain is expressed by a constant designated as "drainability," which is the quotient of the permeability coefficient over the volume of water, per unit head, that can be drained from a unit volume of the material. If this latter quantity should be zero, as it would be in a fine material in which all water is held in the pore spaces by capillarity, the drainability would be infinite, regardless of the magnitude of the permeability coefficient, and the pore pressures would decrease during a drawdown just as fast as they would for a material with a permeability coefficient of infinity. Of course, if the fine material were not well consolidated, so that the capillary forces transmitted to the grain structure resulted in consolidation, the pore pressures would be higher and their rates of decrease would be slower.

Silts and fine sands, which have neither permeability sufficient to permit drainage by a rapid escape of water from the pores, nor yields low enough to offset this drainage, are considered the most difficult materials to drain, and the least desirable materials for the upstream shoulder of a dam.

The data given in this paper indicate that earth slopes subjected to drawdown are more likely to develop local failures from flotation or "piping" action of water draining out of the face of the slope, than to fail by mass sliding. The likelihood of flotation decreases with distance below the surface of the pool, but under certain conditions, it could occur at appreciable distances beneath the surface of the pool, if not protected by an adequate filter.

Although this investigation has not been extended far enough to determine the limits of the analyses, it is tentatively suggested that the analyses will hold as long as the total head inducing drainage does not exceed the head required to remove all drainable water from the material considered. An alternative method of attack for cases beyond this limit has been suggested but not developed. From a practical viewpoint, it is considered that cases beyond this limit would be confined to pervious sands subjected to drawdowns of considerable magnitude.

Acknowledgment

The experimental work described in this paper was done under the auspices of the Tennessee Valley Authority (TVA). The views expressed are those of the writer, and do not necessarily reflect those of the TVA.

APPENDIX I. NOTATION

As far as consistently possible, the following letter symbols conform with "Soil Mechanics Nomenclature," ASCE Manual of Engineering Practice No. 22, and with American Standard Letter Symbols for Hydraulics (Z10.2—1942) and for Heat and Thermodynamics (Z10.4—1943):

B = a simplifying exponent defined by Eq. 29*b*;
b = rate of drawdown;
c_d = drainability analogous to diffusivity or specific heat c'; $c_d = k/w_d$;

D = distance of drawdown at any given time, the total drawdown being designated D';

d = drainage index, expressed as the ratio of the volume drained to a given time to the total volume drained (see Eq. 26):

  d_1 = drainage ratio due entirely to gravity;

  d_2 = drainage ratio due entirely to pressure;

e = base of Naperian logarithms;

F = force—

  F_i = seepage force required for equilibrium, the "actual" or general seepage force being $F'_i = \gamma i$;

  F_d = a force that promotes drainage;

  F_ϕ = internal friction force;

f = any function, as defined in the text; distinction between functions, are designated f_1, f_2 (Eq. 16*b*), or f' (Eqs. 30);

H = total head;

h = head at a given time or section; potential head:

  h_c = height of capillary rise;

  h_L = head loss due to flow;

  h_o = potential head during the initial steady state;

i = hydraulic gradient at the upstream face at a given time;

k = permeability coefficient analogous to the coefficient of heat transmission k';

L = any distance, as defined in the text;

n = an integer having successive values 1, 2, 3, $\cdots$;

q = flow during the initial steady state, between a given point and any convenient fixed datum such as an impervious base;

q_s = quantity of heat required to raise 1 g of transmitting substance 1° C;

q_w = quantity of heat required to raise 1 g of water 1° C;

R = time ratio, t/t_d (Eqs. 30);

r = radius; polar coordinate (see θ);

S = safety factor;

s = length of arc:

  s_f = length of arc, measured along a flow line;

  s_h = length of arc, measured along a head contour;

T = temperature; T' = difference in temperature across an element;

t = time; t_d = duration of drawdown;

u_w = pore pressure reading;

V = volume;

v = velocity of flow;

W = weight;

w = volume of water drained:

  w_d = specific drainage (Eq. 7); the volume that can be drained from a unit volume of porous material by a unit change of head;

  w_y = yield; the volume that is drained, without replacement, from a unit volume of porous material;

x = Cartesian coordinate;

y = Cartesian coordinate;
z = vertical coordinate, or elevation above an indicated base:
  z_d = elevation of drain or outlet;
  z_p = elevation of pore pressure cell;
  z_s = elevation of reservoir surface at any given time;
β = angle with the horizontal, as the slope angle in Fig. 5;
γ = unit weight of water;
θ = polar coordinate of arc (see r);
λ = a substitution factor used to simplify the typography of a complex exponent (see Eq. 23*b*);
ρ = density, in grams per cubic centimeter;
σ_w = surface tension of water;
τ = substitution factor used to simplify the typography of a complex exponent (see Eq. 23*a*);
ϕ = angle of internal friction;
ψ = a parameter equal to q/k; and
∇ = an operator.

APPENDIX II. MATHEMATICAL TREATMENT

Coordinate System.—For a system of any given geometry there is a characteristic flow net; therefore, it would be convenient to consider this flow net as a curvilinear coordinate system, in which the position of any given point is fixed by the intersection of a head contour with a flow line. The coordinates of the point would then be designated by the parameters h_o and ψ, h_o being the numerical value of the head contour during the initial steady state and ψ being equal to q/k—in which q is the flow between the point considered and any convenient datum such as an impervious base (during the initial steady state) and k is the permeability of the medium considered.

The lengths of arc along these curvilinear coordinates are

$$ds_f = dh_o \sqrt{\left(\frac{\delta x}{\delta h_o}\right)^2 + \left(\frac{\delta z}{\delta h_o}\right)^2} = \frac{dh_o}{f_1} \qquad \text{(15a)}$$

and

$$ds_h = d\psi \sqrt{\left(\frac{\delta x}{\delta \psi}\right)^2 + \left(\frac{\delta z}{\delta \psi}\right)^2} = \frac{d\psi}{f_2} \qquad \text{(15b)}$$

From Darcy's law,

$$d\psi = \frac{dh_o}{ds_f} ds_h \qquad \text{(16a)}$$

Therefore:

$$ds_f \, ds_h = \frac{dh_o}{f_1} \frac{d\psi}{f_2} = \frac{(dh_o)^2 \, ds_h}{f_1 \, f_2 \, ds_f} \qquad \text{(16b)}$$

and

$$(ds_f)^2 = \frac{(dh_o)^2}{f_1 \, f_2} = \frac{(dh_o)^2}{f^2_1}; \quad \text{and} \quad f_1 = f_2 \qquad \text{(17)}$$

It can be shown[12] that, in a curvilinear coordinate system of two dimensions, the operator ∇^2 becomes

$$\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta z^2} = f_1 f_2 \left(\frac{\delta}{\delta h_o} \frac{f_1}{f_2} \frac{\delta}{\delta h_o} + \frac{\delta}{\delta \psi} \frac{f_2}{f_1} \frac{\delta}{\delta \psi} \right) \ldots\ldots\ldots\ldots (18)$$

or, from Eq. 17: In one dimension—

$$\frac{\delta^2}{\delta x^2} = f_1 \frac{\delta^2}{\delta h^2_o} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19a)$$

in two dimensions—

$$\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta z^2} = f^2{}_1 \left(\frac{\delta^2}{\delta h^2_o} + \frac{\delta^2}{\delta \psi^2} \right) \ldots\ldots\ldots\ldots\ldots (19b)$$

and, in three dimensions—

$$\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2} = f^3{}_1 \left(\frac{\delta^2}{\delta h^2_o} + \frac{\delta^2}{\delta \rho^2} + \frac{\delta^2}{\delta \psi^2} \right) \ldots\ldots\ldots\ldots (19c)$$

Derivation of Differential Equation.—In a rectangular element, $dx\, dy\, dz$, in which v_x, v_y, and v_z are velocity components parallel to the x-axis, y-axis, and z-axis, respectively, the volume of water flowing in time dt is:

Flow	Formula	
Inflow parallel to the x-axis........	v_x	$\times\, dy \times dz \times dt$
Outflow parallel to the x-axis......	$\left(v_x + \frac{\delta v_x}{\delta x} dx \right)$	$\times\, dy \times dz \times dt$
Net Flow—		
Parallel to the x-axis............	$\frac{\delta v_x}{\delta x} dx$	$\times\, dy \times dz \times dt$
Parallel to the y-axis............	$\frac{\delta v_y}{\delta y} dx$	$\times\, dy \times dz \times dt$
Parallel to the z-axis............	$\frac{\delta v_z}{\delta z} dx$	$\times\, dy \times dz \times dt$
Total volume of flow.............	$\left(\frac{\delta v_x}{\delta x} + \frac{\delta v_y}{\delta y} + \frac{\delta v_z}{\delta z} \right) dx$	$dy\, dz\, dt$

Since Darcy's law may be expressed as

$$v = -k \nabla h \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20a)$$

this total volume can be written as

$$V = -k \nabla^2 h\, dx\, dy\, dz\, dt \ldots\ldots\ldots\ldots\ldots\ldots (20b)$$

From the definition of specific drainage, this volume can also be expressed as

$$V = -w_d\, dh\, dx\, dy\, dz \ldots\ldots\ldots\ldots\ldots\ldots (20c)$$

---

[12] "Higher Mathematics for Engineers and Physicists," by Ivan S. Sokolnikoff and Elizabeth S. Sokolnikoff, McGraw-Hill Book Co., Inc., New York, N. Y., 1934, pp. 361–363.

Equating the two expressions, Eqs. 20*b* and 20*c*, and assuming $k/w_d = c_d$,

$$c_d \nabla^2 h = \frac{\delta h}{\delta t} \tag{21a}$$

From Eqs. 19, Eq. 21*a* becomes, in curvilinear coordinates:

$$c_d f^2_1 \left(\frac{\delta^2 h}{\delta h^2_o} + \frac{\delta^2 h}{\delta \psi^2} \right) = \frac{\delta h}{\delta t} \tag{21b}$$

Solution for Horizontal Drainage.—Consider a saturated porous body with length much greater than height—that is, $x \gg z$, and allow it to drain at the ends, $x = 0$ and $x = L$. Then $\frac{\delta h}{\delta z} = 0$ and Eq. 21*a* becomes

$$c_d \frac{\delta^2 h}{\delta x^2} = \frac{\delta h}{\delta t} \tag{21c}$$

The boundary conditions are: $h = 0$ when $x = 0$, $x = L$, $t = \infty$; and $h = H$ when $t = 0$. The solution is analogous to that used for consolidation tests and is

$$h = \frac{4\,H}{\pi} \sum_{n=1}^{n=\infty} \frac{1}{2\,n - 1}\, e^{-\lambda \tau} \sin \frac{(2\,n - 1)\,\pi\,x}{L} \tag{22}$$

in which, to simplify typography:

$$\tau = \frac{4\,c_d\,t}{L^2} \tag{23a}$$

and

$$\lambda = \frac{(2\,n - 1)^2\,\pi^2}{4} \tag{23b}$$

In Eqs. 22 and 23, n is an integer having successive values 1, 2, 3, $\cdots$. The drainage ratio is the ratio of the volume of water removed up to any given time to the total volume of water that can be drained under the existing conditions. For this case, by analogy with settlement computations,

$$d = 1 - \frac{8}{\pi^2} \sum_{n=1}^{n=\infty} \frac{1}{(2\,n - 1)^2}\, e^{-\lambda \tau} \tag{24}$$

In the solution of Eq. 24, settlement tables may be used to simplify computations.

Solution for Vertical Drainage.—Consider a saturated porous body draining vertically through its base. For this case,

$$d = 1 - \frac{16}{\pi^3} \sum_{n=1}^{n=\infty} \frac{(2\,n - 1)\,\pi + 2\,(-1)^n}{(2\,n - 1)^3}\, e^{-\lambda \tau} \tag{25a}$$

$$\frac{u_w}{\gamma} = f(t, z) - \frac{1 - d}{d}\, z \tag{25b}$$

and

$$f(t, z) = \frac{4\,H}{\pi^2} \sum_{n=1}^{n=\infty} \frac{(2\,n - 1)\,\pi + 2\,(-1)^n}{(2\,n - 1)^2}\, e^{-\lambda \tau} \tag{25c}$$

Now suppose that a constant air pressure is applied at the upper end of the porous body, amounting to a pressure head F_d. Disregarding the effect of gravity, the drainage ratio due to this pressure only can be found from Eq. 24, with z substituted for x. The drainage ratio for the case of gravity drainage alone is given by Eqs. 25. Then, by reasoning analogous to that used in settlement problems, the drainage ratio for the combined pressure and gravity drainage is

$$d = \frac{H\, d_2 + 2\, F_d\, d_1}{H + 2\, F_d} \qquad \text{.....................(26)}$$

in which d_1 is the drainage ratio due to gravity alone and d_2 is that due to pressure alone.

Drainage of an Earth Dam.—In the preceding section, all flow is assumed to take place along straight lines. Then the boundary conditions could be expressed in terms of x-coordinates and z-coordinates, so that solutions of Eq. 21*a* could be expressed as series that would converge to the initial head distribution when the time $t = 0$. In most drainage problems of practical significance, the lines of flow are curved, with curved boundaries, as, for example, the "line of seepage" in an earth dam. Therefore, the coordinate system formed by the resulting flow net is curvilinear. In such a case, f_1, which appears in Eq. 21*b* and is defined in Eqs. 15, is no longer a constant, but varies with h_o and ψ. Hence, Fourier series solutions will no longer satisfy the differential equation. In heat transmission problems, where there are usually boundaries of elementary geometry, polar or elliptical coordinates can be used to express the boundary conditions. Then series converging to $h = h_o$ when $t = 0$ can be obtained by use of Bessel functions or Lamé functions. Unfortunately, flow nets for most drainage conditions of practical significance cannot be so idealized. Often, they defy analysis, and are determined by successive approximation or graphical estimate. Hence, the following approximation method has been used to compute rates of drainage of earth dams.

Consider the condition:

$$f_1 \sim \frac{\Delta h_o}{\Delta s_f} \qquad \text{.........................(27)}$$

in which Δh_o is the head drop between two adjacent equipotential lines in a steady state flow net derived by graphical methods, and Δs_f is the length of a flow line between them. A glance at any of the flow nets in Figs. 2 or 3 will show that, over considerable sections of an earth dam, the ratio is nearly constant. The greatest variations occur near the drain and near the upstream slope, and the greatest errors involved in assuming f_1 as constant will be at these boundaries. However, any equation used to approximate the value of h and conforming to the boundary conditions, $h = 0$ when $h_o = 0$ and $h = H - D$ when $h_o = H$, must approach the same numerical values as these boundaries are approached. Furthermore, the observations described in the text indicate that (at least at appreciable distances below the free surface), $\delta h/\delta\psi = 0$, for earth dams subjected to drawdown. Hence, if the approximate differential equation—

$$c_d \left(\frac{\Delta h_o}{\Delta s_f}\right)^2 \frac{\delta^2 h}{\delta h^2_o} = \frac{\delta h}{\delta t} \qquad \text{...................(28)}$$

—is used to express drainage from an earth dam, the maximum error resulting from the assumption that $f_1 = \Delta h_o/\Delta s_f$ = constant occurs in the boundary regions where the values of h must approach the correct values anyway, as long as the solution satisfies the boundary conditions.

Under these assumptions, the following boundary conditions may be considered: When $t = 0$, $h = h_o$ (initial steady state); when $t = \infty$, $h = \frac{H - D'}{H} h_o$ (final steady state); when $h_o = 0$, $h = 0$; and when $h_o = H$, $h = H - D'$. Considering the drawdown, D', as instantaneous, a solution of Eq. 28 is

$$h = \left(1 - \frac{D'}{H}\right) h_o + \frac{2\,D'}{\pi} \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n} e^{-n^2 B t} \sin \frac{n\,\pi\,h_o}{H} \quad \text{.....(29a)}$$

in which

$$B = \frac{\pi^2\, c\, (\Delta h_o/\Delta s_f)^2}{H^2} \quad \text{.....(29b)}$$

Actually, of course, the drawdown of a reservoir is never instantaneous, but varies as function of time, $D = f(t)$. Still, it can be imagined as the net result of a series of infinitesimal instantaneous drawdowns which immediately follow one another. The magnitude of each of these drawdowns may be considered as $d(D) = f'(t)\,dt$. If any one of these drawdown increments occurs at time t, if t_d is the time during which all of them occur, and if $R\,t_d$ is the time for which a value of h is desired, the drawdown D' in Eq. 29a can be replaced by $f'(t)\,dt$; and t, by $R\,t_d - t$ (the time interval during which the drawdown increment was effective). Thus, the head due to this increment can be computed. The sum of the heads, due to all such increments occurring up to time t_d (or up to time $R\,t_d$, if $R\,t_d < t_d$), is then the head resulting from the gradual drawdown. Hence, when $R \leqq 1$:

$$h = \left[1 - \frac{f(t)}{H}\right] h_o + \frac{2}{\pi} \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n} \left(\sin \frac{n\,\pi\,h_o}{H}\right) \times \int_0^{R t_d} e^{-n^2 B (R t_d - t)} f'(t)\,dt \quad \text{....(30a)}$$

and, when $R \geqq 1$:

$$h = \left[1 - \frac{f(t)}{H}\right] h_o + \frac{2}{\pi} \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n} \left(\sin \frac{n\,\pi\,h_o}{H}\right) \times \int_0^{t_d} e^{-n^2 B (R t_d - t)} f'(t)\,dt \quad \text{....(30b)}$$

Flotation Due to Drawdown.—Assuming that an adequate filter protects the upstream face of an earth dam down to the lowest elevation which the reservoir will reach, then only that part of the face which is always below pool level need be considered. In this region, as suggested by readings of pore pressure cells and as confirmed by watching the flow of dye in models subjected to drawdown, the flow is directed upstream and normal to the face. Therefore, the force polygon is as shown in Fig. 5. The value of the vector F_i in Fig. 5 represents the seepage force at equilibrium. Since the actual seepage force is $\gamma\, i$, in which i is the hydraulic gradient at the upstream face, Eqs. 30 may be used to compare the seepage force induced by a given drawdown of the reservoir with that required for equilibrium.

For zero time, corresponding to an initial steady state, the distribution of head near the upstream face of an earth dam resting on an impervious base may be computed from a conjugate function transformation.[13] The appropriate solution is

$$h_o = H - \frac{2\,\beta}{\pi}\left(\frac{r \sin \beta}{H}\right)^{(\pi-2\beta)/(2\beta)} \cos \beta \cos \frac{\pi\ \theta}{2\,\beta} \ldots\ldots\ldots (31)$$

in which h_o is the potential head referred to the impervious base; H is the maximum potential head; β is the angle made by the upstream face with the horizontal; and r and θ are the polar coordinates of a point referred to the upstream toe as an origin.

The hydraulic gradient along the upstream face is therefore the normal derivative of the head,

$$\text{Gradient } h_o = \left(\frac{r}{H} \sin \beta\right)^{(\pi-2\beta)/(2\beta)} \cos \beta \ldots\ldots\ldots\ldots (32)$$

or, if i is the hydraulic gradient and z is the distance of any point along the upstream face from the impervious base, $r = z \csc \beta$ and

$$i_{t=0} = \left(\frac{z}{H}\right)^{(\pi-2\beta/(2\beta)} \cos \beta \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

During a drawdown, the hydraulic gradient at the upstream face at any time is gradient $h = \frac{dh}{ds_f} = \frac{\delta h}{\delta h_o} \frac{dh_o}{ds_f}$; or:

$$i = \left(\frac{z}{H}\right)^{(\pi-2\beta)/(2\beta)} (\cos \beta) \frac{\delta h}{\delta h_o} \ldots\ldots\ldots\ldots\ldots\ldots (34)$$

The value of $\frac{\delta h}{\delta h_o}$ may be obtained from Eqs. 30. If the drawdown proceeds at a constant rate, $D = b\,t$, then $f'(t)\,dt$ in Eqs. 30 may be replaced by $b\,dt$. Integrating as indicated in Eqs. 30 and then differentiating with respect to h_o, for the upstream face (where $h_o = H$):

$$\frac{\delta h}{\delta h_o} = 1 - \frac{b\,t_d}{H} - \frac{2\,b}{B\,H} \sum_{n=1}^{n=\infty} \frac{1}{n^2} (1 - e^{-n^2 B t_d}) \ldots\ldots\ldots\ldots (35)$$

Placing Eq. 35 in Eq. 34, the hydraulic gradient is

$$i = \left(\frac{z}{H}\right)^{(\pi-2\beta)/(2\beta)} (\cos \beta) \left[1 - \frac{b\,t_d}{H} - \frac{2\,b}{B\,H} \sum_{n=1}^{n=\infty} (1 - e^{-n^2 B t_d}) \right] \ldots (36)$$

In numerical computations, the hydraulic gradient obtained from Eq. 36 is positive when flow is directed away from the upstream face and negative when directed into the dam. Eq. 36, like Eqs. 30, from which it was derived, is not valid above pool level at any given time.

---

[13] "Flow of Homogeneous Fluids through Porous Media," by Morris Muskat, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1937, p. 343.

DISCUSSION

Harry R. Cedergren,[14] Assoc. M. ASCE.—In this basic paper the author has introduced a subject of considerable interest to the designer of earth dams. Not only has he presented data regarding the development of pore pressures within several operating earth dams, but he has developed a mathematical analysis that should be of considerable value to research engineers. Very few data on the actual performance of earth dams have been made available to the engineering profession as a whole, and any contribution in this direction is a significant step forward.

Too often extremely elaborate investigations are made for the design of earth dams, but only meager observations (or none at all) are made of pore pressures within the completed structure. Frequently observation systems and schedules are set up and observed for a short time and then abandoned. It is difficult to convince many persons and agencies responsible for the operation of earth dams that relatively small sums of money spent in observing water pressures in earth dams and their foundations over long periods of time can pay dividends. In order that such observations may be of maximum value, careful field measurements of water pressures should be accompanied by dependable summaries of the soil properties having an important bearing on the rate of movement of the saturation line or changes in pressure within the water phase.

In engineering computations highly refined mathematical theory is often applied to situations in which the possible degree of accuracy of the mathematics is far greater than the limitations imposed in fitting the mathematics to real situations. Although the theory may be capable of a high degree of refinement, factors controlling the practical application of the problem generally can be determined only approximately. Nevertheless the theory can be helpful in establishing probable limits within which the prototype can be expected to perform.

The author has based his mathematical derivation on Darcy's law and consideration of the quantity of water flowing out of an elemental volume of porous soil, with simplifications introduced to make his mathematical treatment possible. Since about 1940 the writer has used a graphical method based on Darcy's law and the flow net to obtain approximate solutions to certain nonsteady seepage problems encountered principally in earth dams. The method can be used in the study of falling saturation lines within embankments subjected to drawdown and other situations in which the position of a saturation line is changing. Although it must be considered only approximate, it is believed to have many practical applications.

Darcy's law provides a method of determining the velocity of a moving saturation line and, when used with the flow net, makes possible a study of the shape of a saturation line at successive positions in a structure. The flow net is derived for conditions of steady seepage, but in the following analysis it is assumed that the flow net can be used to obtain approximate solutions to certain nonsteady seepage problems by considering the moving seepage boundaries to be fixed momentarily. The analysis presented herein is based on the assumptions that no lag in flow or water pressure is caused by inertia or consolidation, and that changes in boundary conditions are transmitted quickly throughout the soil mass. To simplify the discussion, capillarity has been neglected. However, as indicated by the author, negative pore-water stresses, as set up by capillary forces in fine soils, cause an immediate lowering in the effective position of a falling saturation line by an amount not exceeding the effective capillary height. In this discussion, determination of the velocity of a saturation line will be reviewed prior to a discussion of application of the flow net in determining the shape of a saturation line at successive positions within a structure.

---

[14] With U. S. Corps of Engrs., Portland Dist., Portland, Ore.

Darcy's law expresses the relationship between the average, or discharge velocity, v_D, the permeability of the soil, k, and the seepage gradient, i, as indicated by Eq. 37:

$$v_D = k\,i \qquad (37)$$

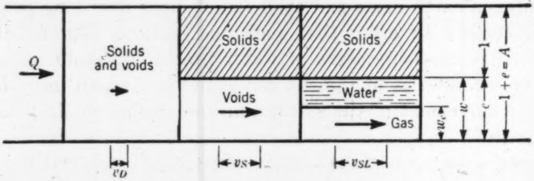


FIG. 7.—SEEPAGE RELATIONSHIPS

The discharge velocity represents the mean velocity in the entire cross section of a soil mass, including both solids and voids (see Fig. 7). The discharge velocity multiplied by the cross-sectional area, A, gives the seepage quantity per unit of time, as follows:

$$q = v_D\,A \qquad (38)$$

The effective velocity in the soil voids (the seepage velocity, v_S, Fig. 7) must be greater than the discharge velocity. Since the quantity of water flowing in the confined tube (Fig. 7) must be constant throughout the length of the tube, it follows that:

$$v_D\,(1 + e) = v_S\,e \qquad (39a)$$

and

$$v_S = v_D\,\frac{1 + e}{e} = k\,i\,\frac{1 + e}{e} \qquad (39b)$$

The seepage velocity, v_S, represents the effective velocity in the soil voids in the direction of flow lines for steady seepage conditions. When considering the rate of movement of a saturation line into or out of a soil mass, the quantity of water required to fill voids already partly filled, or the quantity of water draining out leaving the voids partly filled, must be considered. If the zone of saturation is moving into moist soil, less water must flow in to fill the voids than would be required if all the voids were empty. Likewise, if the zone of saturation is receding in a soil mass, leaving part of the voids filled, the saturation line can drop more rapidly than if all water must be removed. This factor was discussed by the author in terms of "drainability."

The velocity of a moving saturation line (v_{SL}, Fig. 7) is related to the seepage velocity in the following manner:

$$v_{SL}\, w_e = v_S\, w \quad \text{(40a)}$$

and

$$v_{SL} = v_S \frac{w}{w_e} = k\, i\, \frac{1+e}{e} \frac{w}{w_e} \quad \text{(40b)}$$

Grouping all terms dependent on the quantity of water flowing into or out of a soil mass in a drainage factor, *C*, Eq. 40*b* is simplified as follows:

$$v_{SL} = k\, i\, C \quad \text{(41)}$$

in which

$$C = \frac{1+e}{e} \frac{w}{w_e} \quad \text{(42)}$$

In Eq. 42 w is the saturated water content and w_e is the change in water content caused by a change from the moist to the saturated state, or vice versa.

From Eq. 41 the velocity of a moving saturation line can be estimated at any point at which the seepage gradient is known, provided the permeability and drainage factor are known. Knowing the velocity of the saturation line, the time required for a given length of movement may be determined from

$$L = v_{SL}\, t \quad \text{(43)}$$

in which L is the distance traveled in time, t, and v_{SL} is the velocity of the saturation line. For conditions existing during the drainage of water out of an embankment, or during saturation of an embankment, the seepage gradients are in a state of flux, and the seepage velocities are constantly changing. A rigorous solution for these conditions would be highly complex, but approximate solutions can be made by determining the velocity at finite increments of distance in a structure, and summing up the times required for all increments, as follows:

$$\Delta L = v_{SL}\, \Delta t \quad \text{(44a)}$$

and

$$L = \Sigma v_{SL}\, \Delta t \quad \text{(44b)}$$

By use of Eqs. 41 and 44*a* studies may be made of the velocity of a moving saturation line, the time required for a given length of movement, or the distance moved in a given time. Studies of this kind for earth dams can be made in conjunction with determination of the shape of a moving saturation line with the flow net. From the flow net the relative velocities of different points along a moving saturation line at any given moment can be approximated, since the velocities are proportional to the seepage gradient, i, and the gradient can be determined at any point in a flow net from the following fundamental relationship:

$$i = \frac{\Delta H}{\Delta L} \quad \text{(45)}$$

in which ΔH is the drop in head in distance ΔL in the direction of flow.

Application of the foregoing principle to the drainage from an embankment subjected to drawdown is illustrated by an analysis of the cross section in Figs. 8 and 9, which show the same shape as the model in Fig. 2. It is assumed that the section has been subjected to steady seepage with a full reservoir until equilibrium has been established, and the reservoir was lowered instantly to the minimum pool level. Recognizing that the flow net may only represent the approximate true seepage conditions within embankments subjected to nonsteady seepage, a flow net is drawn for the initial steady seepage boundaries (Fig. 8(*a*)). The seepage gradient is then determined at a number of points along the saturation line (points A_1, B_1, and C_1, Fig. 8(*a*)), and the saturation line is moved ahead a moderate amount, with the relative distances moved proportional to the seepage gradients at the various points along the original line. A new flow net is constructed for the new saturation line (Fig. 8(*b*)), a new set of gradients determined, and the saturation line moved to a third position shown by a dashed line in Fig. 8(*b*). This procedure is repeated a sufficient number of times to define the shape of the saturation line over the range desired. Having established the position of the saturation line at a number of intervals, the time required for a given length of movement can be examined in any part of the drawdown zone with the aid of Eqs. 41 and 44*a*. When using Eq. 44*a*, Δt is the time required for movement of the saturation line a distance ΔL, as for example—from the first to the second position in Fig. 8. This procedure, illustrated by a determination of the theoretical time required for the saturation line to move vertically downward at the center of

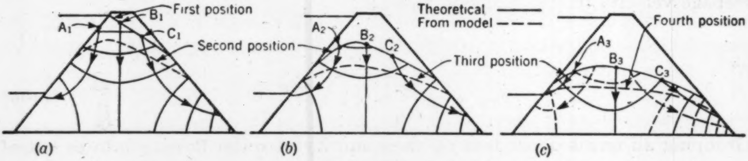


FIG. 8.—FLOW NETS FOR VARIOUS POSITIONS OF THE SATURATION LINE

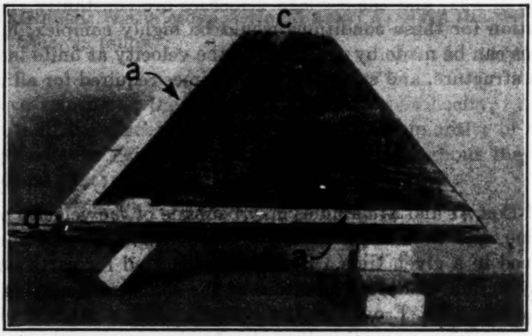


FIG. 9.—SIMPLE VISCOUS FLUID MODEL USED IN DRAWDOWN TESTS

the section in Fig. 8, is shown in Fig. 10. The seepage gradient at the center line is first determined for each position of the saturation line for which a flow net has been constructed, and the downward velocity of the saturation line at the center line computed from Eq. 41, using a drainage factor, *C*, and a permeability coefficient, *k*, which have been determined to be representative of the cross section. The velocity and time curves in Fig. 10 were computed for a small, viscous fluid model having the same shape as the cross section in Fig. 8 (see Fig. 9). The base width of the model was $10\frac{1}{2}$ in.

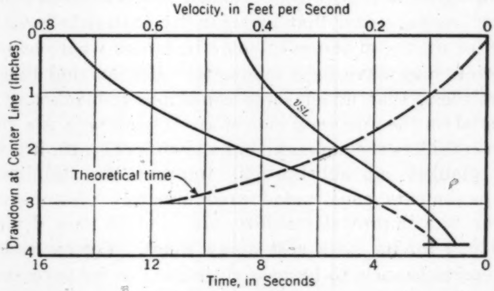
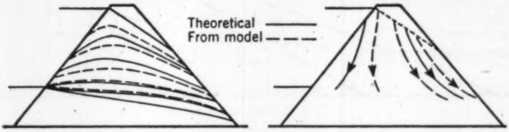


FIG. 10.—DETERMINATION OF THE THEORETICAL TIME FOR DRAINAGE FROM A MODEL

Viscous fluid models provide a valuable supplement to the flow net in studying moving saturation lines. The model shown in Fig. 9 was constructed from plywood and the shape of the saturation line and its direction of movement compared with theoretical determination of these factors. The plywood cross section designated b–b, Fig. 9, is separated (by strips a) about 0.1 in. from a glass plate that can be clamped on the front side of the model (not shown in photograph). The reservoir (c) is filled with a relatively viscous fluid, such as glycerine, which flows through the narrow space simulating flow through an embankment. Drawdown of the reservoir in the model is readily produced by pulling the slide (d) at the left. Determination of the shape and direction of the saturation line from flow nets agreed almost exactly with observations of the model, as indicated by Figs. 11(*a*) and 11(*b*). To the knowledge of the writer, A. Casagrande, M. ASCE, first proposed viscous fluid models for the study of seepage. Professor Casagrande and W. D. Shannon, Jr., have used this type of model extensively in nonsteady seepage studies.



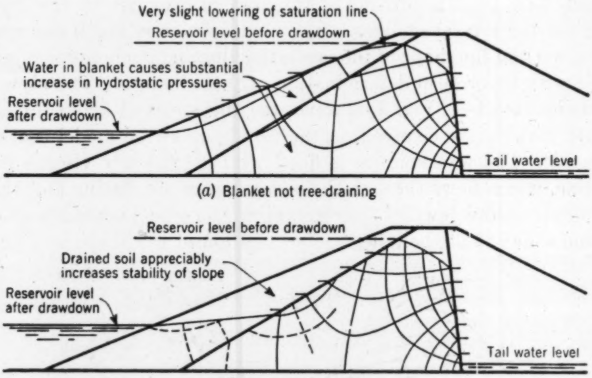
(*a*) Shape of saturation line during drainage

(*b*) Paths followed by random points on saturation line during drainage

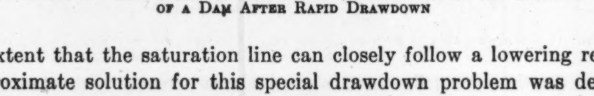
FIG. 11.—DETERMINATION OF THE SHAPE AND DIRECTION OF THE SATURATION LINE

The principles outlined herein are useful not only in studying changes in the position of a saturation line following reservoir drawdown, but also in obtaining approximate solutions to many other nonsteady seepage conditions. If water is flowing into a soil mass (as, for example, during saturation of an embankment), the same basic method[15] can be used; but the drainage factor, C, should be computed using w_c to designate the increase in water content accompanying a change from a moist to a saturated condition.

The author has suggested that high quality materials have been used for the protection of upstream slopes in some instances where less pervious, and cheaper materials may have been adequate. Recognizing this possibility, it is true, nevertheless, that under some conditions a blanket of high quality, pervious material on the upstream face of an embankment is a valuable investment. If one considers the embankment shown in Fig. 12, it is evident that a free draining blanket can substantially improve the stability of upstream slopes of embankments constructed predominately of impervious soil. In some instances, the improved stability provided by free draining, granular material may pay for its extra cost many times. For example, in locations requiring expensive tunnels to by-pass river flow, or for permanent operation, steepened embankment slopes (permissible with the use of free draining blanket material) may permit substantial reduction in the cost of the outlet works. In such locations, a stable, free draining blanket may be a highly valuable investment. To be of maximum benefit, the blanket should be free draining to the extent that the saturation line can closely follow a lowering reservoir.


Very slight lowering of saturation line
Reservoir level before drawdown
Water in blanket causes substantial increase in hydrostatic pressures
Reservoir level after drawdown
Tail water level

(*a*) Blanket not free-draining


Reservoir level before drawdown
Drained soil appreciably increases stability of slope
Reservoir level after drawdown
Tail water level

(*b*) Blanket free-draining

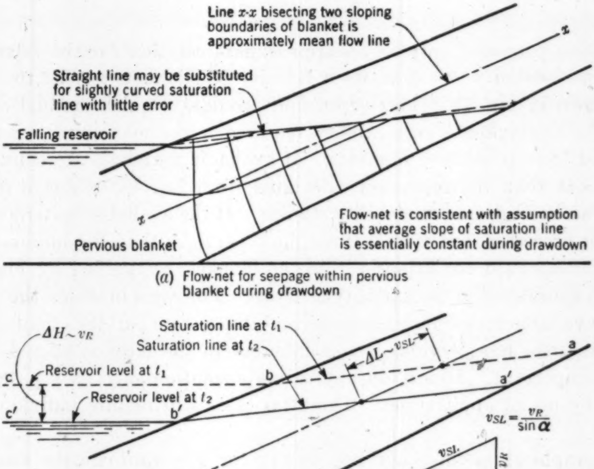
FIG. 12.—INFLUENCE OF PERMEABILITY OF BLANKET ON THE STABILITY OF THE UPSTREAM SLOPE OF A DAM AFTER RAPID DRAWDOWN

An approximate solution for this special drawdown problem was developed by the writer in 1940 using the aforementioned principles.

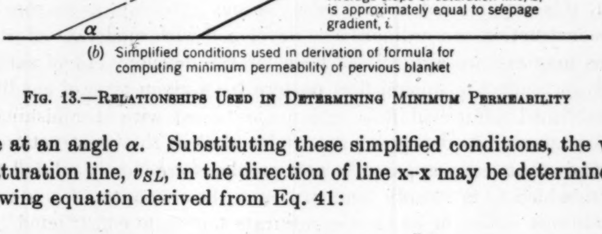
In studying drainage out of thin blankets similar to that shown in Fig. 12, the criterion was established that the saturation line must closely follow the lowering reservoir, lagging on an average slope, S, that remains essentially constant during drawdown. The slope, S, may be established arbitrarily, or it may be the maximum allowable slope of the saturation line established by stability analysis. On the basis of the foregoing criterion a flow net and viscous fluid model study was made. These studies led to the conclusion that simplified conditions (see Fig. 13) could be used in obtaining reasonably accurate solutions to this problem. In the first place, it was found that a straight line could be substituted with little error, for the slightly curved saturation line, and the slope of this line could be used as a measure of the average seepage gradient within the blanket with little error. Also, for thin blankets similar to those shown in Figs. 12 and 13, the mean direction of flow within the blanket is approximately coincident with a line bisecting the two sloping boundaries of the blanket (for example, line x–x, Fig. 13), intersecting the horizontal base line at an angle α. Substituting these simplified conditions, the velocity of the saturation line, v_{SL}, in the direction of line x–x may be determined from the following equation derived from Eq. 41:

---

[15] *Civil Engineering*, August, 1941, p. 499.


Line x-x bisecting two sloping boundaries of blanket is approximately mean flow line
x
Straight line may be substituted for slightly curved saturation line with little error
Falling reservoir
x
Flow-net is consistent with assumption that average slope of saturation line is essentially constant during drawdown
Pervious blanket

(a) Flow-net for seepage within pervious blanket during drawdown


$\Delta H \sim v_R$
Saturation line at t_1
Saturation line at t_2
$\Delta L \sim v_{SL}$
a
c
Reservoir level at t_1
b
a′
Reservoir level at t_2
c′
b′
$v_{SL} = \frac{v_R}{\sin \alpha}$
v_{SL}
v_R
α
Average slope of saturation line, S, is approximately equal to seepage gradient, i
α

(b) Simplified conditions used in derivation of formula for computing minimum permeability of pervious blanket

FIG. 13.—RELATIONSHIPS USED IN DETERMINING MINIMUM PERMEABILITY

$$v_{SL} = k\,S\,C \qquad \text{or} \qquad k = \frac{v_{SL}}{S\,C} \ldots\ldots\ldots\ldots\ldots\ldots (46)$$

in which k is the coefficient of permeability of the soil and S is the average slope of the saturation line. The relationships involved are illustrated by the diagrams in Fig. 13.

Generally the maximum velocity of drawdown is established by the discharge capacity of outlet works, and it is desired to determine the minimum permeability of a protective blanket that will permit the saturation line to drop within a blanket, closely following the elevation of the reservoir. The velocity of drawdown of the reservoir, v_R, is related to the mean velocity of the saturation line, v_{SL}, in the following manner (see Fig. 13(*b*)):

$$v_{SL} = \frac{v_R}{\sin \alpha} \qquad (47)$$

Hence, the permeability that will permit the saturation line to follow the reservoir, with a specified small slope, S, can be approximated by combining Eqs. 46 and 47 as follows:

$$k = \frac{v_{SL}}{S\,C} = \frac{v_R}{S\,C \sin \alpha} \qquad (48)$$

Eq. 48 is proposed only as an approximate solution for the relationship between permeability and drawdown velocity for blankets having the general shape shown in Fig. 12. This expression provides a quick method of determining the approximate permeability required if the maximum benefit is to be derived from protective blankets. If available materials are considerably less pervious than the minimum calculated from Eq. 48, drainage from the blanket can be neglected in stability studies. If the available materials, either natural or processed, are several times more pervious than the minimum indicated, the improved conditions within the blanket, as typified by Fig. 12(*b*), should be considered in the stability analysis. For cases in which the permeability of available materials is either several times more pervious than required, or several times less pervious, errors inherent in the method should be relatively unimportant. Hence, an approximate solution such as Eq. 48 should have many useful applications. Doubtful cases will require individual analysis.

The graphical method outlined herein for the approximate analysis of moving saturation lines requires a knowledge of the construction of flow nets. However, this requirement should be no serious handicap, as reasonable skill in their construction can ordinarily be developed with moderate effort. New situations may require appreciable time for the development of satisfactory solutions; but once the general flow pattern for a given type of condition has been established additional flow nets are obtained with diminishing effort. After an engineer has developed reasonable skill in the construction of flow nets he finds that many apparently new problems are closely related to situations with which he is already familiar; hence, he encounters fewer and fewer new conditions. Also, if he studies accurate flow nets constructed by other engineers he can benefit considerably.

The author has stated that the probable danger of major slides of upstream slopes of dams subjected to rapid drawdown has been overemphasized, and he has indicated the manner in which capillary forces may cause a reduction in pore pressures in earth dams constructed of fine-grained soils. Although laboratory experiments may be helpful in establishing the probable effective height of the capillary zone in proposed embankment materials, observation of actual dams under operating conditions is believed the final answer to the study of this factor. Also, pore pressures caused by the sudden increase in effective weight of large soil masses in embankments subjected to drawdown are factors about which few factual data have been published. Dependable observational data for existing earth dams are needed for an evaluation of the combined result of the many factors effecting pore pressures in embankments.

Ross M. Riegel,[16] M. ASCE.—In the design of earth dams it has been the practice of the Tenessee Valley Authority (TVA) to compute a "factor of safety" for the slope exposed to the reservoir water by the method described by the author (under the heading, "Applications"); that is, it is assumed that the pore-water pressures which would exist under conditions of drawdown would be those due to a complete lack of drainage. (It is assumed also that the pore pressure head existing at any point within the dam is the static head existing at that point, measured either to the surface of the fill, to the saturation line, or to the surface of the reservoir water—as the case may be.) Under these conditions the minimum factor of safety prescribed in the design of new structures has been about 1.25. This rather low factor has been used largely because it has been believed that the assumed criterion of no drainage was actually more severe than the probable true condition; the information necessary for a more accurate appraisal of true conditions was wanting. The author's paper remedies this deficiency to a high degree, and enables the problem of design to be approached with more confidence.

It has also been the practice of the TVA to design the downstream slope, which was usually relatively free from uncertainty as to the effects of drainage, with a factor of safety of at least 1.5. This is an index of a more proper value when the drainage conditions are well understood and can be evaluated through the methods developed by the author. The effect of more accurate drainage treatment on the computed factor is strikingly illustrated by the application to Blue Ridge Dam in northwest Georgia, cited by the author.

The determination of the slope of major earth structures has also been subject to the review of the Board of Consulting Engineers (TVA) whose experience and judgment have been material factors in the final determination of slopes. It is suggested that the factor of safety must still have a material value, even after the most accurate design methods available have been employed. Some allowance must be made for two points. The material actually placed in the fill may depart in its characteristics from those determined from the occasional samples examined in the soils laboratory. The construction process, to some degree, may also fail to develop the anticipated characteristics as to compaction, moisture content, friction angle, and cohesion, which are set up in the laboratory investigation and design assumptions.

The author is to be complimented on shedding much light upon an obscure subject. His investigations of full-size structures have been based on embankment materials which may be generally classified as clays, whereas his model experiments have employed relatively coarse sand. He points out that, from the standpoint of drainage, soils classified as silts and fine sands are probably the least satisfactory embankment materials; yet, embankments have been built from them. It would be most interesting if the application of the author's methods of experiment and analysis to existing conditions in such structures would be undertaken, and it is suggested that such an investigation would be a valuable addition to this subject.

---

[16] Head Civ. Engr., Design Div., TVA, Knoxville, Tenn.

T. W. Lambe,[17] Jun. ASCE.—The method of computing rates of drainage and the magnitudes of total water head in the draining soil mass, based on an analogy of drainage with heat flow or consolidation, is an interesting approach to the drainage problem. In this discussion the writer will present comparative studies of the author's method, and various other theoretical methods, with the results from laboratory tests (made during a research program carried out by the writer which furnished material for a thesis to be submitted in partial fulfilment of the requirements for the degree of Doctor of Science at Massachusetts Institute of Technology in Cambridge), in which a soil was actually drained. Also presented will be a rigorous theoretical derivation of the author's method for vertical drainage, so as to show more clearly some of the assumptions and conditions involved.

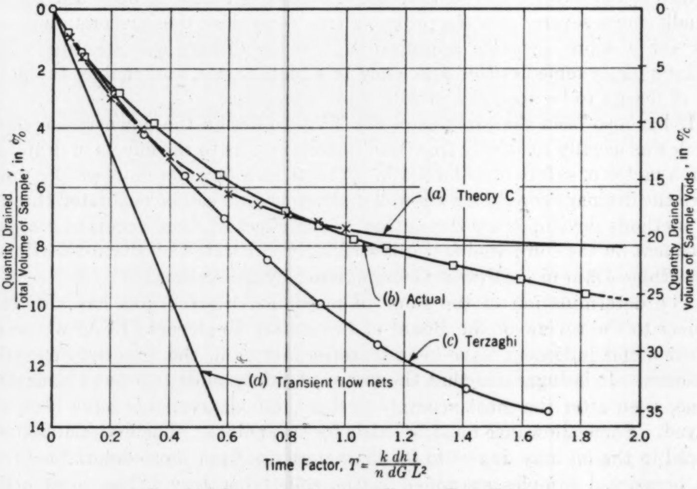


Fig. 14.—Drainage Quantity Versus Time Factor

Fig. 14 is a plot of drainage quantity, expressed as a percentage of both total soil volume and soil void volume of the draining mass, against time, expressed as a dimensionless time factor, T, for a laboratory test and for various theoretical solutions. The methods used to obtain the separate curves are explained first briefly and later analyzed after the theory has been presented so that the meaning of their respective assumptions can be more fully appreciated. Curve (*b*) was plotted from data taken during the bottom drainage of a 6-ft vertical tube of a natural fine, uniform, saturated sand. Theory C is a solution of Eq. 21*a* with a more reasonable assumption of initial total head distribution than that used by the author in obtaining Eq. 25*a*. Curve (*c*) was obtained by use of equations developed by Karl Terzaghi,[18] Hon. M. ASCE. Curve (*d*), for transient flow nets,[19] was plotted using steady state flow nets at the various positions of the falling line of saturation.

---

[17] Instr., Soil Mechanics, Civ. Eng. Dept., Mass. Inst. Tech., Cambridge, Mass.

Eq. 25*a* was not plotted in Fig. 14 because the writer knew no reasonable way to compute the ultimate drainage quantity due to the initial total head assumption used to derive Eq. 25*a*. This point is explained more fully later. However, in Fig. 15, Eq. 25*a* is compared with theory C, for both methods, by plotting T against the percentage of drainage, where the percentage of drainage is assumed equal to the percentage of the average total head which has been dissipated. An examination of Figs. 14 and 15 shows that: (*a*) Theory C more nearly approximates the actual flow curve than does Eq. 25*a*; and (*b*) none of the theoretical methods correctly takes care of the actual time lags that exist.

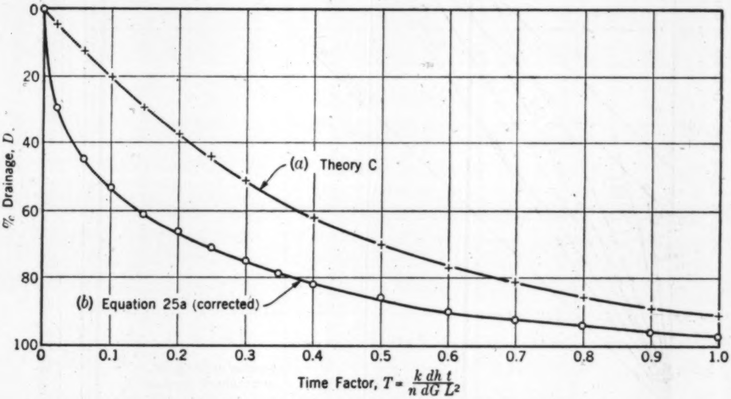


FIG. 15.—PERCENTAGE OF DRAINAGE VERSUS TIME FACTOR

Before considering the theoretical methods, a brief description of the true drainage process is given so that the validity of the various assumptions which are made to obtain solutions can be better evaluated. Fig. 16 is a plot of total heads in the sand sample at various times during drainage. (The datum for measuring elevation heads was taken at the bottom of the soil, which is also the atmospheric pressure line. The convention is followed throughout this discussion.) The times are represented by the dimensionless time factor, T, used in Figs. 14 and 15; T is the same as the author's τ (Eq. 23*a*), without the term 4. These total heads were obtained using a pore pressure measuring system with the same general principle as the one used by D. W. Taylor,[20] Assoc. M. ASCE, in his pore pressure work on undrained shear of clay. Since the pore water pressures were less than atmospheric pressure, it was necessary to prove that the measuring device correctly obtained the true pressures. This proof was secured by measuring the pore pressures for systems in which the values of pressure could be checked by an independent method.

---

[18] "Theoretical Soil Mechanics," by Karl Terzaghi, John Wiley & Sons, Inc., New York, N. Y., 1943, p. 314, Eqs. (9) and (10).

[19] "Retaining-Wall Design for Fifteen-Mile Falls Dam," by Karl Terzaghi, *Engineering News-Record*, May 17, 1934, p. 632.

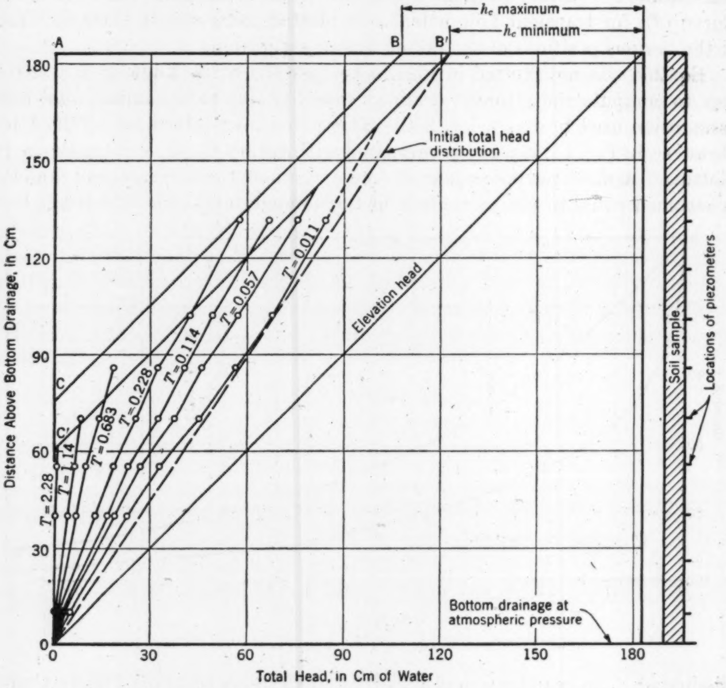


FIG. 16.—PORE PRESSURE DURING DRAINAGE OF SAMPLE

If, at the start of the test, the level of the water is an infinitely small distance above the top surface of soil in the tube, the total water head at every point is equal to its elevation head as shown in Fig. 16. As soon as the water level drops below the top of the soil, menisci are developed, causing the soil to exert a tension (based on a zero pressure of atmospheric pressure) to the water. This tension is transmitted throughout the water and gives the initial total head distribution shown in Fig. 16. As drainage takes place, the total head at all points becomes smaller; at any time the total head distribution has a characteristic shape, as shown in Fig. 16, of smaller gradients at the lower elevations. When the water tension in any small element exceeds the capillary head, h_c, more water is removed than flows into the element and the degree of saturation becomes less than 100%. In Fig. 16, the line BC, or the line B′C′, represents the maximum water tension that can exist at any point. Thus, at any time factor, line B′C′ is the boundary between the partially saturated soil above, and the completely saturated soil below, the line. Drainage from the partially saturated zone, represented by AB′C′, is a large percentage of the drainage and proceeds at a very slow rate because of the greatly lowered permeability.

---

[20] "Shear Research," 10th Rept. to U. S. Engr. Dept., Massachusetts Institute of Technology Soil Mechanics Laboratory, Cambridge, Mass., May, 1944.

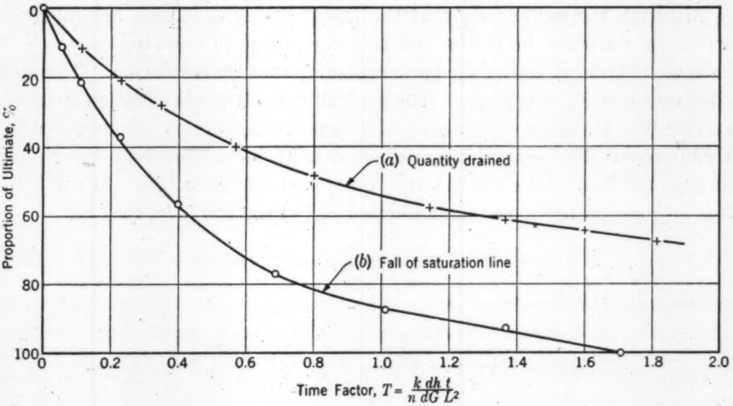


FIG. 17

The time lag of drainage from the partially saturated zone can also be noted in Fig. 17, which is a plot of the percentage of drainage and the relative fall of the line of "saturation" versus T for the laboratory test. The line of "saturation" fall was obtained by observation of the boundary line between the completely saturated and the partly saturated zones. There is some air present just below the visual boundary line, but this has little effect on the plot of fall of line of saturation since it is put on a percentage basis. Fig. 17 shows that the line of saturation has fallen to its final position when only 66% of the

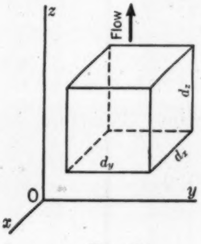


FIG. 18

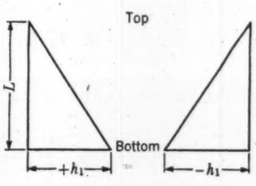


FIG. 19

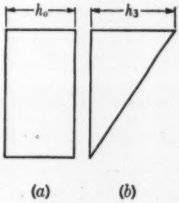


FIG. 20

drainage has taken place, thus 34% of all drainage takes place from the partly saturated zone after the line of saturation has reached its final position; also, while the line of saturation is falling, a large percentage of the water comes from above the line of saturation.

Although the writer feels that the use of analogies is very helpful for instructional purposes, he thinks that true derivations of analytical methods are desirable in order to clarify the various assumptions that are made. Therefore, a derivation of Eq. 4 is given. The general hydrodynamic equation governing laminar flow in soils is set up first. To save space, only one-dimensional flow is considered: The general equation can easily be extended to cover three-dimensional flow. The net flow into the element *dx dy dz* (Fig. 18) will be the flow into the bottom face minus the flow out of the top face; that is,

$$\text{Net } q = \Delta q = q_{\text{bottom}} - q_{\text{top}} \tag{49a}$$

in which, by Darcy's law,

$$q_{\text{bottom}} = k_{\text{bottom}}\, i_{\text{bottom}}\, A = k_z \left(- \frac{\partial h}{\partial z} \right) dx\, dy \tag{49b}$$

and

$$q_{\text{top}} = k_{\text{top}}\, i_{\text{top}}\, A = \left(k_z + \frac{\partial k_z}{\partial z}\, dz \right) \left(- \frac{\partial h}{\partial z} - \frac{\partial^2 h}{\partial z^2}\, dz \right) dx\, dy \tag{49c}$$

In these equations q is the rate of flow; k is the coefficient of permeability; A is the cross-sectional area of soil; h is the total head; and i is the hydraulic gradient.

Substituting Eqs. 49*b* and 49*c* in Eq. 49*a* and combining terms,

$$\Delta q = \left(k_z \frac{\partial^2 h}{\partial z^2} + \frac{\partial k_z\, \partial h}{\partial z^2} + \frac{\partial k_z}{\partial z}\, dz\, \frac{\partial^2 h}{\partial z^2} \right) dx\, dy\, dz \tag{50}$$

In the element *dx dy dz*, the volume of water present is

$$V_w = \frac{G\, e}{1 + e}\, dx\, dy\, dz \tag{51}$$

in which G is the degree of saturation; and e is the void ratio. Therefore,

$$\Delta q = \frac{\partial}{\partial t} \left(\frac{G\, e\, dx\, dy\, dz}{1 + e} \right) = \frac{dx\, dy\, dz}{1 + e} \frac{\partial}{\partial t} (G\, e) \tag{52a}$$

or

$$\Delta q = \left(e \frac{\partial G}{\partial t} + G \frac{\partial e}{\partial t} \right) \frac{dx\, dy\, dz}{1 + e} \tag{52b}$$

Equating Eqs. 50 and 52*b*,

$$\left(k_z \frac{\partial^2 h}{\partial z^2} + \frac{\partial k_z\, \partial h}{\partial z^2} + \frac{\partial k_z}{\partial z}\, dz\, \frac{\partial^2 h}{\partial z^2} \right) dx\, dy\, dz = \frac{dx\, dy\, dz}{1 + e} \left(e \frac{\partial G}{\partial t} + G \frac{\partial e}{\partial t} \right) \tag{53a}$$

or

$$k_z \frac{\partial^2 h}{\partial z^2} + \frac{\partial k_z \, \partial h}{\partial z^2} + \frac{\partial k_z}{\partial z} dz \frac{\partial^2 h}{\partial z^2} = \frac{1}{1+e}\left(e \frac{\partial G}{\partial t} + G \frac{\partial e}{\partial t} \right) \ldots\ldots (53b)$$

—which is the general hydrodynamic equation for laminar flow in soils. By making various assumptions, several of the familiar formulas of soil mechanics can be obtained, for example:

(*a*) If it is assumed that k is constant and $G = 100\%$, Eq. 53*b* becomes

$$k_z \frac{\partial^2 h}{\partial z^2} = \frac{1}{1+e} \frac{\partial e}{\partial t} \ldots\ldots (54)$$

—which is the consolidation equation.

(*b*) If it is assumed that k, e, and G are constant, Eq. 53*b* becomes

$$k_z \frac{\partial^2 h}{\partial z^2} = 0 \ldots\ldots (55)$$

—which is the Laplace equation for one-dimensional steady flow.

To obtain the author's formula (Eq. 4) it must be assumed that k is constant, e is constant, and

$$\frac{e}{1+e} \frac{\partial G}{\partial t} = w_d \frac{\partial h}{\partial t} \ldots\ldots (56a)$$

Assuming the relation expressed by Eq. 56*a*,

$$w_d = \frac{e}{1+e} \frac{\partial G}{\partial h} = n \frac{\partial G}{\partial h} \ldots\ldots (56b)$$

in which n is the porosity of the soil. Eq. 56*b* agrees in principle with the author's paper (under the heading, "Analogy Between Seepage and Heat Flow"), since $n \, \partial G$ is the water drained per unit volume. The writer prefers to use $n \frac{\partial G}{\partial h}$ instead of w_d, as both n and G are widely used terms in soil mechanics literature. If a vertical tube of soil is drained and the degree of saturation, G_z, of the drained soil is obtained at various heights above the atmospheric pressure line, a plot of z versus $n\,(1 - G_z)$ can be made. This is the true curve the author is attempting to plot in Fig. 6. There is some question as to the use of air to obtain the various hydrostatic heads, since it is difficult to know whether or not all the air pressure is carried by the water and whether or not the air picks up water in passing through the sample. Air pressure to obtain plots of $n\,(1 - G_z)$, which is also known as "effective porosity," versus z has been used by other experimenters.[21,22] As recognized by the author, $n \frac{\partial G}{\partial h}$ (or w_d) may be far from constant.

The writer would like to point out that w_d and c_d have units. It would be helpful if the units were given to the values of w_d and c_d listed in the section,

---

[21] "Capillarity of Soils," by Hsien-Hsiang Ku, thesis presented to Purdue University at Lafayette, Ind., in 1940, in partial fulfilment of the requirements for the degree of Bachelor of Science.

[22] "Subsurface Drainage Investigation," New England Div., Corps of Engrs., Dept. of the Army, Boston, Mass., 1947.

"Comparison of Computed and Observed Data," so as to enable the procedure to be better followed. For example, without units, the writer finds it difficult to discover how, by using Eq. 8*b*, the author obtains $c_d = \frac{0.8 \text{ in. per min}}{0.23} = 41.7$.

Making the three assumptions previously mentioned as necessary to obtain Eq. 4, by replacing $\frac{k_z}{n} \frac{\partial h}{\partial G}$ by c_d in Eq. 53*b*,

$$c_d \frac{\partial^2 h}{\partial z^2} = \frac{\partial h}{\partial t} \qquad (57)$$

—which is the author's equation (Eq. 21*a*) for vertical drainage.

Formulas for draining a saturated soil mass vertically through the base are next developed—because it is this flow condition for which the comparison is presented in Figs. 14 and 15.

The solution of Eq. 57 involves the assumption of an initial distribution of total head, *h*. As in the theory of consolidation for the initial distribution of hydrostatic excess, there are many initial total head distributions for which solutions can be obtained. By using boundary conditions of bottom drainage only and of zero total head at the bottom immediately after start of flow, by assuming an initial total head distribution as shown in either Fig. 19(*a*) or in Fig. 19(*b*), and by solving in a manner similar to the solution for the consolidation theory,[23]

$$d = 1 - \frac{16}{\pi^3} \sum_{n=1}^{n=\infty} \frac{(2\,n - 1)\,\pi + 2\,(-1)^n}{(2\,n - 1)^3}\, e^{-\frac{1}{4}\lambda\tau} \qquad (58)$$

—which is Eq. 25*a*, with a correction of 4 in the exponential term. The boundary conditions of bottom drainage and of zero total head at the bottom are in complete agreement with what takes place. On the other hand, the assumption of either initial total head distribution as shown in Fig. 19 is questionable. A solution based on Fig. 19(*a*) gives an infinite initial gradient at the bottom, also, according to this solution, during the early stages of drainage there is upward flow in the upper sections of the soil. Examination of Fig. 16 shows that both these conditions are untrue. A solution based on Fig. 19(*b*) gives water flow into the soil sample and, therefore, has no place in drainage equations.

It is interesting to note that the solution of Eq. 58 can be obtained by substitution of the initial condition of Fig. 19 in Eq. 57, or by correctly combining the better known solutions for initial conditions as shown in Fig. 20.[23] In view of the actual initial total head distribution in Fig. 16, the writer suggests that a solution based on an initial total head distribution as shown in Fig. 20(*b*), in which $h_3 = L - h_c$, is superior to that presented by the author. This solution is given by K. Terzaghi and O. K. Fröhlich[10] and will be called theory C for the comparison in Figs. 14 and 15. The time curves for theory C are given in Fig. 21. A comparison of the theoretical time curves of Fig. 21 with the actual time curves of Fig. 16 shows: (*a*) The theoretical initial total head distribution is correct; (*b*) in the early stages, the actual head dissipation is faster than in theory C (later the reverse is true); and (*c*) the theory does not take into account the partly saturated zone. As one might expect, theory C agrees fairly well with the actual test, as noted in Fig. 14, until the effects of the partly saturated zone become important.

---

[23] "Fundamentals of Soil Mechanics," by D. W. Taylor, John Wiley & Sons, Inc., New York, N. Y., 1948.

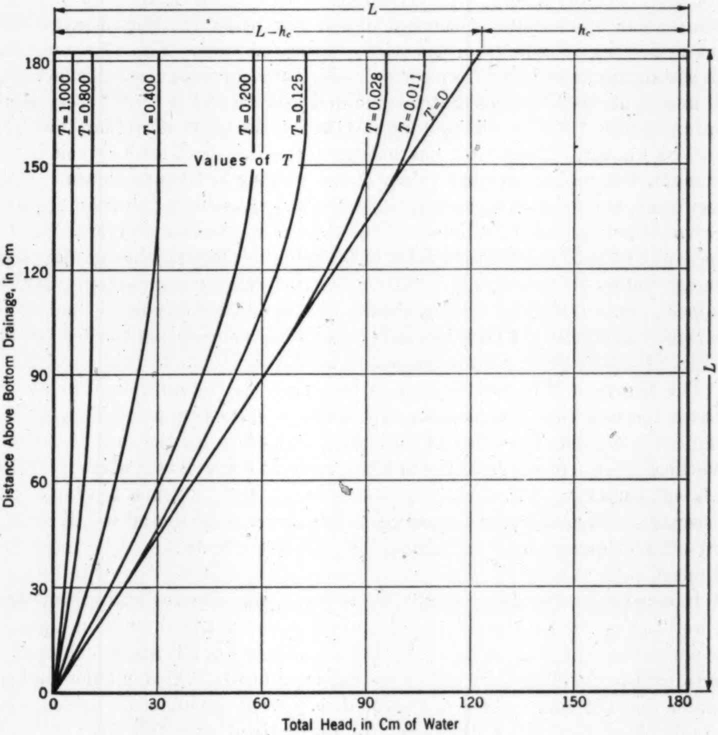


FIG. 21.—TOTAL HEAD AS A FUNCTION OF HEIGHT AND TIME FACTOR

The ultimate total water drained can be computed for theory C by: (1) Setting up the expression for total head as a function of time and distance; (2) differentiating the head expression with respect to z to obtain the gradient and substituting $z = 0$ to obtain the gradient at the bottom; and (3) putting the gradient in Darcy's expression for rate of flow and integrating with respect to time from time = 0 to time = ∞. For the case studied, the ultimate total water obtained by this method was only 57% of the actual. This result is not surprising in view of the fact that the theory does not approximate the actual head dissipation in the partly saturated zone. The ultimate quantity drained was not computed using the author's initial head distribution because the gradients obtained have little meaning. Since the bottom gradient is infinite immediately after starting drainage, a solution, like that applied to theory C, using the author's initial total head distribution, would give an infinite quantity. It would be helpful if the author were to explain how the ultimate quantity can be computed in a manner consistent with his theory, if such a method exists.

The remaining two theoretical solutions plotted in Fig. 14 also require discussion. In his theory, K. Terzaghi assumes that all drained water comes from a small cross-sectional element of soil just below the line of saturation, thus neglecting all flow from the partly saturated zone. Fig. 17 shows that this assumption may be far from correct. K. Terzaghi recognized this fact himself and in his book[24] states that this method may be only a "* * * very crude approximation * * *." In applying this theory, the value of capillary head, h_c, must be known. The writer has not used the definition of h_c given by K. Terzaghi, but he has selected values of the varying h_c from test data. More specifically, the value of h_c used in the gradient expression is the value required to give the actual initial gradient. (This value of h_c was also used for theory C.) The capillary head and the final value of G chosen to compute the quantity were average values so that $A\, n\, (1 - G)\, (H - h_c)$ equaled the measured quantity drained. Such selections of values of G and h_c would not be possible had actual drainage test results not been available; they were made so that the comparison in Fig. 14 would be on a more consistent basis.

The transient flow net solution is based on the assumption that at any instant the flow can be represented by a steady state flow net. The solution completely neglects the effect of capillarity and, thus, for the case of draining a vertical column of soil from the bottom, assumes a constant gradient of unity. It is not surprising, therefore, that this solution gives a much too large rate of seepage. The difference between this solution and the actual would be even larger for a finer-grained soil where the capillary effects would be more pronounced.

In conclusion, the writer would like to emphasize that the author's method, as well as the other theoretical methods considered in this discussion, does not recognize the changing permeability that appears in Eq. 53*b* and which must be considered in draining soils with large capillary heads. Also, all the methods assume that the void content of the soil does not change under the added intergranular load caused by drainage. In fine-grained soils this assumption of constant void content may be questionable.

W. L. SHANNON,[25] Assoc. M. ASCE.—In connection with the design of a rolled fill earth dam in New Hampshire in 1945, the writer became interested in investigating the time required for the drainage of the pervious upstream shell section of a rolled earth fill dam following drawdown of the upstream pool. This problem is of general interest because most earth dams of modern construction contain substantial upstream pervious shell sections. To investigate the stability of such dams it is usually necessary to make some assumption regarding the drainage of this shell section for the condition of drawdown of the upstream pool.

---

[24] "Theoretical Soil Mechanics," by Karl Terzaghi, John Wiley & Sons, Inc., New York, N. Y., 1943, p. 310.

[25] Research Associate in Soil Mechanics, Graduate School of Eng., Harvard Univ., Cambridge, Mass.

At the writer's suggestion an approximate solution for the drainage of an upstream shell section of a dam was developed by A. Casagrande, M. ASCE, and presented in an unpublished report, dated May 23, 1945, to the Boston (Mass.) District, Corps of Engineers, Department of the Army. Professor Casagrande assumes that the upstream pool elevation is suddenly drawn down the depth D (Fig. 22). This is a reasonable approximation providing the time for a small percentage of drainage of the shell section is much greater than the actual time for full drawdown of the pool. It is further assumed that the free water surface during drainage may be approximated by inclined straight lines passing through the toe of the shell. This approximation has been found to be reasonable where the seepage quantity is of interest and the shape of the free surface is not. At the start of drainage the free water surface is at the upstream face of the shell; at any time after drainage starts the free water surface is the line designated t_U in Fig. 22.

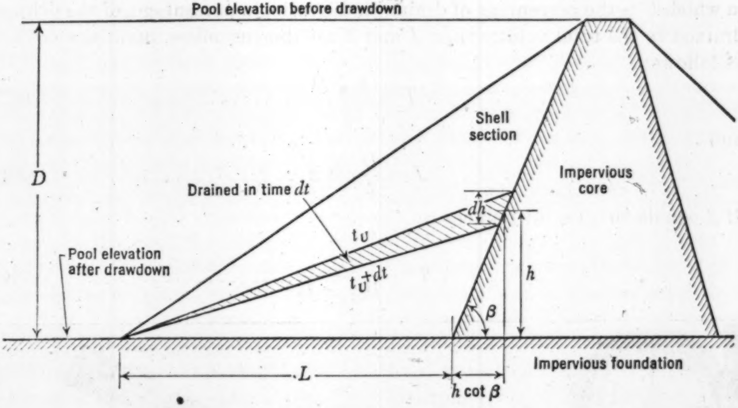


FIG. 22.—DRAINAGE OF UPSTREAM SHELL SECTION OF AN EARTH DAM

The volume of water which will drain from the shell in the time dt (Fig. 22) is equal to the volume of the triangular slice times the yield, w_y:

$$dq = -\frac{dh}{2}(L + h \cot \beta)\, w_y \ldots\ldots\ldots\ldots\ldots\ldots (59)$$

Based on Darcy's law and making Dupuit's assumption,[28] which is reasonable for slopes of the upstream face less than 30°, the quantity of flow per unit of time may be approximated by

$$\frac{dq}{dt} = k\, a\, i = k\left(\frac{h}{2}\right)\left(\frac{h}{L + h \cot \beta}\right) \ldots\ldots\ldots\ldots\ldots (60)$$

---

[28] "Seepage Through Dams," by A. Casagrande, "Contributions to Soil Mechanics 1925–1940," Boston Soc. of Civ. Engrs., Boston, Mass., 1940, p. 304.

Eqs. 59 and 60 when solved and integrated may be reduced to the following form:

$$T = \frac{U}{100 - U} + J \log_e \frac{100}{100 - U} + J^2 \frac{U}{100} \dots\dots\dots\dots (61)$$

in which U is the percentage of drainage, the ratio, in percentage, of the volume drained to the total volume; and J and T are dimensionless quantities defined as follows:

$$T = \frac{t_u \, k \, D}{w_y \, L^2} \dots\dots\dots\dots (62)$$

and

$$J = \frac{D}{L} \cot \beta \dots\dots\dots\dots (63)$$

If β equals 90°, Eq. 61 reduces to

$$T = \frac{U}{100 - U} \dots\dots\dots\dots (64)$$

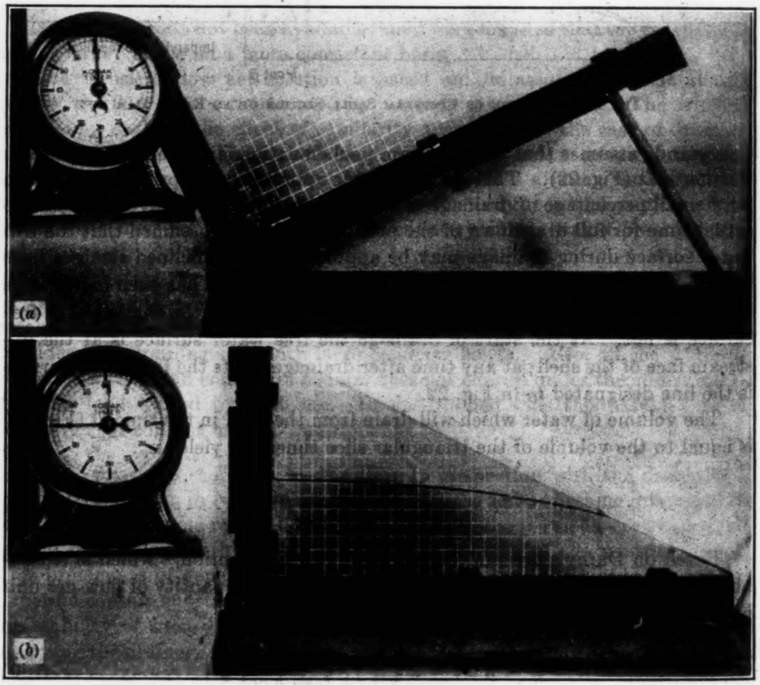


Fig. 23.—Viscous Fluid Model

(a) Inclined for Filling
(b) During Drainage Test

It was suggested by Professor Casagrande that viscous fluid models could be used advantageously to check this theoretical solution. Viscous fluid models have several important advantages over sand models, such as those used by the author, for the study of nonsteady flow problems. In a viscous fluid model all the fluid is drained, whereas in sand models there is a variable quantity of water retained by capillarity. The coefficient of permeability of the viscous fluid model is the same in every direction and at every point. Different coefficients of permeability for different elements of a structure may be obtained by varying the plate spacing. In sand models it is difficult to place the sand so that it has the same coefficient of permeability throughout. To obtain two or more sands having desired ratios between their coefficients of permeability usually is extremely difficult.

A viscous fluid model was designed and constructed at Harvard University (Cambridge, Mass.) and tests were performed under the supervision of the writer at the Soil Mechanics Laboratory of the Boston District, Corps of Engineers. The model, as shown in Fig. 23, represents the upstream shell of a dam resting on an impervious horizontal foundation and having an impervious core with a vertical upstream face. The upstream face of the shell has a slope of 1 vertical and 2 horizontal. The model consists of two triangular parallel glass plates 0.28 cm apart and held in a frame which provides the impervious horizontal and vertical boundary. To fill the model with fluid the frame was tilted so that the sloping open edge was horizontal, as shown in Fig. 23(*a*).

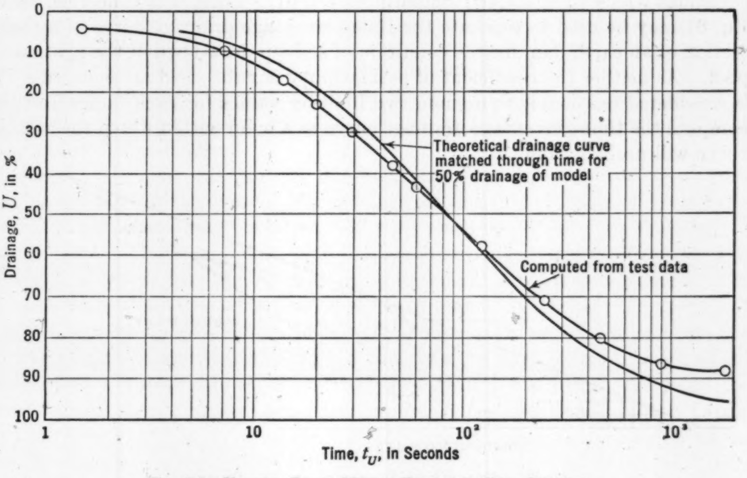


FIG. 24.—VISCOUS FLUID MODEL DRAINAGE TEST RESULTS

The fluid used was glycerine, which is sufficiently viscous to insure laminar flow between the plates. Glycerine has an affinity for glass, and hence, as drainage of the fluid occurs, a thick film will cling to the glass plates in the drained area unless precautions are taken to reduce this affinity. In these tests, the glass plates were given a very thin coating of paraffin, which reduced to a negligible amount the quantity of glycerine clinging to the plates.

A drainage test consists in suddenly tilting the model filled with fluid from the position shown in Fig. 23(*a*) to that shown in Fig. 23(*b*). Periodic observations of the position of the free surface were recorded photographically. In Fig. 23(*b*) the dark line at midheight of the model is the free surface of the glycerine at 45 sec after start of drainage. The results of a drainage test on this model are shown in Fig. 24.

In order to determine a value for the coefficient of permeability of the glycerine flowing between the two paraffin coated glass plates, measurements of the fluid viscosity are necessary. For the model tests performed, measurements of fluid viscosity were not obtained; hence, a direct comparison between Eq. 64 and the model test results is not possible. However, it is possible to match a theoretical curve based on Eq. 64 through any desired point on the drainage curve for the model. In Fig. 24 a theoretical curve has been matched through the time for 50% drainage of the model. The good agreement between the test data and the matched theoretical curve indicates that the derivation is reasonable as far as shape of the time-drainage curve is concerned.

In connection with an extensive viscous fluid model investigation of a very similar problem, a good agreement, both with respect to the shape and the position of drainage curves, was obtained between curves determined from viscous fluid model tests and drainage curves computed from theoretical equations which were derived similarly to Eq. 61. Thus, it is concluded that Eq. 61 may be used to estimate the time-percentage drainage curve of a shell section of an earth dam for the condition of sudden drawdown of the upstream pool. As neither the coefficient of permeability nor the yield are constants, it is considered reasonable to assume two limiting values for each factor and to compute the time-percentage drainage curve as a band within which the actual curve will probably fall.

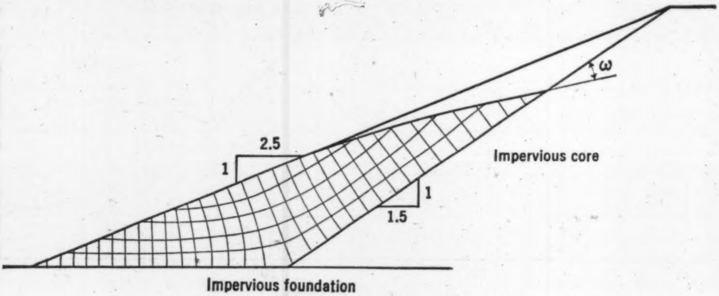


FIG. 25.—EXAMPLE OF FLOW NET AT 16% DRAINAGE

From such a solution, the percentage of drainage corresponding to any desired time after sudden drawdown of the upstream pool may be estimated. This value for percentage of drainage will be reasonable only if the percentage of drainage which occurs during the actual time required for pool drawdown is small.

By making several trial flow nets, one net may be constructed for a desired value of percentage of drainage. Fig. 25 is an example of a flow net representing seepage conditions in an upstream shell of a dam which has drained to $U = 16\%$.

According to Professor Casagrande, the problem of constructing a flow net for the condition of drainage is more complicated than the ordinary steady flow seepage problem, because the free surface is neither a flow line nor an equipotential line. The free surface must fulfil the condition that the intersections with the equipotential lines are all vertically equidistant. In addition, the free surface must be tangent to the upstream slope and must meet the slope of the impervious core at an angle ω which is dependent on the velocity v_β with which this point moves down the slope. This velocity must fulfil the conditions:

$$v_\beta = \frac{k \sin(\beta - \omega)}{w_y \cos \omega} \dots\dots\dots\dots\dots (65)$$

and

$$v_\beta = \frac{k}{100\, w_y \sin \beta} \left(\frac{D}{L}\right)^2 \frac{\partial U}{\partial T} \dots\dots\dots\dots\dots (66)$$

Eq. 65 follows from considerations of the flow net at the point where the free surface meets the impervious boundary of the core, and Eq. 66 represents the rate of progress of drainage.

F. H. KELLOGG,[27] M. ASCE.—The discussions of this paper indicate several of the methods being used in the study of drainage without replacement from an external source. They should point the way toward a rational analysis of this type of ground-water flow which would lead to a much more intelligent use of local experience.

Mr. Cedergren outlines the method of attack by graphical flow nets. As pointed out at the close of Mr. Lambe's discussion, this completely neglects capillarity, which should not invalidate the analysis for pervious shoulders of dams, ground-water supplies, and similar conditions where capillary head is considerably less than the head inducing drainage. Just why the free surface of the water should be considered a source does not seem clear; the volume of water available there is infinitesimal. It appears more rational to consider the unsaturated boundaries of the drained body as a constant source through which air enters, the drains as constant sinks through which water escapes, and the free surface as a transient air-water boundary, as mentioned in the paper. Thus, distinction may be made between free draining materials, in which capillarity plays a minor role, and slow draining materials, in which capillarity plays a major role. The boundary between the two would depend on the ratio of effective capillary head to head inducing drainage, and would thus be variable, just as the boundary between turbulent and laminar flow, as defined by the Reynolds criterion, is variable.

---

[27] Chairman, Dept. of Civ. Eng., Univ. of Mississippi, University, Miss.

Mr. Lambe gives an excellent resume of the mechanics of drainage. He questions the validity of Eq. 25*a*, doubtless because of lack of clarity in the basic paper. The writer hastens to add that he did not fail to plot this equation because he "* * * knew no reasonable way to compute the ultimate drainage quantity * * *"; in support, Fig. 26 is presented, plotting the computed curve, together with observed points for comparison. The test was made for standard Ottawa sand, so that any interested investigator may verify these results.

To clarify Eq. 25, it should be mentioned that it was derived for an initial condition of static saturation. The initial head distribution represents maximum potential head, L, minus maximum capillary head for equilibrium h_c. Because capillary channels are of a variety of sizes, and intercommunicate, the initial value of h_c was taken as varying from zero at the base to L at the top. As drainage proceeds, water flows from smaller into larger channels, accounting for increased net heads at the top of the sample. No flow of water into the sample is involved. An infinite hydraulic gradient occurs for an infinitesimal time at the beginning of drainage, making the volume of drainage during the first instant of flow indeterminate, not infinite. This analysis is similar to one for consolidation under its own weight of clay resting on sand.[28] As the initial head distribution for Mr. Lambe's "theory C" requires initial steady state flow, there is no comparison between this condition and that for which Eq. 25 was derived.

Mr. Riegel moves the discussion from the science of physics to the art of engineering. The importance of the modifying factors he has mentioned has been amply confirmed by field observations, including those of the writer. In fact, until sufficient field observations are available to serve as bases for working hypotheses, the usefulness of models and pure analysis is questioned.

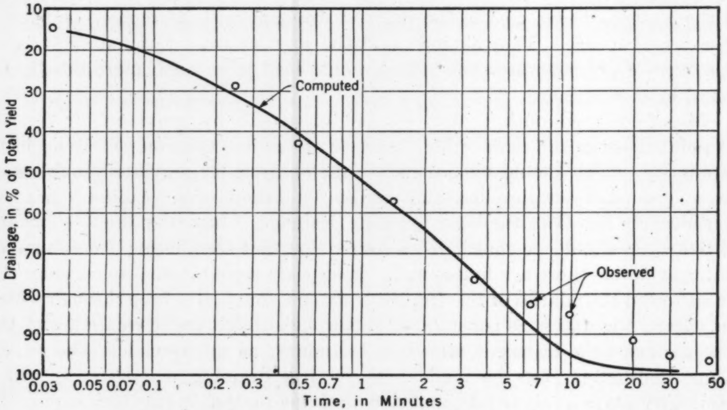


FIG. 26.—COMPARISON OF OBSERVED AND COMPUTED YIELDS FOR VERTICAL DRAINAGE

---

[28] "Theorie der Setzung von Tonschichten," by K. Terzaghi and O. K. Fröhlich, F. Deuticke, Leipzig, 1936, p. 115, section 57.

It is emphasized that the tests described in this paper were performed after field observations to secure a variety of checking conditions, and not to try to duplicate any field conditions.

Mr. Shannon presents an interesting description of drainage study by viscous fluid models. Although these are certainly a technical improvement on sand models, they do not give the heterogeneity of capillary channels leading to a true unsteady state condition in which every point of the body is a source. Like transient flow nets, they illustrate a steady state in which one boundary varies with time—air flowing steadily in at the top and fluid flowing out at the drain. This is not the condition described in the basic paper, nevertheless it is one that is extremely important, particularly in the field of ground-water supply, where the yield w_y, as assumed in Mr. Shannon's equations, is essentially constant. The viscous fluid models described in this discussion should have valuable potentialities under such conditions.

In conclusion, the author wishes to express his appreciation of the discussions submitted, which should be valuable additions to the scanty literature on unsteady state drainage.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2357

APPLICATION OF GEOLOGY TO TUNNELING PROBLEMS

By Ernest E. Wahlstrom,[1] Esq.

With Discussion by Messrs. Berlen C. Moneymaker; Robert S. Mayo; F. A. Nickell; R. H. Keays; Jacob Feld; Thomas W. Fluhr; Portland P. Fox; Hyde Forbes; Roger Rhoades, A. B. Reeves, and W. H. Irwin; and Ernest E. Wahlstrom.

Synopsis

Geological examination of tunnel sites prior to and during tunneling operations has become established procedure. The geologist should study and appraise carefully all surface and subsurface features having any bearing on the engineering problems associated with the drawing of the tunnel. Particular attention should be given to the following: (1) Internal structures, physical characteristics, and distribution of the rocks and overburden; (2) secondary structural features such as folds, faults, and joints; (3) chemical alteration of the rocks; and (4) the distribution and flow of ground water.

Suggestions for overcoming or minimizing difficulties presented by caving ground and other undesirable conditions in tunnels are given, and the causes and effects of caving in selected examples are discussed in detail.

Introduction

Geological investigations play an increasingly important part in the planning and completion of certain types of engineering projects. Geological examination of dam sites, bridge and building foundations, and tunnel sites prior to and during construction has become an established procedure that has yielded substantial or even spectacular returns in the form of safety and reduced costs of construction and maintenance.

The writer has spent considerable time as a consultant studying the geological features of several proposed water diversion tunnels, and in connection with this work has examined many existing tunnels and adits in an effort to discover and evaluate the various geological factors that are important in economical tunneling operations. Mine tunnels yield the most valuable data because, in general, these tunnels follow or cut across ore bodies formed where faulting,

---

Note.—Published in October, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Dept. of Geology, Univ. of Colorado, Boulder, Colo.

fracturing, and alteration are at a maximum. However, much pertinent information has been obtained from railroad tunnels, drainage tunnels, and existing water diversion tunnels.

THE FUNCTIONS OF THE GEOLOGIST

The functions of the geologist in tunneling operations are manifold. Generally, he is called upon to examine the proposed site for a tunnel to determine if it is located to the best advantage. Such an investigation entails a geologic study of the portal areas and the surface over and adjacent to the line of the tunnel. The results of the study may lead to one of several conclusions:

(1) The tunnel is favorably situated and little or no difficulty should be experienced in driving it.

(2) There are geologic indications of the presence of sections of bad ground in the course of the tunnel. (It is the geologist's responsibility to indicate the probable location and nature of the bad ground in these sections.)

(3) The tunnel site is unfavorably situated, and the cost of driving the tunnel will be very high if not prohibitive.

If the location of the tunnel cannot be changed or if conditions demand that the tunnel be driven in spite of the unfavorable prognostication, then the geologist is faced with the responsibility of predicting to the best of his ability the extent and character of the various undesirable conditions that will be encountered.

Many tunnel sites are in areas partly or wholly covered by concealing alluvium or overburden. In covered areas the geologist must decide to what extent and by what methods the subsurface should be explored. Frequently, trenching or drilling is needed, which should be supervised by the geologist with attention to minute detail. Exploratory work of this type is expensive and the geologist generally must be satisfied with a bare minimum of information even though the facts he obtains do not begin to solve the host of unanswered problems that are associated with regions of thick or widespread blanketing deposits. Under certain conditions geophysical investigations yield pertinent information, and the geologist's duty is to determine which geophysical method or methods should be employed and to what extent.

Once the tunnel is started, the geologist should make periodic examinations of the completed sections and the face. He should maintain a detailed written account and a map showing the geological features of the tunnel as well as the day-by-day progress. The study of the rocks and structures encountered in the tunnel may lead to an increase in the accuracy with which the geologist may predict the changes in the rocks or structures beyond the tunnel face. The geologist may give advice as to the type and the amount of support needed in bad sections and as to the most effective manner of drilling and blasting at the face—giving due consideration to the relationship between the structure of the rock and the ease with which it is mined. In the event that unexpected difficulty is encountered because of extremely bad ground, he may offer counsel as to the best method of overcoming the difficulty.

The importance of a detailed geologic map of the tunnel cannot be overemphasized. This map not only should show the geologic structures and rocks encountered, but it also should serve as a lasting record to indicate the location of bad sections, which may require attention many years after the tunnel is completed.

The geologist should act in an advisory capacity to the engineer in charge of the tunneling operation. Too often the geologist's advice does not carry the weight it should because either the engineer does not understand what the geologist is attempting to do or he feels that the geologist's predictions are too vague and indefinite. However, this situation is gradually becoming less serious and the engineer and the geologist are reaching an amicable understanding based on mutual respect and profitable cooperation. A geologist should not attempt to assert authority in any problems facing the engineer except those which bear directly upon, or are the outgrowth of, the geological study. On the other hand, the engineer generally is not qualified by training or experience to make a critical appraisal of the findings of a trained geologist and must accept the advice of the geologist as the best that can be offered.

Physical Characteristics of Consolidated Rocks

Consolidated rocks are those in which the constituent particles or fragments are bound together strongly enough to produce an aggregate with moderate to considerable strength. The rock or mineral components may be held together with a cementing material as in certain clastic sediments, or the contacts of the mineral grains may be interlocked or intermeshed as in igneous rocks and in most metamorphic rocks.

Certain rocks are massive or isotropic. The physical characteristics of these rocks, from the points of view of the geologist and the tunnel engineer, are the same in all directions within the rock. There is no preferred orientation of the mineral particles, and the ease of tunneling in such a rock is the same irrespective of the direction of the tunnel. In this category belong igneous rocks such as certain types of granite, sedimentary rocks such as certain limestones, and metamorphic rocks such as certain marbles and quartzites. Completely isotropic rocks are relatively rare.

Most rocks possess directional structures as a result of the parallel to subparallel orientation of many or most of the mineral particles. Such rocks are anisotropic and show distinct variations in physical properties in different directions. Consideration of the nature of the anisotropy of rocks is important to the tunnel engineer because it has a direct bearing on the cost of driving the tunnel.

In igneous rocks layered or linear structures are commonly present. If the rock contains "platy" minerals such as tabular feldspar or mica, there is a tendency for these minerals to rotate into parallel or subparallel positions during the intrusion and crystallization of the rock. The resulting formation is called "flow layering" or "platy structure." Rocks which show a well-developed platy structure have a gneissoid aspect and commonly contain flat, lenticular, dragged-out inclusions of foreign rocks called "schlieren." Generally, such rocks show a strong tendency to split or break along planes paralleling the flow layering, and they part or break with difficulty in directions at right angles to the layering.

Rocks containing minerals of prismatic or needle-like habit tend to develop a unidirectional formation which is described as "linear structure" or "flow lineation." This formation results from the parallel to subparallel alinement of the prismatic minerals during the intrusion and crystallization of the rock. It is not uncommon to find both platy and linear structures in the same rock.

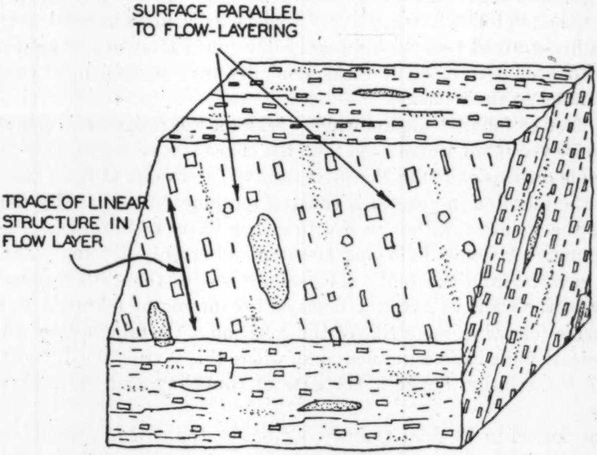

SURFACE PARALLEL TO FLOW-LAYERING
TRACE OF LINEAR STRUCTURE IN FLOW LAYER

Fig. 1.—Block Diagram Showing Flow Layering and Linear Structure in an Igneous Rock; Stippled Areas Are Inclusions of Foreign Rock

If both are present, the needles or prisms lie in the plane of the platy structure Linear structure may be emphasized by the presence of elongated, spindle-shaped inclusions of foreign rocks. Igneous rocks in which linear structure is well-developed generally break easily parallel to the lineation and break with difficulty at right angles to the linear structure.

Fig. 1 is a block diagram of an igneous rock showing platy and linear structures. When the geologist describes these features, he generally gives the strike and dip of the platy structure and the direction of the trace of the linear structure on a horizontal plane. These data permit the easy calculation of the pitch angle of the flow lineation. Where flow lineation and flow layering are steeply inclined, lineation is of slight practical importance in the tunnel operation. However, the direction of the tunnel with respect to the flow layering merits careful consideration. A tunnel being driven across the flow layering probably will have the following characteristics: The face will advance relatively rapidly because in the blasting operation the rocks will break cleanly and easily away from the planes of weakness paralleling the flow layering. The walls, floor, and roof will tend to be rough and blocky. On the other hand, if the tunnel is parallel to the strike of steeply dipping flow layering, the forward progress of the tunnel will be slowed because the tunnel is parallel to a direction of maximum strength and toughness in the rock. The walls of the tunnel will probably be relatively smooth; the floor and roof, more or less irregular.

The irregularity of the walls increases with a decrease in the angle of dip of the layering.

In metamorphic rocks the layering is commonly called "foliation" and is well developed in gneisses and schists. A typical gneiss consists of alternating layers of granular rocks of differing mineral composition and physical properties. A schist, although foliated, is relatively homogeneous in its mineral composition. Slate is a fine-grained rock with a highly developed tendency to split into thin plates. Foliated metamorphic rocks generally tend to split most easily along planes parallel to the foliation.

The effect of foliated metamorphic rocks on the progress of the tunneling operation is even more pronounced than the effect of layering in igneous rocks—for the same reasons and with the same results. A tunnel at right angles to the strike of the foliation in general is constructed much more rapidly than is one following the foliation, although the tunnel is likely to be much more blocky and irregular. A tunnel following the strike of steeply dipping foliated rocks tends to cave in from the roof. The smoothness of the walls lessens and the tendency of the walls to cave increases as the dip of the foliation decreases, if the tunnel is following the strike of the foliation. Certain gneisses and schists are complexly folded and crumpled; and, where this is true, the difficulty of prediction of the relation between the ease of tunneling and the rock structure increases.

In one tunnel in Boulder County, Colorado, the cost of driving the tunnel was as much as 25% greater in the sections paralleling the foliation than in the sections cutting across the foliation. However, the overbreak in the latter was about 20% greater than in the first instance because of the irregularity of breakage along the walls and roof.

Typical conditions are illustrated in Fig. 2, which shows one tunnel cutting across steeply dipping, foliated rocks, and another following the foliation of steeply inclined layers. The latter condition should be avoided if possible.

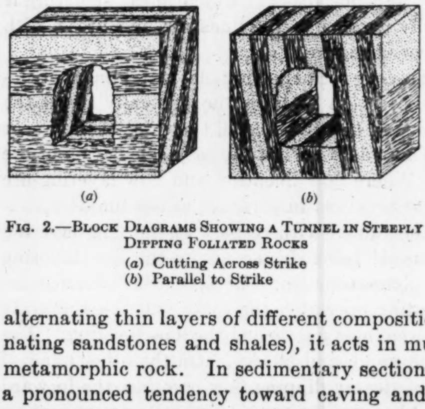

(a) (b)

Fig. 2.—Block Diagrams Showing a Tunnel in Steeply Dipping Foliated Rocks

(a) Cutting Across Strike
(b) Parallel to Strike

In sedimentary rocks bedding or stratification may be conspicuous, but the strength of an individual layer parallel to the bedding is not much different from the strength across the bedding. However, if the rock is thinly stratified or consists of alternating thin layers of different composition (for example, thin-bedded alternating sandstones and shales), it acts in much the same manner as a foliated metamorphic rock. In sedimentary sections where shale is abundant, there is a pronounced tendency toward caving and slumping, and considerable supporting timber may be required as the tunnel is driven.

If a tunnel is driven through inclined layered rocks at right angles to their strike, an additional factor should be considered—that is, the direction in which the tunnel is being driven relative to the dip of the layers. Preferably, the tunnel should be driven so that the layers will dip away from the heading. If this is done, drilling and blasting operations will be expedited because the cleavage in the rocks tends to produce an overhanging face rather than one which is sloping back toward the already completed section of the tunnel. This suggestion applies equally well to tunnels being driven in faulted or strongly jointed rocks. In general, the tunnel should be headed in the down dip direction of the planes of weakness in the rock.

These conclusions purposely are generalized. No attention is paid to the modifying effects of fracturing, alteration, and other features because these are considered separately. It is not claimed that the generalizations noted are without exception because various conditions have modifying effects.

Folds, Joints, and Faults

Folding in layered rocks requires consideration by the geologist because, if folded structures are present, a correct interpretation permits prediction of the attitudes of the layers in different sections of the tunnel. In foliated metamorphic rocks, particularly, the prediction of variation of the attitudes of the layers at the tunnel level may call for detailed surface studies and all the ingenuity, skill, and experience at the command of the geologist.

Actual or incipient fractures are present in all rocks. Fractures along which there has been little or no movement are called "joints." "Faults" are fractures along which there has been appreciable to considerable movement. Joints result from various causes, most important of which are expansion or contraction of the rock due to thermal, chemical, or mechanical causes, or fracturing of the rock in response to deformation by stresses accompanying movements in the outer part of the earth's crust. Faults show relative displacement of the rocks on either side of the fault plane. Fault displacements range from a few inches to several miles. A fault may be very narrow and sharply defined or may involve rocks in a zone hundreds of feet wide.

Joints and faults are a major cause of bad, caving ground in tunnels and deserve careful consideration. Joint surfaces commonly are rough and irregular in coarsely granular rocks, particularly if the joints developed as a result of shrinkage or decompression. Blocks bounded by such joints rarely cause trouble because friction along the rough surfaces prevents disengagement of blocks from the walls or roof of the tunnel. However, if the irregularities of the joint surfaces have been smoothed by slight movements between adjacent blocks or if the joints are smooth for other reasons, the tendency to cave is increased. Moreover, if water has permeated closely jointed rocks and has resulted in alteration and formation of clay minerals along the joints, a particularly bad, "running" or raveling type of ground is produced.

A careful study of a rock mass may show that several joint systems are present and, moreover, that certain of the joint systems are better developed than others. If possible, the tunnel should be located so that its direction is at right angles to the most prominent, most steeply inclined joint system. This course should be chosen because in drilling and blasting the ease of breaking the rock in the heading away from joints or "slips" is advantageous. Moreover, if a tunnel follows a major joint system, the danger of overhead caving is constantly present. In igneous rocks showing flow layering it has been observed that a very prominent joint system commonly parallels the flow layering.

Engineers and geologists have developed a real fear of faults in tunnels, and justly so. Of course, a tunnel should not be driven through or along an active fault; but, fortunately, most faults are not active. The fear of encountering a fault in a tunnel should be justified or tempered by a detailed study of the faults in the region where it is proposed to drive the tunnel. Some faults offer no difficulty, others present distressing problems.

Among the faults which cause little or no trouble are those which show a sharp, clean offset of the rocks and are not accompanied by decaying alteration of the wall rocks. Certain faults or fault zones have served to localize the circulation of mineralizing solutions and examples are known where the substances deposited from the solutions have produced a tight, well-cemented rock stronger than the original rock. However, mineralizing solutions may produce the opposite effect, and a broad zone within and on either side of the fault may be converted into rotten, swelling, or squeezing ground. Alteration of feldspars to clay minerals or micaceous minerals often produces a weak, slippery rock which will cause considerable trouble.

Fault movements commonly develop selvages of weak, pulverized rock flour or "gouge." In wide fault zones where the gouge is present in weaving, crisscrossed, or braided fractures, extensive caving is likely to result. In some faults the rock are brecciated into loose, porous aggregates. Additional movements may cause the rounding of the angular fragments. Generally, such faults are filled with water and the combination of loose rock and water under hydrostatic pressure has caused more than one tunnel to be rerouted or even abandoned. Such fault zones were encountered in the Elton tunnel under the Oquirrh Range in Utah and in the Leadville drainage tunnel in Colorado. Fig. 3 shows typical joints and Fig. 4 shows several common types of faults.

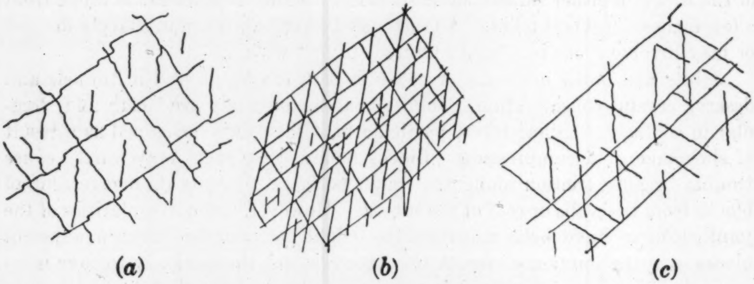


Fig. 3.—Typical Joint Patterns; Joint Surfaces
(a) Rough and Irregular, Minimum Caving
(b) Smooth Surfaced, Maximum Caving
(c) Smooth and Rough, Considerable Caving

In summary, it may be stated that a study of the joint systems and faults in an area where it is proposed to drive a tunnel should be included in the geological investigations over the line of the tunnel and in the area adjacent to it.

Altered Rocks and Swelling Ground

In the preceding section brief reference was made to altered rocks in the vicinity of joints and faults. It was suggested that alteration might produce effects of a serious nature in tunneling operations.

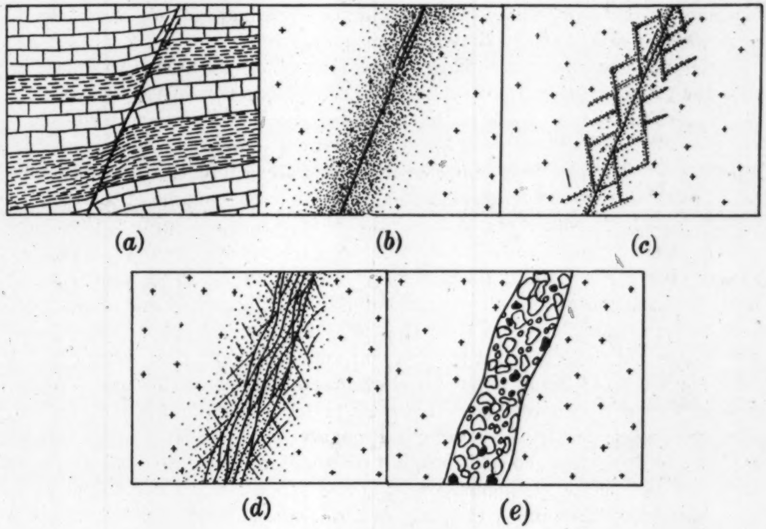


Fig. 4.—Types of Faults

(a) Clean Sharp Break, No Alteration of Wall Rocks
(b) Sharply Defined, No Subsidiary Fractures, Pervasive Alteration
(c) Sharply Defined, with Subsidiary Fractures, Localized Alteration
(d) Braided, with Subsidiary Fractures, General Alteration
(e) Open, Porous, Filled with Breccia

One of the minerals most susceptible to alteration in rocks is feldspar. Attack on feldspar by reactive solutions of surface origin or of deep-seated derivation may cause feldspar to be converted from a fresh, physically strong substance to a weak, pulverulent aggregate of clay minerals or micaceous minerals, and of fine-grained silica. This change, essentially a chemical one, generally is most pronounced in and near zones of intensive jointing or faulting, where altering solutions move with maximum freedom.

Grain size controls the amount of alteration to a certain extent. Fine-grained feldspathic rocks in general alter more readily than coarse-grained rocks, probably in part because fine-grained rocks may be more closely jointed than coarse-grained rocks and in part because the small size of each grain promotes more rapid, pervasive alteration. Fault movement may pulverize the rocks along the fault and cause the development of a multitude of tiny fractures in the rocks in the vicinity of the fault, thus producing ideal conditions for rock alteration. Moreover, faulting causes the minerals in the rock along the fault to be strained and, in general, strained minerals decompose more readily than do unstrained ones.

If the altering solutions are of deep-seated origin, alteration may not change the strength of the rock. For example, hydrothermal solutions commonly deposit condiderable silica in the wall rocks of fissures, and the silicified rock will stand in the tunnel as well or even better than the original rock. Sulfides deposited in fractures or in the wall rocks along fractures tend to oxidize on exposure so that mineralized areas containing abundant sulfides have a tendency to swell slowly and to slough after exposure to air.

Swelling or squeezing ground in tunnels is much to be feared. In squeezing rock the tunnel cross section is reduced by the plastic flow of very weak rocks in response to the pressure of the overlying rocks. Rocks that act in this manner generally are highly altered, are rich in clay minerals, and contain considerable water. The plasticity increases with the water content.

In swelling ground there is an actual increase in the volume of the rock resulting from hydration or chemical reaction or from a combination of these processes. Certain clay minerals and anhydrite have a remarkable affinity for water. If rocks containing these minerals are encountered in the tunnel, water from the atmosphere and other sources tends to be absorbed and swelling results. Tunnel sections have been studied where the rock when first encountered appeared to be reasonably fresh and unaltered; but, after being exposed for a short time, it began to swell and eventually was converted into a sticky, plastic mass of clay minerals. This same material broke down on the dump into an almost impalpable powder. Probably, the rock was partly altered at depth by hydrothermal solutions, but the confining pressure of the overlying rocks did not permit complete hydration and expansion. When the pressure was released by the tunnel, the rock, in a metastable state, rapidly reached a state of chemical equilibrium in response to the release of the confining pressure. Pressures as high as 20 tons per sq ft have been encountered in swelling ground.

The most spectacular example that has come to the writer's attention was the swelling of a dike of fine-grained porphyry cut by a tunnel in a gold mining operation. Within a few days after exposure, the originally fresh looking rock had almost completely filled the tunnel with sticky, relatively cohesive mud. In the Moffat tunnel water bore (in Colorado), a deceptively fresh looking granite irresistibly splintered Oregon fir timbers, 12 in. by 18 in. Many similar examples may be cited.

Rocks in a metastable condition may swell by chemical reaction as well as by simple hydration. It is an established principle that the release of pressure on a rock tends to produce a new mineral assemblage which is in equilibrium with the new environment. Fortunately, however, chemical changes in response to pressure changes in most rocks are so slow that they are of no importance in tunneling operations.

UNCONSOLIDATED ROCKS

The range of composition, texture, and structure in unconsolidated rocks is extreme. Texturally these rocks grade from the finest muds to coarse aggregaates such as are found in rock slides and bouldery glacial deposits. Unconsolidated rocks are characterized by the lack of interlocking contacts between adjacent grains or by the lack of a strong, binding cement or matrix. Some rocks, as, for example, certain sandstones, have been converted into loose aggregates of mineral grains through the removal of the cement by dissolution.

In tunnel operations unconsolidated rocks generally are encountered at the portals, where the bedrock is blanketed with rock waste or alluvium. Loose sand and gravel, wet or dry, present a considerable obstacle as does loose material consisting of large, round, or angular boulders with no cementing material. The difficulty of tunneling through unconsolidated rocks consisting partly or largely of very fine-grained or clayey materials increases in direct proportion to the amount of water present. Dry or slightly moist clayey aggregates stand fairly well in portal areas; and, despite a slumping tendency, the walls and roof generally will stand long enough to permit support and lining to be set before caving starts. As the water content increases, however, the plasticity and mobility of the clay mineral aggregate increase, and considerable difficulty may be experienced in excavating and supporting such ground.

One of the most vexing situations arises when a tunnel is driven through water-saturated volcanic ash beds. In wet ash beds a mobile, gelatinous slurry is generally present which is held with only the greatest of difficulty. On the other hand, if the material is compact and dry, it offers little trouble except that it may swell on exposure to water.

A railroad tunnel near Leadville, Colo., was driven through dry, bouldery glacial till without mishap. The clayey matrix of the till was dry enough to have considerable strength and to serve as a reasonably strong binding medium. On the other hand, a near-by drainage tunnel driven through water-saturated glacial deposits encountered so many difficulties that part of the tunnel had to be abandoned and the cost of the tunnel greatly exceeded the most pessimistic cost estimates.

GROUND WATER

Prediction of ground-water conditions in tunnels is one of the most baffling problems confronting the geologist. Any prediction must be based on a consideration of the porosity and permeability of the rocks and the availability of ground water. Rapid movement of water in rocks generally is determined by the presence of connected open joints, fissures, or faults. It is a common experience of the tunnel engineer to drive through a fault containing little or no water and then to encounter a narrow crack or a section of jointed ground from which water issues in great volume and with destructive velocity. This phenomenon is explained in terms of relative permeability. Many faults, but not all of them, contain abundant clayey gouge which effectively prevents the rapid movement of water. Joints and small fractures, however, generally are not clogged with impermeable material and will permit the easy flow of water.

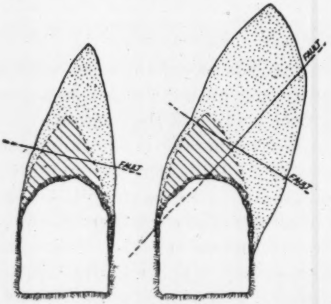
Faults characterized by abundant breccia and open spaces generally are permeable and contain ample water. Tunnels driven in limestone are likely to encounter underground caverns filled with water. The location of these caverns commonly is determined by the presence of fractures which have permitted the easy circulation of water with a consequent dissolution of the limestone.

A knowledge of the configuration of the ground-water table is useful; but, in general, the accurate determination of the depths of the upper surface of the saturated zone is obtained only by extensive drilling or geophysical study. In compact, consolidated, relatively impermeable rocks, the configuration of the ground-water table is of no great significance, because impermeable rocks do not permit the circulation of water. In alluvium or any permeable or unconsolidated rock, however, the story is quite different.

The Tunnel Cross Section

Too little attention has been paid to the relationship between the tunnel cross section and the ease or difficulty with which rocks of the wall and roof cave. In certain tunnels, much time and expense can be saved if careful consideration is given to details of rock composition and structure and to methods of preventing or reducing the hazard of caving ground. It is the purpose of this section briefly to describe the nature of the caving and to suggest a remedy in certain selected examples. The generalizations do not apply to tunnel sections in squeezing or swelling ground.

The caving tendency is present in all tunnels, but the extent of caving depends on the strength and structure of the rocks. It is well known that in average caving ground the caving tendency does not extend for a great distance above the tunnel unless the ground is almost completely without strength and is of the free running type. Observations in hundreds of caved sections of rock tunnels indicate that the caved area commonly extends to a height above the roof nearly equal to the width of the tunnel. If the rock is highly fractured or altered, the caved section may involve a mass of rock above the tunnel roof with a height equal to as much as twice the width of the tunnel. As the caving proceeds, the tunnel tends to assume a cross section similar to that of a Gothic arch. When the arch is finally developed, it acts to support the walls and roof of the tunnel and to prevent further collapse. Except in shallow tunnels or in tunnels in free running ground the caving effects almost never extend to the earth's surface. In supporting the average tunnel roof, then, the maximum strength of the supporting structure, even in the worst kind of caving ground, in general need not exceed that necessary to support an arch-shaped block of ground above the tunnel roof having a maximum height equal to twice the width of the tunnel. For average rock the potential dead weight on the roof of the tunnel is considerably less than in very bad ground. However, this dead weight needs support only when the rock has small strength. In massive, unaltered rocks, the potential dead weight of the roof rocks is more than compensated for by the inherent strength of the rock, the rock load is reduced to zero, and little or no support is necessary. Also, if the caving is not allowed to begin, even in bad ground, the ground may be held by a relatively weak supporting structure because the caved arch can develop only by progressive slumping starting at or near the roof of the tunnel and extending upward.



(a) (b)

Fig. 5.—Tunnel Sections Showing How Faults May Affect the Caving Tendency

The shape assumed by the arch resulting from caving is to a certain extent a function of the geologic structures in the roof rocks. In general, the height of the caved area is greater in faulted and highly fractured or altered rocks than in rocks in which these features are absent or poorly defined. Fig. 5 shows the effects of faults on the upward extension of caving ground in two selected examples from tungsten mines in Boulder County, Colorado.

In driving tunnels through caving ground special attention should be paid to the tunnel cross section. Much time and effort can be saved by arching the roof in such a manner as to utilize what strength there is in the caving rock until supporting structures can be set in place. Obviously, it is not generally feasible to drive a tunnel in which the height is twice or three times the width; but, nevertheless, it is profitable to arch the roof somewhat and, if possible, to make the arch resemble in shape the Gothic-like arch that will result from caving. In nonlayered rocks the tunnel cross section is made as symmetrical as possible; but, in layered or foliated rocks, an asymmetrical arch may serve the purpose to better advantage.

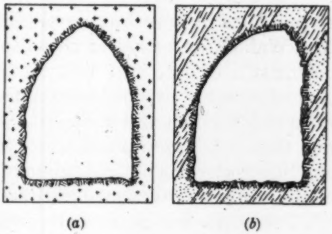


FIG. 6.—SUGGESTED TUNNEL SECTIONS FOR CAVING GROUND IN
(a) Relatively Homogeneous Unlayered Rock
(b) Inclined Layered Rocks

Fig. 6 shows two suggested tunnel cross sections for caving ground. Of course, in spite of all precautions and in spite of the maintenance of an arched cross section, some ground will continue to cave. However, if this caving tendency is offset to even a slight degree, the precautions suggested will pay off in the more rapid forward advance of the tunnel and in the need for less temporary or permanent supporting structure. Of course, if the rock in the tunnel is strong, a cross section is preserved which will be most economical considering the purpose of the tunnel and the immediate requirements of the engineers driving it.

It may be argued that altering the cross section of the tunnel in response to changes in the rocks and rock structures might in the long run cost more than attempting to preserve a standard section and taking a chance on caving ground. However, even a slight concession in the direction of tailoring the tunnel cross section to the rocks will pay dividends. Thus, from a practical point of view, the actual cross section employed necessarily must be a compromise between what is wanted by the geologist and what is desired by the engineer.

In squeezing or swelling ground a circular cross section with closely spaced circular supporting ribs appears to overcome the difficulty to best advantage.

ACKNOWLEDGMENT

The writer wishes to express his appreciation to Prof. W. O. Thompson, Department of Geology, University of Colorado, in Boulder, who read the manuscript and offered several helpful suggestions.

DISCUSSION

Berlen C. Moneymaker,[2] Assoc. M. ASCE.—The application of geology to problems in tunneling, as presented in this paper, is interesting and timely. Mr. Wahlstrom places at the disposal of the engineer and the inexperienced geologist a concise and well-written account of the manner in which certain geologic conditions affect tunneling. The author not only presents a very clear, rather elementary discussion of the characteristics of the more common rocks and their primary and secondary structures, but also indicates several geologic conditions that may affect tunneling adversely and suggests ways of eliminating or reducing the resultant difficulties.

The writer has made both preliminary and final geologic studies of many miles of tunnels, both within and without the United States. Although no issue is taken with either the author's statements (under the heading, "The Functions of the Geologist"), or the facts he states regarding rock characters and rock structures, the paper tends to convey the idea that the geologist has more to do with planning a tunnel than is actually the case. Generally, he has little to do with determining the alinement, and even less to do with determining the size of the bore and the shape of its cross section, especially if the tunnel is for the diversion of water for power, irrigation, or water supplies. In this case the engineer determines the points to be connected by a tunnel and the amount of water to be carried and then calculates the size and the shape of the cross section of the bore which best meet the requirements. In the case of highway and railroad tunnels, the engineer also determines the alinement, the size, and the shape of the cross section. The location of such proposed tunnels may be surveyed and staked before the geologist is asked to make his studies. The geologist may or may not have more to do with the planning of tunnels in mining districts, where the tunnels are for exploration, mine development, or drainage.

In 1937 the writer was asked to make a geologic study of the Garzas tunnel in Puerto Rico. This tunnel, for the diversion of water from the Garzas reservoir for power and irrigation, was already beyond the planning stage and had been surveyed. It is 11,640 ft long and extends in a north-south direction across the rugged Cordillera Central, which forms the insular divide. The ground to be penetrated had not been explored by core borings, test pits, trenches, or any other method and, apparently, no funds were available for exploration. As most of the tunnel was to be from 400 ft to 1,200 ft below the surface, sufficient exploration to yield detailed information was not feasible. There was a good exposure of bedrock at the intake portal and small outcrops in the beds of two of the several small streams crossed by the tunnel line. Elsewhere bedrock was concealed by as much as 200 ft of residuum. The entire tunnel location was covered by a dense jungle of tropical vegetation, except along the cleared parts of the survey line. Under such conditions the geologist is at an acute disadvantage. By a very careful study of the stratigraphy and the orogenic and geomorphic history of the region as a whole, the writer was able to predict accurately the nature of the geologic problems to be encountered. A painstaking study of the residuum along the survey line revealed the character of the underlying rock with a very high degree of accuracy, and a study of the only outcrops on the line revealed the location of a few faults. Although it was believed that hundreds of faults would be crossed by the tunnel, there was no method of determining their location, their magnitude, or their seriousness as tunnel problems.

---

[2] Chf. Geologist, TVA, Knoxville, Tenn.

The tunnel, a 7-ft horseshoe in cross section, was driven from the two portals and was "holed-through" near the midway point. As predicted, the northern 10,486 ft of the bore penetrated water-laid volcanic ash shales and associated limestones; the remaining 1,188-ft section penetrated massive tuff and agglomerate of andesitic composition—both types of rock being cut by numerous intrusions of andesite porphyry. Hundreds of faults were encountered, but most of them presented no serious problems. Many involved sizable zones of closely fractured rock and resulted in considerable overbreakage. These structures required timbering during construction and lining afterward. Two major faults, both apparently active, were crossed with some difficulty and after considerable delay. Both yielded copious quantities of water under considerable head. One resulted in the caving of the heading and the partial filling of the bore with 1,000 cu yd of fine gouge and 500 cu yd of broken rock. This fault zone was 152 ft wide and yielded 1,300 gal of water per min. The other fault also resulted in the failure of the heading and the partial filling of about 40 ft of the bore with broken rock and gouge. The fault zone was only 21 ft wide and yielded 200 gal of water per min under a head of 160 lb per sq in.

The location of the spillway tunnel at Garzas dam was also examined by the writer before its construction was undertaken, but after it was beyond the planning stage and after it had been surveyed. This tunnel, 1,100 ft long, was designed as a large bore of horseshoe cross section. Bedrock was exposed at the outlet portal and near the intake, but it was concealed elsewhere by a thick residuum. The ground to be penetrated was explored inadequately by a few drill holes. Geologic studies, as detailed as could be made under the circumstances, afforded no indication of serious problems, but, as the construction neared completion, a fault zone was encountered and occasioned a long delay and considerable difficulty.

The Apalachia tunnel, constructed by the Tennessee Valley Authority to divert water from the Apalachia reservoir to the powerhouse 12.6 miles downstream, is located on the south side of Hiwassee River in Polk County, Tennessee. Although the terrain to be crossed by this tunnel was thoroughly studied geologically and mapped on a scale of 1 in. to 500 ft before the project reached the planning or designing stage, geologic conditions played little or no part in its alinement and design. It is 43,637 ft long, and was driven as a bore of circular cross section, from 20 ft to 22 ft in diameter. It crosses the strike of the Great Smoky (Thunderhead), Pigeon, Citico, and Wilhite formations, which consist of altered fine to coarse clastic sediments. The geologic structure is moderately complex and is characterized mainly by folds and thrust faults. Exposures of bedrock were sufficiently large, numerous, and continuous in the vicinity of the tunnel alinement to permit accurate projection to the tunnel grade of all formation contacts and changes in lithology and in the larger structural features. Although numerous faults were encountered, no difficulty of any kind was experienced in driving the tunnel.

Another tunnel constructed by the Tennessee Valley Authority in the highly deformed metamorphic rocks of the Southern Applachians is the Ocoee No. 3 tunnel, which diverts water from the Ocoee No. 3 reservoir to the powerhouse 4.2 miles downstream. This 13,000-ft-long tunnel was driven as a bore of horseshoe cross section. From a geologic study of the surface along the survey line, all the larger geologic structures and changes in lithology were projected accurately to the tunnel grade. Even though the geologic structure was characterized by numerous folds, faults, and joints, no problems were encountered. There were a small amount of weathered rock and a few zones of shattered rock which required support, but these conditions presented no difficulties.

In all previously mentioned tunnels, and numerous others on which the writer has done geologic work, it would not have been feasible to follow some of the author's recommendations. First, the alinements were determined by engineering and other requirements. Moreover, the direction of strike and dip of the strata and foliation changed so many times in each of the tunnels that it would have been impossible to keep the bore advancing normal to such structures and as stated (under the heading, "Physical Characteristics of Consolidated Rocks") "* * * so that the layers will dip away from the headings." The rocks penetrated by these tunnels are cut by numerous joints and in most cases the most prominent joints are dip joints, which are parallel to the dip and normal to the strike. Where the most prominent joints are dip joints, a tunnel normal to the layers will be parallel to the joints That no tunnel could be driven normal to two structures which intersect at right angles is obvious. Furthermore, in spite of the advantages of altering the cross section of a tunnel, as Mr. Wahlstrom suggests (under the heading, "The Tunnel Cross Section") "in response to changes in the rocks and rock structures," it would not be practicable to do so in the case of tunnels where hydraulic considerations are important.

In large diameter tunnels, especially those constructed for use in hydroelectric projects, such features as bedding planes, flow lineation, foliation, joints, and minor faults are of relatively minor importance. Large faults and fault zones of badly weathered and weak rock, swelling ground, large flows of water under high head, and high temperatures are much more serious. In a great majority of cases, some of these conditions cannot be accurately predicted, even by experienced tunnel geologists.

In pointing out the difficulty of locating and designing tunnels so as to take full advantage of geologic conditions, the writer does not question the validity of the author's conclusions. Rock conditions and rock structures do affect tunneling precisely as he states. Although, generally, it is not possible to meet all engineering and other requirements and at the same time to locate and design the tunnel to the best advantage geologically, an experienced tunnel geologist can undoubtedly contribute substantially to the economy of funds and of time in the planning, design, and construction stages.

Robert S. Mayo,[3] **M. ASCE.**—The field that the geologist should occupy during the planning and construction phases of tunnels is clearly indicated by this excellent paper. Professor Wahlstrom's experience has apparently been gained in the deep mines of the west in connection with the driving of haulage and drainage tunnels. In this type of work, where the tunneling is generally done by company forces, the advice of a trained geologist in selecting the route for the safest and most economical tunnel is of utmost importance.

To the tunnel engineer on the eastern seacoast of the United States, such opportunities for profiting by the advice of the geologist are few. The tunnel route is fixed in most cases—the railroad engineer has his portals all staked out for him, regardless of the intervening ground; in the larger cities, the sewer engineer must run an intercepting tunnel down the center of certain streets from trunk sewers to the sewage treatment plant. Only in the rare case of a diversion or aqueduct tunnel in an unsettled area can the tunnel be shifted an appreciable amount to secure better ground.

This does not mean that the engineer cannot use the services of the geologist. The specifications prepared for the information of bidders would be of much greater value if there were a section in which a trained geologist analyzed the borings and other available information in order to forecast the types of ground to be found, the possibility of encountering faults, and how the rock will break; but this introduces an objection. The engineer has found, by sad experience, that it is rather expensive to hazard a guess to the contractor as to what type of ground he will encounter. In presenting the log of borings in the bidding documents, most engineers specifically state that the findings shown in the log are not guaranteed and that the contractor must make his own interpretation of the borings. To incorporate in the specifications a geologic monograph stating what types of ground are likely to be encountered, what difficulties might be expected, and what overbreak to anticipate, would result in a lower bid by the contractors in most cases. The courts have been extremely liberal in interpreting implied geologic data in the contractor's favor and, therefore, many engineers would be hesitant to include such forecasts in the specifications, when it might implicate the owners in expensive lawsuits, if the geologist's opinion were wrong.

This an unfortunate situation, but it does not lessen the value of geologic advice during the planning and bidding stages of a job. Consultation with the geologist would certainly enable the designing engineer to establish the "A" and "B" lines of the tunnel more intelligently. These lines govern the thickness of the concrete lining, and at present they are established by "rule of thumb."

The writer has stated that, in the past, the routes of tunnels are selected by factors other than geological ones; but, nevertheless, should the geologist indicate that extremely bad ground is to be expected, in most cases the engineer would call for a re-examination of the entire project with major relocation of the tunnels to secure better ground.

---

[3] Proprietor and Chf. Engr., Mayo Tunnel Equipment, Lancaster, Pa.

In spite of the present legal pitfalls that make it unwise for the engineer to add to the bidding specification the opinion of a trained geologist, the geologist can be of great value to the owner's engineer. By reading the daily reports of earlier tunnels, he can deduce the type of rock penetrated. He can forecast, with fair accuracy, the results of the drilling, the type of ground to be penetrated, and the difficulties (particularly "swelling ground") to be anticipated. He can be of real help in setting the "A" and "B" lines, in addition to recommending whether grouting should be specified and just what quantity of grout each type of ground will take.

The geologist can be of immense value on all tunnel projects, not only to the engineer, but to the contractor. To the end that every job can benefit from geologic science, it is recommended that, when an organization is not large enough to warrant a full-time geologist, that a consulting geologist be engaged on a part-time basis to visit the tunnel at least weekly to report on the ground and its effect on progress and overbreak. Complete engineering and geological reports on every tunnel job will be of immense value to the next generation.

F. A. NICKELL,[4] ESQ.—The feasibility of tunnels is dependent on numerous factors, and the geological conditions of the area that affect suitability of the rock are of immediate importance. Rarely, however, are special geological features pertaining to an earlier project so similar in detail that methods adopted to overcome them can be applied strictly elsewhere. Unlike engineering, founded on mathematical principles and largely concerned with materials of prescribed physical properties, geology is not an exact science. Its value to engineering projects results from skilful interpretation based on experience in construction problems. Generalization that limits discussion in technical articles leads to oversimplification and the inferences are not always applicable.

Choice of a tunnel in preference to some other structure is usually determined by economic and engineering data in which geological conditions are a consequential, but perhaps not the most significant, factor. Presumably, the plan would be altered only if insurmountable geological difficulties and greatly increased costs can be demonstrated. Nevertheless, geological proof showing superiority of one route over an alternative line, or the occurrence of questionable ground, including active fault zones, would be of major importance.

The foregoing comment leads to the identification of geological effort in engineering projects, a subject discussed in several published articles. The relationship between the engineer and the geologist has grown harmoniously on the basis that the ultimate responsibility for success of the undertaking rests with the engineer, who would want to make decisions on the best information obtainable. The geologist serves in an advisory capacity, responsible for the disclosure and practical interpretation of all pertinent features of bedrock that affect construction.

The significance of many conditions is influenced commonly by nongeological factors. For example, rock suitable for excavation of small tunnels might not stand well if the tunnel were larger. In addition, the seriousness of objectionable items beyond effect on construction must be judged from the viewpoints of repair and maintenance as well as of the magnitude of damage in case of interrupted service. This implies that the geologist ought to know and to evaluate related aspects of the problem.

---

[4] Cons. Geologist, Pasadena, Calif.; formerly Head Geologist, Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

The amount of study needed depends on the geological complexity of the area, as well as on the length and the size of the tunnel. Investigation is customarily done in stages, starting with general regional examination as a proper background in explanation of local conditions. Detailed mapping of formations along the tunnel line almost always proves valuable providing as it does facts on the distribution of rock types, the nature of doubtful features, and the geologic structure. These can then be projected into approximate position at tunnel level.

Special features are explored by pits, drifts, and core drilling, particularly in the cases of portal locations and of conspicuously bad zones. Ordinarily, the cover of rock above grade is too great for an elaborate program of core drilling down to the tunnel level. In fact, mechanical difficulties, slow progress, and diminished core recovery with depth limit the number of holes that will be drilled.

In most cases, surface mapping provides much of the required geological information on which to base plans for construction. Fracture zones, water-bearing horizons, or elevation of bedrock concealed by thick overburden can be located by one or more geophysical methods of investigation, sometimes with very accurate results.

Geological problems revolve around questions of distribution and physical character of rocks, their probable stability during excavation, and the nature of objectionable features. The fields of interest and responsibility for the geologist, engineer, and contractor may overlap. The geologist should not hesitate to give his views to focus attention on construction problems. The value of suggestions, of course, depends on experience and judgment.

Formations free of defect are probably nonexistent. From a practical standpoint, many kinds of materials can be tunneled easily. Nearly all hard, massive rocks, unless disturbed, excavate readily and stand without support. These include the crystalline group (as granite and gneiss); the indurated, fairly thick-bedded sediments (for example, sandstone and limestone); and many of the lavas, although the latter are commonly jointed and may contain bodies of poorly compacted deposits of fragmental origin.

With diminished hardness, and with consequent lower physical strength, formations of a wide variety become less able to stand unsupported. The ease of excavation is offset by problems of stability, by required changes in construction methods, and by need for protective lining or timbering. These rocks belong to or are analogous to the clayey-shaly series. Sweeping generalization is unwarranted. Adequate recognition of physical limitations will permit safe construction, presumably at a higher cost. In such cases, the geologist, besides identifying the rocks, should propose supplemental treatment to insure successful excavation. Timbering, guniting, or spraying with asphalt will protect the fresh surface.

Unconsolidated deposits—including the normal overburden on bedrock in preparing approach cuts to the portals, valley fill, and alluvial side wash; and the scarcely compacted silts, clay, and coarser debris of stream and glacial origin—commonly offer the most difficult problems. The task is simplified by accurate knowledge of the nature of loose materials and selection of suitable tunneling methods. The presence of some interstitial clay in sandy deposits, mentioned by Professor Wahlstrom, adds to stability, but good drainage and close support are usually necessary. It seems reasonable that the better informed the geologist is regarding construction methods, the more dependable will be his opinions. Underriver tunnels are practical through unstable deposits by use of caissons and compressed air; certain loose materials may be consolidated by grouting beyond the tunnel heading; and stage mining is ordinarily successful in clays (and shales), under which only a section of the tunnel face is removed at one time so that support can be replaced immediately.

Physical features that tend to control the pattern of breakage during blasting operations, or that reduce stability of the rock, deserve particular attention. Massive bodies like granite have nearly equivalent strength in every direction. As a result usually of metamorphism with increasing intensity, the rock assumes a foliate structure, passing gradationally from gneiss to schist, to slate, and finally to phyllite, progressively losing uniformity and coherence. Similar conditions arise from the inclusion of abundant platy minerals such as mica or chlorite. Physical properties are varied according to orientation, and the rocks tend to break along the banding or to slacken on exposure to air. The array of additional problems due to the banded character ranges from overbreakage to caving. Such difficulties can generally be limited by controlled excavation and support. Extremely difficult ground may be found in highly micaceous zones, in which the unconfined material will flow if wet.

The laminar structure of shale and, to a lesser extent, the bedding planes of thinly stratified sediments, facilitate parting on exposure. Initially stable rock may cave after prolonged contact with air, so that a protective asphalt coat is helpful. Tunneling methods are successful with adequate support, much in the manner described for unconsolidated materials. The difficulties increase with the diameter of the tunnel.

Joints are fairly universal in all types of rocks, occuring in sets of different orientation and importance. Weathering is facilitated on the surface by admission of water into the cracks, resulting in accentuation of the patterns. At a depth, beyond the immediate effects of atmospheric conditions, joints become tight. Dependent on prominence, joints have a varied influence on tunneling operations generally expressed by slight overbreakage guided by circumstances that permit dislodgment of rock in blocks outlined by intersecting fractures. That section of the paper describing problems related to joints may be needlessly alarming—unless the excavation is chiefly in the zone of weathered rock. It is doubtful that the alinement in any case would be changed merely to cross the joint pattern at a favorable angle. Practically all tunnel problems traceable to jointing can be remedied by timbering the bad sections and trimming loosened rock elsewhere.

Excavation and stability of formations are influenced by the trend and dip of beds in a stratified series cut by the tunnel. It is usually less troublesome to cross the bedding nearly at right angles, whereas the rock is most secure if the dip is steep or vertical. If the strata are gently inclined or horizontal, the tendency for separation of unsupported masses, with consequent caving, is comparable to the experience with fissile or closely jointed rock. Liberal usage of timbering may be required. The difficulties are greater according to the incompetency of the beds and to the size of the tunnel.

Faults are common in many regions as a result of strong movement between crustal masses. Tunnels are subject to extreme difficulty in excavation of badly mashed rock of the fault zone, particularly where the zone consists of plastic material charged with water. If a fault zone is unavoidable in alinement, the tunnel should be located to cross the feature nearly at right angles to minimize the amount of difficult operation.

The hazards involved in tunneling through faults are twofold: The actual difficulties caused by the instability of the crushed rock, and the questions introduced by the prospect of renewed movement. Excavation can be carried on by methods comparable to those used in unconsolidated materials with adequate support, perhaps by steel liners, kept abreast of the heading. In some instances, stage mining yields results. Grouting to consolidate fragmented rock might stabilize the zone sufficiently for normal tunnel operations. The broken area generally carries considerable water, which must be provided for either by grouting or by special drainage features.

In addition to the practical problems, the geologist must consider the seriousness of future disturbance, which may close the tunnel. The time needed for repair in event of rupture could be disastrous for a metropolitan water system, but might be only a calculated risk for a vehicular tunnel, from which traffic can be diverted. These factors enter into the evaluation of the geological significance.

Variation in geological conditions is endless, so particular problems will inevitably escape the discussion. Nevertheless, Professor Wahlstrom outlines many of the common difficulties encountered in excavation for tunnels. A careful geological study prior to final location and design will afford greater assurance of success, and routine inspection of the work during construction allows both a check on surface interpretation and a modification of ideas required by changed conditions underground. A permanent record of geological features disclosed by the excavation should be filed for reference in the event repairs are needed later.

R. H. Keays,[5] M. ASCE.—Greater cooperation between tunnel engineers and geologists and greater respect for the opinions of one another on problems that arise in tunnel construction should result from this valuable paper. Several instances may be mentioned where the lack of correct geological information in regard to the character of the rock has been a great disadvantage in tunnel excavation.

---

[5] Inspecting Engr., Reconstruction Finance Corp., Brooklyn-Battery Tunnel, New York, N. Y.

Reference has been made in the paper to the Moffat tunnel in Colorado. This tunnel has been outstanding in several ways, particularly so, in that the final cost was several times the engineers' preliminary estimate of cost, which was said to be based on the opinion of an eminent geologist that the rock throughout was solid. On the basis of this preliminary estimate of cost the Colorado legislature had authorized a bond issue of $6,720,000, which included the cost of electrification of the tunnel.

The writer, on taking office as chief engineer of the Moffat Tunnel Commission in July, 1923, was asked by the commission to make an estimate of the cost of the tunnel construction on the foregoing assumption, concerning the equality of the rock. No borings whatever had been made, nor had any test pits been dug to verify the statement of cost. Basing costs on past experience with many tunnels, the writer made an estimate of $5,250,000 for the tunnel work alone. This estimate was to serve as a so-called "upset" price in a proposed fixed fee form of contract. The contract was awarded in September, 1923. In the meantime, on the writer's recommendation, the west portal of the tunnel had been moved south a short distance to the only rock outcrop in the vicinity.

A geological report[6] on the tunnel, after its completion in 1928, designated the formation as that of the pre-Cambrian age. The rock itself has been defined as granite, injection gneiss, and metamorphic granite with some pegmatite, quartz monzonite, and schist for two miles east of the west portal. A badly faulted area of the same kind of rock, approximately one half mile in extent, followed. Gray metamorphic granite with quartz monzonite and injection gneiss seams was encountered the rest of the way—about 3½ miles.[7] The rock proved to be excellent on the eastern side of the continental divide, as was to be expected from the inspection of many outcrops. On the western side of the divide, however, where the rock surfaces were buried beneath unknown depths of glacial drift, the quality of the rock could hardly have been worse for tunnel driving purposes, and it was fortunate that at no place did the glacial drift extend downward to the tunnel line. An estimate of the final cost, made after completion of the work, indicated that the work on the eastern half of the tunnel had not cost more than the original estimate,[6] and that all the extra cost had been on the western half of the tunnel, which was heavily timbered, and, on which, heavy steel supports were used for short distances.

The author speaks of squeezing ground and swelling ground. By these terms it is understood that squeezing ground is mainly the result of pressure on plastic materials, and that swelling ground is that due to chemical changes or absorption of water. It is difficult to admit that the pressures that developed on the tunnel support of the Moffat tunnel for the two miles next to the west portal were due to either of these causes. All the rock was badly faulted and broken into blocks. On removal of the support given by the tunnel rock while in place, these blocks tended to slip out of place and deprive further outlying blocks of their support. It was necessary, therefore, to place tunnel support immediately after blasting and to maintain it in place, so as not to allow the rock to start moving. An easily compressible local cordwood, loblolly pine, together with some rock packing, was used to pack the space between the timber support and the rock.[8] This permitted additional movement, and no doubt was the cause of some movement which would not have occurred had better packing been used. The writer was in Europe and not connected with the work later on when the worst of the bad rock was excavated in the tunnel.

---

[6] "Completion of Moffat Tunnel of Colorado," by C. A. Betts, *Transactions*, ASCE, Vol. 95, 1931, p. 334.

[7] "Geology of the Moffat Tunnel, Colorado," by T. S. Lovering, *Transactions*, A.I.M.M.E., Vol. 76, 1928, p. 337.

Another situation wherein a geologist undoubtedly would have been of service occurred on the excavation of the Shandaken tunnel, an important feature of the Catskill Aqueduct.[9] This tunnel was excavated in Hudson River bluestone, a shaly sandstone of excellent quality, and a thin-bedded sandy red shale. These formations were bedded approximately horizontally. In this instance trouble was experienced with the thin-bedded shale which tended to cave up to a point forming a Gothic arch; this type of difficulty was described in the paper (under the heading, "The Tunnel Cross Section").

It was estimated that only a nominal stretch in this 18-mile tunnel would require timbering, but the final result was that nearly 50% of the tunnel length was timbered—practically all due to the thin-bedded shale. This is a result certainly not to be expected in a tunnel which is only 10 ft 3 in. wide and 11 ft 6 in. high inside the concrete lining. This shale was really very dangerous. Construction was carried out by hand methods, mostly using the old top heading and bench system. There were about twelve headings being excavated simultaneously, and the timbering where used had to be kept as close to the heading as possible. This prevented heavy blasting, as the timbering would have been shot out.

The permanent timbering was designed as a three-piece or five-piece set of 10-in. by 10-in. material resting on wall plates which were supported by posts. As an alternate design, a three-piece set was to be supported on hitches cut in the rock walls. Soon both schemes were shown to be impractical, as too much unprotected tunnel existed and progress was altogether too slow. It was a day's job to cut properly the two hitches necessary for one set of timber, and the men cutting them were quite unprotected from rock falls.

At this time the contractor developed a scheme whereby the three-piece set was to be used exclusively, but instead of supporting the timbers on hitches, the inclined legs of the set were each to be supported on a horizontal row of three 1½-in. steel pins set in holes drilled in the rock. On top of these three pins a wooden block was set. The inclined legs of the three-piece timber set were cut on their lower ends in such a way that they would be supported largely by being forced against the rock through wedge action. This was important, as the pins themselves were not intended to take bending moment. The design required that the timbers for each set be specially cut due to the varying width of the tunnel. Those who are familiar with recent developments in tunnel support will recognize that this pin support idea is now in general use, but not for exactly the same purpose as it was originally intended.

---

[8] "Rock Tunneling with Steel Supports," by R. V. Proctor and T. L. White, Commercial Shearing and Stamping Co., Youngstown, Ohio, 1946.

[9] "The Shandaken Tunnel," by Roy W. Gaussman, *Transactions*, ASCE, Vol. 92, 1928, p. 233.

A good example of squeezing ground was an extensive stretch of altered black chlorite schist in the 8½-mile Boyati tunnel of the Athens (Greece) water supply, constructed from 1926 to 1931. Some of the timber support in this small diameter tunnel had to be replaced as many as four times. It was finally replaced by a concrete block lining, reinforced by thoroughly grouted dry rock packing.

JACOB FELD,[10] M. ASCE.—A rock tunnel is one of the simplest engineering structures for the designer and the draftsman, but it often develops into a troublesome job for the field engineers and supervisory staff. As far as the design is concerned, the location is determined by the geometric requirements of the future use of the tunnel (whether for railroad, highway, or water) and sometimes subject to conclusions drawn from geological explorations and analysis. The actual design of the tunnel section is based on assumed empirical or extrapolated experimental values of the physical characteristics of rock, used in formulas derived from mathematical analysis which are based on the assumption of rock uniformity. That such uniformity never exists is admitted—but any assumption of nonuniformity which results in a much more complicated design analysis will probably be just as far from the actual conditions encountered as the assumption of uniformity.

In regard to construction procedure, the matters which must be determined before work is actually started are, as follows:

1. The question whether a full-face section can be drilled and mucked, and whether the progress must be based on a pilot tunnel with either longitudinal or radial enlargement, or on multiple-step benching, must be answered.
2. Having determined the answer to the first question by guesswork or by other means, the drilling support and bracing erection framework must be designed and procured on an estimated amount of use. In addition, it must be flexible enough to permit rapid modification, if the assumed conditions are not found.
3. The general pattern of drilling (and sequence of blasting) is usually the controlling item of ultimate cost in tunneling. Not only all the required excavation must be moved with a minimum of overbreakage, but the pattern must be such that it can be modified from day to day as the feel of the actual rock is obtained.
4. An amount and a type of explosive to insure economical mucking is chosen so that a minimum amount of scaling will be required and still all the rock will fall in the desired position.

If the geological advice carefully described by the author is proposed for use in the design stage, and if such advice can be of use, then, of course, the paper is of great value. However, the information presented can be obtained in greater detail in a textbook by Robert F. Legget,[11] M. ASCE, and in several others of similar, recognized quality. If the author presents his paper as advice to use when tunneling is actually in operation, most men engaged in such work will agree that the information has come too late.

---

[10] Cons. Engr., New York, N.Y.

[11] "Geology and Engineering," by Robert F. Legget, McGraw-Hill Book Co., Inc., New York, N. Y., 1940.

It is difficult for the writer to discount the impression that such statements as the author's, defining the division of work and authority between geologists and engineers (under the heading, "The Functions of the Geologist"), are parallel to the dicta one encounters in the construction field (such as the rule that only a dues-paying electrician holding a proper union card may turn on a switch). Similar advice and instruction was recently published to cover highway design and construction.[12,13]

Such a division of professional control of closely allied fields of endeavor can only lead to chaos and an end to progress. The writer received formal training in mathematics and the physical, chemical, and geological sciences; yet, the fact that he received a college degree in engineering has never reduced his interest in geology nor has it discouraged his study of that field, when related to the work at hand. In common with many other practicing engineers, the writer resents the attempt of similarly trained men, who happened to have received degrees under the sponsorship of geology or some other academic faculty, to circumscribe, or in any way limit, the field of his endeavors. In parallel fashion, he considers that men similarly trained can be of service in the engineering field, even though they do not have engineering degrees.

The main part of the paper (under the heading, "Physical Characteristics of Consolidated Rock") contains informative data and conclusions. It should, however, be read by the engineer with the reminder that it is not written by an engineer. For instance, the statement, "Certain rocks are massive or isotropic," indicates that the two descriptions are synonymous—a policy which is not in accordance with recognized engineering terminology. Certainly, anyone who has worked in seamy granite which was massive did not find any isotropic characteristics.

Fig. 6 summarizes the recommendations of tunnel cross sections in caving rock. In Fig. 6(*a*), apparently a uniform caving material is pictured; any shape can be cut through if properly timbered and no shape is safe if it is not timbered. If the indication is intended to represent a uniform rock with no cleavage or seams, then any shape can be cut without timbering. In Fig. 6(*b*), the recommended shape will result after the blasting and the scaling operations are complete, no matter what pattern of drilling is followed; such a condition is immediately taken advantage of in the next round of drilling.

Finally, every tunnel engineer should carefully read and heed the concluding sentence of the paper:

> "In squeezing or swelling ground a circular cross section with closely spaced circular supporting ribs appears to overcome the difficulty to best advantage."

If the reader has been through a similar operation, he should take his own experience preferably as a guide and install any shape of bracing as fast as possible. For, in such ground, it is not feasible to excavate a full section, circular or otherwise, and to expect to place the recommended closely spaced circular supporting ribs.

---

[12] "Geology and Highway Engineering," by M. T. Huntting, *Transactions*, ASCE, Vol. 110, 1945, p. 271.

[13] "Engineering Geology and Highway Location and Design," by L. W. Currier, Daniel Linehan, and George H. Delano, presented at a meeting of the Highway Research Board, National Research Council, Washington, D. C., December, 1947.

Thomas W. Fluhr,[14] Assoc. M. ASCE.—In spite of the fact that the works of the engineer are founded on, or pass through, the materials of the earth's crust, papers on geology seldom appear in engineering publications, and too few engineers have sufficient acquaintance with the subsoil and bedrock that concern their projects so vitally. The subsoil and bedrock vary to an extreme degree in quality and condition. The engineer designs his steel and concrete structures with the indicated degree of precision and then founds them on natural materials, the quality and condition of which are known only approximately and, unfortunately, in some cases not at all. This paper therefore serves an extremely useful purpose in that it delineates some of the geological features encountered in tunneling in such a manner as to be easily comprehended by an engineer, and in such form that he may make practical use of them in planning and construction. Since the kinds of rock and soil and their characteristics vary considerably in different locations, opinions may differ on the emphasis to be placed on various geologic features.

The engineering geologist is most valuable to the engineer prior to the beginning of construction. Reconnaissance of a proposed tunnel line often is sufficient to discover the critical points of the line. Geologic mapping and structural studies will permit the geologist to predict subsurface conditions and anticipate difficulties with greater assurance. Programs of test borings, test pits, or other methods of exploration can be devised and their progress guided by geological considerations.

When the subsurface features have been explored, the engineer and the geologist, working together, may find it possible to shift a tunnel line to advantage. For example, one section of the Delaware aqueduct tunnel line was shifted laterally to avoid passing through the intersection of two fault zones. In another case, part of the tunnel was placed much deeper than planned, and thus decayed rock present at the original level was successfully avoided. However, such shifts in position can seldom be made, because they usually involve increasing the tunnel length which results in more expense than would crossing bad ground.

The engineering geologist, by predicting the location, quality, and extent of bad tunneling ground, makes it possible to plan for such eventualities when cost estimates are prepared. His reports are frequently of great service to contractors bidding on proposed work. Geologic forecasts should not be included in specifications. On the other hand, the proper manner of presentation of geologic data is of advantage both to the engineer and to his client, and to the contractor, and the withholding of such geologic data as are available may be construed by a court as evidence of bad faith. One practical aspect of making geological data available to the contractor is the possible elimination of future claims.

Once construction has begun, the geologist can furnish advisory service to the engineer regarding tunneling conditions and can record the geology of the tunnel for future reference, but unless wet or caving ground or serious fault zones are encountered, he can do little to change the course of events.

It would appear that in the paper too much weight has been placed on "flow layering" of igneous rocks. The directions and intensity of the jointing are of much greater importance. Not everyone will agree that a greater amount of overbreak occurs in tunnels driven normal to the foliation or bedding. The overbreak depends to a great extent on the jointing, as well as on the manner of drilling and shooting. Similarly, the problem of driving tunnels so that the layers dip away from the heading is of little practical importance, and can be taken care of by a skilful heading boss.

---

[14] Senior Geologist, Board of Water Supply, City of New York, Downsville, N. Y.

One point emphasized by Mr. Wahlstrom is the effect of ground water on tunneling. In the eastern section of the United States, the writer has observed that less difficulty is usually encountered with the badly decayed central crush zones of faults, than with the jointed and water-bearing zones on each side of the decayed zone. Control of water is of first importance both in hard rock and soft ground tunneling. In general, when an inflow of water is encountered in a tunnel, the average engineer tries to shut it off by grouting, usually without success. Most inflows diminish or disappear if allowed to flow; if shut off, the hydrostatic pressure builds up to that corresponding to the ground-water level, and eventually the water breaks through elsewhere.

When a water-bearing fault, sometimes accompanied by a run-in of soft material is encountered, the contractor, anxious for progress, often feels that he has to take some immediate action, and it is difficult, if not impossible, for the geologist to persuade him to wait until the water has drained off and the soft ground has become stabilized. One must recognize that in underground, as well as in living bodies, nature tries to heal her wounds.

Modification of the cross section of a tunnel to conform with the geologic structure is theoretically desirable, as Mr. Wahlstrom points out. However, tunnel driving today is a highly systematized procedure, and any change in section is expensive in terms of time and progress. Moreover, since a large proportion of tunnels are concrete lined, changing the section would necessitate the use of special and expensive forms for lining, or else the use of excessive amounts of concrete to fill the additional excavation.

One problem, not mentioned by the author, is that of popping or bursting rock, which often necessitates the use of roof support.

Mr. Wahlstrom is to be commended for having presented his material in a form which is simple and direct, as well as technically correct, thus enhancing its value to those engaged in tunneling work.

PORTLAND P. FOX,[15] Assoc. M. ASCE.—Many civil engineers will find this paper interesting and instructive, but experienced engineering geologists will feel that it is somewhat academic. The civil engineer will have difficulty in interpreting many of the geologic structures from outcrops, drill cores, and other explorations. In many regions there are no rock outcrops from which a geologist or engineer can determine the character of the joints and faults and only information obtained from core drill holes will be practical or possible.

If a core drill hole was drilled through faults or fault zones as represented in Fig. 4, few geologists or engineers would be able to recognize the fault (much less appraise its significance) unless a nearly perfect core recovery was obtained, which is rare in such zones. Such fault zones could be represented by a few feet of core loss and pass completely unsuspected until encountered in the tunnel. Geologists may easily classify the faults after they are encountered in the tunnel, but it is difficult to do it before.

---

[15] **Chf. Geologist, Brazilian Traction, Power & Light Co., Ltd., São Paulo, Brazil.**

The author states (under the heading, "Physical Characteristics of Consolidated Rocks") in discussing the relative merits of the geologic structure represented in Figs. 2(*a*) and 2(*b*) that "the latter condition should be avoided if possible." The writer has found, in tunneling in the Brazilian Archean complex, that the structural conditions represented in Fig. 2(*a*) will cause much more overbreakage and many more tunneling problems than those shown in Fig. 2(*b*) if the dip joints are well developed and if there is slight weathering along them. This is especially true if the dip joints strike at an acute angle with the tunnel.

It is not practical in the planning and construction of most tunnels for civil engineering works to aline the tunnel to suit best the geologic structures. The writer assisted in the planning and exploration of the 7,100-ft tunnel for the relocation of the Chicago, Burlington and Quincy Railroad Company in the Wind River Canyon in Wyoming for the construction of Boysen Dam. A detailed geologic study was made of the area and it was immediately realized that several large fault zones would be encountered by the tunnel, but even so the tunnel was more feasible than the alternates.

The author states (under the heading, "The Tunnel Cross Section"):

> "Much time and effort can be saved by arching the roof in such a manner as to utilize what strength there is in the caving rock until supporting structures can be set in place."

Generally speaking, this would not be practical because it would involve a different tunnel section in caving and noncaving ground, and, if the tunnel had to have a permanent concrete lining, the forming of such an arch would be objectionable.

The writer has usually found that coarse-grained rocks alter more rapidly than fine-grained rocks, contrary to the author's statement (under the heading, "Altered Rocks and Swelling Ground"). For instance, in the Wind River Canyon in Wyoming the fine-grained granite gneisses underlying the Flathead sandstone are rarely altered, but the coarse-grained pink granites are usually weathered several feet below the sandstone. Pegmatites are usually more deeply weathered in Brazil than are the adjoining finer-grained wall rocks.

HYDE FORBES,[16] M. ASCE.—This well-considered outline of the application of geology to tunneling problems is directed more to the student geologist or to the geologist employed in such work than it is to the engineer responsible for design and construction. It is to be hoped, therefore, that discussion of this comprehensive paper will yield field experiences so that the engineer, although possibly not fully able to recognize the rock conditions with which he is confronted, will know when to seek the advice of a geologist who is able to do so, and learn how to appreciate the importance of obtaining a geologic interpretation.

---

[16] Cons. Engr. and Geologist, Palo Alto, Calif.

It is true that the preliminary investigation and the actual construction of tunnels involve a most direct and detailed application of the science of geology within the field of activity of the engineering geologist. Unfortunately, however, this is usually limited to the preliminary investigation of ground conditions by interpretation of diamond drill cores, or as revealed through shaft and other type exploration. From these all too limited "peeks" into the subsurface the geologist is supposed to give the engineer a complete geologic cross section of the formations to be bored, and geologic interpretation or determination of the character of the ground as to the occurrence or nonoccurrence of fractured, seamy, or noncoherent rock; water; and gases from point to point. In addition, he must solve the practical problem of the load to be carried by the tunnel support or lining, and the magnitude and distribution of ground or water pressures on which to predicate plans and specifications for the tunnel.

Excavation for tunnels through sound, hard rock has not presented any exceptional features. The section generally can be taken out with a center top heading and one or more benches (depending on the height and width of the section heading) with the spacing and number of drill holes depending on the hardness of the rock. The engineer achieves a confidence in his ability to judge and handle rock. It is, therefore, well to place emphasis on the necessity for "full coverage" through the employment of a geologist throughout the life of a job. In nearly every length of tunnel some modifications of the rock due to geologic processes are to be expected; these may be enumerated as follows:

(1) Heavy ground, wherein hard but "blocky" rock exists with the blocks of hard rock separated by bedding planes, planes of schistocity, and fractures or joints opened either through geologic processes in the zone of surface weathering or through faulting and intense folding, or through blasting in the tunnel itself, subject to fall if not supported;

(2) Seamy ground, wherein the rock walls of fracture or other planes are found separated by varying thicknesses of the products of rock decomposition, produced by the action of water and in which water is encountered;

(3) Swelling ground of fault zones and that due to chemical or physical action when the bore exposes the rock to contact with air and moisture and when it removes retention and support; and

(4) Flowing ground, which includes plastic clay of high moisture content and saturated sand.

Designers and constructors can cope with these types of ground by methods which have been developed through experience. It is the function of the engineering geologist to interpret the field conditions and to provide working data of value for design and construction in the event that these forms of rock weakness, disintegration, and decomposition can be anticipated or found to occur after construction starts.

In discussing the physical characteristics of consolidated rock the author presents a picture too involved with geologic terminology to mean much to the engineer. The writer's contact with the profession has shown him that the civil engineer studied the prescribed geology courses in college, and that the professor generally maintained the position that the science is an end in itself, rather than that it is of extreme value to the engineer as a means to an end; and so the student understood and retained little of the "book thrown at him." The same thing applies to a general discussion of rock structure. Generalities on the subject are of little value and it is difficult to state which particulars have wide application or avoid being misleading if not correctly identified. The writer has found that when rock cores show slickensides when broken, or swell when they are removed from the core barrel, it is an indication that actively swelling ground will be encountered in the tunnel bore. Fractures of any kind that carry gouge seams or oxidized products with more than a minimum thickness require strong support, and probably will carry water when pierced by the bore. The clayey gouge of fault zones and moist kaolinized (clayey) walls of faults and fractures act as viscous fluids under the load pressures produced by overburden. These materials are seldom recovered in the core.

Actively swelling ground due to chemical causes depends on the combination of minerals in the rock with the chemically active substances in the air. Air slacking is the general term used to cover oxidation, carbonization, and hydration. Much can be predetermined in that regard by tests on rock cores. The difference between the density of a rock upon its removal from the core barrel, and after it has stood several days exposed to the oxygen and carbon dioxide of the air, indicates the volume of swell taking place. The action of some rocks upon submergence in water is an indication of the amount of hydration to which they will be subject upon contact with moisture. Swelling ground results from the integration of minute slippages between extremely small fragments or crystals in rock material. This may be due to hydration, as in serpentinization, resulting in a slickensided rock. In driving a tunnel through serpentine under Parker Avenue (San Francisco, Calif.) blasting was necessary to remove the massive rock, but within a short time the further hydration of the rock produced clay which could be scraped from the walls and roof to the extent that the workers classed all the material as clay.

Swelling results from pressure (essentially hydrostatic) in fault zones characterized by crushed and ground-up rock (gouge). The outlet tunnels from the San Andreas reservoir of the San Francisco water department are bored through the San Andreas fault zone, the most extensive and active rift in California. The material making up the zone is gouge and breccia, the latter consisting of blocks of unground rock (called boulders by the miners), which are segments of the original rock that have slid in the finely ground gouge in adjustment to the pressures and have suffered only compression. The gouge is made up of a multitude of minute dry particles, between which slippage occurred as soon as the bore pierced it. The initial swell amounted to as much as 8 in. and the working material developed slickensided planes in the wall and at the contact between gouge and breccia. Swell continued for 36 hours as the pressure acting on the gouge at tunnel level was gradually relieved by the shearing stresses or "arching" developed in the gouge, until a new condition of pressure equilibrium was achieved in the mass around the bore. The material was cut back to line when the swelling ceased and the load was successfully carried by the timbers. When the tunnel was completed and approved for lining, blasting in a near-by quarry regenerated movement as the stress-strain equilibrium was upset by shock waves. The result was a new "working" of the material; new slickensided planes developed, and breccia boulders fell from the roof as shown in Fig. 7.

Rock falls and excessive overbreak due to blocky ground, characteristic of the weathered zone of all formations as well as where folding and faulting has been geologically recent and intense, are well described by the author (under the heading, "Folds, Joints, and Faults"). This can be foretold through a surface geological survey and mapping. A case in point was the Broadway Low Level tunnel in Oakland, Calif., approaching and through the Wild Cat fault zone. The rock formation exposed in road cuts was sheared to the extent that its original structure was hardly recognizable. This led to the conclusion that overbreak in the tunnel would be excessive and rock falls would be a common occurrence.

In this regard, as well as in geologic mapping during the construction of a tunnel, it is well to place emphasis on the employment of a geologist throughout the life of the job. The writer wishes to underscore the author's statement (under the heading, "The Functions of Geologist"):

> "The importance of a detailed geologic map of the tunnel cannot be overemphasized. This map not only should show the geologic structures and rocks encountered, but it also should serve as a lasting record to indicate the location of bad sections, which may require attention many years after the tunnel is completed."

An example of neglecting to provide such a map is found in the Coast Range tunnels of the Hetch Hetchy aqueduct system of San Francisco; nor is the writer aware of such a record for other important tunnels bored in California in the past twenty years.

As to the location of a tunnel, the geologist is rightly limited to the selection of the best of alternate locations which will serve the proposed engineering requirements, and frequently there is no alternative in relation to the purpose of the tunnel. It is seldom possible to change the direction of the tunnel to meet changes in rock structure as the author suggests, but the author briefly mentions (under the heading, "Folds, Joints, and Faults") that the geologist can advise in connection with the important feature of rock structure in relation to blasting. Forty years ago during precollege and college vacations the writer worked in California gold mines. Then, much was learned of tunneling from the experienced, practical miner, but the knowledge of rock and the application made of that knowledge, which was most interesting and impressive to this student of geology, was found in studying the rock to determine its "cleavage." The location, number, depth, and powder charge of drill holes were dictated by this cleavage, and the foot or hanging wall of waste rock would be cleanly stripped from the gold bearing quartz vein without fracturing or damaging the vein in any way. After the tunnel was mucked out, the vein would be drilled and shot down on canvas without breaking into or carrying down waste rock.

The writer has found it difficult in his professional work to impress upon engineers the idea that most rock has incipient cleavage, because of compressive stresses to which it has been subjected in crustal movement, or swell of adjacent igneous masses in geochemical change. In addition, bedding planes of sedimentary formations, planes of schistocity, or other structural features of rock character contribute prominent lines along which rock cleaves. These cleavages should be given consideration as a guide for using powder to break the rock. In considering the dam sites for the State Wide Plan of California (later the Central Valley Project) from 1928 to 1930, the writer endeavored to present specifications covering stripping of the sites. The reaction is best

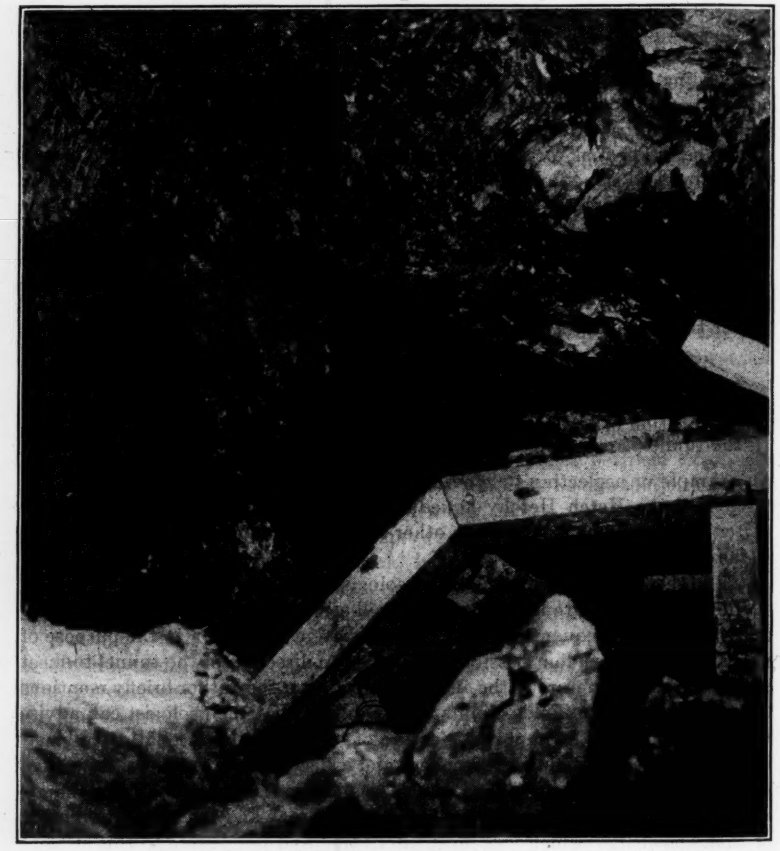


FIG. 7.—ROOF AND WALL OF SAN ANDREAS OUTLET TUNNEL AFTER QUARRY BLASTING

recorded in the remark of one consulting engineer that a contractor will not strip a dam site with a nut pick.

In connection with the sites selected in 1934 for debris dams on the Yuba, Bear, and American rivers to be constructed by the United States Army Engineers in California, the writer presented a definite stripping pattern on maps of the sites, meeting the cleavage of the rock and utilizing the banding due to compressive stress so that a minimum of rock had to be removed to provide a sound, unyielding, strong, watertight foundation and abutment. This was not understood, nor was its significance appreciated by the designing engineer, and it was unwelcome as an unwarranted trespass by the geologist into matters within the prerogative of the designer.

The ruination of good dam sites and the damage to tunnels observed by the writer have been largely due to "poor rock" caused by improper blasting. Much rock in California does not require blasting with the modern excavation equipment now available. In a recent rock excavation project in a built-up area in San Francisco the writer prescribed slopes to meet the rock conditions, based on boring and sampling, and specified that no blasting was required. It was believed by the engineers that bids could not be obtained on that basis, and a compromise was reached permitting the drilling of holes no deeper than 3 ft and loading with not more than 2 lb of powder. The contract is nearing completion without resorting to this limited use of blasting, except in the case of two large rock masses, one rolling down the slope, and another projecting above finished grade.

The observation of the diamond drill operations and the study of (and tests on) core recovered in connection with the prospective Broadway tunnel in San Francisco led to the recommendation that a tunnel be driven without blasting. The "chittering" of the drill so shattered the rock cores that less than 25% core recovery was effected. The recovered cores had a compressive strength ranging from 530 lb to 1,750 lb per sq in. when tested unconfined, but they shattered to small fragments under a percussive force of from only 5 ft-lb to 9 ft-lb. It was clearly shown that cleavages were so numerous that no piece of rock larger than a 1-in. cube would resist a sharp hammer blow. An important function of the geologist is to determine the presence of lines of structural weakness in rock and the methods of treating them in the process of tunneling, as well as in other excavation. It will take considerable persuasion to convince the engineering profession, if the writer's experience is any criterion.

The author states that the prediction of ground-water conditions in tunnels is one of the most baffling problems confronting the geologist. Experience will soon teach the observer that much can be predetermined in relation to the source, occurrence, and effect of water in tunneling. Tunnels driven under lakes, rivers, or other bodies of surface water can be expected to tap considerable volume of flow. Those driven under regions in which wells or springs are used by surface landowners for water supply purposes are found to dry up those sources, and tunnels driven into some formations release highly mineralized stored ground water that contaminates streams into which the tunnel drainage flows. Surface mapping of all water sources in a considerable area on both sides of the tunnel line should be carried out, and the probable water table beneath the area should be plotted. The relation of water bodies or courses to faults or other geologic structural features should be noted on the map. During exploration, drilling notes made of the depths at which water is lost or rises in the drill hole are of value, and measurements of water level in the hole at the end of each day's run and before the next run, to determine the loss or gain, will be found helpful.

Some water troubles occur where fissures filled with water are encountered. These fissures may range in width from hair thickness to several inches, and the water pressures may be the full hydrostatic head to the surface at the highest point of the water table. The Santa Barbara tunnel in California was bored through almost vertical dipping, sedimentary rock beds and water was encountered along bedding planes that had been opened by blast shock; the water pressures were sufficient to cause the rock to "pop" dangerously (fatal to several workmen) before the water seam was reached by the bore.

The city tunnel of Santa Barbara was driven a total of 18,000 ft through sandstone and shale formations characteristic of the Coast Range of California. Because the rock was not of a porous nature, it would have yielded little water except that, as the work progressed, the incipient, closed fractures (due to distortion suffered by the formation in crustal movement) and bedding planes were made crevices through blasting. These opened to the ground surface and filled with water from rainfall and soil water percolation. The tunnel portal is at El. 1500; the range rises to El. 5000. The writer made a survey of the springs that had been dried up by drainage through the tunnel in 1915 and found that the drained area extended approximately 3 miles east and west of the tunnel section at the crest of the range, in the form of an inverted cone with the tunnel at the apex. The tunnel was lined with concrete, but, because of water pressures, weep holes were installed, and the tunnel developed a considerable inflow—ranging up to 250 gal per min with the amount of seasonal rainfall. The flow from the tunnel, since used for city water supply, was fairly constant in that the sections nearer the portals discharged the greatest amount soon after (and during) the winter rains, whereas the flow from sections at greater depth below ground surface increased as that at the portal sections diminished.

The same experience was encountered in the boring of the Lafayette tunnel of the East Bay Municipal Utility District of Oakland, through a geologic formation of the same character. There, wells of dairies located on the ridge were deprived of water, whereas formerly the wells, more than 100 ft in depth, had a water level always standing between 10 ft and 20 ft from the surface—about 800 ft above sea level. The tunnel at El. 370 passed about 1,500 ft south of one well and down the dip of the strata from it, and the blasting within the bore opened the bedding planes beyond the well, as indicated by the drainage of water from the formation.

Similar conditions were brought about in driving the Mono tunnel of the Los Angeles (Calif.) Aqueduct in 1935. The formation through which the tunnel was driven, and from which springs issued at the surface, was volcanic in origin, varying greatly in mineral composition, and occurring as beds of lava, tuff, and agglomerate. It was dislocated by faulting, and the members of the volcanic series had a decided dip from their original horizontal position so that the tunnel penetrated them in succession. The bore, 9 ft by 10 ft, was driven approximately 300 ft below the spring area. Blasting in the tunnel opened the fractures and contact planes in the various formations and drained the water into the tunnel at rates in excess of 100 gal per min, drying up the spring area.

The effect of opening planes of schistocity by blasting was found when a state highway was carried over a tunnel of the Western Pacific Railroad near Keddie, Calif. The tunnel was bored and lined and had been in use several years. The blasting of the rock for the highway cut over the tunnel opened the formation along the planes so that segments of the rock were loosened and slid along the self-lubricating micaceous schist, buckling the tunnel lining. The character of the rock and the dip of the planes directly over the tunnel are shown in Fig. 8. In this instance, lubrication of the sliding plane through moisture was not required to facilitate movement. In tunnels, however, most rock slips occur when clayey or talclike alteration products resulting from the action of ground water on the rock and filling joints, bedding, or fracture planes, or those making up the gouge of fault seams or walls of joints, are softened when tunneling operations allow water to move down the slightly opened crevice.

The author suggests the alteration of the tunnel section to meet the caving of the walls or roof of a tunnel. The walls and roof of the tunnel become unstable due to conditions brought about by the tunnel, after it has been bored through the troublesome formation, and little can be predetermined which would indicate the necessity for a change in cross section, even if the engineering requirements should allow such change. Moist clayey decomposition products form a consequential part of a thoroughly fractured rock mass through which

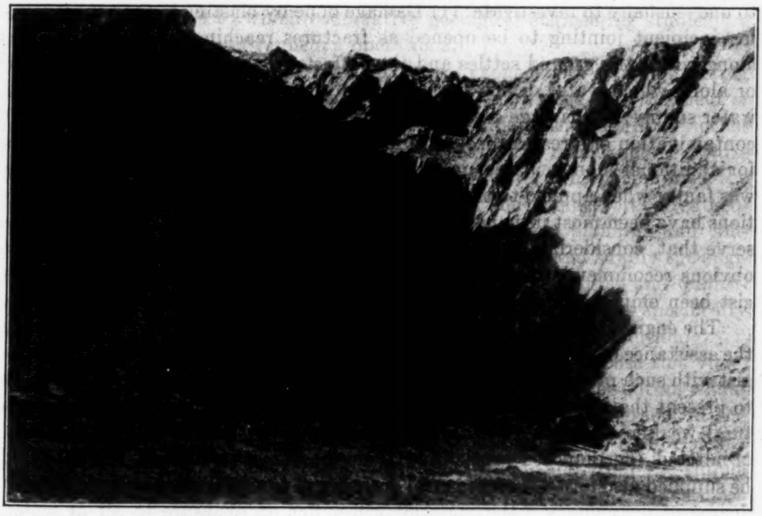


FIG. 8.—FACE OF HIGHWAY CUT EXCAVATION OVER TUNNEL

water has circulated, and the mass will be incapable of withstanding even moderate stresses when restraint is removed by the bore. Faults that carry gouge seams more than a few inches thick in humid regions are usually found to have carried water, even if no water is found in the preliminary drill hole. The rock hydrates to the extent of taking up the water trapped in the fractures, and the presence of water is not revealed. Strong timbers will be required to meet the imposed forces, similar to hydrostatic pressure, as the clay gouge and kaolinized walls of a fault or fracture will act as viscous fluids under the pressures at tunnel level regardless of the cross section.

The author states (under the heading, "The Functions of the Geologist"), that in relation to cooperation between the engineer and the geologist:

> "* * * this situation is gradually becoming less serious and the engineer and the geologist are reaching an amicable understanding based on mutual respect and profitable cooperation."

This is true among the salaried employees or civil service employees, in large corporations and governmental agencies. Nevertheless, the "awareness" of the importance of geological advice from a consultant of broad experience is not so well developed in the administrative head or the engineer in responsible charge. The writer's experience is that he has been employed after a tunnel is constructed, rather than before and during construction, on a ratio of five to one—usually to investigate: (1) Damage done by blasting in the tunnel causing incipient jointing to be opened as fractures reaching to ground surface, along which the ground settles and thus affects the foundations of structures, or along which water drains to the tunnel depriving overlying lands of their water supply; (2) damage done to tunnels due to blasting in their vicinity; (3) contamination of streams through tunnel drainage; and (4) contractor's claims for additional money, based upon the contention that the engineering design was faulty when applied to the particular ground conditions. These investigations have been most instructive, but have also afforded the opportunity to observe that, considering the money involved in the project, only one of several obvious recommendations was needed, had an experienced engineering geologist been employed, to save the cost of the geologist's fee several times over.

The engineer in responsible charge of tunnel projects should insist on having the assistance of an engineering geologist. In turn, he should provide the geologist with such preliminary exploration as he recommends and believes necessary to present the occurrence or nonoccurrence of consequential fault zones, structural weaknesses in the form of extensively jointed, fractured, disintegrated, or altered (decomposed) rock, and their locations. Sufficient funds should be supplied also for tests on material occuring at tunnel level to determine what degree of decrepitation the material will suffer on exposure to air or water. Provision should also be made to take sufficient rock samples or cores representative of the rock profile above the tunnel, from which the pressures at the tunnel arch can be determined, as well as to ascertain such properties of the rock as elastic expansion upon release from restraint, swell, compressive strength, and resistance to percussive force; and such observations and tests should be progressively carried on during the driving of the tunnel as a basis for current recommendations and final record. All these are well worthwhile, to the end that safe and economical tunnel design and construction will be better founded on fact.

ROGER RHOADES,[17] ESQ., A. B. REEVES,[18] M. ASCE, AND W. H. IRWIN,[19] ESQ.—It is now accepted as a truism that geology can contribute helpfully to civil engineering. The author has restated some aspects of the geologist's role in tunneling, has described or mentioned several basic geologic principles which may be applied profitably to tunneling operations, and has alluded to a few specific examples of such application.

The object of this discussion is not to disagree with either the general thesis or the specific content of the paper; it is written soundly, from the geologic standpoint, and, if tunnel engineers uninitiated in geologic matters obtain from it a more sympathetic understanding of the medium—the rocks and soil—with which they work, the author's objective has presumably been achieved, and a useful purpose has been served. However, a large and complex subject is involved and the author, constrained to brevity, has dealt only with the basic fundamentals. The purpose of this discussion is to point out the difficulty of compressing a subject so large into a presentation so brief, and to warn of the danger that the uninitiated—among both engineers and geologists—may fail to comprehend the full extent of the potential contribution of geology to tunneling and the degree to which cooperative geological-engineering studies are being applied currently to tunneling operations.

The extensive and increasing participation of geologists in civil engineering operations in general is now a matter of common knowledge. In the field of tunneling this geologist-engineer partnership should be closer than in any other kind of civil engineering because the problems met by the tunnel engineer arise almost wholly from the rocks, soil, and ground water which comprise the geologist's special field of professional interest. The engineer requires geological information for the most effective and economical planning and construction of tunnels; the geologist is challenged to procure and supply the engineer with the information he needs, in terms that are as precise and quantitative as possible—not in the form of generalizations—and in terms specifically applicable to the engineering problem and directly useful to its solution.

Nothing in Mr. Wahlstrom's paper can be construed to oppose this viewpoint; it may be inferred from the paper's general context that the author would agree fully. However, the paper is brief and is purposely concerned, in the main, with elementary fundamentals; it would be easy to derive the erroneous conclusion (from the few concepts and examples cited, and from the simple geological principles described) that the scope of geology in tunneling is more limited than is actually the case. In reality, the applications of geology to tunneling are manifold and complex; its application to tunneling problems and operations is a matter of long standing, although it has become greatly expanded in the last one or two decades.

---

[17] Chf. Geologist, U. S. Bureau of Reclamation, Denver, Colo.

[18] Head, Structures and Tunnels Section, Canals Design Div., U. S. Bureau of Reclamation, Denver Colo.

[19] Asst. Chf. Geologist, U. S. Bureau of Reclamation, Denver, Colo.

The paper describes, for example, the association of soft or squeezing ground with the geological processes of chemical alteration, and the "arching" effect of caving ground. Nevertheless, if an uninitiated reader is to be protected from gaining the mistaken impression that the geologist can supply no more than such academic explanations of tunneling problems, allusion must be made to other important considerations. For instance, it should be mentioned at least that the mechanics of such matters have been comprehensively analyzed by engineering geologists and engineers. Through these analyses such geological phenomena have been translated into specific stress-strain interpretations, and the interpretations, in turn, have been applied quantitatively to answer such questions as: How much and what kind of support will be required? Will permanent lining be required—what kind and how strong?

Such questions may seem to some to be beyond the proper sphere of a geologist's interest, but it is essential in any description of the relationship of geology to engineering to point out that the engineering and geological aspects of these problems are merged and interlocking and cannot be disentangled. The engineer must, of course, retain responsibility for the decision as to what kind of structure is most appropriate, and how it is to be built.

The geologist is responsible for interpreting the geologic conditions to which the structure must be adapted, and for defining the problems which these conditions will present. The geologist's data and interpretations cannot be pertinent and directly useful unless he has comprehended fully the engineering viewpoint and objective, the design alternatives, and the construction procedures that are available for use—he must understand what can and what cannot be done. The geologist must be conversant at least with the technical facilities and limitations of engineering procedure if the information he obtains is to be of any practical value to the engineer. Similarly, the geologic studies must be intimately coordinated with the economic considerations, which, in the long run, dictate what can or cannot be done within the controlling limits of cost and value.

Simply stated, the geologic studies must be practical as well as scientifically sound. Thus, although the engineering geologist must recognize that the ease or difficulty of tunnel driving may be controlled to a large extent by the relationship between the direction of the tunnel and the direction and inclination of the schistosity, foliation, or stratification of the rocks (as succinctly described in the paper), he must also recognize that, in practice, it is only occasionally practicable to vary the course of a tunnel very much from a predetermined line to take advantage of such factors. Similarly, although the theoretical advantage of skewed or asymmetric tunnel sections shaped to conform with the structure of the rock is well described in the paper, the geologist should be equally aware of possible, serious engineering disadvantages to such shapes.

These disadvantages, even though commonly of negligible importance in mining tunnels, are frequently critical in the tunnels of the civil engineer, where the shape is often predetermined and controlled by factors directly related to the function of the tunnel. For example, in unlined tunnels designed to carry water, overbreak or irregular shape may be seriously detrimental to hydraulic efficiency, and may even so reduce the efficiency that a larger tunnel diameter is required with attendant increased costs. If the tunnel is to be lined, and has been designed with a shape and diameter controlled by the capacity requirements, any overbreak represents so much rock needlessly removed and replaced by expensive concrete and steel. The geologist cannot remain ignorant of these considerations but must take them into account specifically if the geologic studies are to be of maximum practical use to the engineer.

Although the paper acknowledges the desirability of applying geology to actual tunneling operations and recommends periodic geologic examinations and the assembly of geologic records as the work progresses, the reader may get the impression that the author is primarily concerned with the exploration of prospective sites. The experience of the United States Bureau of Reclamation attests to the value of thorough explorations continuously coordinated with the progress of design studies; on the basis of that experience—not only with tunnels, but also on engineering undertakings in general—one could recommend even stronger emphasis on drilling, ground-water testing, geophysical explorations, and the application of any other technique for searching beneath the earth's surface and securing useful reliable data bearing on the engineering plan.

It is difficult to recall a structure site that has been "overexplored"; money spent on judicious exploration is inevitably saved many times over in cheaper and more efficient design and construction. However, the applications of geology during construction are at least as important and, in some respects, more productive of useful results. The preceding comments have stressed the necessity for quantitative geological determinations, but it is recognized that this quantitative ideal is usually unattainable during the stage of exploration—no matter how many holes are drilled, or how exhaustively routine and special investigational techniques are applied. At best, explorations serve only to reduce the guesswork and increase the assurance of advance predictions.

During construction, when the rock is exposed for precise study and when the water flow can be measured and interpreted, the measurements, observations, and tests necessary for quantitative appraisals can be made. Moreover, when precise studies made during construction (particularly if compared closely with the predictions made during the explorations) become the subject of comprehensive records, they serve as the principal method by which the procedures of tomorrow may profit by the experience of yesterday and today. The literatures of both engineering and geology need to be augmented by more "case histories," carefully and fully described.

The paper suffers from two important omissions: No mention is made of the problems of (1) residual rock stress, so important in relation to rock bursts and "heaving ground," which is sometimes critical to the design of tunnel linings, and (2) rock temperature, so important because of its technical and economic implications with respect to ventilation. The extent of recent researches in both problems attests to the current interest of engineers and geologists in these important considerations. Related minor or special problems, such as the occurrence of gas or hot water, should be mentioned even though space limitations have prevented their thorough discussion by both Mr. Wahlstrom and the writers.

There is a dearth of critical, fundamental discussions of the geological principles applicable to tunneling problems. Such data and written discussion as may apply are widely scattered and not easily accessible, but papers of the type presented by Mr. Wahlstrom represent an encouraging step toward the rectification of that deficiency, and deserve encouragement.

Engineers and geologists desiring a more extended and analytical treatment of some aspects of tunnel geology and related construction problems may refer profitably to a treatment prepared by Karl Terzaghi,[20] Hon. M. ASCE.

Ernest E. Wahlstrom,[21] **Esq.**—The carefully considered discussion by several well-qualified experts emphasizes the diversity of opinion as to the exact functions of the geologist on engineering projects. This is only natural because geologists serve in different capacities from one project to another and have been assigned differing degrees of responsibility by engineering firms and governmental agencies. Moreover, the discussion clearly demonstrates the difficulty of writing a paper that places emphasis on general principles rather than on specific examples. Generalizations are always subject to exceptions.

Geology is a very complex, fluid science, and a geological paper written in a condensed, more or less academic manner should not be construed by the reader to mean that geologists work by rule of thumb or make interpretations which are based on elementary considerations only. Actually, a successful engineering geologist is a scientist who interprets geologic and engineering data artfully in the light of knowledge gained from long years of rigorous academic training and diversified field and laboratory experience. The engineering geologist attempts to apply the complex science of geology to the equally complex problems of engineering. He should remain primarily a geologist and should not attempt to usurp the place of the engineer, and first and last should think and act as a geologist and not as an engineer with a smattering of geological training.

There is an urgent need for publications that clearly detail and discuss "case histories." The vast amount of data in files of engineering firms and governmental agencies should be collated and made available to the geological and engineering professions as expediently as possible. Free interchange of ideas and widespread dissemination of the published results of geologic investigations of engineering projects will enable the practicing professional man measurably to improve both the qualitative and quantitative aspects of his work.

---

[20] "Rock Tunneling with Steel Supports," by R. V. Proctor and T. L. White, The Commercial Shearing and Stamping Co., Youngstown, Ohio, 1946, p. 17.

[21] Dept. of Geology, Univ. of Colorado, Boulder, Colo.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

TRANSACTIONS

Paper No. 2358

VARIATION OF COEFFICIENTS OF SIMULTANEOUS LINEAR EQUATIONS

By Bernard L. Weiner,[1] M. ASCE

With Discussion by Messrs. T. F. Hickerson, A. Floris, Frederick S. Merritt, I. F. Morrison, Sigmund Roos, and Bernard L. Weiner.

Synopsis

Simultaneous equations are of great importance in modern structural engineering, and the purpose of going through a solution is, obviously, to obtain numerical results for a specific set of conditions. It is often of equal importance, however, to know the effect of a change in the geometric and elastic properties of the structure. Without such knowledge it is impossible to study the design with a view toward obtaining the most in economy and certain other desirable qualities. Such studies, except for the simplest structures, can be made only at the expense of re-solving the equations each time a change is made in the coefficients, and the labor involved is so great that studies cannot ordinarily be made except very roughly and sketchily. With the theory presented in this paper, the effect on the solution produced by a change in any number of the coefficients in any one equation, or in any number of coefficients of any one unknown, can be computed quickly and simply in one step.

Although the variation of coefficients is the primary purpose of the paper, other useful properties of the theory, such as the solution of equations and the rapid correction of errors, follow simply as corollaries. In the actual operation of the method, however, the procedure is the same whether coefficients are being varied in a set of equations that have already been solved, or whether the purpose is to solve the equations. It is, as a matter of fact, possible to vary the coefficients endlessly and any particular "solution" therefore becomes merely a stopping point in an endless series of variations—from which it follows that the method is very flexible.

Variation of coefficients as a method of solving simultaneous equations has several marked advantages over the classical methods and leads also to the derivation of several useful properties of such equations. The numerical

---

Note.—Published in October, 1947, *Proceedings*. Positions and titles given are those in effect when the paper or discussion was received for publication.

[1] Associate Engr., D. B. Steinman, Cons. Engr., New York, N. Y.

work required for a solution by the method presented in this paper is very much less than that required by standard methods; and, what is perhaps more important, fewer significant digits are necessary in the intermediate computations to obtain any desired degree of accuracy in the final solution. In fact, with a little care and by using a few fairly obvious arithmetical manipulations, results sufficiently accurate for design purposes often may be obtained by slide rule for the bulk of the computations. The method is an exact one, however, and computations may be carried out to any degree of accuracy but with much less labor than that required by the classic methods.

In addition, the method of solution has another very important property. As will be shown, the solution falls into distinct steps, or partial solutions, each and every one of which is complete in itself. As a result, the computation may be checked completely and absolutely at the conclusion of each step or variation, so that it is not necessary to wait for the end of the solution to check for an arithmetical error. Should the computer, however, either neglect to check his work at intermediate points, or discover an error in the original setup of the equations, the basic theory of the solution is such that it offers a quick method for the correction of an error without the necessity of changing the entire computation from the point at which the error was made to the end of the solution. Furthermore, the method is self-checking in many different ways, so if it is found that the intermediate computations at some particular point in the solution have not been carried out to a sufficient number of significant digits, the defect may be remedied by using one of the checks to recompute the defective quantity.

INTRODUCTION

Structural analysis and many other engineering problems occasionally require the solution of simultaneous equations. Since classical methods of solution involve small differences resulting from the subtraction of large numbers, computations must be carried to many significant digits, thus making the work very laborious. Study of the effect of varying the characteristics of a structure (which involves, among other things, changing the coefficients of the unknowns) by these same methods is so time consuming as to make such studies prohibitive except in the simplest cases. The value of a direct and simple method for finding the effect of varying coefficients of simultaneous equations is therefore obvious.

Since an error in coefficients can be interpreted to mean that, in effect, a structure has been analyzed whose characteristics are expressed by the incorrect coefficients, the correction of such errors becomes a problem in "variation of coefficients." It was to meet just such a problem that the methods outlined in this paper were developed and later extended to other applications. An obvious extension of the concept involved leads directly to a new method for solving this type of equations—for, as will be shown, a "solution" is merely a stopping point in an endless chain of variations.

The theory presented, as it applies to the solution of simultaneous equations, differs in one respect from the classical theories in that it is developed from a different point of view. The usual method of solution for this and other types of equations consists, essentially, of what might be termed a process of "breaking down"—that is, the complete equations are converted to such forms that the extraction of a root or the elimination of other unknowns will result in a simple equation with only one unknown or root. This method, on the contrary, consists of a process of "building up"—that is, the solution begins with simple, reduced equations that are easily solved or need no solution, and are then built up, step by step, to the required equations by converting the solutions to conform to the new equations at each step. In this process, it is necessary to vary the coefficients of the unknowns and to find the resulting effect on the values of the unknowns. This process is direct, and the theory for the variation of the coefficients makes the method of solution almost self-evident.

DERIVATION OF THEORY FOR VARIATION OF COEFFICIENTS

Each one of a set of simultaneous equations takes the form:

$$a_{1n} X_1 + a_{2n} X_2 + a_{3n} X_3 + \cdots = c_n \ldots\ldots\ldots\ldots\ldots (1)$$

in which the X_m's are unknowns; the a_{mn}'s are coefficients dependent only on the elastic and geometric properties of the structure; and the c_n's are constants determined by the position and magnitude of the loads in addition to the properties of the structure.

For any given structure the coefficients are, of course, constant, but c_n will have as many sets of values as there are loading conditions to be analyzed. It has often been erroneously stated that the equations must be solved for each one of these sets of constants—a statement that is only a half-truth. If the work is properly arranged and the equations are solved for one set of values of c_n, the solution for any other set requires comparatively little additional labor. In practice, the equations are solved in a routine manner for all sets of values of c_n at the same time and the computation of influence lines is therefore no great additional hardship. A simple illustration of such a procedure is presented in Table 1, which gives a solution for the simultaneous equations:

$$\left.\begin{array}{r} +\,4\,x + 6\,y = \\ +\,2\,x + 5\,y = \end{array} \begin{array}{|c||c|c|} 42 & a & 0 \\ 31 & 0 & b \end{array}\right|\Bigg\} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Attention is called to Col. 7, from which it is easily seen that if a set of equations is solved for one set of values of c_n a solution for another set can be obtained simply by using the factors and performing on the new constants only the operations indicated in this column. However, if the number of sets of values for c_n is large, it is often advantageous to solve for sets of constants such as are shown in Cols. 5 and 6. The numerical solution $y = 5$, for example may then be obtained by setting $a = 42$ and $b = 31$, from which (line 5) $y = -\frac{1}{4} \times 42 + \frac{1}{2} \times 31 = 5$. Similarly (line 8), $x = +0.625 \times 42 - 0.75 \times 31 = 3$.

Any number of solutions may therefore be found either by following the operations indicated in Col. 7 or by using the factors in Cols. 5 and 6. It should be noted that these latter factors are actually solutions of Eqs. 2 for $c_n = a,0$ and $0,b$. However, if the coefficients of x and y were changed, it would give rise to a new set of equations. By the classical methods, solutions for the new set of equations can be found only by actually re-solving the equations. The factors in Cols. 5 and 6 can be used to avoid this difficulty and therefore are termed "variation coefficients."

TABLE 1.—SOLUTION OF EQS. 2 BY THE PROCESS OF ELIMINATION

Line No.					VARIATION COEFFICIENTS		Remarks
(1)	(2)	(3)		(4)	(5)	(6)	(7)
1	$+4x$	$+6y$	=	42	a	0	
2	$+2x$	$+5y$	=	31	0	b	
3	$+2x$	$+3y$	=	21	$0.5a$	0	½ (line 1)
4		$+2y$	=	10	$-0.5a$	b	(line 2) − (line 3)
5		y	=	5	$-0.25a$	$0.5b$	½ (line 4)
6		$5y$	=	25	$-1.25a$	$2.5b$	5 (line 5)
7	$2x$		=	6	$+1.25a$	$-1.5b$	(line 2) − (line 6)
8	x		=	3	$+0.625a$	$-0.75b$	½ (line 7)

Let it be supposed that it is desired to change the coefficient of x in line 1, Table 1, from 4 to 7. The new set of equations then becomes:

$$\left.\begin{aligned} 7x' + 6y' &= 42 \\ 2x' + 5y' &= 31 \end{aligned}\right\} \qquad (3)$$

which may be rewritten as

$$\left.\begin{aligned} 4x' + 6y' &= (42 - 3x') \\ 2x' + 5y' &= 31 \end{aligned}\right\} \qquad (4)$$

Now let $(42 - 3x'),31$ be treated as new values of c_n—that is, as additional constants of Eqs. 2 for which solutions are to be found. Substitution of $a = 42 - 3x'$, and $b = 31$, in line 8, Table 1, gives $x' = 0.625(42 - 3x') - 0.75 \times 31 = 3 - 1.875x'$; which gives

$$x' = \frac{3}{1 + 1.875} = \frac{24}{23} \qquad (5)$$

From line 5, Table 1, $y' = -0.25(42 - 3x') + 0.5 \times 31 = 5 + 0.75x'$; from which

$$y' = 5 + 0.75 \times \frac{24}{23} = \frac{133}{23} \qquad (6)$$

These values of x' and y' obviously satisfy Eqs. 3 and are therefore solutions of these equations. Furthermore, since the variation coefficients are also solutions of Eqs. 2, they too can be revised to satisfy Eqs. 3 for $c_n = a',0$ and

$0,b'$. Thus, by setting $a = a' - 3\,x'$, and $b = 0$,

$$x'_a = 0.625\,(a' - 3\,x'_a); \qquad \text{or} \qquad x'_a = \frac{0.625\,a'}{2.875} = \frac{5}{23}\,a' \ldots\ldots (7)$$

and

$$y'_a = -\,0.25\,(a' - 3\,x'_a); \qquad \text{or} \qquad y'_a = -\,\frac{a'}{4} + \frac{3}{4} \times \frac{5}{23}\,a' = -\,\frac{2}{23}\,a' \ldots (8)$$

Likewise, by setting $a = -\,3\,x'$, and $b = b'$,

$$x'_b = 0.625\,(-\,3\,x') - 0.75\,b'; \qquad \text{or} \qquad x'_b = -\,\frac{0.75\,b'}{2.875} = -\,\frac{6}{23}\,b' \ldots (9)$$

and

$$y'_b = -\,0.25\,(-\,3\,x') + 0.5\,b'; \qquad \text{or} \qquad y'_b = -\,\frac{3}{4} \times \frac{6}{23}\,b' + \frac{1}{2}\,b' = \frac{7}{23}\,b' \ldots (10)$$

The complete solution of Eqs. 3 for all values of c_n, as obtained by revision of Eqs. 2, is given in Table 2.

Eqs. 3 can be changed back to Eqs. 2—or may be "varied" to any other equations—by using the new variation coefficients, since it obviously does not matter whether one set of equations was solved by revising another set or was solved directly.

For simplicity, only one coefficient was varied in the previous problem. It will now be shown that any number of coefficients of the same unknown can be varied at the same time.

TABLE 2.—SUMMARY OF SOLUTION OF EQS. 3

Line No.					VARIATION COEFFICIENTS	
(1)	(2)	(3)		(4)	(5)	(6)
1.....	$+7\,x'$	$+6\,y'$	=	42	a'	0
2.....	$+2\,x'$	$+5\,y'$	=	31	0	b'
3.....	x'		=	$\frac{24}{23}$	$\frac{5}{23}\,a'$	$-\,\frac{6}{23}\,b'$
4.....		y'	=	$\frac{133}{23}$	$-\,\frac{2}{23}\,a'$	$\frac{7}{23}\,b'$

GENERAL FORMULAS FOR VARIATION OF COEFFICIENTS

Let it be assumed that the equations—

$$\left.\begin{array}{l} A_1\,x + B_1\,y + C_1\,z = \\ A_2\,x + B_2\,y + C_2\,z = \\ A_3\,x + B_3\,y + C_3\,z = \end{array} \left|\begin{array}{c} K_1 \\ K_2 \\ K_3 \end{array}\right\|\begin{array}{c|c|c} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array}\right|\right\} \ldots\ldots\ldots\ldots\ldots (11)$$

—have been solved by some method, no matter what, with results as shown in Table 3.

Also, suppose that the coefficients A_1, A_2, A_3 are increased by d_1, d_2, and d_3, respectively. Following the procedure used in varying Eqs. 2:

$$\left.\begin{array}{l} a = K_1 - d_1\,x' \\ b = K_2 - d_2\,x' \\ c = K_3 - d_3\,x' \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Substituting in line 4, Table 3,

$$x' = a_1\,(K_1 - d_1\,x') + a_2\,(K_2 - d_2\,x') + a_3\,(K_3 - d_3\,x') \ldots\ldots (13a)$$

from which

$$x' + a_1 d_1 x' + a_2 d_2 x' + a_3 d_3 x' = a_1 K_1 + a_2 K_2 + a_3 K_3 \ldots\ldots (13b)$$

and

$$x' = \frac{a_1 K_1 + a_2 K_2 + a_3 K_3}{1 + a_1 d_1 + a_2 d_2 + a_3 d_3} \ldots\ldots (14)$$

As previously shown (Table 1), a solution for any set of constants, such as K_1, K_2, and K_3, may be found by setting $a = K_1$, $b = K_2$, and $c = K_3$. Therefore,

$$a_1 K_1 + a_2 K_2 + a_3 K_3 = S_1 \ldots\ldots (15a)$$

Similarly, by substituting $a = d_1$, $b = d_2$, and $c = d_3$:

$$a_1 d_1 + a_2 d_2 + a_3 d_3 = x_d \ldots\ldots (15b)$$

in which x_d is the solution for x, for a set of values of $c_n = d_1$, d_2, and d_3.

Substituting Eqs. 15*a* and 15*b* in Eq. 14,

$$x' = \frac{S_1}{1 + x_d} \ldots\ldots (16)$$

However, since S_1 is any solution for x,

$$x' = \frac{x}{1 + x_d} \ldots\ldots (17)$$

In a similar manner, substituting Eq. 12 in line 5, Table 3,

$$y' = b_1 (K_1 - d_1 x') + b_2 (K_2 - d_2 x') + b_3 (K_3 - d_3 x') \ldots\ldots (18a)$$

$$y' = (b_1 K_1 + b_2 K_2 + b_3 K_3) - (b_1 d_1 + b_2 d_2 + b_3 d_3) x' \ldots\ldots (18b)$$

$$y' = S_2 - x' \sum b_n d_n \ldots\ldots (18c)$$

Since S_2 is any solution for y, and $\sum b_n d_n$ is a solution for y when $c_n = d_1$, d_2, and d_3,

$$y' = y - y_d x' \ldots\ldots (19)$$

Similarly,

$$z' = z - z_d x' \ldots\ldots (20)$$

TABLE 3.—SUMMARY OF SOLUTION OF EQS. 11

Line No.						VARIATION COEFFICIENTS		
(1)	(2)	(3)	(4)		(5)	(6)	(7)	(8)
1...	$+ A_1 x$	$+ B_1 y$	$+ C_1 z$	=	K_1	a	0	0
2...	$+ A_2 x$	$+ B_2 y$	$+ C_2 z$	=	K_2	0	b	0
3...	$+ A_3 x$	$+ B_3 y$	$+ C_3 z$	=	K_3	0	0	c
4...	x			=	S_1	$a_1 a$	$a_2 b$	$a_3 c$
5...		y		=	S_2	$b_1 a$	$b_2 b$	$b_3 c$
6...			z	=	S_3	$c_1 a$	$c_2 b$	$c_3 c$

Three equations with three unknowns were chosen for simplicity. The derivation would obviously be the same for any number of unknowns. Also, in this derivation the coefficients of x were varied; but this is evidently immaterial, as no restriction is involved. Revised solutions for the unknown whose coefficients are being varied are given by Eq. 17; revised values for the solutions of all other unknowns are given by Eqs. 19 and 20.

Generalizing:

$$u' = \frac{u}{1 + u_d} \qquad \text{(21}a\text{)}$$

and

$$v' = v - v_d u' \qquad \text{(21}b\text{)}$$

in which u and u' represent, in the original and revised equations, respectively, the unknown whose coefficients are being varied; v and v' represent any other unknown in the original and final equations, respectively; d_n is the amount by which A_n, any coefficient of u, is increased; and u_d and v_d are solutions for u and v, respectively, when $c_n = d_1, d_2, \cdots, d_n$.

Application of basic Eqs. 21 to the solution of the equations—

$$\left.\begin{array}{r} 4x + 3y + 2z = \\ x + 4y + 3z = \\ 5x + 2y + 6z = \end{array} \left| \begin{array}{c} 28 \\ 26 \\ 49 \end{array} \right\| \begin{array}{c|c|c} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array} \right\| \right\} \qquad (22)$$

—is illustrated in Table 4.

EXPLANATION OF TABLE 4

In Table 4, it is assumed that the set of simultaneous equations on lines 1, 2, and 3 (Eqs. 22) has been solved by some method, no matter what. The solutions corresponding to the four sets of constants are given on the next three lines. Let it be supposed that it is desired to reduce the coefficients of these equations by the amounts shown in brackets on the first three lines of the table. For example, on line 1 the coefficient of x, which totals 4, is to be reduced by 3 to a net value of 1; the coefficient of z, on the same line, is to be reduced by 2—that is, it is to be eliminated entirely. The order of variation is immaterial; by arbitrary choice the coefficients of z will be varied first.

From Eq. 21a, $u = z$, $u' = z'$, and $u_d = z_d$. Also, from Col. 4, lines 1, 2, and 3, $d_1 = -2$; $d_2 = -3$; and $d_3 = -5$. By definition, $u_d = z_d$ is a solution of the original equations for z corresponding to a set of constants $c_n = d_n$—that is, $c_1 = d_1 = -2$; $c_2 = d_2 = -3$; and $c_3 = d_3 = -5$.

To obtain this solution, z_d, it is necessary only to substitute the set of constants in line 6, setting $a = -2$, $b = -3$, and $c = -5$, as indicated in Col. 15, lines 7 to 9. Making this substitution, $u_d = z_d = \left(-\frac{2}{7}\right)(-2) + \frac{1}{9}(-3) + \frac{13}{63}(-5)$, from which $z_d = \frac{4}{7} - \frac{1}{3} - \frac{65}{63} = -\frac{50}{63}$, as shown on line 9, Cols. 9 to 12. Also, $1 + z_d = 1 - \frac{50}{63} = \frac{13}{63}$, and $\frac{1}{1 + z_d} = \frac{63}{13}$ as shown on line 9, Cols. 13 and 14.

Having computed the reciprocal of the denominator in Eq. 21a, the next step is to find the revised values of z. Thus, for $z = 5$, $z' = 5 \times \frac{63}{13} = \frac{315}{13}$; for $z = -\frac{2}{7}a$, $z' = \left(-\frac{2}{7}a\right) \times \frac{63}{13} = -\frac{18}{13}a$; that is, the original values on line 6 are multiplied by $\frac{1}{1 + z_d} = +\frac{63}{13}$ to obtain the revised values, z', on

TABLE 4.—WORK SHEET FOR THE APPLICATION OF EQS. 21a AND 21b TO VARIATION OF COEFFICIENTS OF EQS. 22

Line No.						VARIATION COEFFICIENTS			CONVERSION FACTORS						Substitution factors
									$a_n d_1$	$b_n d_2$	$c_n d_3$	u_d or v_d	$1 + u_d$	$\frac{1}{1 + u_d}$	
(1)	(2)	(3)	(4)		(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
1.....	$+x + [3x]$	$+ [3y]$	$+ [2z]$	=	28	a	0	0							
2.....	$+ [x]$	$+y + [3y]$	$+ [3z]$	=	26	0	b	0							
3.....	$+ [5x]$	$+ [2y]$	$+z + [5z]$	=	49	0	0	c							
4.....	x			=	$+3$	$+\frac{2}{7}a$	$-\frac{2}{9}b$	$+\frac{1}{63}c$							
5.....		y		=	$+2$	$+\frac{1}{7}a$	$+\frac{2}{9}b$	$-\frac{10}{63}c$							
6.....			z	=	$+5$	$-\frac{2}{7}a$	$+\frac{1}{9}b$	$+\frac{13}{63}c$							
7.....					$+\frac{5}{13}$	$-\frac{2}{91}a$	$+\frac{1}{117}b$	$+\frac{1}{63}c$	$-\frac{4}{7}$	$+\frac{2}{3}$	$-\frac{5}{63}$	$+\frac{1}{63}$			$a = d_1 = -2$
8.....					$-\frac{50}{13}$	$+\frac{20}{91}a$	$-\frac{10}{117}b$	$-\frac{10}{63}c$	$-\frac{2}{7}$	$-\frac{2}{3}$	$+\frac{50}{63}$	$-\frac{10}{63}$			$b = d_2 = -3$
9.....					$+\frac{315}{13}$	$-\frac{18}{13}a$	$+\frac{7}{13}b$	$+c$	$+\frac{4}{7}$	$-\frac{1}{3}$	$-\frac{65}{63}$	$-\frac{50}{63}$	$+\frac{13}{63}$	$+\frac{63}{13}$	$c = d_3 = -5$
10.....	x'			=	$+\frac{34}{13}$	$+\frac{4}{13}a$	$-\frac{3}{13}b$	0							
11.....		y'		=	$+\frac{76}{13}$	$-\frac{1}{13}a$	$+\frac{4}{13}b$	0							
12.....			(z')	=	$+\frac{315}{13}$	$-\frac{18}{13}a$	$+\frac{7}{13}b$	$+c$							

TABLE 4.—(*Continued*)

Line No. (1)	(2)	(3)	(4)		(5)	Variation Coefficients			Conversion Factors						Substitution factors (15)
						(6)	(7)	(8)	$a_n d_1$ (9)	$b_n d_2$ (10)	$c_n d_3$ (11)	u_d or v_d (12)	$1 + u_d$ (13)	$\frac{1}{1 + u_d}$ (14)	
13.....					$-\frac{57}{13}$	$+\frac{3}{52}a$	$-\frac{3}{13}b$	0	$-\frac{12}{13}$	$+\frac{9}{13}$	0	$-\frac{3}{13}$			$a = d_1 = -3$
14.....					$+19$	$-\frac{1}{4}a$	$+b$	0	$+\frac{3}{13}$	$-\frac{12}{13}$	0	$-\frac{9}{13}$	$+\frac{4}{13}$	$+\frac{13}{4}$	$b = d_2 = -3$
15.....					$+\frac{133}{13}$	$-\frac{7}{52}a$	$+\frac{7}{13}b$	0	$+\frac{54}{13}$	$-\frac{21}{13}$	-2	$+\frac{7}{13}$			$c = d_3 = -2$
16.....	x''			=	$+7$	$+\frac{1}{4}a$	0	0							
17.....		(y'')		=	$+19$	$-\frac{1}{4}a$	b	0							
18.....			z''	=	$+14$	$-\frac{5}{4}a$	0	c							
19.....					$+28$	$+a$	0	0	$-\frac{3}{4}$	0	0	$-\frac{3}{4}$	$+\frac{1}{4}$	$+4$	$a = d_1 = -3$
20.....					-7	$-\frac{1}{4}a$	0	0	$+\frac{3}{4}$	-1	0	$-\frac{1}{4}$			$b = d_2 = -1$
21.....					-35	$-\frac{5}{4}a$	0	0	$+\frac{15}{4}$	0	-5	$-\frac{5}{4}$			$c = d_3 = -5$
22.....	(x''')			=	$+28$	$+a$	0	0							
23.....		y'''		=	$+26$	0	b	0							
24.....			z'''	=	$+49$	0	0	c							

line 12. For reasons that will be obvious later, the values of z' are also shown on line 9, Cols. 5 to 8.

With z', the revised values of z determined, the revised values of the other unknowns can be computed. With $v = x$, and $v_d = x_d$, x_d is the only value on the right-hand side of Eq. 21*b* that has not already been computed. Again by definition, $v_d = x_d$ is a solution for x of the original equations corresponding to the same sets of constants $c_n = d_n$. Therefore, substituting in line 4 the values $a = -2$, $b = -3$, and $c = -5$, $v_d = x_d = +\frac{2}{7}(-2) + \left(-\frac{2}{9}\right)(-3) + \frac{1}{63}(-5)$ from which $x_d = -\frac{4}{7} + \frac{2}{3} - \frac{5}{63} = +\frac{1}{63}$, as shown on line 7, Cols. 9 to 12.

Having obtained $v_d = x_d$, the term $v_d\, u' = x_d\, z'$ may be computed. Thus, for $x = 3$ and $z' = +\frac{315}{13}$, $x_d\, z' = \frac{315}{13} \times \frac{1}{63} = +\frac{5}{13}$. Likewise, for $x = +\frac{2}{7}a$ and $z' = -\frac{18}{13}a$, $x_d\, z' = \left(-\frac{18}{13}a\right)\frac{1}{63} = -\frac{2}{91}a$; that is, the values of z' on line 9 are multiplied by $x_d = +\frac{1}{63}$ to obtain the term $x_d\, z'$ on line 7 corresponding to each value of x (in the original equations) on line 4. Since, by Eq. 21*b*, $x' = x - x_d\, z'$, values on line 7, Cols. 5 to 8, are subtracted algebraically from corresponding values on line 4 to give the revised values, x', on line 10. Thus, for $x = 3$, $x' = 3 - \frac{5}{13} = +\frac{34}{13}$. Likewise, for $x = \frac{2}{7}a$, $x' = +\frac{2}{7}a - \left(-\frac{2}{91}a\right) = +\frac{4}{13}a$.

Similarly, setting $v = y$, $v' = y'$, and $v_d = y_d$ in Eq. 21*b* and substituting $a = -2$, $b = -3$, and $c = -5$ in line 5, $y_d = +\frac{1}{7}(-2) + \frac{2}{9}(-3) + \left(-\frac{10}{63}\right)(-5)$, from which $y_d = -\frac{2}{7} - \frac{2}{3} + \frac{50}{63} = -\frac{10}{63}$, as shown on line 8, Cols. 9 to 12. Multiplying the values of z' on line 9 by $y_d = -\frac{10}{63}$, the values of the terms $y_d\, z'$ are obtained as shown on line 8, Cols. 5 to 8. Thus, for $y = 2$ and $z' = +\frac{315}{13}$, $y_d\, z' = +\frac{315}{13}\left(-\frac{10}{63}\right) = -\frac{50}{13}$. Also, for $y = \frac{1}{7}a$ and $z' = -\frac{18}{13}a$, $y_d\, z' = \left(-\frac{18}{13}a\right)\left(-\frac{10}{63}\right) = +\frac{20}{91}a$.

Since, by Eq. 21*b*, $y' = y - y_d\, z'$, values on line 8, Cols. 5 to 8, are subtracted algebraically from corresponding values on line 5 to give the revised values, y', on line 11. For example, for $y = 2$, $y' = 2 - \left(-\frac{50}{13}\right) = +\frac{76}{13}$; for $y = +\frac{1}{7}a$, $y' = \frac{1}{7}a - \frac{20}{91}a = -\frac{1}{13}a$.

The computation for the variation of the coefficients of z is now complete, and lines 10, 11, and 12 are the solutions for the revised set equations—

$$\left.\begin{array}{rc|c||c|c|c|}
4x' + 3y' & = & 28 & a & 0 & 0 \\
x' + 4y' & = & 26 & 0 & b & 0 \\
5x' + 2y' + z' & = & 49 & 0 & 0 & c
\end{array}\right\}\dots\dots\dots\dots\dots(23)$$

In the next step of Table 4, Eqs. 23 are further revised by varying the coefficients of y' from 3, 4, and 2 to 0, 1, and 0, respectively. For this step, referring again to Eqs. 21, $u = y'$, and $v = x'$ or z' as the case may be. The computations follow, step by step, the procedure described in detail for varying the coefficients of z. Since the new set of equations are now being varied—that is, Eqs. 23—their solutions on lines 10, 11, and 12 are used to compute x'', y'', and z'' on lines 16, 17, and 18.

These solutions correspond to another new set of equations—

$$\left.\begin{array}{rc|c||c|c|c|}
4x'' & = & 28 & a & 0 & 0 \\
x'' + y'' & = & 26 & 0 & b & 0 \\
5x'' + z'' & = & 49 & 0 & 0 & c
\end{array}\right\}\dots\dots\dots\dots\dots(24)$$

In the last step of Table 4, Eqs. 24 are revised further by varying the coefficients of x'' from 4, 1, and 5 to 1, 0, and 0, respectively. For this variation, $u = x''$, and $v = y''$ or z''. The new solutions are given on lines 22, 23, and 24, corresponding to the new equations:

$$\left.\begin{array}{rc|c||c|c|c|}
x''' & = & 28 & a & 0 & 0 \\
y''' & = & 26 & 0 & b & 0 \\
z''' & = & 49 & 0 & 0 & c
\end{array}\right\}\dots\dots\dots\dots\dots(25)$$

Attention is called to the fact that each step in Table 4 is entirely independent and complete. Once the solution to a set of equations is known, it is immaterial whether it was obtained independently or by varying the coefficients of some other set of equations.

Solution of Simultaneous Equations by Variation of Coefficients

In Table 4, the coefficients have been varied in a more or less arbitrary manner. This numerical example illustrates clearly that, once the variation coefficients are known for any set of simultaneous equations, the coefficients can be varied at will and the order of the operations is almost entirely arbitrary. Also, the process is obviously reversible—which leads directly to another method for solving such equations.

In Table 4, it was assumed that Eqs. 22 were solved in some way, no matter how—the solutions being given on lines 4, 5, and 6. By a series of variation operations, all the coefficients in brackets in this original set of equations were removed, resulting in the corresponding revised solutions given on lines 22, 23, and 24. However, this last result obviously can be obtained directly by inspection. Since the process is reversible, the results shown on these last three lines can be varied back to the original set of equations—or to any other set of equations, for that matter. This reverse process is illustrated in Table 5, and thus another method for solving simultaneous linear equations is obtained.

TABLE 5.—WORK SHEET FOR SOLUTION OF EQS. 22 BY VARIATION OF COEFFICIENTS

Line No.						VARIATION COEFFICIENTS			CONVERSION FACTORS						Substitution factors
									$a_n d_1$	$b_n d_2$	$c_n d_3$	u_d or v_d	$1 + u_d$	$\frac{1}{1 + u_d}$	
(1)	(2)	(3)	(4)		(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
1.....	$+x + [3x]$	$+ [3y]$	$+ [2z]$	=	28	a	0	0							
2.....	$+ [x]$	$y + [3y]$	$+ [3z]$	=	26	0	b	0							
3.....	$+ [5x]$	$+ [2y]$	$+z + [5z]$	=	49	0	0	c							
4.....	x			=	28	a	0	0							
5.....		y		=	26	0	b	0							
6.....			z	=	49	0	0	c							
7.....					+7	$+\frac{1}{4}a$	0	0	+3	0	0	+3	+4	$+\frac{1}{4}$	$a = d_1 = +3$
8.....					+7	$+\frac{1}{4}a$	0	0	0	+1	0	+1			$b = d_2 = +1$
9.....					+35	$+\frac{5}{4}a$	0	0	0	0	+5	+5			$c = d_3 = +5$
10.....	(x)			=	+7	$+\frac{1}{4}a$	0	0							
11.....		y		=	+19	$-\frac{1}{4}a$	b	0							
12.....			z	=	+14	$-\frac{5}{4}a$	0	c							

TABLE 5.—(*Continued*)

Line No.						VARIATION COEFFICIENTS			CONVERSION FACTORS						Substitution factors
									$a_n d_1$	$b_n d_2$	$c_n d_3$	u_d or v_d	$1 + u_d$	$\frac{1}{1 + u_d}$	
(1)	(2)	(3)	(4)		(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
13.....					$+\frac{57}{13}$	$-\frac{3}{52}a$	$+\frac{3}{13}b$	0	$+\frac{3}{4}$	0	0	$+\frac{3}{4}$			$a = d_1 = +3$
14.....					$+\frac{76}{13}$	$-\frac{1}{13}a$	$+\frac{4}{13}b$	0	$-\frac{3}{4}$	3	0	$+\frac{9}{4}$	$+\frac{13}{4}$	$+\frac{4}{13}$	$b = d_2 = +3$
15.....					$-\frac{133}{13}$	$+\frac{7}{52}a$	$-\frac{7}{13}b$	0	$-\frac{15}{4}$	0	2	$-\frac{7}{4}$			$c = d_3 = +2$
16.....	x			=	$+\frac{34}{13}$	$+\frac{4}{13}a$	$-\frac{3}{13}b$	0							
17.....		(y)		=	$+\frac{76}{13}$	$-\frac{1}{13}a$	$+\frac{4}{13}b$	0							
18.....			z	=	$+\frac{315}{13}$	$-\frac{18}{13}a$	$+\frac{7}{13}b$	c							
19.....					$-\frac{5}{13}$	$+\frac{2}{91}a$	$-\frac{1}{117}b$	$-\frac{1}{63}c$	$+\frac{8}{13}$	$-\frac{9}{13}$	0	$-\frac{1}{13}$			$a = d_1 = +2$
20.....					$+\frac{50}{13}$	$-\frac{20}{91}a$	$+\frac{10}{117}b$	$+\frac{10}{63}c$	$-\frac{2}{13}$	$+\frac{12}{13}$	0	$+\frac{10}{13}$			$b = d_2 = +3$
21.....					$+5$	$-\frac{2}{7}a$	$+\frac{1}{9}b$	$+\frac{13}{63}c$	$-\frac{36}{13}$	$+\frac{21}{13}$	$+5$	$+\frac{50}{13}$	$+\frac{63}{13}$	$+\frac{13}{63}$	$c = d_3 = +5$
22.....	x			=	$+3$	$+\frac{2}{7}a$	$-\frac{2}{9}b$	$+\frac{1}{63}c$							
23.....		y		=	$+2$	$+\frac{1}{7}a$	$+\frac{2}{9}b$	$-\frac{10}{63}c$							
24.....			(z)	=	$+5$	$-\frac{2}{7}a$	$+\frac{1}{9}b$	$+\frac{13}{63}c$							

Furthermore, it is not necessary to begin with lines 22, 23, and 24 of Table 4, which are duplicated in lines 4, 5, and 6 of Table 5. If a solution of any set of equations is known—from some previous computation or in any other way—it can be varied to any other set of equations desired. "Knowing a solution" means that the "variation coefficients"—which result from solving the equations for a set of constants a, 0, 0; 0, b, 0; and 0, 0, c—are known. The process is the same for any number of unknowns.

Explanation of Table 5

Referring to Table 5, Eqs. 22 (lines 1, 2, and 3) are to be solved by a series of steps which is the reverse of that in Table 4. All the coefficients and parts of coefficients in brackets are considered removed, giving rise to the new set of equations:

$$\left.\begin{array}{lllr|c||c|c|c|} x & & & = & 28 & a & 0 & 0 \\ & y & & = & 26 & 0 & b & 0 \\ & & z & = & 49 & 0 & 0 & c \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

This new set of equations can obviously be solved by inspection; the solutions shown on lines 4, 5, and 6 are merely the repetition of Eqs. 26. To solve the original set of equations, it is necessary to vary the coefficients of this reduced set until all the coefficients and parts of coefficients which have been removed are replaced. By arbitrary choice and also to furnish a comparison between Table 5 and Table 4, the coefficients are varied in the order of x, y, and z.

To replace the coefficients of x, $d_1 = +3$, $d_2 = +1$, and $d_3 = +5$, since comparison of Eqs. 26 with lines 1, 2, and 3 in Col. 2 shows that these values are the coefficients and parts of coefficients which have been removed. Substitution in line 4 of $a = d_1 = +3$, $b = d_2 = +1$, and $c = d_3 = +5$ gives $x_d = +3$, since x has a value other than zero only when $c_n = a$.

From this point to the end of the computation the procedure is exactly similar to that used in Table 4 and needs no further explanation. The final solution is given on lines 22, 23, and 24.

Reciprocal Properties of Simultaneous Equations

For ready reference, and also to bring out certain relationships, the operations shown in Tables 4 and 5 have been summarized in Table 6. It is customary to consider x, y, and z as the unknowns and a, b, and c as the constants; that is, the equations would be set up on this basis in a practical problem. However, once the equations are solved—considering lines 1 to 6 of Table 6—it is equally correct, mathematically, to consider a, b, and c as the unknowns and x, y, and z as the constants.

From this point of view, considering a, b, and c as the unknowns, lines 1, 2, and 3 give the solutions for the equations on lines 4, 5, and 6. To illustrate: Substitution in line 4 of the values $a = 4x$, $b = x$, and $c = 5x$, for example, gives $\frac{2}{7}(4x) - \frac{2}{9}(x) + \frac{1}{63}(5x) = x$, which checks. Similarly, substitution at random (in line 5, for example) of $a = 28$, $b = 26$, and $c = 49$ gives $\frac{1}{7} \times 28 + \frac{2}{9} \times 26 - \frac{10}{63} \times 49 = 2$, which checks again.

It should be remembered that, after a variation operation is completed, the set of resulting equations with their solutions are complete in themselves. The equations may not be those that are finally desired, but they are nevertheless complete. It therefore follows that any set of equations is merely a stopping point in an endless series of variations. Lines 7, 8, and 9, for example, are the solutions for a set of equations in which the coefficient of z is zero in the first two and equals unity in the third, but which is otherwise the same as the original set on lines 1, 2, and 3 (Eqs. 23). Either x, y, and z or a, b, and c may be considered as the unknowns and if the corresponding solutions are substituted the results will check in the new equations.

The reciprocity of the equations plus the fact that at each step in a variation process the results are complete makes the operations self-checking. Furthermore, there are so many methods of checking that some of the checks may be used to reduce the numerical work and still leave a sufficient number for actual verification.

A consideration of the fundamental theory of equations will bring to light other ways of checking not mentioned. For example, the fact that rows and columns of determinants may be interchanged leads to the conclusion that other sets of equations exist which may be obtained by "rotating" the coefficients to positions consistent with the corresponding "rotated" determinants.

Furthermore, in a practical problem in structural analysis, most coefficients occur in pairs as a consequence of the Maxwell theorem of reciprocal deflections. The final "variation coefficients" also occur in pairs and in the same order as the coefficients of the unknowns—furnishing an automatic check, which may also have other uses besides checking.

An examination of Table 6(*c*) will show that, on any line, terms above the heavy line are merely repetitions of the terms on the line immediately above. It will also be noted that the values of the solution on line 24 are not needed until the last step, those on line 20 are not needed until the next to last step, and so on, if there are more than three equations. Furthermore, once a value appears in the computation, it must be revised at every subsequent step. Hence, if there were a method of obtaining these values at the time they are actually required for the next step, they could be omitted from all the previous computations, thus saving part of the work of revising them. This abbreviation can be achieved by using one of the checks to obtain the omitted values.

The revised variation operations illustrated in Table 7 show that there is a considerable saving in work even though it is partly offset by the necessity of making a substitution computation at every step in the process.

THEORY OF ABBREVIATED SOLUTION

The procedure followed in Table 5 for solving a set of simultaneous equations may be simplified by taking advantage of certain characteristics of this type of solution.

Reference to Eqs. 21*a* and 21*b* will show that u', the revised value of the unknown whose coefficients are being varied in the particular step or "variation" under consideration, is computed first and that it also enters in the formula for the revised values of the other unknowns. The term u' may thus be considered the critical one for its step. It will be shown that, in a sense, a solution reduces to the computation of critical values only—that is, values of the unknowns that have not been critical in previous steps, and that are also not critical in the step under consideration or in the one immediately following, need not be computed.

TABLE 6.—COMPARISON OF RESULTS OF TABLE 4 AND TABLE 5

Line No.	EQUATIONS IN x, y, AND z →				Constants		← EQUATIONS IN a, b, AND c		
(1)	(2)	(3)	(4)		(5)		(6)	(7)	(8)
					(*a*) STATEMENT OF EQS. 22 WITH SOLUTIONS (FROM TABLE 4)				
1.....	$+4x$	$+3y$	$+2z$	=	28	=	a	0	0
2.....	$+x$	$+4y$	$+3z$	=	26	=	0	b	0
3.....	$+5x$	$+2y$	$+6z$	=	49	=	0	0	c
4.....	x	0	0	=	3	=	$+\frac{2}{7}a$	$-\frac{2}{9}b$	$+\frac{1}{63}c$
5.....	0	y	0	=	2	=	$+\frac{1}{7}a$	$+\frac{2}{9}b$	$-\frac{10}{63}c$
6.....	0	0	z	=	5	=	$-\frac{2}{7}a$	$+\frac{1}{9}b$	$+\frac{13}{63}c$
					(*b*) VARIATION OF COEFFICIENTS OF EQS. 22 (FROM TABLE 4)				
7.....	x	0	0	=	$+\frac{34}{13}$	=	$+\frac{4}{13}a$	$-\frac{3}{13}b$	0
8.....	0	y	0	=	$+\frac{76}{13}$	=	$-\frac{1}{13}a$	$+\frac{4}{13}b$	0
9.....	0	0	$(z)^a$	=	$+\frac{315}{13}$	=	$-\frac{18}{13}a$	$+\frac{7}{13}b$	c
10.....	x	0	0	=	$+7$	=	$+\frac{1}{4}a$	0	0
11.....	0	(y)	0	=	$+19$	=	$-\frac{1}{4}a$	b	0
12.....	0	0	$[z]^b$	=	$+14$	=	$-\frac{5}{4}a$	0	c
13.....	(x)	0	0	=	$+28$	=	a	0	0
14.....	0	$[y]$	0	=	$+26$	=	0	b	0
15.....	0	0	$[z]$	=	$+49$	=	0	0	c

In Table 5, for example, the solutions on lines 16, 17, and 18 correspond to Eqs. 23, in which the values of x and y are obviously determined by the first two equations and are independent of the third. Mathematically, this fact becomes evident in the computation for x_a and y_a on lines 7 and 8, respectively, Table 5; that is, $c = d_3 = +5$ is substituted in lines 4 and 5, where it is multiplied by zero; d_3 is obviously the coefficient of x in the third of Eqs. 23. The corresponding result occurs in the computation for x_d and y_d in lines 13 and 14, wherein $c = d_3 = +2$ is multiplied by zero when substituted in lines 10 and 11; $d_3 = +2$ is obviously the coefficient of y in this same third equation.

TABLE 6.—(*Continued*)

Line No.	EQUATIONS IN x, y, AND z →				Constants		← EQUATIONS IN a, b, AND c		
(1)	(2)	(3)	(4)		(5)		(6)	(7)	(8)
(*c*) SOLUTION OF EQS. 22 BY VARIATION OF COEFFICIENTS (FROM TABLE 5)									
16.....	x	0	0	=	$+28$	=	a	0	0
17.....	0	y	0	=	$+26$	=	0	b	0
18.....	0	0	z	=	$+49$	=	0	0	c
19.....	$(x)^a$	0	0	=	$+7$	=	$+\frac{1}{4}a$	0	0
20.....	0	y	0	=	$+19$	=	$-\frac{1}{4}a$	b	0
21.....	0	0	z	=	$+14$	=	$-\frac{5}{4}a$	0	c
22.....	$[x]^b$	0	0	=	$+\frac{34}{13}$	=	$+\frac{4}{13}a$	$-\frac{3}{13}b$	0
23.....	0	(y)	0	=	$+\frac{76}{13}$	=	$-\frac{1}{13}a$	$+\frac{4}{13}b$	0
24.....	0	0	z	=	$+\frac{315}{13}$	=	$-\frac{18}{13}a$	$+\frac{7}{13}b$	c
25.....	$[x]$	0	0	=	$+3$	=	$+\frac{2}{7}a$	$-\frac{2}{9}b$	$+\frac{1}{63}c$
26.....	0	$[y]$	0	=	$+2$	=	$+\frac{1}{7}a$	$+\frac{2}{9}b$	$-\frac{10}{63}c$
27.....	0	0	(z)	=	$+5$	=	$-\frac{2}{7}a$	$+\frac{1}{9}b$	$+\frac{13}{63}c$

[a] (u) Unknown whose coefficients are being varied in step under consideration. [b] $[u]$ Unknown whose coefficients have been varied in previous steps.

Since the values of x and y are independent of the third of Eqs. 23, all the coefficients in this equation may be neglected in the computation up to this point. However, once x and y are known, the values of z can be computed from the third equation by direct substitution. It therefore follows that the values of z need not necessarily be computed in the previous steps. The values of z must be computed at this point, however, since the coefficients of z are varied in the next step of Table 5 and the critical value $u' = z'$ could not otherwise be computed.

Similarly, the solutions on lines 10, 11, and 12, Table 5, correspond to Eqs. 24, in which x is determined by the first equation and is independent of

TABLE 7.—WORK SHEET FOR ABBREVIATED SOLUTION OF EQS. 22 BY VARIATION OF COEFFICIENTS

Line No.						VARIATION COEFFICIENTS			CONVERSION FACTORS					Substitution factors
									$a_n d_1$	$b_n d_2$	$c_n d_3$	u_d or v_d	$\frac{1}{1+u_d}$	
(1)	(2)	(3)	(4)		(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
1.....	$+x+[3x]$	$+[3y]$	$+[2z]$	=	28	a	0	0						
2.....	$+[x]$	$+y+[3y]$	$+[3z]$	=	26	0	b	0						
3.....	$+[5x]$	$+[2y]$	$+z+[5z]$	=	49	0	0	c						
4.....	x			=	28	a	0	0						
5.....		y		=	26	0	b	0						
6.....			z	=	49	0	0	c						
7.....	(x)			=	$+7$	$+\frac{1}{4}a$	0		3			$+3$	$\frac{1}{4}$	$(a = d_1 = +3)$
8.....		y		=	$+19$	$-\frac{1}{4}a$	b							
9.....					$+\frac{57}{13}$	$-\frac{3}{52}a$	$+\frac{3}{13}b$		$+\frac{3}{4}$	0		$\frac{3}{4}$		$a = d_1 = +3$
10.....					$+\frac{76}{13}$	$-\frac{1}{13}a$	$+\frac{4}{13}b$		$-\frac{3}{4}$	$+3$		$+\frac{9}{4}$	$\frac{4}{13}$	$b = d_2 = +3$
11.....	$+5x$			=	$+\frac{170}{13}$	$+\frac{20}{13}a$	$-\frac{15}{13}b$	0						
12.....		$+2y$		=	$+\frac{152}{13}$	$-\frac{2}{13}a$	$+\frac{8}{13}b$	0						
13.....					$+\frac{322}{13}$	$+\frac{18}{13}a$	$-\frac{7}{13}b$	0						
14.....	x			=	$+\frac{34}{13}$	$+\frac{4}{13}a$	$-\frac{3}{13}b$	0						
15.....		(y)		=	$+\frac{76}{13}$	$-\frac{1}{13}a$	$+\frac{4}{13}b$	0						
16.....			z	=	$+\frac{315}{13}$	$-\frac{18}{13}a$	$+\frac{7}{13}b$	c						

TABLE 7.—(*Continued*)

Line No.						VARIATION COEFFICIENTS			CONVERSION FACTORS					Substitution factors
									$a_n\ d_1$	$b_n\ d_2$	$c_n\ d_3$	u_d or v_d	$\frac{1}{1+u_d}$	
(1)	(1)	(3)	(4)		(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
17.....					$-\frac{5}{13}$	$+\frac{2}{91}a$	$-\frac{1}{117}b$	$-\frac{1}{63}c$	$+\frac{8}{13}$	$-\frac{9}{13}$	0	$-\frac{1}{13}$		$a = d_1 = +2$
18.....					$+\frac{50}{13}$	$-\frac{20}{91}a$	$+\frac{10}{117}b$	$+\frac{10}{63}c$	$-\frac{2}{13}$	$+\frac{12}{13}$	0	$+\frac{10}{13}$		$b = d_2 = +3$
19.....					$+5$	$-\frac{2}{7}a$	$+\frac{1}{9}b$	$+\frac{13}{63}c$	$-\frac{36}{13}$	$+\frac{21}{13}$	5	$+\frac{50}{13}$	$\frac{13}{63}$	$c = d_3 = +5$
20.....	x			=	$+3$	$+\frac{2}{7}a$	$-\frac{2}{9}b$	$+\frac{1}{63}c$						
21.....		y		=	$+2$	$+\frac{1}{7}a$	$+\frac{2}{9}b$	$-\frac{10}{63}c$						
22.....			(z)	=	$+5$	$-\frac{2}{7}a$	$+\frac{1}{9}b$	$+\frac{13}{63}c$						

the other two. Mathematically, this fact is again evident in the computation—the appropriate d_n-values are multiplied by zero in the computation for x_d. Hence, all the coefficients of the second and third equations may be neglected in the computation up to this point—that is, in the computation for $u' = x'$.

The values of both y' and z' may be computed by direct substitution, once x' is known. However, it is necessary to compute only the values of the unknown whose coefficients are to be varied in the next step—in this case y' is the next unknown to become critical. The revised values of z can be computed later, as already explained.

The abbreviated procedure may now be summarized. In Table 7, lines 1, 2, and 3, all the coefficients below the lower heavy line in each column may be neglected in each step or "variation." The values of u', which is critical, are computed by Eq. 21*a*; the values of v', corresponding to the unknowns whose coefficients have already been varied in previous steps, are computed by Eq. 21*b*; and the values of v', corresponding to the unknown whose coefficients are to be varied in the next step, are computed by direct substitution. The values of the remaining unknowns are neglected because they are not essential to the final solution, and are not computed until they are about to become critical in the step immediately following.

EXPLANATION OF TABLE 7

In Table 7, the order of variation of the coefficients is the same as in Table 5: x, y, and z.

In the first step, it is only necessary to use $d_1 = 3$, corresponding to $[3\,x]$ in line 1, Col. 2, as previously explained. With this exception, nothing new is presented by line 7, which gives the first revised value of x. The primes are omitted for simplicity—the x in line 7, for instance, is really x'. Since the solution in this step will correspond to Eqs. 24, the values of y, which is critical in the next step, may be computed from the second of these equations—$y = c_2 - x$.

Values of x on line 7, Cols. 5, 6, and 7 subtracted from the corresponding values on line 2 result in the revised values of y in line 8. As previously explained, it is not essential to compute the revised values of z at this point.

In the second step, variation of the coefficients of y, it is only necessary to use $d_1 = +3$ and $d_2 = +3$. With this exception, the computations on lines 9 and 10, and lines 14 and 15 present nothing new. Since the solution in this step will correspond to Eqs. 23, the values of z can be computed from the third of these equations—$z = c_3 - (+5\,x + 2\,y)$. The quantities in parentheses are computed on lines 11 and 12 and are summed up on line 13. Subtraction of the values on line 13 from the corresponding values on line 3, Cols. 5, 6, 7, and 8, gives the revised values of z on line 16.

The third and last step presents nothing new since d_1, d_2, and d_3 are used and there is no substitution to be made; the last variation in Table 7 is exactly the same as that in Table 5.

MULTIPLE VARIATION

If the number of unknowns is large, the solution still requires considerable work—there will be as many steps or variation operations as there are unknowns. A further simplification is possible in such a case. If the coefficients of two or more unknowns are varied at the same time, an auxiliary set of simultaneous equations must be solved. If the coefficients of all the unknowns were varied simultaneously, for instance, the auxiliary equations would simply be a repetition of the original equations. However, since it is fairly easy to solve a set of two or even three equations, it does save considerable work to vary two or three unknowns at a time. The result of such a procedure is, in effect, that one large set of equations is broken up into several smaller sets. It is well known that, when a similar situation occurs in an analysis as the result of symmetry, the resulting equations are much easier to solve.

Referring to Eqs. 11, let it be supposed that the coefficients A_1, A_2, and A_3 are to be increased by d_1, d_2, and d_3, respectively, as before. In addition, however, the coefficients B_1, B_2, and B_3 also are to be increased by amounts e_1, e_2, and e_3, respectively. Following the same procedure used for varying the coefficients of only one unknown:

$$\left.\begin{aligned} a &= K_1 - d_1 x' - e_1 y' \\ b &= K_2 - d_2 x' - e_2 y' \\ c &= K_3 - d_3 x' - e_3 y' \end{aligned}\right\} \qquad (27)$$

Substituting these values in line 4, Table 3, gives

$$\begin{aligned} x' = a_1(K_1 - d_1 x' - e_1 y') &+ a_2(K_2 - d_2 x' - e_2 y') \\ &+ a_3(K_3 - d_3 x' - e_3 y') \end{aligned} \qquad (28a)$$

or

$$\begin{aligned} (1 + a_1 d_1 + a_2 d_2 + a_3 d_3)x' &+ (a_1 e_1 + a_2 e_2 + a_3 e_3)y' \\ &= a_1 K_1 + a_2 K_2 + a_3 K_3 \end{aligned} \qquad (28b)$$

Therefore,

$$(1 + x_d)x' + x_e y' = x \qquad (29)$$

Similarly, substitution of the values for a, b, and c, in line 5, Table 3, gives

$$y_d x' + (1 + y_e)y' = y \qquad (30)$$

and substitution in line 6 gives

$$z' = z - z_d x' - z_e y' \qquad (31)$$

To obtain x' and y', Eqs. 29 and 30 must be solved simultaneously; the values of z' are then computed directly from Eq. 31.

If there were more than three unknowns, the equations for the remaining ones would take the same form as Eq. 31; for example:

$$r' = r - r_d x' - r_e y' \qquad (32)$$

Generalizing,

$$(1 + u_d)u' + u_e w' = u \qquad (33a)$$

$$w_d u' + (1 + w_e)w' = w \qquad (33b)$$

$$v' = v - v_d u' - v_e w' \qquad (34)$$

TABLE 8.—WORK SHEET FOR SOLUTION OF EQS. 35 BY MULTIPLE VARIATION OF COEFFICIENTS

Line No.							VARIATION COEFFICIENTS				CONVERSION FACTORS						Substitution factors
											$a_n d_1$	$b_n d_2$	$c_n d_3$	$f_n d_4$	u_d	$1 + u_d$	
(1)	(2)	(3)	(4)	(5)		(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)
1.....	$+10w$	$+9x$	$+8y$	$+7z$	=	$+114$	a	0	0	0							
2.....	$+9w$	$+6x$	$+5y$	$+4z$	=	$+76$	0	b	0	0							
3.....	$+8w$	$+5x$	$+3y$	$+2z$	=	$+53$	0	0	c	0							
4.....	$+7w$	$+4x$	$+2y$	$+z$	=	$+39$	0	0	0	f							
5.....	w				=	$+114$	a	0	0	0							
6.....		x			=	$+76$	0	b	0	0							
7.....			y		=	$+53$	0	0	c	0							
8.....				z	=	$+39$	0	0	0	f							
9.....	$+10w$	$+9x$			=	$+114$	a	0	0	0	$+9$	0			$+9$	$+10$	$d_1 = +9$
10.....	$+9w$	$+6x$			=	$+76$	0	b	0	0	0	$+9$			$+9$		$d_2 = +9$
11.....	w				=	0	$-\frac{2}{7}a$	$+\frac{3}{7}b$	0	0	$+9$	$+0$			$+9$		$e_1 = +9$
12.....		x			=	$+\frac{266}{21}$	$+\frac{3}{7}a$	$-\frac{10}{21}b$	0	0	0	$+5$			$+5$	$+6$	$e_2 = +5$
13.....	$+8w$				=	0	$-\frac{16}{7}a$	$+\frac{24}{7}b$	0	0							
14.....		$+5x$			=	$+\frac{1,330}{21}$	$+\frac{15}{7}a$	$-\frac{50}{21}b$	0	0							
15.....	$+7w$				=	0	$-2a$	$+3a$	0	0							
16.....		$+4x$			=	$+\frac{1,064}{21}$	$+\frac{12}{7}a$	$-\frac{40}{21}a$	0	0							
17.....	w				=	0	$-\frac{2}{7}a$	$+\frac{3}{7}b$	0	0							
18.....		x			=	$+\frac{266}{21}$	$+\frac{3}{7}a$	$-\frac{10}{21}b$	0	0							
19.....			y		=	$-\frac{217}{21}$	$+\frac{1}{7}a$	$-\frac{22}{21}b$	c	0							
20.....				z	=	$-\frac{245}{21}$	$+\frac{2}{7}a$	$-\frac{23}{21}b$	0	f							

TABLE 8.—(*Continued*)

Line No.							VARIATION COEFFICIENTS				CONVERSION FACTORS						Substitution factors
											$a_n d_1$	$b_n d_2$	$c_n d_3$	$f_n d_4$	u_d	$1 + u_d$	
(1)	(2)	(3)	(4)	(5)		(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)
21.....			$-23y$	$-25z$	=	-217	$+3a$	$-22b$	$+21c$	0	$-\frac{16}{7}$	$+\frac{15}{7}$	0	0	$-\frac{1}{7}$		$d_1 = +8$
22.....			$-25y$	$-29z$	=	-245	$+6a$	$-23b$	0	$+21f$	$+\frac{24}{7}$	$-\frac{50}{21}$	0	0	$+\frac{22}{21}$		$d_2 = +5$
23.....											$+\frac{8}{7}$	$-\frac{110}{21}$	2	0	$-\frac{44}{21}$	$-\frac{23}{21}$	$d_3 = +2$
24.....			y		=	$+4$	$+\frac{3}{2}a$	$+\frac{3}{2}b$	$\left(-\frac{29c}{42}\right)21$	$\left(+\frac{25f}{42}\right)21$	$+\frac{16}{7}$	$-\frac{115}{21}$	0	2	$-\frac{25}{21}$		$d_4 = +2$
25.....				z	=	$+5$	$-\frac{3}{2}a$	$-\frac{1}{2}b$	$\left(+\frac{25c}{42}\right)21$	$\left(-\frac{23f}{42}\right)21$							
26.....											-2	$+\frac{12}{7}$	0	0	$-\frac{2}{7}$		$e_1 = +7$
27.....	$w_d y$				=	$-\frac{4}{7}$	$-\frac{3}{14}a$	$-\frac{3}{14}b$	$+\frac{29}{14}c$	$-\frac{25}{14}f$	$+3$	$-\frac{40}{21}$	0	0	$+\frac{23}{21}$		$e_2 + = 4$
28.....	$w_e z$				=	$-\frac{10}{7}$	$+\frac{3}{7}a$	$+\frac{1}{7}b$	$-\frac{25}{7}c$	$+\frac{23}{7}f$	$+1$	$-\frac{88}{21}$	2	0	$-\frac{25}{21}$		$e_3 = +2$
29.....											$+2$	$-\frac{92}{21}$	0	0	$-\frac{50}{21}$	$-\frac{29}{21}$	$e_4 = 0$
30.....		$x_d y$			=	$+\frac{88}{21}$	$+\frac{11}{7}a$	$+\frac{11}{7}b$	$-\frac{319}{21}c$	$+\frac{275}{21}f$							
31.....		$x_e z$			=	$+\frac{115}{21}$	$-\frac{23}{14}a$	$-\frac{23}{42}b$	$+\frac{575}{42}c$	$-\frac{529}{42}f$							
32.....	w				=	$+2$	$-\frac{1}{2}a$	$+\frac{1}{2}b$	$+\frac{3}{2}c$	$-\frac{3}{2}f$							
33.....		x			=	$+3$	$+\frac{1}{2}a$	$-\frac{3}{2}b$	$+\frac{3}{2}c$	$-\frac{1}{2}f$							
34.....			y		=	$+4$	$+\frac{3}{2}a$	$+\frac{3}{2}b$	$-\frac{29}{2}c$	$+\frac{25}{2}f$							
35.....				z	=	$+5$	$-\frac{3}{2}a$	$-\frac{1}{2}b$	$+\frac{25}{2}c$	$-\frac{23}{2}f$							

in which both u and w represent the unknowns whose coefficients are being varied, and e_n is the amount by which each of the coefficients of w is being varied. The notation is otherwise the same as that of Eqs. 21*a* and 21*b*, and the primes have the same meaning.

Revised values of the unknowns whose coefficients are being varied simultaneously are computed by solving an auxiliary set of simultaneous equations such as Eqs. 33. The remaining unknowns are then computed by Eq. 34. Eqs. 33*a*, 33*b*, and 34 can obviously be extended to include the simultaneous variation of any number of unknowns and the solution of a large set of simultaneous equations can thus be broken up into the solution of several smaller sets. The optimum number of simultaneous variations is more or less a matter of individual preference. The procedure is illustrated in Table 8 for a set of four equations split into pairs of two each.

Explanation of Table 8

The procedure for the variation of the coefficients of more than one unknown at one time is similar, in general, to that illustrated in the previous tables for variation of the coefficients of only one unknown.

In Table 8, the set of four equations—

$$\left.\begin{array}{r} +\,10\,w + 9\,x + 8\,y + 7\,z = \\ +\;\;9\,w + 6\,x + 5\,y + 4\,z = \\ +\;\;8\,w + 5\,x + 3\,y + 2\,z = \\ +\;\;7\,w + 4\,x + 2\,y + \;\;z = \end{array} \begin{array}{|r||c|c|c|c|} 114 & a & 0 & 0 & 0 \\ 76 & 0 & b & 0 & 0 \\ 53 & 0 & 0 & c & 0 \\ 39 & 0 & 0 & 0 & f \end{array}\right\}\cdots\cdots\cdots(35)$$

—is broken up into a pair of sets with two unknowns each, as indicated by the heavy lines in lines 1 to 4, Cols. 2 to 5. Since Table 8 illustrates a solution, the procedure and abbreviations of Table 7, suitably modified, are also used in Table 8.

Remembering that Eqs. 33*a*, 33*b*, and 34 are used instead of Eqs. 21*a* and 21*b*, reference to Table 7 will clarify the work in Table 8. Coefficients below the lower heavy line, for example, are omitted from the variation computations in both cases for reasons previously explained. However, since two sets of coefficients are being varied in the same step, the numerical equivalents of Eqs. 33 are shown in lines 9 and 10. These two auxiliary equations are solved simultaneously by inspection—as will be explained later—and the result is given in lines 11 and 12. Since the coefficients of both x and y are to be varied in the next step, their revised values are obtained at this point by direct substitution as shown in lines 13 to 16, and lines 19 and 20. It happens that there are no other unknowns, but if there were, their revised values would not have to be computed at this point; they can be computed later by direct substitution when needed. The complete solution for the first step is shown in lines 17 to 20.

The next step is similar: The auxiliary equations are shown in lines 21 and 22, their solution (by inspection) in lines 24 and 25, and part of the numerical equivalents of Eq. 34 in lines 30 and 31. The final solution for this step, which is the last of this computation, is shown in lines 32 to 35. Since this is the last step and there are no other unknowns, there is no direct substitution to be made.

An over-all check of the entire computation may be made by adding the four equations on lines 1 to 4, Table 8, and substituting, in the resulting equation, the sum of the solutions on lines 32 to 35, Table 8, setting $a = b = c = f = 1$. For example, $w = 2 - \frac{1}{2} + \frac{1}{2} + \frac{3}{2} - \frac{3}{2} = 2$, as shown in line 1, Col. 8, Table 9. The result of this substitution (line 5, Col. 7, Table 9) should equal the sum of the constants in lines 1 to 4, Table 8 (again setting $a = b = c = f = 1$), all as shown on lines 1 to 5, Table 9.

TABLE 9.—CHECK FOR SOLUTION OF EQS. 35 BY MULTIPLE VARIATION

Line No. (1)	(2)	(3)	(4)	(5)		(6)	(7)	Substitution factors (8)
1.......	$34w$				=	+68	+115	$w = 2$
2.......		$24x$			=	+72	+77	$x = 3$
3.......			$18y$		=	+90	+54	$y = 5$
4.......				$14z$	=	+56	+40	$z = 4$
5.......						+286	+286	$a = b = c = f = 1$
6.......			c		=	+71	+3	$c = +71$
7.......				$-f$	=	−53	+4	$f = +53$
8.......						+18	+5	$w = x = y = z = 1$
9.......							+6	
10......							+18	

The over-all check on lines 6 to 10, Table 9, is similarly made by considering a, b, c, and f as the unknowns and setting $w = x = y = z = 1$; that is, lines 32 to 35, Table 8, are considered as the simultaneous equations, with lines 1 to 4, Table 8, as the solutions.

As previously stated, the solutions of the auxiliary equations on lines 11 and 12, and on lines 24 and 25, Table 8, may be found practically by inspection. Considering a set of simultaneous equations such as is given in lines 1 to 4, Table 8, and its solutions on lines 32 to 35, it can be shown that the determinant formed from the coefficients of a, b, c, and f on lines 32 to 35 is the reciprocal of the determinant formed from the coefficients of w, x, y, and z on lines 1 to 4. Either determinant can be obtained from the other. An element a_{mn} of one determinant is equal to the cofactor of the element b_{nm} of the other determinant divided by the value of the latter determinant. The cofactors of a second order determinant are simply elements—in each case, simply the element not crossed out. It is also a simple matter to find the value of the determinant itself. The solution of the auxiliary equations can therefore be found practically by inspection.

As an illustrative example, reference is made to lines 9 and 10, Table 8, where the determinant formed by the coefficients of the auxiliary equations is: $\begin{vmatrix} 10 & 9 \\ 9 & 6 \end{vmatrix}$. The value of this determinant is $(10 \times 6) - (9 \times 9) = -21$.

The cofactor of the element 10 (which is a_{11}) is 6, and the element b_{11} of the reciprocal determinant is therefore $-\frac{6}{21} = -\frac{2}{7}$. The cofactor of a_{12}, or 9, is -9; the element b_{21} is $+\frac{3}{7}$; etc. The reciprocal determinant is, therefore, $\begin{vmatrix} -\frac{2}{7} & +\frac{3}{7} \\ +\frac{3}{7} & -\frac{10}{21} \end{vmatrix}$. The elements of this determinant are the variation coefficients on lines 11 and 12, Cols. 7 and 8.

The values shown in parentheses in lines 24 and 25, Cols. 9 and 10, are similarly obtained from the determinant formed from the coefficients on lines 21 and 22, Cols. 4 and 5.

In both cases, the other values are obtained by using the variation coefficients to obtain other solutions. For example, the set of constants from lines 21 and 22, Col. 8 (that is— $-22\,b$ and $-23\,b$), are substituted for c in line 24, Cols. 9 and 10. Thus, $y' = \left(-\frac{29}{42}\right)(-22\,b) + \frac{25}{42}(-23\,b) = +\frac{3}{2}\,b$, which is the value of y' in line 24, Col. 8. The primes are not indicated in Table 8 for simplicity.

Similarly, substitution of the constants -217 and -245 for c, line 25, gives $z' = +\frac{25}{42}(-217) + \left(-\frac{23}{42}\right)(-245) = +5$, which is the value of z' in line 25, Col. 6. The other values are computed in similar manner.

It will be noticed that the coefficients on lines 1 to 4, Table 8, occur in pairs, as they would in a practical design problem as a result of the Maxwell theorem of reciprocal deflections. The variation coefficients, lines 32 to 35, also occur in pairs and in the same order.

ACCURACY OF COMPUTATIONS

In practical design, the question of the number of significant digits to which the computations must be carried is a difficult one to determine. It is, however, a very important consideration from the point of view of time. The method of variation of coefficients makes the determination of the required accuracy a simple matter.

DISCUSSION

T. F. HICKERSON,[2] M. ASCE.—This interesting paper is presented elaborately and clearly, except in certain instances where the author generalizes rather abruptly while introducing a multiplicity of unfamiliar symbols.

Two sets of simultaneous equations will be solved independently with a twofold purpose—(1) to help elucidate inductively the application of the author's key formulas and (2) to make comparisons of the relative advantages claimed by the author. Of the so-called classical methods the writer would eliminate determinants and include only the method of successive elimination by addition or subtraction after the coefficients of each unknown, in turn, have been made equal by multiplication or division—as was done in Table 1.

The solution of the set of three equations (Eqs. 22) given in Table 10 is believed to be self-explanatory. Anticipating a later variation of the coefficients, the letters a, b, and c are used here in a manner similar to that in Table 4. It will be seen that the equations are not renumbered when multiplied, or divided by any quantity, because in effect they are unchanged. Thus,

$$4\,x + 3\,y + 2\,z = 28 = a \tag{36a}$$

is equivalent to

$$12\,x + 9\,y + 6\,z = 84 = 3\,a \tag{36b}$$

In Table 10, let the coefficients of z in (1), (2), and (3) (Eqs. 22) be reduced by 2, 3, and 5, respectively, as was done in Table 4. Then a, b, and c will be replaced by $(a - 2\,z')$, $(b - 3\,z')$, and $(c - 5\,z')$, respectively. Hence, z', the revised value of z, becomes

$$z' = -\frac{2}{7}(a + 2\,z') + \frac{1}{9}(b + 3\,z') + \frac{13}{63}(c + 5\,z') \tag{37a}$$

Rearranging terms,

$$z' + \left[\frac{2}{7}(2) - \frac{1}{9}(3) - \frac{13}{63}(5)\right] z' = -\frac{2}{7}a + \frac{1}{9}b + \frac{13}{63}c \tag{37b}$$

It must be seen that the right side of Eq. 37*b* represents z (= 5), the initial (unrevised) value of z; the expression in the brackets equals z'_d and is comparable to x_d in Eq. 15*b*, the numerical value of which is $-\frac{50}{63}$, as given in Col. 12, line 9, Table 4. By factoring Eq. 37*b*, and solving for z', $z' = \dfrac{5}{1 + \dfrac{50}{63}}$ (see Eq. 21*a*) $= \frac{315}{13}$, as recorded in Col. 5, line 12, Table 4.

---

[2] Prof. of Applied Mathematics, Univ. of North Carolina, Chapel Hill, N. C.

To determine the corresponding revised values of x (designated x') due to the variation in the three coefficients of z, $x' = \frac{2}{7}(a + 2z') - \frac{2}{9}(b + 3z') + \frac{1}{63}(c + 5z') = x - \left(-\frac{4}{7} + \frac{2}{3} - \frac{5}{63}\right)z' = 3 - \left(\frac{1}{63}\right)\left(\frac{315}{13}\right) = \frac{34}{13}$, as

TABLE 10.—WORK SHEET FOR SOLUTION OF EQS. 22

Equation	x	y	z	Constant term
(1)	4	3	2	$= 28 = a$
(2)	1	4	3	$= 26 = b$
(3)	5	2	6	$= 49 = c$
(1)	12	9	6	$= 84 = 3a$
(2)	2	8	6	$= 52 = 2b$
(4)	7	7	0	$= 35 = 3a - c$
(5)	10	1	0	$= 32 = 3a - 2b$
(4)	1	1		$= 5 = \frac{3}{7}a - \frac{1}{7}c$
(6)	9	0		$= 27 = \frac{18}{7}a - 2b + \frac{1}{7}c$
(6)	1 (x)			$= [3] = \frac{2}{7}a - \frac{2}{9}b + \frac{1}{63}c$
(4)		1 (y)		$= [2] = \frac{1}{7}a + \frac{2}{9}b - \frac{10}{63}c$
(1)			1 (z)	$= [5] = -\frac{2}{7}a + \frac{1}{9}b + \frac{13}{63}c$

Check: $x + y + z = 10 = \frac{1}{7}a + \frac{1}{9}b + \frac{4}{63}c = \frac{28}{7} + \frac{26}{9} + \frac{4 \times 49}{63} = 10$

TABLE 11.—WORK SHEET FOR SOLUTION OF EQS. 35

Equation	w	x	y	z	Constant term
(1)	10	9	8	7	$= 114 = a$
(2)	9	6	5	4	$= 76 = b$
(3)	8	5	3	2	$= 53 = c$
(4)	7	4	2	1	$= 39 = f$
(1)	10	9	8	7	$= 114 = a$
(2)	15.75	10.5	8.75	7	$= 133 = 1.75b$
(3)	28.0	17.5	10.5	7	$= 185.5 = 3.5c$
(4)	49.0	28.0	14.0	7	$= 273.0 = 7.0f$
(5)	−5.75	−1.5	−0.75	0	$= -19.0 = a - 1.75b$
(6)	−12.25	−7.0	−1.75	0	$= -5.25 = 1.75b - 3.5c$
(7)	−21.0	−10.5	−3.5	0	$= -87.5 = 3.5c - 7.0f$
(5)	7.667	2.0			$= 25.333 = -1.333a + 2.333b$
(6)	7.0	4.0	1		$= 30.0 = -b + 2c$
(7)	6.0	3.0	1		$= 25.0 = -c + 2f$
(8)	0.667	−2.0	0		$= -4.667 = -1.333a + 3.333b - 2c$
(9)	1.0	1.0	0		$= 5.0 = -b + 3c - 2f$
(8)	0.333	−1.0			$= -2.333 = -0.667a + 1.667b - c$
(10)	1.333	0			$= 2.667 = -0.667a + 0.667b + 2c - 2f$
(10)	1 (w)				$= [2] = -0.5a + 0.5b + 1.5c - 1.5f$
(9)		1 (x)			$= [3] = 0.5a - 1.5b + 1.5c - 0.5f$
(7)			1 (y)		$= [4] = 1.5a + 1.5b - 14.5c + 12.5f$
(4)				1 (z)	$= [5] = -1.5a - 0.5b + 12.5c - 11.5f$

Check: $w + x + y + z = 14 = c - f = 53 - 39 = 14$

recorded in Col. 5, line 10, Table 4. Likewise, $y' = \frac{1}{7}(a + 2z') + \frac{2}{9}(b + 3z') - \frac{10}{63}(c + 5z') = y - \left(-\frac{2}{7} - \frac{2}{3} + \frac{50}{63}\right)z' = 2 - \left(-\frac{10}{63}\right)\left(\frac{315}{13}\right) = \frac{76}{13}$, as recorded in Col. 5, line 11, Table 4. The revision due to the variation of the coefficients of z alone is now complete, and the revised set of equations is given by Eqs. 23. The variation will not be continued as was done by the author in Table 4. Instead of breaking the four equations (Eqs. 35) into a pair of sets with two unknowns each, as is indicated in Table 8, the classical method is applied for purposes of comparison (Table 11).

TABLE 12.—WORK SHEET FOR SOLUTION OF EQS. 35 AND THE REVISION —WHEN THE COEFFICIENTS OF w ARE VARIED

Equation	w	x	y	z	Constant
(1)	10 (8)	9	8	7	= 114
(2)	9 (7)	6	5	4	= 76
(3)	8 (6)	5	3	2	= 53
(4)	7 (5)	4	2	1	= 39
(1)	10 (8)	9	8	7	= 114
(2)	15.75 (12.25)	10.5	8.75	7	= 133
(3)	28.0 (21.0)	17.5	10.5	7	= 185.5
(4)	49.0 (35.0)	28.0	14.0	7	= 273.0
(5)	−5.75 (−4.25)	−1.5	−0.75	0	= − 19.0
(6)	−12.25 (−8.75)	−7.0	−1.75	0	= − 5.25
(7)	−21.0 (−14.0)	−10.5	−3.5	0	= − 87.5
(5)	7.667 (5.667)	2.0	1		= 25.333
(6)	7.0 (5.0)	4.0	1		= 30.0
(7)	6.0 (4.0)	3.0	1		= 25.0
(8)	0.667 (0.667)	−2.0	0		= − 4.667
(9)	1.0 (1.0)	1.0	0		= 5.0
(8)	0.333 (0.333)	−1.0			= − 2.333
(10)	1.333 (1.333)	0			= 2.667
(10)	1 (w)				= 2.0 [2.0]
(9)		1 (x)			= 3.0 [3.0]
(7)			1 (y)		= 4.0 [8.0]
(4)				1 (z)	= 5.0 [1.0]

Check: $26w + 24x + 18y + 14z = 282$ and $52 + 72 + 144 + 14 = 282$

Omitting a, b, c, and f would shorten the work 50% or more, hence at least one of the coefficients may be varied and the corresponding equations resolved before starting the computation of the conversion factors proposed by the author. This point is illustrated in Table 12, in which varied coefficients of w and values of the constants are shown in parentheses. If, however, the constants are to be varied while the coefficients remain fixed—that is, if the loading conditions are varied while the framework remains unchanged—it is advantageous to exprdss the values of w, x, y, and z in terms of a, b, c, and f.

A. FLORIS,[3] ESQ.—The proposed method of solving simultaneous equations is a modification of the process of successive elimination of unknowns. This is clearly seen from the "Remarks" in Col. 7, Table 1. From a practical point of view the author's method is of value only if a numerical error has been made in the statical analysis, necessitating the change of one or several coefficients of the unknowns. Hence, it is of restricted usefulness in the design of structures, since a change in the shape will require changing all the coefficients of the unknowns, including the terms that depend on the loading.

---

[3] Los Angeles, Calif.

The method presented is not readily adapted to the solution of ill-conditioned equations; that is, equations in which different values for the unknowns can be obtained that will satisfy the equations.

The following equations, given by T. S. Wilson,[4] are cited as examples:

$$5x + 7y + 6z + 5u = 23 \tag{38a}$$

$$7x + 10y + 8z + 7u = 32 \tag{38b}$$

$$6x + 8y + 10z + 9u = 33 \tag{38c}$$

and

$$5x + 7y + 9z + 10u = 31 \tag{38d}$$

The exact values of the unknowns in Eqs. 38 are

$$x = y = z = u = 1 \tag{39}$$

There are other solutions besides Eq. 39, however, as may be demonstrated by substituting the following three sets of values, successively, in Eqs. 38:

x	y	z	u
14.6	−7.2	−2.5	3.1
2.36	0.18	0.65	1.21
1.136	0.918	0.965	1.021

The result is three additional values for the right-hand side of Eq. 39, as follows:

Eq. 38a	Eq. 38b	Eq. 38c	Eq. 38d
23.1	31.9	32.9	31.1
23.01	31.99	32.99	31.01
23.001	31.999	32.999	31.001

Similar cases of ill-conditioned equations also can be found among those amenable to iteration.

FREDERICK S. MERRITT,[5] ASSOC. M. ASCE.—The writer has used the method of variation of coefficients on several occasions for solving the simultaneous linear equations arising in engineering practice. As a result of this experience, he has reached the conclusion that the method is especially well adapted for the purpose.

In the linear equations encountered in structural engineering, for example, the coefficients of the unknowns on the left side of the equation are generally functions of the dimensions of the structure and the constant terms on the right side are usually functions of the magnitude and position of the load. Classical methods of solution may readily be applied when the coefficients and the constants are unchanged, or even when the constants have several sets of values. However, when any of the coefficients are changed, say, because of change in the dimensions of a part of the structure, classical methods leave no recourse other than to re-solve the equations. On the other hand, the method of variation of coefficients makes it possible to eyaluate, directly, the effect of a change in the coefficients.

---

[4] "The Escalator Method in Engineering Vibration Problems," by Joseph Morris, John Wiley & Sons, Inc., New York, N. Y., 1947, p. 42.

[5] Associate Editor, *Engineering News-Record*, New York, N. Y.

I. F. MORRISON,[6] ESQ.—In this interesting paper the author has developed a systematic method for the solution of simultaneous linear equations which offers "several marked advantages over the classical methods" without specifying which of the several classical methods. The method developed in the paper somewhat resembles the Doolittle method, which is a modification of the method developed by C. F. Gauss. The writer has always found this to be a most satisfactory method for the solution of such equations. It is certainly classical. In so far as he is aware, no difficulty concerning the small difference of quantities arises, as it so often does in the classical Cramer method. The introduction of the idea of a change in some of the coefficients is novel and of merit. The method developed in the paper possesses the advantages inherent in all well-planned computation procedure.

In the analysis of any structure, indeterminate to the nth degree, a set of simultaneous linear equations eventually must be solved in one way or another. Even the modern relaxation methods are simply a way of doing this. These equations may be written in condensed form as

$$a\,x = C \dots\dots\dots\dots (40)$$

in which a is a square matrix of order n, based solely on the size, shape, and elastic properties of the structure; in fact, it may be said to represent the structure in mathematical form. The elements of this matrix, a_{ij}, are most conveniently computed numerically from assumed self-stressed conditions. The elements of x, which is a column matrix, are the redundants—that is, the unknown thrusts, moments, shears, etc., which are sought.

The elements of C, also a column matrix, are the loading terms. They depend on the loads and the structure but, nevertheless, are independent of a. They may be changed to meet any number of local requirements.

The solution of Eq. 40 is

$$x = a^{-1}\,C \dots\dots\dots\dots (41)$$

and since a^{-1} is not dependent on C, it will be constant for all loadings provided there be no change in a.

The paper presents the following mathematical problem: Given a set of simultaneous equations in the form of Eq. 40, in which the symbols represent matrices, which have been solved for x, it is required to find from that solution what new values of x (say, x') will result from a change of the matrix a into $[a + b]$. The assumed solution of Eq. 40 is given by Eq. 41, which presupposes that a^{-1} is known. According to the conditions of the problem,

$$[a + b]\,x' = C = a\,x \dots\dots\dots\dots (42a)$$

which, multiplied through by a^{-1}, becomes

$$a^{-1}\,[a + b]\,x' = [I + a^{-1}\,b]\,x' = x \dots\dots\dots\dots (42b)$$

In Eq. 42b, I is the unit matrix. To solve Eq. 42b, multiply through by $[I + a^{-1}\,b]^{-1}$, so that,

$$x' = [I + a^{-1}\,b]^{-1}\,x \dots\dots\dots\dots (43)$$

In this solution, all the factors on the right-hand side are known so that x' is determined from the original x.

---

[6] Prof. of Applied Mechanics, Dept. of Civ. and Municipal Eng., Univ. of Alberta, Edmonton, Alberta, Canada.

By way of further explanation, it should be mentioned that the reciprocal matrix is best determined by finding the adjoint of the matrix and dividing by the determinant of the matrix. Thus, Eq. 43 may be written

$$x' = \frac{\text{adjoint}\ [I + a^{-1}\, b]}{|I + a^{-1}\, b|}\, x \ldots\ldots\ldots\ldots\ldots\ldots (44)$$

The adjoint A of a matrix a is another matrix of the same order in which the element a_{ij} is the cofactor of the element a_{ji}. The cofactor of a_{ji} is $(-1)^{i+j}$ times the $(n - 1)$-row determinant obtained by striking out row j and column i of $|a|$. It is not necessary to work out the value of the determinant because, if one multiplies the first row of the matrix by the first column of the adjoint, the result is the numerical value of the determinant. It is advisable to avoid large determinants.

In the first example (which, of course, has been chosen for simplicity), $a = \begin{bmatrix} 4 & 6 \\ 2 & 5 \end{bmatrix}$; $b = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$; $a^{-1} = \frac{1}{8}\begin{bmatrix} 5 & -6 \\ -2 & 4 \end{bmatrix}$; and $x = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. According to the foregoing theory,

$$a^{-1}\, b = \frac{1}{8}\begin{bmatrix} 5 & -6 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 15 & 0 \\ -6 & 0 \end{bmatrix}$$

$$I + a^{-1}\, b = \frac{1}{8}\begin{bmatrix} 23 & 0 \\ -6 & 8 \end{bmatrix}$$

and

$$[I + a^{-1}\, b]^{-1} = \begin{bmatrix} \frac{8}{23} & 0 \\ \frac{6}{23} & 1 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{8}{23} & 0 \\ \frac{6}{23} & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{24}{23} \\ \frac{133}{23} \end{bmatrix}$$

These results check those given in the paper. The first row of a-values multiplied by the first column of the cofactor of a^{-1} is 8, which is the value of the determinant $\begin{vmatrix} 4 & 6 \\ 2 & 5 \end{vmatrix}$.

The foregoing theory is quite general, however. Suppose, for example, all the elements of the original matrix are changed so that $b = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix}$; and let the other data remain the same. Then,

$$a^{-1}\, b = \frac{1}{8}\begin{bmatrix} 5 & -6 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 23 & -17 \\ -14 & 10 \end{bmatrix}$$

$$I + a^{-1}\, b = \frac{1}{8}\begin{bmatrix} 32 & -17 \\ -14 & 18 \end{bmatrix}$$

$$[I + a^{-1}\, b]^{-1} = \frac{1}{40}\begin{bmatrix} 18 & 17 \\ 14 & 31 \end{bmatrix}$$

and

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 18 & 17 \\ 14 & 31 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{139}{40} \\ \frac{197}{40} \end{bmatrix}$$

Referring to Eqs. 22 of the paper,

$$\text{Adjoint}\begin{bmatrix} 4 & 3 & 2 \\ 1 & 4 & 3 \\ 5 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 18 & -14 & 1 \\ 9 & 14 & -10 \\ -18 & 7 & 13 \end{bmatrix}$$

from which the reciprocal matrix and solution of the equations can be written immediately.

Whether this general method offers any advantage over a direct solution starting fresh with the revised, or new, matrix is open to doubt. The author has developed a step-by-step process, in work sheet form, which provides periodic checks. It is likely that this method will possess a definite advantage, especially for those not familiar with matrix calculations. For large sets of equations the method of submatrices[7] will be found very useful.

The author has correctly stated that in structural analysis certain coefficients occur in pairs. For any structure, the analysis of which is based on the assumption of Hooke's law, the matrix a, as a consequence, will always be symmetrical. This fact affords a check on computations because the reciprocal matrix a^{-1} will also be symmetrical. Moreover, the product of row 1 of a by column 1 of a^{-1}, row 2 by column 2, etc., will always be one; and row 1 by column 2, etc., will be zero. This affords a check on the computation of the reciprocal matrix.

SIGMUND ROOS,[8] M. ASCE.—The need for an efficient method of solving simultaneous linear equations is of primary importance in all fields of engineering. The method of variation of coefficients presented in this paper was developed for use in solving the equations encountered in structural engineering work, but it can be used for other equations to great advantage. The method is applied easily and gives exact answers rapidly.

The author is to be commended highly for his work, as he has employed the principles of advanced mathematics to formulate procedures which may be used without knowledge of the underlying mathematics. In this connection, it is suggested that the author add to his closing remarks a rigorous mathematical proof of the method. More engineers would be inclined to use the method if they knew its scientific basis.

It is difficult to make comparisons between the present paper and others that go so far toward satisfying a universal need among all engineers. In the field of structural engineering, however, it will be found that Mr. Weiner's paper has certain distinct advantages. It can be checked at each step and it can be adjusted easily to changes in design.

---

[7] "Some Devices for the Solution of Large Sets of Simultaneous Linear Equations," by W. J. Duncan, *Philosophical Magazine*, Vol. 35, 1944, p. 660.

[8] Structural Engr., Clarke, Rapuano and Holleran, Cons. Engrs., New York, N. Y.

It is the last mentioned feature of the method of variation of coefficients which makes this method particularly attractive. A system of equations which represents the geometric and elastic properties of a statically indeterminate structure may undergo considerable variation as the problems of design and detail are more completely studied. Minor changes of any one member or group of members of a continuous framework usually have an insignificant effect on the entire structure. An experienced designer knows the limits within which the length and cross section of a member can be varied without changing the elastic properties of the structure as a whole. Nevertheless, these limits are often exceeded, and the present method provides a quick procedure of accurately determining the resultant effects without resolving the entire system of equations. The writer has found the method of variation of coefficients to be a remarkably efficient tool for this purpose. In one case, it was found that, when the foundations for a rigid frame bridge were excavated, the elevation of rock was considerably lower than indicated on the boring record. The necessary adjustments in design were easily performed by using the method presented in this paper.

The method also presents a possibility not mentioned by the author, in that it is possible to use this method for the direct design of statically indeterminate structures. If one frame of a group of similar frames is designed, the other frames may be designed directly by merely varying the coefficients of the system of equations for the first frame.

In addition, the method of variation of coefficients possesses a very dynamic quality. Any system of equations can be extended far beyond the original conditions, thus giving the user great freedom of action. For these reasons, many new and important applications will undoubtedly be found as the method is used.

Bernard L. Weiner,[9] M. ASCE.—It is unfortunate that most of the discussers have stressed only one feature of the paper—the application of the theory to the solution of simultaneous equations—and have compared the method of solution by the variation of coefficients with other methods. The writer has never felt this application to be the salient feature of the theory. In fact, the theory was developed by the writer when it was necessary to find some way of correcting several errors in coefficients in a rather complicated analysis when there was insufficient time to re-solve the equations with the corrected coefficients. Application of the theory to the solution of equations came later as a rather obvious corollary.

As far as the solution of equations is concerned, comparisons may be made between the method of varying coefficients and other methods—both new and old. However, as far as the writer knows, there is no other method for finding the effect on the values of the unknowns resulting from a change in the coefficients—excepting, of course, the obvious one of re-solving the equations. It is well known that the analysis of statically indeterminate structures consists of the process of assuming the sizes of the various members—either from previous experience or from preliminary approximate designs—and then analyzing the structure to check the adequacy of the assumed sizes. If the designer is either sufficiently experienced, or lucky, his first assumption may be close enough for design purposes. However, should he find that his "guesses" are too far off, he must run another complete check analysis to test the revised sections.

---

[9] Associate Engr., D. B. Steinman, Cons. Engr., New York, N. Y.

With the method presented in this paper, the effect of the changes can be determined without going through an entirely new analysis. For frames and other structures in which the moments are large in comparison to the thrust—that is, in which the thrust has a large eccentricity—rather drastic changes may be made in the sections without seriously affecting the value of the redundants. However, in arches and other structures in which the thrust has a small eccentricity, the stresses are sensitive to changes in section and a check analysis becomes imperative. Variation of coefficients is a valuable tool in such cases.

However, in most instances, even if the designer finds that his first assumption of member sizes is sufficiently close, he cannot be sure that he has achieved the maximum in economy unless he tries to change sizes to achieve a net reduction in material. This, of course, involves changing the coefficients. Such studies, in which the number of redundants is large, are too time consuming to permit anything more than a very rough and inadequate investigation if the equations are re-solved for every change.

The writer believes that this application of the paper is its most important feature. Although it is believed that the theory offers many advantages in its application to the solution of equations, there are other methods. The fact remains, however, that variation of coefficients is the only known quick method for making the investigations previously described.

It is to be noted that, for the variation of any number of the coefficients of one unknown, only two simple formulas are necessary for the application of the theory. The factors such as u_d and v_d are actually solutions of the equations for a set of constants $c_n = d_1, d_2, d_3, \cdots, d_n$. Hence, no cumbersome formulas are necessary for their evaluations. This fact makes the theory easy to apply and the procedure also lends itself readily to work sheet form.

The writer disagrees with Mr. Floris' comment that Col. 7, Table 1, headed "Remarks," proves that the method of variation of coefficients is a modification of the process of successive elimination of coefficients. Table 1 is merely introductory and its heading states that it gives the solution of Eqs. 2 by the process of elimination. The purpose of this table is to obtain a solution by any method whatever, in order to illustrate the properties of certain solutions—namely, solutions for the values of $c_n = a$, 0, 0; 0, b, 0; and 0, 0, c. Later in the paper, it is shown that these solutions can be used to obtain the necessary conversion factors for varying the coefficients and are therefore designated "variation coefficients."

Mr. Floris states that the theory is not readily adaptable to the solution of certain types of equations and gives Eqs. 38 as an example. These equations are rather "tricky" and their solution would be difficult by any method unless special devices were used. This situation occurs fairly often in computations when the equations under consideration approach certain limits. Under such conditions, if certain special methods are used, the solution becomes very simple—more simple, very often, than in more "normal" cases; if they are not used, the solution becomes very difficult. A common example is the problem of finding middle ordinates or tangent offsets to a circular arc with very large radius. The usual method of the difference of the squares becomes very cumbersome; successive approximations however make the solution very simple—so simple in fact that slide rule computations are often sufficiently accurate. If the radius is very large, even the first correction is small enough to be neglected.

Mr. Floris shows that Eqs. 38 have the solution $x = y = z = u = 1$, but states that they also have other solutions. He then gives a series of solutions and shows that substitution of these values in Eqs. 38 results in a series of constants on the right-hand side of the equations which differ only slightly from the original constants.

The writer must disagree with Mr. Floris' statement that Eqs. 38 are satisfied by more than one value for each unknown. The determinant formed from the coefficients of Eqs. 38 is equal to 1. Since this determinant does not equal zero, it follows from the theory of equations that there exists only one solution for each unknown for any given set of constants. In spite of the fact that the values of the constants given in the second set of numbers differ only slightly from the original values (23.1, 23.01, and 23.001 instead of the original 23, for example), these constants are not the same, as would be the case if it were merely a question of accuracy of computations. The values given are absolutely exact and are different sets of constants. The difficulty lies in the fact that the equations are nearly "parallel" to each other and are therefore sensitive to slight changes—in this region, at least.

If similar equations with only two unknowns were solved, the solution would represent the coordinates of the intersection of two straight lines. If these lines were nearly parallel, their intersection point would be very sensitive to the constants on the right-hand side of the equations. Properly transformed —as shown in analytical geometry—these constants could be made to represent the distance from the origin of the points where the lines intersect the x-axis. If the two lines were made more and more nearly parallel, a slight change in this distance for one of the lines would make a tremendous difference in the point of the intersection. A change of, say, one unit in this distance could be made to result in a change of a million units—or any other large number in the location of the point of intersection by making the lines sufficiently parallel to each other. If they were absolutely parallel, the change, of course, would be infinity.

This conclusion can readily be proved. If Eqs. 38 are solved for a set of constants,

$$c_n = 1, -1, -1, 1 \ldots\ldots\ldots\ldots\ldots\ldots (45)$$

the results are

$$x = 136, \quad y = -82, \quad z = -35, \quad \text{and} \quad u = 21 \ldots\ldots\ldots (46)$$

Mr. Floris' results for the new constants of Eqs. 38 can be obtained by superposition by combining the original constants with multiples of Eq. 45. Combining Eq. 39 with the same multiples of Eqs. 46, the corresponding new values for x, y, z, and u given by Mr. Floris can be obtained; for example, taking

0.1 of Eq. 45 and adding to the constants of Eqs. 38 gives: $23 + 0.1 = 23.1$; $32 - 0.1 = 31.9$; $33 - 0.1 = 32.9$; and $31 + 0.1 = 31.1$.

Taking 0.1 of Eqs. 46 and combining them with Eq. 39 gives: $x = 1 + 0.1 \times 136 = 14.6$; $y = 1 - 0.1 \times (-86) = -7.6$; $z = 1 + 0.1 \times (-35) = -2.5$; and $1 + 0.1 \times 21 = 3.1$.

The remaining values on the second and third lines of the two sets of numbers given by Mr. Floris may be similarly obtained by using factors 0.01 and 0.001, respectively, instead of the factor 0.1 used in this computation.

This computation clearly illustrates that Eqs. 38 can be solved for any set of constants and the solution will have only one value for each unknown. The method of variation of coefficients may be used as well as any other method. This example of "ill-conditioned equations," as Mr. Floris calls them, is susceptible to easy solution with the aid of special devices (as are other equations) when they approach certain limits which make them difficult to handle by "routine" computations.

The contention of Mr. Floris that variation of coefficients is of limited usefulness is contradicted by some of the other discussers, especially by Mr. Roos who tells of some applications of the theory to problems he has solved.

Professor Hickerson varies the coefficients of the writer's Eqs. 22 as illustrated in Table 4 by re-solving the equations by successive elimination of the unknowns. The work is illustrated in Table 10; and, by using a, b, and c for the constants in addition to the numerical values 28, 26, and 48, it is shown how the basic formulas (Eqs. 21) come into the computation. The letter constants, a, b, and c, give rise, of course, to the solutions designated by the writer as "variation coefficients." After completing the solution, this method leaves the computer no advantage whatever. He has to re-solve the equations once more for another variation—unless he takes advantage of the variation coefficients as explained in the paper.

Professor Hickerson also solves Eqs. 35, by successive elimination of the unknowns. The advantage of breaking up a large set of equations into smaller sets as was done by the writer is not apparent in this example for two reasons. Throughout the paper, simple numerical examples were chosen in order to illustrate principles, thus eliminating the distractions produced by the mechanical difficulty of handling large numbers. Secondly, four equations with four unknowns only begin to make the solution of such equations cumbersome. However, as the number of terms in a set of simultaneous equations is equal to the square of the number of unknowns, this number increases very rapidly. In larger sets of equations (equations with twelve and eighteen unknowns) the simplifications become very marked.

The writer notices that Professor Hickerson has omitted the "Remarks" column from his work sheets. The writer on numerous occasions has found this column invaluable and has been grateful to the unknown person who suggested its use. The writer agrees wholeheartedly with one of the other discussers (Professor Morrison) that systematic arrangement is the very essence of any computation. It not only helps eliminate errors, but also reduces the labor involved in any computation more than practically any other one single feature. The writer has made a statement (under the heading, "Introduction") to this effect in which it is also stated that the computation for multiple loading constants (such as influence lines) is no great hardship if the work is systematically arranged. The writer fails to see how anything but numerous loading conditions can be used in the design of modern structures.

Since the computation of variation coefficients is a simple matter if the work is systematically arranged, the device suggested by Professor Hickerson for varying the coefficients of at least one unknown by re-solving becomes unnecessary. In fact, the writer sees no need at all for re-solving the equations. That 50% or more of the work would be saved by omitting the constants *a*, *b*, and *c* is greatly to be doubted in view of the arguments previously given. The question whether or not to use *a*, *b*, *c*, $\cdots$, *n*, therefore becomes academic, and their use amounts to computing "variation coefficients." In any case, if the constants are "varied" (in Professor Hickerson's words) there is no problem—the procedure for accomplishing this is well known. If this were the only question involved, this paper would never have been written. If the coefficients are varied, there is a problem which this paper has attempted to solve. From either point of view, therefore, it is desirable to compute the variation coefficients.

Professor Morrison's discussion is very interesting from several points of view. Most important, in the writer's judgment, is the point which this discusser makes as to the value of a well-planned computation procedure.

It cannot be reiterated too often that, just as it is necessary to plan a construction job in the field, it is also necessary to plan the work in the office. It has been the writer's sad experience to have to wade through computations spread over innumerable pages, without page numbers, dates, or titles. The usual result is that such computations are worthless a week or even a day after they are made. They are unintelligible even to the computer—if he has not neglected to add his initials and can be identified. More often than not, such haphazard computations contain serious errors. Wherever possible, computations should be tabulated, thus saving much repetition of explanatory matter. Tabulated computations are concise, are easy to follow, save much labor, and reduce errors. If a series of results is in tabular form, an error can very often be detected simply by the fact that it looks erratic—out of proportion—when compared with similar results adjacent to it.

The writer notes that Professor Morrison considers the idea of a change in some of the coefficients as novel and of merit. To writer's knowledge, this idea is unique in the theory of this type of equations. Professor Morrison develops the fundamental mathematical basis of the paper by the use of matrix algebra and gives the final result in general terms in Eq. 44. It is well worth the effort to follow through this discussion in detail, but, as matrix algebra is not too familiar a subject, a simple explanation of the high lights—sufficient to enable one to follow Professor Morrison's mathematics—is readily available in a paper by Stanley U. Benscoter, Assoc. M. ASCE.[10]

Professor Morrison then proceeds to apply Eq. 44 to some of the problems worked out in the paper; in one case, he solves a problem in which he varies the coefficients of two unknowns. He admits, however, that it is open to doubt whether this general method has any advantage over the direct solution offered by the writer, and again calls attention to the fact that the method presented in the paper lends itself to a step-by-step process in work sheet form, always a distinct advantage in itself.

---

[10] "Matrix Analysis of Continuous Beams," by Stanley U. Benscoter, *Transactions*, ASCE, Vol. 112, 1947, p. 1109.

The writer, too, doubts that this general method based on matrix algebra has any advantages for numerical solutions. For research purposes, however, the method is invaluable. Every time a familiar analysis is studied from a new point of view or by another method of attack, the results will often appear in a new form, hitherto unknown. By making the proper algebraic or other mathematical transformations, the new form can always be converted to the more familiar forms, but a gain has nevertheless been made. Such studies are usually worth the effort.

For this reason, and also to anticipate a point raised by Mr. Roos, the writer has extended Professor Morrison's analysis (Eq. 44) to derive Eqs. 17, 19, and 20, which, when generalized, give Eqs. 21, the basic equations of the paper. Extending the analysis further, the writer has also derived the equations used in multiple variations (Eqs. 33 and 34).

A system of n equations with n unknowns takes the form:

$$\left.\begin{array}{l} a_{11}\,x + a_{12}\,y + a_{13}\,z + + + = c_1 \\ a_{21}\,x + a_{22}\,y + a_{23}\,z + + + = c_2 \\ a_{31}\,x + a_{32}\,y + a_{33}\,z + + + = c_3 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ a_{n1}\,x + a_{n2}\,y + a_{n3}\,z + + + = c_n \end{array}\right\} \dotfill (47)$$

It is desired to increase the coefficients of x by $d_1, d_2, d_3, \cdots, d_n$, respectively, and to find the resulting effect on all the unknowns.

For the present purpose, it will be sufficient to take three equations with three unknowns; the derivation will be exactly the same for any number of unknowns.

Referring to Eq. 44,

$$a = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \dotfill (48)$$

$$a^{-1} = \frac{\text{adjoint } a}{a} = \frac{1}{a}\begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix} \dotfill (49)$$

and

$$b = \begin{vmatrix} d_1 & 0 & 0 \\ d_2 & 0 & 0 \\ d_3 & 0 & 0 \end{vmatrix} \dotfill (50)$$

Therefore,

$$a^{-1}\,b = \frac{1}{a}\begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix} \; \begin{vmatrix} d_1 & 0 & 0 \\ d_2 & 0 & 0 \\ d_3 & 0 & 0 \end{vmatrix} \dotfill (51)$$

and

$$a^{-1}\,b = \frac{1}{a}\begin{vmatrix} (d_1\,A_{11} + d_2\,A_{21} + d_3\,A_{31}) & 0 & 0 \\ (d_1\,A_{12} + d_2\,A_{22} + d_3\,A_{32}) & 0 & 0 \\ (d_1\,A_{13} + d_2\,A_{23} + d_3\,A_{33}) & 0 & 0 \end{vmatrix} \dotfill (52)$$

However,

$$\left.\begin{aligned} \frac{d_1 A_{11} + d_2 A_{21} + d_3 A_{31}}{a} &= x_d \\ \frac{d_1 A_{12} + d_2 A_{22} + d_3 A_{32}}{a} &= y_d \\ \frac{d_1 A_{13} + d_2 A_{23} + d_3 A_{33}}{a} &= z_d \end{aligned}\right\} \ldots\ldots (53)$$

Substituting Eqs. 53 in Eq. 52,

$$a^{-1} b = \begin{vmatrix} x_d & 0 & 0 \\ y_d & 0 & 0 \\ z_d & 0 & 0 \end{vmatrix} \ldots\ldots (54)$$

and

$$I + a^{-1} b = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} x_d & 0 & 0 \\ y_d & 0 & 0 \\ z_d & 0 & 0 \end{vmatrix} \ldots\ldots (55a)$$

from which

$$I + a^{-1} b = \begin{vmatrix} 1 + x_d & 0 & 0 \\ y_d & 1 & 0 \\ z_d & 0 & 1 \end{vmatrix} \ldots\ldots (55b)$$

and

$$[I + a^{-1} b]^{-1} = \frac{\text{adjoint } [I + a^{-1} b]}{I + a^{-1} b} = \frac{\text{adjoint } [I + a^{-1} b]}{1 + x_d} \ldots (56a)$$

Therefore,

$$[I + a^{-1} b]^{-1} = \frac{1}{1 + x_d} \begin{vmatrix} 1 & 0 & 0 \\ - y_d & 1 + x_d & 0 \\ - z_d & 0 & 1 + x_d \end{vmatrix} \ldots\ldots (56b)$$

and

$$\begin{vmatrix} x' \\ y' \\ z' \end{vmatrix} = \frac{1}{I + x_d} \begin{vmatrix} 1 & 0 & 0 \\ - y_d & 1 + x_d & 0 \\ - z_d & 0 & 1 + x_d \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} \ldots\ldots (57)$$

from which

$$\left.\begin{aligned} x' &= \frac{x}{1 + x_d} \\ y' &= y - \frac{y_d x}{1 + x_d} \\ z' &= z - \frac{z_d x}{1 + x_d} \end{aligned}\right\} \ldots\ldots (58)$$

Eqs. 58 will be recognized as being the same as Eqs. 17, 19, and 20, respectively.

To derive Eqs. 33 and 34, the value of b in Eq. 50 becomes

$$b = \begin{vmatrix} d_1 & e_1 & 0 \\ d_2 & e_2 & 0 \\ d_3 & e_3 & 0 \end{vmatrix} \ldots\ldots (59)$$

and

$$a^{-1} b = \frac{1}{a} \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix} \begin{vmatrix} d_1 & e_1 & 0 \\ d_2 & e_2 & 0 \\ d_3 & e_3 & 0 \end{vmatrix} \ldots\ldots (60)$$

Therefore,

$$I + a^{-1}\,b = \begin{vmatrix} 1 + x_d & x_e & 0 \\ y_d & 1 + y_e & 0 \\ z_d & z_e & 1 \end{vmatrix} \quad \text{.................(61)}$$

$$[I + a^{-1}\,b]^{-1} = \frac{\begin{vmatrix} (1+y_e) & -x_e & 0 \\ -y_d & (1+x_d) & 0 \\ y_d\,z_e - (1+y_e)\,z_d & x_e\,z_d - (1+x_d)\,z_e & (1+x_d)\,(1+y_e) - x_e\,y_d \end{vmatrix}}{(1+x_d)\,(1+y_e) - x_e\,y_d} \quad \text{..(62)}$$

and

$$\begin{vmatrix} x' \\ y' \\ z' \end{vmatrix} = \frac{1}{D} \begin{vmatrix} (1+y_e) & -x_e & 0 \\ -y_d & (1+x_d) & 0 \\ y_d\,z_e - (1+y_e)\,z_d & x_e\,z_d - (1+x_d)\,z_e & (1+x_d)\,(1+y_e) - x_e\,y_d \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} \quad \text{..(63)}$$

in which $D = (1 + x_d)\,(1 + y_e) - x_e\,y_d$. Therefore,

$$\left.\begin{aligned} x' &= \frac{(1 + y_e)\,x - x_e\,y}{(1 + x_d)\,(1 + y_e) - x_e\,y_d} \\ y' &= \frac{(1 + x_d)\,y - y_d\,x}{(1 + x_d)\,(1 + y_e) - x_e\,y_d} \end{aligned}\right\} \quad \text{...............(64)}$$

$$z' = z - z_d\,x' - z_e\,y' \quad \text{....................(65)}$$

Eqs. 64 are the solutions of Eqs. 29 and 30, and Eq. 65 is the same as Eq. 31.

If all the coefficients were varied at once, Eqs. 50 would become

$$b = \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix} \quad \text{.........................(66)}$$

and

$$I + a^{-1}\,b = \begin{vmatrix} 1 + x_d & x_e & x_f \\ y_d & 1 + y_e & y_f \\ z_d & z_e & 1 + z_f \end{vmatrix} \quad \text{.............(67)}$$

The matrix of Eq. 67 is a full one; that is, it contains no zeros and is equivalent to the matrix involved in re-solving the equations.

It is again shown that, if the coefficients of all the unknowns are varied at one time, this procedure simply results in the complete solution of the equations by one of the classic methods—in this case, by determinants. In fact, this is exactly the result of Professor Morrison's numerical example illustrating the general solution. The simplicity of this numerical general solution is deceiving; the examples used both by the discusser and by the writer were chosen deliberately for the purpose of illustrating principles with the least amount of distraction from the mechanical details of manipulating difficult numbers.

For this reason the writer agrees with Professor Morrison that it is doubtful whether the general method based on matrix algebra offers any advantage since it is really a solution by determinants. The writer is grateful to Professor Morrison for submitting his very interesting and enlightening discussion.

Mr. Roos' comments relating how he has used the theory presented in this paper indicate that variation of coefficients is not a mere mathematical curiosity, but a useful engineering tool. When the millennium comes, there may be a paradise in which designers will have the foresight to anticipate all contingencies and the final design will be exactly the same as the preliminary design. Adjustments will then not be necessary. The fact remains, however, that normally changes are made and adjustments are necessary.

If the structure under consideration is indeterminate to a large degree, an accurate determination of the exact effect of the changes is too time consuming to permit anything more elaborate than a "guesstimate" to be made. Mr. Roos corroborates the writer's experience that variation of coefficients is an efficient tool for this purpose and for the other purposes he mentions.

Mr. Roos states that, with the method, he has been able to adapt one basic design to several other structures. The writer has stated (under the heading, "Solution of Simultaneous Equations by Variation of Coefficients") that if a solution of any set of equations is known—from some previous computation or in any other way—it can be varied to any other set of equations desired. Thus, the writer, from his own experience, has anticipated Mr. Roos' findings.

Mr. Roos requests the mathematical basis for the method. It is supposed that what is desired is a more rigorous proof, but the writer feels that the proof given under the heading, "General Formulas for Variation of Coefficients," leading to the derivation of Eqs. 21, is sufficiently rigorous. Nevertheless, he has complied with Mr. Roos' request by extending Professor Morrison's Eq. 44 to the derivation of Eqs. 21, 33, and 34. Matrix algebra deals of course with determinants, hence this latter proof in terms of determinants constitutes a rigorous mathematical proof.

The writer is grateful to Mr. Roos for presenting evidence that the paper is a useful engineering tool and not a mere problem in abstract mathematics.

Mr. Merritt has also used the theory of the paper and calls attention to the fact that, when any of the properties of the structure are changed, resulting in a change in coefficients, classical methods permit no other alternative than re-solving the equations. If the loading conditions are changed, no particular problem is involved—provided the work is arranged according to a well-laid plan. The writer is grateful also to Mr. Merritt for his brief discussion presenting additional evidence of the practical value of the method as an engineering tool.

The writer feels that there are simplifications still possible in the evaluation of simultaneous equations and that further studies will reveal them. One difficulty encountered in the study of these equations is the fact that, when expanded, the algebraic forms rapidly become very cumbersome. On the other hand, if the solutions are expressed symbolically, it is very difficult to see relationships not already known. Matrix algebra and similar devices have been developed to overcome the first difficulty and the writer feels that further study of the theory of variation of coefficients with the aid of such methods should reveal other simplifications. Although modern procedures are most helpful in practical design of structures having a high degree of indeterminancy, the general methods necessitating sets of linear simultaneous equations are still the primary ones for research. It is to be hoped that such studies will be undertaken by the members of the profession and that further simplifications will be discovered and reported.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2359

THE SOCIETY'S FUTURE AND ITS ECONOMIC LIMITATIONS

ANNUAL PRESIDENTIAL ADDRESS

BY R. E. DOUGHERTY,[1] PRESIDENT, ASCE

In accordance with the provisions of the Constitution, it is the privilege of the President to address the Society on important phases of its affairs at the Annual Convention. Although I would very much prefer to discuss ethereal or perhaps technical questions with you, nevertheless, the most important problem before us is financial, prosaic though it may be.

The dues of the Society were fixed in accordance with the present constitutional provisions in 1921. There is no need to expatiate on the changing world that has since evolved, with its expansion of problems and the decrease in the value of the dollar.

In recent years, your various Boards of Direction, in order to meet the increasing obligations and requirements of the Society, passed deficit budgets and based their judgment primarily on the cash position. This procedure could not be followed indefinitely, and the 1947 Board endeavored to persuade the Society to secure additional revenue by amendments authorizing an increase in dues, and—failing in that effort— wisely decided to balance its budget in any event. This caused serious curtailment in many activities, including the Technical Divisions and Committees, the Local Sections and their ramifications, the Juniors and Student Chapters, and the publications, already reduced beyond the satisfaction of many members preparing papers—in fact, a general, all-around reduction in the efficiency of our functioning.

I would like to pause for a moment to seize upon this opportunity—and I am sure every member of the Board will concur—to pay tribute to the careful and efficient work of the Budget Committee consisting of George W. Burpee as Chairman, Roy W. Crum, and Irving V. A. Huie.

The present problem has received the careful attention, not only of the Board but likewise of the Sections, and two petitions have been submitted offering amendments permitting the dues to be increased. These amendments will be very shortly submitted to you and may be briefly summarized as follows:

---

[1] Vice-Pres.-Asst. to Pres., New York Central System, New York, N. Y.

Amendment Based on Petition No. 1:

> "(1) An increase of five dollars ($5.00) a year in the dues of Members, Associate Members and Affiliates.
>
> "(2) An increase of two dollars and fifty cents ($2.50) a year in the dues of Juniors."

This amendment, if passed, would still retain the existing differential between the dues paid by the members in District No. 1 and the rest of the Society.

Petition No. 1 was signed by 148 members in Zone I, 348 members in Zone II, 485 members in Zone III, and 379 members in Zone IV, a total of 1,360.

Amendment Based on Petition No. 2:

> "The annual dues payable by Members, Associate Members, Affiliates and Junior Members within District No. 1, shall be the same as those payable by corresponding members residing outside District No. 1."

This amendment would accord to the membership in District No. 1 the same treatment as that enjoyed by the rest of the Society. Please note most carefully that the second amendment is to be made effective only if the first passes.

Petition No. 2 was signed by 138 members in Zone I, 203 members in Zone II, 338 members in Zone III, and 237 members is Zone IV, a total of 916. It did not originate in Zone I.

An increase of $85,000 of revenue would be made available under the first amendment. If both amendments became effective, the additional amount realized would be reduced to $67,000.

In either event the immediate financial problem would be solved and the functioning of the Society restored substantially to the status prior to the reductions of the budget adopted in 1947.

The question has been raised as to what would happen if these amendments were not passed. I, for one, dislike to think of any further loss of efficiency in our functioning, but that is just exactly what it would mean. The Board has reached no conclusion as to just what form this further curtailment might take, but the Budget Committee thinks that it might mean an additional cut of $25,000. Various suggestions have been considered as to the form of further reductions, but I am sufficiently optimistic and confident to believe that that unfortunate contingency will not prevail. Let me call your attention to the very pertinent views expressed by Director Samuel A. Greeley of Chicago, Ill., in *Civil Engineering*[2] for July, 1948, very well stating the case in greater detail.

I shall endeavor to present a factual analysis of Petition No. 2 and the accompanying amendment.

Many members residing outside of District No. 1 may have the idea that the differential now existing should continue, although others consider it to be an inheritance of past thinking, when the activities of the Society primarily centered in the old building at Fifty-seventh Street and before the days of Local Sections. In those days there was an advantage, although now largely dissipated as a sequel to the transfer of activity from Fifty-seventh Street to the Engineering Societies Building and the initiation of Local Sections. The Metropolitan Section pays for the use of its rooms in the Engineering Societies Building.

---

[2] *Civil Engineering*, July, 1948, p. 55.

None of the other founder societies makes any distinction as to dues charged to members in the Metropolitan area as against dues charged to the rest of the membership.

Geographical expediency necessitates calling upon the membership in District No. 1 for the brunt of such work as membership on United Engineering Trustees, Engineering Foundation, Engineering Societies Library, consideration of questions involving securities and investments, the recurring responsibility for the Annual Meetings, and other duties of a similar character.

On the other hand, I have heard it stated that this is one of the distinct advantages for service to the profession which is enjoyed by members in District No. 1, and is less available to members living at greater distances. It has also been stated that it is an advantage to the membership of District No. 1 to have the Annual Meetings in New York.

It is sometimes claimed that there is a distinct advantage in close contact with the Engineering Societies Library. Analysis indicates that a large proportion of the service, possibly 60%, at least, is furnished to the membership outside of New York State, let alone District No. 1. Furthermore, the service rendered by the Library is primarily of advantage to employing organizations rather than to individual members.

I have endeavored in the foregoing to present impartially the various arguments "for" and "against," and will leave it for you to strike the proper equation.

The question has also been raised as to certain functions of the Society being outside the limits of technical activities and beyond the scope of the Constitution and its original purpose. I disagree emphatically with that idea, and let me quote the following from Article I of the Constitution:

> "3. The objects of the Society shall be the advancement of the sciencel of engineering and architecture in their several branches, the professionas improvement of its members, the encouragement of intercourse between men of practical science, and the establishment of a central point of reference and union for its members."

The Society is facing a different world from that of the founding fathers. The great problems of the world today are in economics and in public and human relations, and the Society should take its proper place in these considerations. What better example can we have of this theory of service than that offered by the unselfish response of our Past-President E. M. Hastings to an invitation to go to China as a member of a special survey mission to China appointed by Economic Cooperation Administrator Paul Hoffman. It is a great tribute not only to our Past-President but likewise to the Society. The day has long since passed when any organization or individual can face these basic problems with complacency and with the thought that others will assume the burden. It must be remembered that inertia at various times through the years has prompted the formation of other organizations which it now seems might well have been avoided. It is easy enough to second guess the past, but it is unpardonable if we do not use such experience as guidance for the future.

In considering the problems of the Society, let me emphasize the importance of the work of the Student Chapters, now numbering 125, with a total membership of approximately 8,600, which originated in 1920, only one year before the dues were last raised. Similarly, an increased interest in the Juniors is another distinct manifestation of progress as compared with the older days, certainly applying to the period when I first became a Junior in 1903. In 1921 there were 500 Juniors out of a total membership of 10,000. In 1948 there are 7,250 out of a total membership of 23,600. The importance of these particular activities cannot be overestimated. Let me commend to the attention of the Local Sections careful consideration of the advisability of placing a Junior on their Boards.

The question has arisen as to whether we should eliminate the field offices, particularly the two in Chicago and Los Angeles, Calif. A careful review of this question by the Board and discussion with many Presidents of Local Sections, including the coordination of these considerations by the Vice-Presidents and Directors of Zones III and IV, leads to the inescapable conclusion that the Regional Representatives, Walter E. Jessup and George S. Salter, are serving a most useful purpose, particularly in dealing with the Local Sections, Juniors, and Student Chapters. It must be remembered that we are a far-flung organization and although the initiation of the idea originally, largely but not entirely, had to do with collective bargaining, nevertheless, that is now very much of a minor consideration in so far as time is concerned and the only functioning of Messrs. Jessup and Salter in that respect has to do with keeping headquarters "posted" as to what is going on in their respective territories. I have never heard anyone raise the slightest question as to the necessity for the Regional Office in Washington, D. C. The service of E. L. Chandler in keeping the Board and the Executive Secretary informed as to legislation and other federal activities, as well as his other work in connection with the Local Sections, Juniors, etc., has justified that position in the eyes of everyone familiar with this work. One has only to discuss that particular subject with members of the District of Columbia Section to realize the appreciation that we should extend to Mr. Chandler.

There was a great deal of confusion for several years in connection with collective bargaining, and the 1943 Board came in for considerable criticism when it took cognizance of that very controversial issue, which was particularly acute as it involved the membership of the Society in the western, central, and southern states and, to a more limited extent, in the east. Any pioneer movement in an uncharted field is bound to be accompanied by certain actions which subsequently have to be modified and this instance is no exception. Like most of you, I was not too familiar with its ramifications when I came back to the Board as a Vice-President in 1944. Although the original procedure had to be changed radically, the 1943 Board was faced with a very serious situation and I am convinced that, controversial though the subject and procedure may have been, the Society owes a debt of gratitude to that Board for its courage in endeavoring to meet a critical problem. By the very differences of opinion that arose out of it, the action of that Board of Direction focused active attention on the subject in a way that could hardly have been accomplished. Fortunately, the Taft-Hartley Act has eliminated some of that worry, but there are even presently problems in that same connection that must be watched carefully. Is there anyone who will argue that careful attention to national legislation and policy should not be given the direct and continuing consideration of our Society in cooperation with others involved?

The question has been raised with me on several occasions that we should more clearly state the aims and intent of our organization. In casting about to develop that suggestion, I find that some of the Sections, following the leadership of the Kentucky and Illinois Sections, are using a statement of reasons for membership that I would be willing to adopt as a platform. You will find this effort more fully outlined on the editorial page of the July, 1948, issue of *Civil Engineering.*[3]

> "To encourage and perfect discoveries in basic and applied science and to bring these discoveries to all members of the profession, it has published technical papers and discussions.
>
> "To stimulate professional activity at the local level, it has created and supported local sections.
>
> "To afford the beginnings of professional associations in universities and colleges, it has established and encouraged student chapters.
>
> "To advance the economic standing of the profession, it has formulated and adopted classification and compensation schedules for civil engineering positions.
>
> "To encourage the payment of higher salaries by employers, it has recommended adequate fees for consulting engineering services.
>
> "To foster the professional standing of its members, it has promoted national registration laws.
>
> "To improve engineering education standards, it has cooperated with other national engineering societies in the accrediting of engineering curricula.
>
> "To free its members from enforced collectivism, it has taken effective action on national labor legislation affecting the profession.
>
> "To the older engineer, membership in the Society provides the professional associations and contacts by which success in his profession usually has been attained.
>
> "To the Junior engineer, membership is the medium for providing wider sources of technical information, more varied personal relations, and less formal contacts with his superiors than he usually finds in his daily work.
>
> "And to all engineers, membership in the Society carries a recognition of professional standing in a degree second to no other organization."

I have endeavored to analyze the economies that might be effected if we were to be confined entirely to technical activities, which certainly should be considered as including the functioning of the Society with its Local Sections, Juniors, and Student Chapters, and have reached the conclusion that a large proportion of that which many call professional activity involves functioning on the executive level for both the Board and the staff. It is difficult to place a finger on any material saving that could be accomplished by even a literal compliance with that suggestion. In addition, and on the minus side, it certainly would entail a tremendous loss of prestige.

---

[3] *Civil Engineering*, July, 1948, p. 54.

In order better to meet the many problems before us, it was wisely concluded about four years ago to cooperate actively with the other Founder Societies and with the American Chemical Society in the formation of what is now known as Engineers Joint Council, which presently consists of the president, immediate past-president, and secretary of each organization. This combination is actively watching over the general affairs of the profession from a national and international standpoint and in company with other organizations. The general provisions of its Constitution were approved at a meeting of the Council on July 8 and at the Seattle (Wash.) Meeting of your Board of Direction, and may be stated as follows:

The objectives of the Council shall be:

> "(*a*) To advance the general welfare of mankind through the available resources and creative ability of the engineering profession.
>
> "(*b*) To promote cooperation among the various branches of the engineering profession.
>
> "(*c*) To develop sound public policies respecting national and international affairs wherein the engineering profession can be helpful through the services of the members of the engineering profession."

A Committee of the Engineers Joint Council has been created consisting of one member appointed by each Constituent Society to prepare a plan for the increased unity of the engineering profession.

Let me emphasize the advisability of a more intensive cooperation with other organizations than those now included in Engineers Joint Council. The Executive Secretary and I have conferred with the president and executive director of the Associated General Contractors, and our Board has just authorized the designation of a joint committee to work with the Associated General Contractors with the objective of close cooperation. I certainly hope the day will come when there will be complete understanding among the various engineering organizations that should have the same eventual purpose in mind.

I have heard the statement made that the Society "is at the crossroads." I am not inclined toward such a comparison. I am more given to an analogy with travel on a highway as it may have existed in 1921, probably not more than two lanes wide and lucky if it were even of gravel or macadam, but which now has four or six lanes. I cannot help but compare the "brass-nosed flivvers" and similar automobiles of those days with the equipment that we now enjoy. I sincerely hope that the membership will permit its Board to operate with what corresponds to a modern automobile on an improved through thoroughfare, and not require it to attempt to keep its proper place among the professions with out-of-date equipment or outmoded concepts. The days of the 1921 roads and motor vehicles have gone. Crossroads? No! The same road of the engineering profession modernized and a road upon which I hope we may travel with up-to-date equipment.

MEMOIRS OF DECEASED MEMBERS

Honorary Members

FRANCIS TRENHOLM CROWE, Hon. M. ASCE[1]

Died February 26, 1946

Francis Trenholm Crowe, the son of John and Emma (Wilkinson) Crowe, was born on October 12, 1882, at Trenholmville, Que., Canada. John Crowe, born in Huddisfield, England, came to the United States in 1869 and was married to Emma Wilkinson in Brooklyn, N. Y., in 1880. They resided in Sherbrook, Que., Canada, for two years, and then moved to Trenholmville where Francis was born.

John Crowe had established a woolen mill in Trenholmville in 1882, and in 1888 he sold his interests and moved his family to Fairfield, Iowa, where he invested in another mill. This venture was unsuccessful, so, in 1890, the family moved back to Kezar Falls, Me., where the father worked as superintendent in a large woolen mill. In 1892 still following his interests in woolen mills the family again moved—this time to Picton, N. J., where they stayed for seven years. In 1899 they settled at Byfield, Mass.

At Byfield, Francis completed his elementary studies and started high school. The town did not have a high school, but sent its students to Dummer Academy (the oldest school of its type in the United States, founded by Governor Dummer, who was appointed by the English throne in prerevolutionary times). He was graduated from Dummer Academy in June, 1901, and that fall he entered the University of Maine at Orono.

Mr. Crowe often mentioned that his father had wanted him to follow the ministry or practice medicine, but the young man's mind had long been made up to be an engineer, so it was that he studied civil engineering at the University of Maine. At the university he excelled in mathematics and engineering and took an active part in military work; in his junior year he was also selected as artist for the year book, "The Prism."

During Mr. Crowe's junior year at college, in January, 1904, Frank E. Weymouth,[2] M. ASCE (Class of 1896), had returned from the west where he was employed by the United States Reclamation Service, and he gave a series of lectures on the plans for reclaiming vast regions of land. Mr. Crowe's enthusiasm was immediately aroused, so much so that, after the course of lectures, he approached Mr. Weymouth and asked, "If I show up on your project this spring, will you give me a job?" Mr. Weymouth assured him of work and so in the spring of 1904 Mr. Crowe borrowed enough money to make the trip. He worked the summer of that year on a survey party for the Reclamation Service on the Lower Yellowstone River near Glendive, Mont.

---

[1] Memoir prepared by S. O. Harper, Walker R. Young, and W. V. Greeley, Members, ASCE.

[2] For memoir, see *Transactions*, ASCE, Vol. 107, 1942, p. 1712.

He returned to the university in November, 1904, a most enthusiastic booster of the west and of the great work the Reclamation Service was doing in reclaiming vast, arid regions. He had fallen in love with that country and had learned much about its rugged life in the few short months he spent there; he could make camp, ride a bronco, and spin and throw a lariat (Will Rogers' style). He earned his trip back to Chicago (Ill.) as a drover of several carloads of horses; his drover's ticket enabled him to cover partly the cost of a ticket from Chicago to Bangor, Me. Mr. Crowe brought back with him a large buffalo skull that adorned the fireplace in the Sigma Alpha Epsilon house for many years. It was a common sight to see him showing the students how easy it was to throw the rope and to catch the boys as they ran by. His interest in what he had seen showed in his every thought and action and he was extremely impatient to return to the west.

In January, 1905, he took and passed the civil service examinations at Bangor. Mr. Weymouth offered him a job, but it was necessary for him to start work about the middle of May, a month before graduation. He immediately arranged with Harold Boardman, M. ASCE, dean of the civil engineering department, for special examinations, which he passed with high marks. So, not waiting for commencement exercises or his diploma, he started west again—this time to stay and to begin a career which was to bring him fame as a great construction engineer and dam builder. This enthusiasm showed during his last year at Maine; it spread throughout the senior and junior classes, and many students followed him west, the most notable being Frank A. Banks, Assoc. M. ASCE (Class of 1906).

His first work with the Reclamation Service, after leaving the university, was as engineering aid and instrumentman on construction of the headworks for the Lower Yellowstone irrigation system. He was assigned to this project during 1905 and 1906. In 1906, having met James Munn, M. ASCE, a general contractor, he left the Reclamation Service to join Mr. Munn as superintendent of construction, in charge of several large reinforced concrete siphons, sluiceways, and spillways.

He left Mr. Munn in 1908 and went back to work for the Reclamation Service as engineer and superintendent of construction on the second lift pumping station on the Minidoka project in Idaho. In 1910 and 1911 he was engineer and, later, superintendent of construction on the Snake River storage project, involving investigation of the storage possibilities at Jackson Lake in Wyoming, the location and construction of 35 miles of wagon road through the Teton Range of the Rocky Mountains, and construction of Jackson Lake Dam. It was at this time that Mr. Crowe was married to Marie Sass, who passed away in 1911.

In December, 1911, Mr. Crowe was appointed assistant superintendent of construction on the Arrowrock Dam in Idaho under Mr. Munn, who had given up his contracting firm and joined the Reclamation Service. This dam, then 349 feet high, was the highest concrete dam in the world at the time. While at Arrowrock Mr. Crowe showed his inherent skill in the mechanical field by designing a comprehensive concrete conveying and distributing system, as well as a unique pipe system for transporting cement pneumatically. It was during this period that he saw the great possibilities for the use of cableways in the construction of concrete dams, and he spent considerable time in studying them and developing their efficiency. He later became a strong advocate of the cableway system and recommended its use wherever possible.

From 1913 to 1915 Mr. Crowe was superintendent of construction on the Boise project in Idaho in charge of the building of 50 miles of large open drains, returning in 1915 to superintend the Jackson Lake Dam enlargement. In 1916 he was appointed project manager of the Flathead project in Montana, remaining there until 1920. The same year he left the Reclamation Service and formed a partnership (under the name of Rich, Marcus, and Crowe) in Missoula, Mont., with Messrs. Rich and Marcus. This partnership was dissolved after a year's operation and Mr. Crowe returned to the Reclamation Service in 1921 as construction engineer at Tieton Dam (235 feet high) on the Yakima project in Washington.

On the completion of Tieton Dam in 1924, he was appointed general superintendent of construction in the office of the chief engineer of the Bureau of Reclamation (formerly Reclamation Service) in Denver, Colo. In this position he had general supervision of all construction activities of the bureau in seventeen western states, and as principal assistant to the chief engineer, he was in charge of the Denver office in the latter's absence. Thus it was that he became one of the top-ranking men of the bureau, having acquired an enviable reputation for organizing ability and ingenious and rapid prosecution of construction work.

In 1925 the policy of handling the construction of its major projects by government forces, which the bureau had followed for many years, was changed, and this type of construction was abandoned in favor of construction by contract as current jobs were brought to completion. This change in construction methods resulted in a radical change in Mr. Crowe's duties. He was always happiest in the open spaces on the front line of a tough construction job, and the limitations of an executive office were most irksome to him. On one occasion he remarked, "Do you know what I feel like sitting at this desk shuffling papers? I feel like a bull in a china shop."

So it was that Mr. Crowe came to an important turning point in his career. He was only beginning to realize his greatest ambition to build bigger and bigger dams. During the years past his close association with such engineers as F. H. Newell,[3] M. ASCE, A. P. Davis,[4] Past-President, ASCE, A. J.

---

[3] For memoir, see *Transactions*, ASCE, Vol. 98, 1933, p. 1597.

[4] *Ibid.*, Vol. 100, 1935, p. 1582.

Wiley,[5] M. ASCE, D. C. Henny,[6] M. ASCE, C. H. Paul,[7] M. ASCE, F. E. Weymouth,[2] and others had fired his imagination with dams of unprecedented size, and he was determined he could build them. He loved his work and associations in the Bureau of Reclamation, but he had a conviction that he could best develop his own abilities and perform a more useful service by directing the building of dams on the ground rather than by long-range supervision of construction. Therefore, when the opportunity came to enter the construction field he made his decision without reservation to cast his lot with contractors.

Mr. Crowe resigned in 1925 from the bureau and accepted the position of superintendent in charge of construction of Guernsey Dam on the North Platte River in Wyoming for the Morrison-Knudsen Company of Boise. This was a young and aggressive organization, which was just starting out in the heavy construction field, and which was later to become one of the largest construction companies in the world. Mr. Crowe soon hit his stride in successfully bidding and building dams throughout the west. After completing Guernsey Dam in 1927, he built Van Giesen (Coombe) Dam on the Bear River in California in 1927–1928, and Deadwood Dam in Central Idaho in 1929–1930.

Hoover Dam (formerly Boulder Dam), 726 feet high, was open for bids by the Bureau of Reclamation in 1931. This dam, located in the rugged Black Canyon of the Colorado River, was of such magnitude and presented such unusual hazards and seemingly unsurmountable problems that most contractors hesitated to tackle it. Mr. Crowe, however, was determined to build it—this was his life's ambition. He had waited for this job ever since its conception by the bureau engineers years before. With his enthusiasm, confidence, and prodding, he inspired H. W. Morrison of the Morrison-Knudsen Company to promote the organization of Six Companies, Incorporated, a syndicate of six large construction companies, to bid on the job. In preparing the bid for the dam each member of the combine made its individual estimate. Mr. Crowe made the cost estimate for the Morrison-Knudsen Company. With a slight adjustment of approximately one per cent, his figures were accepted as the official cost estimate of Six Companies, Incorporated, and proved to be the low bid for the job. So great was the confidence of the syndicate in Mr. Crowe that he was appointed immediately as general superintendent in charge of building the dam. His genius at organizing, designing the construction plant, and moving materials in the most efficient manner was demonstrated by the fact that Hoover Dam was completed twenty-five months ahead of schedule. This record started Mr. Crowe and Six Companies, Incorporated, well on the way to fame.

In 1935 he took enough time off to make a trip back to his alma mater, the University of Maine, where he was awarded the honorary degree of Doctor of Engineering in recognition of his outstanding contributions to the profession.

---

[5] For memoir, see *Transactions*, ASCE, Vol. 96, 1932, p. 1577.

[6] *Ibid.*, Vol. 101, 1936, p. 1577.

[7] *Ibid.*, Vol. 107, 1942, p. 1809.

In 1936 Mr. Crowe and Six Companies, Incorporated, bid and were awarded the construction of Parker Dam on the Colorado River. The group "left over a million dollars on the table" in bidding the job, which made most of the members uneasy. Mr. Crowe, however, was still so confident of his bid that he went into partnership with Charles Shea of the J. F. Shea Company (one of the Six Companies group) and subcontracted the job. From the outset it was very difficult because of the unprecedented depth of foundation excavation, and the subcontractor was constantly threatened with heavy loss. The dam, although 320 feet high from the foundation, required excavation to a depth of 235 feet below the original river bed in sand and gravel heavily saturated with water. The treacherous Colorado and Bill Williams rivers threatened disaster on several occasions, but Mr. Crowe stayed with the job day and night, fighting all the threats of nature and a very difficult foundation problem. His determination and his efforts were rewarded by completing the job well ahead of schedule, with a profit for the contractors. He also constructed the Copper Basin and Gene Wash dams in near-by California for the Metropolitan Water District of Southern California in 1937 and 1938.

The next dam to be undertaken by Mr. Crowe was Shasta Dam on the Sacramento River in California, the key structure in the immense Central Valley project of the Bureau of Reclamation. Although the group he was associated with in bidding for this job was unsuccessful by a small margin, the low bidders, Pacific Constructors, Incorporated, had such confidence in his ability that they employed him as general superintendent. Shasta Dam, 602 feet high, the second highest and second largest dam in the world, presented many difficult problems. It was here that Mr. Crowe, an advocate of the cableway system, devised a system of handling concrete that was unique in construction practice. He designed a 460-foot central tower from which radiated seven 25-ton cableways, delivering concrete and materials to all parts of the job. The cableway system was so well planned and so well built that full coverage on the entire dam was possible from the first to the last bucket of concrete placed. In 1944 Shasta Dam was completed—another monument, and a final one, to a great construction engineer.

He received the crowning distinction of honorary membership in the Society in 1944. In recognition of his great work the "Moles," a New York (N. Y.) organization of tunnel and heavy construction men, gave him an honorary award in 1945 for outstanding contributions to the engineering and construction profession.

Mr. Crowe was pre-eminent as a builder of dams. Associated with some of the largest construction firms in the west, he had never been submerged in any company. He and his men were an institution within themselves; all were picked and trained over a period of years for the difficult tasks involved in dam building. They followed him about, from job to job, some as far back as Arrowrock Dam thirty years before and some even into the second generation, with father and son working together in crews. He directed his work through personal contact with his men. He knew them by name, and was known to all as "The Old Man." By his friends and associates in the construction industry he was invariably called "Frank," and he carried the name "Frank Crowe" on his personal letterhead in preference to his given name, "Francis."

He was a man of few words and a master of terse and forceful language which construction men understand so well. He disliked long letters and reports, and his own letters seldom consisted of more than two or three sentences. A typical example was a letter to the Bureau of Reclamation when he felt that the progress at Shasta Dam was being unnecessarily delayed —it read: "Herewith a picture of the Hay-Wire Dam," and the following caption was on the picture: "A view from Point Despair where Time Marches On and the dam stands still."

Mr. Crowe probably knew more about dam construction than any other man of his generation. He designed, built, and used much of the equipment which has become standard for construction men today. He was the outstanding authority on the cableway method of building dams. The system he designed for Shasta Dam, using a central tower as a hub with cables radiating out to cover the entire structure, was a radical departure from prevailing practice and involved previously unknown stresses and strains. Although many authorities questioned the feasibility of the system, Mr. Crowe never doubted that it was fundamentally sound and this was proved by its efficient operation over a period of four years.

As construction methods progressed from the days of the horse drawn scrapers through the era of the steam shovel on into the modern period of robot operation of electrical equipment, he was a leader in devising new and better methods of obtaining faster operation and more efficiency. As specifications for building became more rigid he pioneered many new schemes in order to keep up production and at the same time meet all the stringent requirements set forth by the designers.

Hoover Dam may well be considered Mr. Crowe's crowning achievement, as well stated in a magazine article: [8]

> "To the extent it can be said to belong to any single man, Boulder was Frank Crowe's triumph. For five long years the heat, the mud, the remorseless current, the swift floods, the jangling telephones were his responsibility. 'We had 5,000 men jammed in a 4,000-foot canyon,' Crowe says. 'The problem, which was a problem in materials flow, was to set up the right sequence of jobs so they wouldn't kill each other off.' "

In 1945 he was asked by the War Department to take complete charge of all rehabilitation work in the American occupation zone in Germany. He was ready to tackle this tough assignment, but the forty years he spent in pioneering the construction of the west and in becoming the world's outstanding dam builder were slowly exacting their toll, and he was forced to refuse this assignment on doctor's orders.

---

[8] "The Earth Movers," *Fortune*, August, 1943.

During the construction of Shasta Dam Mr. Crowe acquired extensive cattle ranches in northern California and at his death he was engaged in managing the property, as well as acting as a construction consultant for the Morrison-Knudsen Company. He died suddenly on February 26, 1946, at Redding, Calif., near one of his greatest achievements—Shasta Dam.

He is survived by his widow, the former Linnie Kortz of Boise, now residing in Redding, and two daughters, Patricia (Mrs. Stanley Lee) of San Francisco, Calif., and Bettee (Mrs. Robert Parry) of Palo Alto, Calif.

Mr. Crowe belonged to Sigma Alpha Epsilon fraternity, the engineering honor society of Tau Beta Pi, and the Episcopal Church. In addition, he was a Thirty-Second Degree Mason, a Shriner, an Elk, and a Rotarian.

His projects will stand forever as monuments to his great ability as a constructor, but he will be best remembered by his many friends and close associates for his ever-present human understanding, his extreme fair-mindedness, his wonderful sense of humor, and his absolute integrity—once his word was given it was carried out, no matter what his personal sacrifice might be.

Mr. Crowe was elected an Associate Member of the American Society of Civil Engineers on May 3, 1910; a Member on October 11, 1915; and an Honorary Member on October 11, 1943.

GEORGE WATSON KITTREDGE, Hon. M. ASCE[1]

DIED AUGUST 22, 1947

George Watson Kittredge was born on December 11, 1856, at North Andover, Mass. His parents were of English ancestry, resident in New England since 1660.

His father, Joseph Kittredge, M.D., was born in North Andover and his mother, Henrietta Frances Watson, was born in Belfast, Me. Dr. Kittredge was the seventh generation of an unbroken line of physicians, dating back to John Kittredge, a sea captain, who was born in Lowestoft, England, and settled in Billerica, Mass., in 1660. The fifth generation physician was Thomas Kittredge who was a surgeon in Col. James Frye's regiment at the battle of Bunker Hill and who built the North Andover home in 1784. The ninth in this unbroken line is Edwina Kittredge, M.D., of Yonkers, N. Y., a niece of George Watson Kittredge.

The boyhood of George Kittredge was spent on the family's fifty-eight-acre farm and home in North Andover where he attended the public schools. He was graduated from the Massachusetts Institute of Technology at Boston in 1877 with the degree of Bachelor of Science in Civil Engineering. After graduation he engaged in private engineering practice and was associated with the South Boston Flats Improvement in 1878 and 1879.

---

[1] Memoir prepared by Boynton S. Voorhees and George A. Noren, Members, ASCE.

He began his railroad career in 1880, at the age of twenty four, as rodman on the lines of the Pennsylvania Railroad Company west of Pittsburgh, Pa. He rose, successively, to division engineer on the Peoria Division, between Peoria, Ill., and Terre Haute, Ind.; the Louisville Division, between Indianapolis, Ind., and Louisville, Ky., and the Panhandle Division, between Pittsburgh and St. Louis, Mo. It was while working on the division last mentioned that the Johnstown (Pa.) flood occurred in 1889. Mr. Kittredge and his bridge, building, and track men were called upon for assistance and were the first reconstruction forces on the scene.

In June, 1890, the Cleveland, Cincinnati, Chicago and St. Louis Railway Company (the Big Four) offered him the job of assistant chief engineer at $2,000 a year. He accepted the offer and ended ten years of service with the Pennsylvania Railroad. During this period, and for several years subsequently, he acted as consultant chief engineer for the Louisville and Jeffersonville Bridge Company.

Mr. Kittredge became engineer, maintenance of way, of the Big Four on December 1, 1890. As this road later, like Mr. Kittredge, joined the New York Central System, it might be said that the last thirty-seven years of his professional life were dedicated to that single employer. At the age of thirty four (on July 1, 1891), he was promoted to the position of chief engineer of the Big Four with headquarters at Cincinnati, Ohio. During the succeeding fifteen years in which he held this position, extensive rebuilding was under way all over the Big Four system under his direction and supervision. A large part was double tracked; wooden bridges were replaced by structures of masonry and steel; alinement was straightened; curvature and gradients were reduced; and several lines of new railroad were built to shorten distance. One of these last projects was the construction of 44 miles of double track into St. Louis to handle the traffic for the World's Fair. In 1904 and 1905 the line from Cincinnati to Indianapolis was rebuilt between Lawrenceburg and Sunman in Indiana, reducing grades and rectifying alinement.

On April 9, 1906, at the age of forty nine, Mr. Kittredge was made chief engineer of the New York Central and Hudson River Railroad (later the New York Central, Buffalo (N. Y.) and East). The construction of the Grand Central Terminal was just starting; the electrification out of New York, N. Y., had been ordered; the freight situation on the west side of New York City was acute; at Albany, N. Y., and Buffalo, congestion of traffic was developing; at Rochester, Utica, and Rome in New York, improved facilities were needed; projects for four-tracking between New York and Albany were under study. These and other varied improvements (engine terminal, passenger station, grade revision, freight yard, etc.) were consummated under his direction and supervision, including the Hudson River Connecting Railroad (Castleton Cutoff) which eliminated the Albany bottleneck. Before his retirement on January 1, 1927, plans for the Buffalo Station, the Syracuse Improvement, and the West Side Improvement in New York City were well advanced. Among his other activities during this busy period, he was valuation engineer for the New York Central Railroad (Buffalo and East) and a member of the

Engineering Committee of the New York Central System. From 1911 to 1927 he was vice-president of the Eastern Railroad Association and a member of its executive committee.

Mr. Kittredge was a charter member of the American Railway Engineering Association, serving as president for two years (1901–1902 and 1902–1903), and was made an honorary member in 1933. He was also active on its committees and was chairman of Committee No. 18—Electricity, from 1909 to 1913. Prominent in Society affairs, he served as Director from 1908 to 1910 and as Vice-President in 1917 and 1918. He was active in committee work and on the Library Board of United Engineering Trustees. In 1931 Mr. Kittredge was a member of a committee to assemble authentic information relative to engineering opportunities in Russia. Always keenly interested in engineering society activities, he encouraged younger men to join, to participate in discussions, and to state their opinions and experiences. He well knew that such interest and activity were the surest way to produce valuable members and to increase engineering knowledge.

In addition to the foregoing societies, he was a member of the New York Society of the Order of Founders and Patriots of America, the Society of the Mayflower Descendants, the American Association for the Advancement of Science, and the Fortnightly Club for the Study of Anthropology. He attended church consistently and served as trustee for the Unitarian Church of Yonkers.

Mr. Kittredge interested himself deeply in the advancement of education and was active especially in the Halsted School of Yonkers, serving as president and finally president emeritus. He was coauthor of the early editions and author of the 1923 and 1926 editions of "The Metal Workers Pattern Book." After his retirement he was a consulting engineer for railroads in the United States and Canada.

He was an opera lover and enjoyed social functions. He played golf and for a number of years was a member of the Ardsley-on-Hudson (N. Y.) Club. His great hobby was the breeding and racing of homing pigeons. His love for animals, and pigeons especially, dated back to his boyhood days spent on a New England farm. His pigeon "Old Dearborn" held the one thousand-mile record; "Old Chatty" was released at Chattanooga, Tenn., at 6 a.m. and was in Yonkers at 9 a.m. the second morning; a pure white cock made 150 miles in 132 minutes. His pigeons were in both world wars and in World War I his pigeon carried the first carrier message from the front line trenches.

On October 17, 1888, Mr. Kittredge was married to Georgia Davis of Louisville. Mrs. Kittredge died on December 29, 1921. He is survived by a daughter, Mary Henrietta (Mrs. Denman McNear) of Petaluma, Calif.; and a son, George Davis Kittredge, a civilian employee with the Army Engineers in Puerto Rico. Also surviving are a grandson, Denman Kittredge McNear, a student at the Massachusetts Institute of Technology; and two granddaughters, Mary McNear (Mrs. James McNab) of San Francisco, Calif., and Georgia Kittredge (Mrs. Kenneth Schick) of Buffalo.

Mr. Kittredge was elected a Member of the American Society of Civil Engineers on January 6, 1886, and an Honorary Member on October 5, 1931.

LUIGI LUIGGI, Hon. M. ASCE[1]

DIED FEBRUARY 1, 1931

Massauwa, Tobruk, Derna, Bengasi, Homs, Palermo, Messina, Leghorn, Spezia, Genoa—ports bombed by Americans and Britons in World War II—were the life work of one of the world's outstanding harbor engineers, Luigi Luiggi.

With the practical nature, thrift, and conservatism characteristic of the Genoese, he abhorred waste—especially the waste of war. For forty years he had endeavored to build and to strengthen cordial relations between Italy and the two countries he admired deeply—England and the United States. Those who loved him—that is to say those who knew him—are glad he died before Italy was enchained to her enemy of two thousand years. Invariably when the name Germany came up in private conversation he had repeated, "It is a nation not to be trusted."

Luigi Luiggi was born on August 3, 1856, at Genoa. His grandfather, Pasquale Luiggi, was a captain of artillery in the port of Spezia. His father, Settimio Luiggi, had a stationery shop in Genoa which is still in the family. His mother, Francesca Bruzza, died when he was a young child. His father then married Paola Rossi; her stepson never forgot her angelic patience and serenity while bringing up the two families each of four children.

His preliminary education was received in Genoa, including two years at the University of Genoa. In 1875 he entered the Royal Civil Engineering College in Turin and was graduated in 1878 with the degree of Civil Engineer.

In 1880 he was appointed junior engineer in the Royal Corps of Italian Engineers and was sent to England to study harbors and lighthouses for two years under the late Sir James Douglas. Part of the basis for his appointment was the high rating he received in the English language courses. He recalled with amusement in later years that to his chagrin on landing at Dover the speech of Englishmen bore so little resemblance to that taught in Turin, he could make himself understood only in writing.

While in England he met Annie East of Lincoln, to whom he was married in 1887. The relationship was a happy one and her death on February 28, 1911, left for twenty years a void in his life that intensive work or travel could never fill. The home they planned together on Via Nomentana in Rome, in which many engineers from all over the world found welcome, bears their combined names in Spanish "Villa Luis-y-Ana." They had a daughter, Luise, and a son, M. L. Luiggi, an engineer, who has lived in the United States since 1920.

When he returned to Italy in 1882, he was sent to build a harbor in steaming Massaua. In connection with this project he spent some time at

---

[1] Memoir prepared by Mrs. M. L. Luiggi, New York, N. Y.

Famagusta on the island of Cyprus, where he was associated with the silent and pious young engineer who later became Lord Kitchener.

On again returning to Italy he was appointed a member of a staff of engineers to modernize the port of Genoa. His special work was constructing two dry docks behind the old mole and two great breakwaters, one a mile long, extending from 80 to 100 feet into the water. Fifty years later when the port was again enlarged, part of one of his breakwaters had to be removed. To his secret satisfaction, this was done with greatest difficulty.

Plans for enlarging the port, he found, called for the destruction of the beautiful Bank of St. George, built in the thirteenth century and believed to be the first building designed especially for banking purposes. In accordance with his lifetime motto, "Do not tear down the old; build new," he worked out a scheme to save the building. Original arrangements had called for a narrow road and a railroad running parallel to the harbor's edge, through the bank site. He relocated the railroad on the seaward side of the bank, below road level, on land made by building the harbor toward the sea and filling in. On the city side he tore down old tenements and laid out a wide road which he foresaw Genoa would eventually need along the sea. This great road was slowly extended so that, thanks to his forethought in the 1880's, Genoa has the most outstanding water-front highway of any Riviera city.

Similarly, by use of imagination, he preserved for his wife's home city of Lincoln, an irreplaceable antiquity, the "Stonebow," an arched gateway, remaining from the Roman city wall. To widen its main street the city planned to demolish this arch. Mr. Luiggi prepared a plan to route traffic one way and to build a cutoff road. The Stonebow remains one of Lincoln's attractions.

To obtain fill for the harbor of Genoa, heretofore brought by sea, Mr. Luiggi opened a quarry in a hill in the middle of the city. The citizenry, indignant at first, were delighted when a wide boulevard finally emerged through the hill to relieve congestion on narrow streets and connect Genoa with neighboring towns to the west.

In 1892–1893 he served as technical secretary to the ministry of public works in Rome. To study the working of railway ferryboats he went to Denmark and to New York (N. Y.)—the first of innumerable trips to the United States. On his return he directed the installation of railway ferries from the mainland across the Straits of Messina to Sicily.

During 1894–1895, as chief engineer of the ports of Tuscany, he overhauled and enlarged the harbors of Leghorn and Spezia. On the island of Tino, near Spezia, he installed the first electrically operated lighthouse in Italy. Power came from Ericsson hot air engines.

In 1896 Mr. Luiggi was chosen by the government of Argentina to select a site, prepare plans, and supervise construction of a military harbor for the upkeep of its nascent navy. After a detailed survey of the entire coast he chose a zone in an uninhabited sandy waste along the estuary of Bahia Blanca.

Within ten years this had been transformed into the most important harbor in the southern hemisphere. Among projects completed were a 1,000-

foot graving dock with a very low sill to allow entry of warships in sinking condition; wharves; a breakwater; lighthouses; buoys in the dredged channel; defense batteries at the harbor entrance and a railway connecting them with the main line 30 miles away; roads, sewers, a power station, and a water supply from sand dunes 5 miles away. To provide labor, he brought several thousand picked workmen from Italy, for whom hospitals and barracks were built. During these ten years he preferred to stay at Puerto Militar, close to his job, rather than to live at ease in Buenos Aires and supervise the work through resident engineers.

On concluding this work he was made an honorary citizen of Bahia Blanca and a principal street was named for him. Although a school in Bahia Blanca bears the name of Umberto I, Mr. Luiggi always thought of it as a memorial to the charm and determination of his British wife. When the king of Italy was assassinated in 1900, Italian workmen contributed a considerable sum of money to buy a bronze wreath for his tomb. Hearing of this during the absence of her husband, Mrs. Luiggi went before the final meeting and, in halting but obviously effective Italian, persuaded the donors to build a school for their children with the money.

At this school, Mr. Luiggi introduced Arbor Day in Argentina and presented trees for the pupils to plant on their school grounds. Always an ardent advocate of reforestation, he planted 100,000 eucalyptus trees around Puerto Militar (later called Puerto Belgrano) to prevent shifting of sand dunes. He set up extensive nurseries of eucalyptus and tamarisk from which the government made massive plantings along the dunes far down the coast.

In connection with Puerto Militar, he erected a chain of lighthouses and semaphores from Buenos Aires to the southernmost tip of Patagonia, some 800 miles distant. All were connected by a telegraph line. With only primitive means of transportation then available, it was a formidable undertaking. These, as well as the tower on the administration building overlooking Puerto Belgrano, bear the triple motif from the papal crown which only Genoese engineers may rightfully use. Pope Innocent III permitted its use by Genoese builders when, during a crusade, a Genoese engineer, Guglielmo Embriacus, devised the tenth century version of a tank. Armored scaffolds were pushed against the walls of Jerusalem and the walls were scaled and breached from inside the scaffolds. As a loyal son of Genoa, Mr. Luiggi always worked the motif somewhere into his constructions.

During his years in Argentina, he acted as consultant for new harbor works of Buenos Aires, Mar del Plata, Rosario de Santa Fe, Montevideo, and Rio Grande do Sul. He also reported on several proposed canals for inland navigation and for irrigation of the pampas.

He always referred to his years in Argentina as the happiest of his life. It was with great pleasure that he returned in 1910 as representative of the Italian government to the centenary celebration of Argentine independence. At this time he prepared plans for expanding the harbor of Buenos Aires, following his customary method—building out into the sea, then dredging sand from the approach channel and using it as fill. Thus, land area was increased and deep cuttings and heavy earth movements were avoided. In his honor, a railhead town in the Pampa Central was named "Ing. Luiggi." A library to his memory has been built in this flourishing community by his son.

Again, in September, 1928, he returned, amid warm welcomes, to unveil a monument of General Belgrano by the Italian sculptor, Arnaldo Zocchi, the gift of citizens of Genoa to the city of Rosario de Santa Fe. On this last visit to the country for which he had so strong an attachment, he was asked by the government to complete the expansion project of his plan of 1896 for Puerto Militar.

To symbolize the close relationship between the people of Argentina and Italy, a graceful white marble lighthouse which he designed stands on the Janiculum hill behind the Vatican. It was built in 1911 by popular subscription from Italians in Argentina. Nightly it flashes the red, white, and green of the Italian flag onto the dark perennial oaks in the Pincio gardens across the Tiber.

It gave him much pleasure to install a similar light in 1912 when planning improvements in navigation of Lake Garda. Before World War I the north half of the lake belonged to Austria, and although the population was chiefly Italian, the Italian flag could not be displayed. From a new lighthouse he built at Sermione, the forbidden red, white, and green swept over the facade of the city hall in Austrian Riva di Garda.

On returning to Italy in 1906, Mr. Luiggi became inspector general of the Royal Corps of Civil Engineers and was appointed to the seven-member board of directors of the Italian State Railways. In 1906 when the contracts leasing the railways to private operating companies expired, the government decided to appoint a board of directors responsible to parliament for their rehabilitation and operation. Rolling stock was insufficient, rights of way had long been poorly maintained, and marshalling yards were inadequate. Until 1911 he was in charge of personnel, equipment, and electrification.

Especially important was the line through the Alps between Turin and Modane. This link with France had vital strategic importance since the only other connection was the highly vulnerable one through Ventimiglia, along the Mediterranean coast. The French government had decided to double-track the steam access railway on its side to the 8-mile-long tunnel at the summit of Montcenis.

Mr. Luiggi, on the other hand, decided to electrify the single-track section from Turin to the Italian portal. The expense of building a double track on this winding, mountainous line with heavy gradients would have been prohibitive. He decided on a synchronous, three-phase system to give constant speed both in upgrade and downgrade so that trains would meet exactly on time at crossings. The wisdom of his choice was shown during World War I, when heavy military traffic was handled much more easily on the Italian than on the French side. Among the many marshalling yards which he built, the large ones at Genoa and at Mestre near Venice were particularly important in World War I.

To effect quick and economical improvement of rolling stock, he bought second-hand locomotives and immediately began expanding locomotive and car building works in Italy. Jointly he and Sen. Riccardo Bianchi developed a series of locomotives ranging from light tank engines for lines where traffic was sparse to heavy decapods for hauling freight on the Porretta line between Bologna and Florence. A new electrified line was begun at a much lower level, but World War I halted construction. When finished in 1932, it included the 11.5-mile "tunnel of the Apennines" at the summit, the longest double-track tunnel in the world.

Early, Mr. Luiggi foresaw the importance of electrical development and was active in building new hydroelectric plants and tying all schemes into one network. At first, plants had to be built to furnish current solely for railways. As soon as possible, he arranged for them to supply power to industry also. Likewise he insisted that all new railway bridges of a certain importance should be built to carry public roads as well. Also, he installed on all trains in Italy, except international expresses, at least one third-class coach.

In 1912 when Italy occupied Tripolitania, he took over projects and works there for the harbors of Tripoli, Bengasi, Derna, and Homs. These, including the building of a system of lighthouses along the Lybian coast, he carried out with exceptional rapidity. At various times he inspected and prepared plans for improving the harbors of Massaua and Assab on the Red Sea, Ras Filuk in Somaliland, and Brava and Benadir on the African east coast.

After this, he was in charge of all Italian harbors, among the more important of which were Genoa, Naples, Palermo, Leghorn, Spezia, and Imperia. An especially interesting project, never carried out, was for a new harbor at Ostia, the port of Rome 25 miles away. He planned it to be built like an atoll in the Mediterranean, connected with the mainland by a causeway. Since Roman days various harbors had been built at Ostia, but had eventually been buried by silt from the Tiber. He estimated that it would be at least 1,000 years before the delta of the Tiber could reach this atoll harbor.

From 1913 until his death he was a member of the International Advisory Board for the Suez Canal and for the harbor of Alsa in Egypt, and he belonged to the European board for navigation of the Danube River. In 1912 he acted as consultant to the Russian government for improving navigation of the Volga River. Collaborating with Sir Cyril Kirkpatrick, he acted for the Egyptian government in the decade before his death, and was responsible for port and harbor undertakings at Alexandria, Damietta, Kossier, and Mersa Matru. In 1925 he visited Rhodes and Durazzo to confer on improving harbors and roads.

Although Mr. Luiggi was one of the outstanding authorities of the past generation on port and harbor works, his interests embraced all types of civil engineering. He carried out important work on navigable and irrigation canals and built or acted as consultant on aqueducts at Bahia Blanca and Puerto Militar in Argentina, as well as those in Leghorn, Lucca, Pontedera, Chiavari, and Apulia, and the Val Noci and Entella aqueducts of Genoa.

In his native commune of Genoa he was honorary consultant for the Genoa-Milan-Stelvio and Genoa-Piacenza railways, for the Lavagna-Cavi road, and for the hospitals of Pammantone and Albera. He gave his services to the commune of Chiavari for defense against inroads of the sea.

An illustration of his ingeniousness in solving problems is found in the borax works at Lardarello in central Italy, where he harnessed several small volcanoes to produce electric power. During World War I, with coal scarce, he exerted every effort toward maximum utilization of Italy's electrical resources. Since Roman times, borax has been extracted from these volcanoes by injecting water into the ground. It is returned as superheated steam, rich in boric acid. Mr. Luiggi, to preserve the heat in this steam, passed it through an aluminum heat exchanger so that as the boric acid deposited, the superheat was utilized to produce low pressure steam for operating turbines. As much as 10,000 kilowatts were produced.

Another increase in Italy's resources which he aided on every possible occasion was the replanting of denuded hills. Particularly in southern Italy he encouraged tree planting as a control to violent flash-floods that swept away railway bridges year after year. He presented trees for planting by school children of many localities (accompanying the trees with a treat for each child). He reiterated:

> "Whoever plants new trees or protects those already planted, contributes to the prosperity of the country. He helps regulate water, produces building material, beautifies the landscape, and renders life more healthful and pleasanter. Whoever plants new trees serves his country."

He was author of monographs on the ports of Antwerp, Southampton, Hull, Genoa, and Tripoli; on Argentine lighthouses and the electric lighthouse in the harbor of Spezia; and on the advantages of electric traction. One report described the great construction difficulties in building the Apulian aqueduct in Southern Italy, the longest in the world when it was completed in 1913. He published various articles on dams, both of concrete—principally the multiple-arch type—and of earth. These latter types he considered the most suitable for Italian needs and did much to develop their use.

A series of sunken dams which he built in riverbeds in the Tessenei region of Italian Somaliland were successful in storing up water to raise the subsurface level so that cotton could be successfully grown in the area. During summer months, water coming from the mountains had lost itself in the wide, sandy river bottoms. He built clay cutoffs in trenches across the river beds to impound the water. Heavy loss from evaporation was averted by keeping the level below the surface.

Perhaps his most important publications are those on his research as to the force of sea waves and their effects on breakwaters. Others were on concrete, reinforced concrete, pozzuolanic cement, and other materials best adapted to resist corrosive action of sea water. He also published the results of experiments on adhesion of concrete to iron, both clean and rusted, and on the type of reinforced concrete best adapted for piers, caissons, and other maritime constructions. His reports are valued because he was an authority on the subject of pozzolana and pozzolana mortars and cement for making concrete blocks that are cheaper and more durable in sea water than those made with ordinary Portland cement.

Endowed with an expansive and sympathetic personality as well as a keen mind, Mr. Luiggi made himself more than an accomplished linguist. "When I speak French with a Frenchman, I think as a Frenchman—I really am a Frenchman," he often said. He felt equally able to think in English or Spanish. Within six months of landing in Argentina he had acquired fluent Spanish. He spoke German less easily.

Because he was a real citizen of the world, his country relied upon him for four decades to represent it with dignity and charm in international congresses. He served as delegate to such gatherings in Frankfurt, Manchester, Paris, Chicago, London, Petrograd, Stockholm, Buenos Aires, Philadelphia, San Francisco, Geneva, Lausanne, and Tokyo, among others.

He was particularly happy in his relations with English-speaking countries. In July, 1914, he visited Australia as honorary guest of the British Association for the Advancement of Science and delivered several lectures on Italian irrigation at the universities of Melbourne, Adelaide, and Sydney. From Melbourne he received the honorary degree of Doctor of Science. At the request of the Australian government he made recommendations for unification of railway gages. The University of London in 1922 invited him to give a course of lectures on characteristic Italian hydroelectric works.

He headed the Italian section of the International Trade Congress on its visit to various cities in the United States in 1919, and remembered Denver (Colo.) and Kansas City (Mo.) with special pleasure. He was vice-president of the International Chamber of Commerce soon after it was organized. At the meeting in Rome in 1922 he was unofficial host, a role which he enjoyed and in which he was unsurpassed. Many who came from various parts of the world will recall luncheons in the sunny room overlooking the Roman campagna toward Tivoli and Castel Gandolfo, where often several languages were being spoken at the long table. When especially well-known guests were expected, he invariably included some obscure acquaintance. "It will give her pleasure the rest of her life to remember that she sat next to a Cardinal," he answered when a relative remonstrated with him for placing an elderly filing clerk beside His Eminence.

In 1930 he was given by the Hon. Herbert Hoover, Hon. M. ASCE, then president of the United States, one of sixteen gold medals struck off at the fiftieth anniversary of the American Society of Mechanical Engineers. From Rensselaer Polytechnic Institute in Troy (N. Y.), he received the honorary degree of Doctor of Engineering in 1925.

He was a member of the institutes of Civil Engineers and Structural Engineers of London, the College of Engineers of Genoa, and the Centro Nacional de Ingenieros of Buenos Aires. He was honorary correspondent member of the Sociedad Cientifica Argentina. For three terms of three years each he served as president of the Italian Society of Civil Engineers in Rome.

From the king of Italy he received three high honors: Grand Commander of Saints Maurice and Lazzarus, Grand Commander of the Star of Italy, and Grand Commander of the Crown of Italy. He also received a medal for engineering services during the Abyssinian campaign of 1882–1884 and that of Lybia in 1911–1912. A gold medal was given him for his work as member of the munition board of World War I, in which his title was honorary colonel of engineers. The Italian Red Cross also gave him a gold medal for his aid in organizing transportation.

The crimson rosette of the Legion d'Honneur was awarded him by the Republic of France for arbitrating disputes between French and Italian industries. For acting as consultant on the ports of Candia, Piraeus, and Salonika, the Greek government gave him the cross of Saint Salvatore. The title that he prized most, however, was the one on his hospitable doorway—Ing. Luiggi—Engineer Luiggi.

In 1921 he undertook the thankless job of acting as member of parliament from Genoa, hoping to help restore order after postwar political disturbances. He resigned with relief on being appointed to the Italian Senate in 1924. Members of this supreme advisory body are chosen for life from leaders in various fields.

Although he retired from active public life in 1923 as honorary president of the superior council of the corps of engineers, some of his most useful advisory work was done in his last eight years. Whenever Italy needed a representative at international congresses and conferences—where few faces were more familiar than his—he would continue to go, although his health became increasingly less robust.

At the time of his retirement he was president of the royal commission of irrigation; member of the council of administration of state railways; member of the higher council of public instruction; and professor of hydraulics and maritime construction at the University of Rome.

He continued to give a course of lectures each year at the university because he had always taken keen pleasure in associating with young engineers and in passing on his accumulated knowledge to them. His library of some twenty thousand engineering books, pamphlets, maps, and plans was presented by his son to that engineering school. It is housed in a bright, frescoed room and a bronze bust of him, created by his friend, the sculptor Arnaldo Zocchi, keeps paternal watch over the students.

He last saw his many American friends in September, 1930, when he headed the Italian delegation to the International Roads Congress in New York. In 1926 he had served as president of the congress at its first meeting in Milan.

Senator Luiggi had looked forward with eagerness to going to Hull (England) in April, 1931, to consult with his colleague of many years, Sir Cyril Kirkpatrick, on the widening of the Humber River, but death came on February 1.

He had been on a leisurely drive with an old friend from Argentina on a January morning when spring flowers were already in bloom in his garden.

"The day is too perfect to go indoors—" he said, and was suddenly silent. Cerebral thrombosis had struck and he died five days later.

To muted dirges from a military band he was carried across the street to the eighth century church of Saint Agnes. Vatican singers chanted softly during the simple rites. Then he was laid in the chapel tomb he had designed twenty years before, on which were entwined details from the cathedrals of Lincoln and Genoa.

Mr. Luiggi was elected a Member of the American Society of Civil Engineers on February 7, 1906, and an Honorary Member on October 10, 1921.

Members

FRANK HOOKER ALFRED, M. ASCE[1]

Died October 6, 1947

Frank Hooker Alfred was born on December 24, 1866, at Logan, Ohio. His parents were George Washington Alfred, a native of Burlington, Vt., and Mary Loraine Hooker, who was born near Baltimore, Md.

Later the family moved to a farm on the outskirts of Lancaster, Ohio, in which city the father established a law practice. Frank Alfred attended the public schools of Lancaster and was graduated from the local high school. At that time, a sewerage system was being installed in Lancaster. Young Frank, much intrigued by the doings of the engineers, decided that he would like such work and shortly obtained a position as chainman and rodman on the corps. However, when he enrolled at the University of Michigan at Ann Arbor in 1886, he planned to study law, his father's profession.

When the college year ended, Mr. Alfred obtained a job as rodman on the construction of the Columbus, Lima and Milwaukee Railroad. He found himself in his element, and it was not long before he became resident engineer. The idea of leaving active and responsible work to return to college made no appeal whatever to him, so he continued with the railroad until its completion in 1889, going then to the Norfolk and Western Railway as assistant engineer. In 1892 and 1893 he had charge of field work and construction on the Columbus (Ohio) terminal, and during that time contrived to study engineering for a year at Ohio State University at Columbus.

In May, 1894, an event occurred which Mr. Alfred always considered the most fortunate of his career. He met Maud Bush, a native of Mansfield, Ohio, whose family was then living in Daytona, Fla. They were married in Daytona on October 17, 1894. It was a union that endured in happiness for fifty-three years. To it Mrs. Alfred brought qualities of sagacity, poise, and thoughtful devotion that made their home a haven of peace that was no small factor in his success.

---

[1] Memoir prepared by Paul Chipman, M. ASCE.

After a brief period with the Denver and Rio Grande Western Railroad Company, Mr. Alfred become engineer, maintenance of way, of the Cleveland, Akron and Columbus Railway Company, remaining in that position until November, 1899, when he joined the Wheeling and Lake Erie Railway Company as engineer, maintenance of way.

In the latter part of the year 1900, the State of Michigan began an appraisal of the railroad property within its borders with a view toward uniform taxation. This was the first railroad valuation to be based on inspection and, to a large extent, it set the pattern for the federal valuation begun in 1913. This appraisal was made under the direction of Mortimer E. Cooley,[2] Hon. M. ASCE, with Henry Earle Riggs, Past-President and Hon. M. ASCE, in charge of the roadway division. Time was limited and a force of about one hundred and fifty men was hastily recruited—for the most part, engineers from the middle west. Mr. Alfred was offered a position on this work and was assigned the important task of determining unit prices and supervising the computing office. As the project drew to a close in 1901, Mr. Riggs was asked by the president of the newly formed Pere Marquette Railroad Company to select some young man from his force who would make a competent chief engineer and would qualify for promotion. He named Mr. Alfred, who in 1902, after a brief trial period as district engineer, became the chief engineer of the railroad that was eventually to become the field of his most constructive efforts.

The Pere Marquette was then making some extensive grade revisions and in 1903, by acquiring its Canadian lines, extended its operations to the Niagara frontier; at the same time a short extension to the southwest made possible a great improvement in its Chicago (Ill.) terminal arrangements. Numerous surveys were made to determine the feasibility of line and grade changes and new routes. All this work, as well as maintenance of way, was under the supervision of the new chief engineer. At about this time the seemingly bright prospects of the rapidly expanding Pere Marquette made it the object of financial manipulation which saddled it with heavy fixed charges. Mr. Alfred saw the consequences of this situation and later was to have an active hand in correcting it.

In 1905 Canada was entering a period of rapid development. J. G. White and Company, engineers and contractors of New York, N. Y., formed a subsidiary known as Canadian White Company, Limited, with headquarters at Montreal, Que. Mr. Alfred was chosen general manager of this company, and filled the position until the depression of 1907–1908 caused the company's discontinuance. During this period he directed the construction of a number of important projects; among them were a large hydroelectric plant on the Soulanges Canal near Montreal, a section of the Grand Trunk Pacific Railway in Saskatchewan, and a sewer system at Calgary, Alberta.

---

[2] For memoir, see *Transactions*, ASCE, Vol. 110, 1945, p. 1619.

Mr. Alfred returned to his primary interest of railroading by becoming assistant to the president of the Cincinnati, Hamilton and Dayton Railway Company in 1908 and two years later was made general superintendent. In April, 1912, he returned to the Pere Marquette as assistant general manager and in December of that year he was made general manager for the receivers, who leaned heavily upon him in shaping their policy. On the reorganization of the Pere Marquette in April, 1917, he was chosen president and general manager.

His talents were admirably suited to the task before him—the rehabilitation of a property, the resources of which had been drained by excessive fixed charges, in order to meet the demands of a territory that was rapidly developing industrially. Abhorring waste, he kept a tight rein on expenditures, but welcomed every justifiable improvement. The track structure was improved, new yards were constructed and old ones extended, signal facilities were installed and improved, and new cars and locomotives were bought. This work was delayed at its start by the entry of the United States into World War I and the consequent taking over of the railroads by the federal government. During this period, Mr. Alfred was a regional director for the railroad administration, having in his charge the Pere Marquette, the Grand Trunk Western Railway, the Ann Arbor Railroad, and the Detroit and Mackinac Railway.

As the result of the improvement program and of the growth and prosperity of the industries in the territory, the net earnings of the Pere Marquette soon became very satisfactory. Mr. Alfred had attractive offers to enter broader fields, but he did not like to quit an unfinished job nor did he like to break his associations in Detroit, Mich. Wistful glances were now being cast toward the Pere Marquette from various quarters. The Van Sweringen interests failed to secure approval of a merger of the Chesapeake and Ohio, the Nickel Plate, the Erie, and the Pere Marquette roads; but the Interstate Commerce Commission, recognizing the mutual advantage of affiliation, authorized the purchase by the Chesapeake and Ohio of a controlling interest in the Pere Marquette. This was soon accomplished and the affiliation continued in this form until 1947, when the Pere Marquette was merged with the Chesapeake and Ohio by an exchange of securities.

Mr. Alfred retired from active duties on the Pere Marquette in August, 1929, but continued for a time as consultant on special assignments. Undoubtedly, the rehabilitation of the Pere Marquette was his greatest achievement. Making all due allowance for better business conditions, it was the policies he advocated and his leadership in putting them into effect that made it possible within a few years to transform a bankrupt railroad into a highly profitable one.

Mr. Alfred was more widely known in Detroit as a civic leader than as a railroad administrator. He was a director and one of the founders of the Governmental Research Bureau (later the Citizens' League), the function of which was to supply voters with information regarding candidates and public issues. Starting in 1917 he served for many years on the Detroit Board of

Education, a part of the time as its president, and was for several years a member of the Detroit Library Commission. A trustee of the Young Men's Christian Association, he headed a campaign in 1920 to provide adequate facilities for that organization in rapidly growing Detroit. He was a trustee of the First Baptist Church of Detroit, of Kalamazoo College (Kalamazoo, Mich.), and of the Salvation Army.

In business, he was a director of the Mid-West Casualty Company, the Selected Securities Corporation, and the Commonwealth-Federal Bank, one of two Detroit commercial banks that weathered the storm of 1933 without reorganization. Like many a man brought up on a farm, he had a liking for country life which led him in 1920 to purchase a six hundred-acre estate near Milford, Mich. His family spent part of the year there for a number of seasons, and he would join them for such limited periods as his duties permitted.

In 1933 he was appointed by Franklin D. Roosevelt, President of the United States, as chairman of the three-man Michigan Advisory Board of the Public Works Administration and served until the duties of that board were completed in the following year. This was the only political appointment that he ever held, and it was accepted with the understanding that politics would have no part in the activities of the board. This stipulation was strictly observed, in contrast with the functioning of similar boards in many other states.

After his retirement from active work, he declined opportunities in broad administrative fields in both railroad and public service. This was fortunate, for in a short time he was warned that the state of his health would not permit any considerable activity. Thereafter, he disposed of his country estate and he and Mrs. Alfred divided their time between their summer cottage at Walloon Lake, near Charlevoix, Mich., and Orlando, Fla., where they purchased a home which eventually became their yearlong residence. He now had ample opportunity to indulge his fondness for good reading and made many additions to his already extensive library.

Perhaps his outstanding attribute was courage, moral and physical—"Courage, the footstool of the virtues, upon which they all stand." Once a conviction was formed, he did not hesitate to give it voice on due occasion. Aggressive but cautious, he planned his course with care, after due counsel, and held it with determination, with little regard for personal consequence. A more apparent trait was intense concentration on the task in hand. A job was something to be done, and done well, but without delay. Of great mental and physical vigor, he never acquired an easy pace in his work. The tension was often relieved by flashes of humor, however, for his ingrained jocularity was another marked characteristic. He kept it under reasonable control during serious discussion, but in his home and among friends it had free rein. In work or play, he relished a contest, preferably as a participant rather than as a spectator. As a young man he was fond of baseball, and in the rather infrequent hours of relaxation in his later active life he played a good game of golf and an excellent game of billiards.

A keen judge of men and motives, his discernment and sincerity commanded the respect of all. He inspired loyalty by his fairness, his enthusiasm, and his consideration for those about him. He prized originality and initiative, but valued the patience and persistence of the routine worker, and had high praise for good craftsmanship in any field. Interested in any practical plan for social betterment, he was generous in promoting self-help and self-reliance, but did not approve of indiscriminate charity. Underlying all his deeds and decisions, hidden from public view and revealed to but few associates, was an enduring faith in the great Engineer of the Universe, whose guidance he sought throughout his life. The confinement of his last years was borne with the same courage and cheerfulness that distinguished his entire career.

When living in Detroit, Mr. Alfred was a member of the Detroit Club, the Detroit Golf Club, the Oakland Hills Country Club, and the Ingleside Club. He was a life member of the American Railway Engineering Association.

He is survived by his widow; a daughter, Dorothy (Mrs. Wilton W. Minto), of Detroit; and three grandchildren, Wilton, Maud, and Alfred Minto.

Mr. Alfred was elected a Member of the American Society of Civil Engineers on June 3, 1903.

EDWARD ANDERBERG, M. ASCE[1]

DIED JANUARY 18, 1947

No one could have been associated with Edward Anderberg for long without recognizing his essential goodness, uprightness, and integrity. Associated with these elements of character was a keen, analytical mind, and a store of technical knowledge gained from many years of work on projects of magnitude in various parts of the United States.

Edward Anderberg was born into a well-to-do family at Mount Vernon, N. Y., on September 23, 1881. His parents were Erland and Mary (Drews) Anderberg. Mr. Anderberg was graduated from the Mount Vernon High School in 1899, and in September of that year, he entered New York University in New York, N. Y., where he received the degree of Bachelor of Science in Civil Engineering in 1904, and the degree of Civil Engineer in 1909. He was a member of Delta Upsilon fraternity.

Due to ill health, he spent the year 1902–1903 in the west, and for a period of nine months, he worked with a survey party of the Atchison, Topeka and Sante Fe Railroad Company in New Mexico, Texas, Oklahoma, and Kansas. In 1904 and 1905 he was employed in Basking Ridge, N. J., on surveying work.

---

[1] Memoir prepared by Leslie G. Holleran, Stanley H. Wright, and Samuel Rosenberg, Members, ASCE.

From 1905 to 1925, Mr. Anderberg worked for the State of New York on surveys and designs for the Barge Canal, and on the supervision of Barge Canal construction contracts of considerable magnitude. His design work included locks and other structures, as well as the Barge Canal terminals in New York City, and elsewhere. He supervised construction of the Barge Canal through difficult terrain in upstate New York, as well as the Lockport (N. Y.) locks and bridges. As senior assistant engineer, he was in charge of the New York City residency of the state engineers' office from 1917 to 1925, and directed the construction of eight Barge Canal terminals and a grain elevator there.

In 1925 Mr. Anderberg entered the employ of the Westchester County Park Commission as right-of-way engineer, but was soon promoted to division engineer, in charge of the commission's construction work in the easterly half of the county, during the development of its extensive park and parkway program. The projects on which he supervised construction were: The Hutchinson River Parkway, the Cross-County Parkway, and part of the Saw Mill River Parkway; also the Glen Island bathing beach and bathhouse, together with the approach road and the bascule bridge; and the Saxons Woods Golf Course, parts of Playland at Rye Beach; and numerous smaller projects.

The parkway project included numerous stone-faced reinforced concrete bridges, several of which were of rigid-frame design as adapted to bridge construction shortly before by the engineering department of the Westchester County Park Commission. At that time, the design and construction of parkways was a new development, and Mr. Anderberg took part in the pioneer work of the commission in planning them.

In 1933, due to the depression, construction work of the park commission was suspended and Mr. Anderberg was appointed engineer examiner in the Office of Director of the Public Works Administration (P. W. A.) for New York State. Later, he was promoted to assistant chief engineer for the state. In these positions, Mr. Anderberg reviewed or supervised the review of numerous and varied municipal projects for which funds had been requested from the P. W. A.

On completion of this assignment in 1937, he became assistant manager, and afterwards, manager of the planning division for the Works Progress Administration in New York City. From 1938 to 1940, he was responsible for the review and administration of the federal supervision of important municipal projects in New York State for the P. W. A.

In February, 1941, Mr. Anderberg was recalled from retirement by Col. H. M. Waite, who was organizing the Defense Projects Unit, which was later called the War Projects Unit of the Bureau of the Budget. Mr. Anderberg's former experience in governmental service with the P. W. A. had especially equipped him for war service with a group of engineers who were to prepare "confidential" and "secret" reports on war activities. His construction engineering experience was used during 1941, 1942, and 1943, in connection with critical surveys of contracts, covering a wide range of facilities—from airfields and aircraft plants to various types of ammunition and ammunition component manufacturing plants, hospitals, and shipbuilding and industrial plants of all kinds—constructed with governmental funds to meet the war emergencies. When these facilities were completed and placed in operation, progress and production reports were prepared in the War Projects Unit for various purposes. Mr. Anderberg had prepared himself to take over field reporting on the production of airplanes, aviation gasoline, synthetic rubber, ammunition components, and other articles of war. With the passing of Colonel Waite, his associate, Col. E. W. Clark, assumed the direction of the War Projects Unit. Colonel Clark had only the highest praise for Mr. Anderberg's work, and he continued with the War Projects Unit until it was terminated on December 31, 1945.

Following the retirement of Colonel Clark, this group of engineers was reorganized into the inspection group, Division of Estimates, in the Bureau of the Budget. The change from wartime to peacetime activities necessarily changed the scope of the work of the engineers in the inspection group, but without lessening the need for men with technical experience and training. Mr. Anderberg continued his activities with this group until his death, which occurred shortly after his return from a meeting with the United States Army Engineers in Columbus, Ohio. The nature of this confidential work required the utmost tact, diplomacy, analytical ability, technical skill, and a consideration for others. Mr. Anderberg was noted for all these traits.

While engaged on the construction of the Barge Canal at Lockport, Mr. Anderberg met and was married to Winifred Huntley in 1911. His widow and four children, Mary Ralston, Jannet Elizabeth, Elinor Drews, and Edward, Jr., survive, as well as two sisters—Clara G. and Augusta H. Anderberg.

For many years prior to his death, Mr. Anderberg lived with his family at 26 Oakwood Avenue, White Plains, N. Y. He was a vestryman at Christ Protestant Episcopal Church, in Lynbrook, N. Y., for two years prior to moving to White Plains in 1925. He was also a life member of the Red Jacket Lodge of the F. and A. M. in Lockport.

Mr. Anderberg had great respect for the right of other people to choose their own way of life, either individually or as nations. This was evident from the wide latitude he gave his children in their selection of subjects at college, and in the life work they wished to follow. He was concerned about his former associates, some of whom he felt could not readily adjust themselves to new associations during the depression and after World War II, and he made it his business to interest people in their behalf. He was friendly, but somewhat reserved in his association with others, and not many were privileged to know him intimately. Those who knew him will feel keenly the loss of this kindly, considerate, and understanding friend.

Mr. Anderberg was elected a Junior of the American Society of Civil Engineers on March 5, 1907; an Associate Member on March 1, 1910; and a Member on April 4, 1922. He became a Life Member in January, 1945.

RUPERT ANDREW ANDEREGG, M. ASCE[1]

Died July 18, 1947

Rupert Andrew Anderegg, the eldest son of James Albert and Lily May (Turrittin) Anderegg, was born in Kasota, Minn., on February 8, 1894. The family consisted of one daughter and six sons.

Rupert received his early education in the public schools of Le Sueur, Minn. In September, 1913, he entered the University of Cincinnati (Cincinnati, Ohio) as a cooperative student in civil engineering. While a cooperative student, he performed surveying work, in the various capacities of chainman, rodman, and instrumentman, which was followed in succession by railroad trackwork, concrete viaduct construction, simplex pile construction, and inspection work on concrete building erection. In addition to his academic studies and cooperative employment Mr. Anderegg participated in college athletics and was a member of the football squad.

On August 13, 1917, young Mr. Anderegg was commissioned a Second Lieutenant in the Corps of Engineers, United States Army. He served as a Second Lieutenant with the First Engineers until November, 1918. From November, 1918, to February, 1919, he was a First Lieutenant in the Chemical Warfare Service and Assistant Division Gas Officer of the Ninety-Second Division; one year of this service was in Europe.

Following the termination of this military duty, Mr. Anderegg accepted a commission in the Officers' Reserve Corps. He was active in the reserve until ill health made it mandatory for him to go on an inactive status in 1941. He held successive reserve commissions as Second Lieutenant, First Lieutenant, and Captain in the Corps of Engineers, and those of Captain, Major, and Lieutenant Colonel in the Coast Artillery Corps (antiaircraft division). In 1941 he was ordered to active duty with the Second Army Headquarters as a Lieutenant Colonel in the Coast Artillery Corps. Ill health prevented him from assuming this duty.

On conclusion of his active military service in World War I, Mr. Anderegg returned to the University of Cincinnati and was awarded the degree of Civil Engineer in June, 1919. From June, 1919, to July, 1921, he was superintendent of construction for the Texas Company at Port Arthur, Tex. He laid out and supervised $7,000,000 worth of new work, which included twenty-four units of pressure stills, eight units of crude stills, a steam powerhouse, a pump house, tanks, and an office building.

In September, 1921, Mr. Anderegg accepted an appointment as instructor in the Civil Engineering Department at the University of Cincinnati. He received promotions to assistant professor in September, 1929, associate professor in June, 1930, and a full professorship in 1941.

From January to September, 1929, Professor Anderegg was on leave of absence from the university in order to perform special work with the Cin-

---

[1] Memoir prepared by R. W. Renn, M. ASCE.

cinnati District of the United States Engineer Office. He was placed in charge of the Flood Control Section and made power, flood, navigation, and irrigation investigations and reports on the Miami, Licking, and Kentucky rivers. From January to April, 1934, in conjunction with his academic duties, Professor Anderegg served as engineering supervisor with the United States Coast and Geodetic Survey for the Southwest Ohio Civil Works Administration project.

Professor Anderegg was granted a leave of absence in July, 1944, from the University of Cincinnati to serve on the staff of the Price Adjustment Section, Cincinnati Ordnance District. Ill health forced his retirement from this work in the spring of 1945, and he resumed his academic duties at the University of Cincinnati in the summer term of that year.

During his university career Professor Anderegg participated in many activities both in the academic field and in the professional field of civil engineering. He served as faculty adviser to the band from 1924 to 1946, to the orchestra from 1929 to 1946, and to the Varsity Vanities from 1936 to 1946. He was President of the Cincinnati Section of the Society in 1939, and was an active member of the Society for the Promotion of Engineering Education (later, the American Society for Engineering Education), the American Association of University Professors, and the Reserve Officers' Association.

On November 29, 1917, Professor Anderegg was married to Mabelle L. Davis. They had four sons, James Albert and Rupert Andrew, Jr., Juniors, ASCE, Richard Davis, and David Louis, who survive him. He was a member of the Knox Presbyterian Church of Cincinnati, serving as an Elder of that church.

Throughout his years at the University of Cincinnati, Professor Anderegg was a staunch supporter of the cooperative system of education, and a firm believer in his chosen field of civil engineering. The two eldest sons followed in their father's footsteps and were graduated from the cooperative course in civil engineering at the University of Cincinnati. At his death, the two youngest sons were attending Withrow High School in Cincinnati. The faculty of the university has lost one of its hardest workers, the students have lost a true friend and adviser, and the civil engineers of the community have suffered a real loss in the passing of Professor Anderegg.

Professor Anderegg was elected a Member of the American Society of Civil Engineers on February 23, 1932.

FRANK ALEXANDER BARBOUR, M. ASCE[1]

DIED MAY 24, 1947

Frank Alexander Barbour was born at St. John, N. B., Canada, in 1870. His parents were George Ludlow and Elizabeth Augusta Barbour. He was graduated from the University of New Brunswick at Fredericton in 1888, and received his first training in engineering in the Dominion of Canada. Early in his career he joined the engineering staff of the water department of the City of Boston (Mass.), where he was associated with Desmond FitzGerald,[2] Past-President and Hon. M. ASCE, and Frank McInnis,[3] Assoc. M. ASCE (the latter, a graduate of New Brunswick University).

---

[1] Memoir prepared by Arthur D. Weston, M. ASCE.

In 1892 Mr. Barbour was employed by the City of Brockton (Mass.) in the design, construction, and operation of a sewage disposal works for that city, working under the direction of F. H. Snow,[4] M. ASCE, city engineer. In 1900 Mr. Barbour and Mr. Snow opened a consulting office in Boston which the former conducted until his death. Frank Barbour was a specialist in matters of water supply and water purification, sewerage, and sewage disposal practice, and appraisal of water works property. This work he loved, and, although president of the New England Pressed Steel Company, most of his time was devoted to the practice of sanitary engineering.

Other works of interest include appraisals of property of the Passaic (N. J.) Consolidated Water Company, the Long Island (N. Y.) Water Corporation, and the New Rochelle (N. Y.) Water Company. With W. E. Fuller,[5] M. ASCE, he was responsible for reconditioning the Passaic consolidated system. He collaborated with John Gregory,[6] M. ASCE, to prepare a report on the subject of a water supply for the Northern Metropolitan District of New Jersey for the Water Policy Commission. In the interstate case of Connecticut versus Massachusetts he made investigations and appraisals of water power and process water damages, resulting from the diversion of the Ware and Swift rivers by the Metropolitan District Water Supply Commission of Massachusetts.

Mr. Barbour served as president of the American Water Works Association, the Boston Society of Civil Engineers, and the New England Water Works Association, being an honorary member of both water works associations. He was a member of the Board of Direction of the Society from 1934 to 1936, during which time he served on the Committee on Publications, including one year as chairman. Mr. Barbour was a member of the Engineering Institute of Canada, and he served on the executive committee of the American Society for Testing Materials. In addition he was on the editorial advisory board of *Water Works Engineering* and was the author of numerous technical articles.

He was an engineer of rare ability and sound judgment; and those working with him could not but profit from the association. His death has resulted in a great loss to the engineering profession, to all the engineering societies with which he was associated, and especially to the field of sanitary engineering.

---

[2] For memoir, see *Transactions*, ASCE, Vol. 92, 1928, p. 1656.

[3] *Ibid.*, Vol. 102, 1937, p. 1879.

[4] *Ibid.*, Vol. 108, 1943, p. 1634.

[5] *Ibid.*, Vol. 101, 1936, p. 1556.

[6] *Ibid.*, Vol. 102, 1937, p. 1552.

In April, 1916, he was married to Dorothy Boswell in Philadelphia, Pa. He is survived by his widow.

Mr. Barbour was elected a Member of the American Society of Civil Engineers on September 5, 1900. He became a Life Member in January, 1935.

WALTER LACY BOYD, M. ASCE[1]

Died June 11, 1947

Walter Lacy Boyd was born in Owingsville, Bath County, Ky., on November 9, 1874. Mr. Boyd, whose parents were James William and Anna Elizabeth (Robinson) Boyd, was the eldest of five children.

Most of his life, including much of his engineering experience, was spent in Florida—particularly in Polk County, at Bartow, the county seat. There, he was graduated from high school, completing the courses of Summerlin Institute. Following one year at the Florida Agricultural College in Lake City, he spent two years at the South Florida Military College in Bartow, receiving the degree of Bachelor of Science in 1896—an award which was later confirmed by the University of Florida at Gainesville. During these years of study, he was, at the same time, serving as deputy tax collector of Polk County. Upon graduation Mr. Boyd continued this work for the following five years, until 1901, as a regularly appointed deputy.

His first actual engineering work began during the latter part of 1901, with the position of rodman for the Atlantic Coast Line Railroad Company in southeast Georgia. Wishing to obtain the contractor's point of view, he left this work in November, 1901, to become foreman of railroad construction on a grading contract for the Atlanta, Birmingham and Atlantic Railway Company between Waycross and Cordele in Georgia. At the request of the chief engineer of the railroad he was released after a few months to become assistant engineer. This work comprised the location of some 200 miles of road from Cordele to Birmingham, Ala., and it was completed about a year later, in April, 1903. At this time Mr. Boyd accepted a similar assignment with the Illinois Central Railway Company at Jackson, Tenn. After considerable survey and drafting work, he was transferred to Memphis, Tenn., where he became engineer on construction of a large freight yard.

After about a year with the railroad, Mr. Boyd had a longing to return to Florida. For a few months in the spring of 1904 he was assistant superintendent on construction of an extensive plant for the International Phosphate Company at Jane Jay, Fla., and, for several months during the remainder of 1904, Mr. Boyd set up an office as a real estate broker at Bartow. This, however, did not appeal to him and in January, 1905, he established his own office at Bartow as a professional engineer. With a number of short interruptions this professional practice continued for almost forty years until his death.

---

[1] Memoir prepared by Harry Cornell, Florida State Board of Engr. Examiners, Bartow, Fla.

Among the interruptions to this general practice were a number of engagements. For most of 1909 he was secretary to the Byrom Corporation in Byromville, Ga., and was employed on farming, industrial, and banking work. Similarly, most of 1912 was devoted to business for himself as an electrical contractor in Miami, Fla. During 1929 and 1930 Mr. Boyd was engaged in making mineral surveys in North Carolina and Georgia including engineering work for a marble quarry at Andrews, N. C. For the first half of 1934 he was chief of party for the United States Bureau of Soils, making base map soil surveys in Madison County, Florida, as well as a road survey of Washington County, Florida, under the Federal Emergency Administration. The remainder of 1934 was spent with the Florida State Road Department in making general and detailed surveys of roads in Hillsboro and Polk counties.

In his private consulting work Mr. Boyd was engaged in making land surveys and in general civil engineering, much of it in the vicinity of Bartow. This private work included: Phosphate mine railroad location and construction; highway design, location, and construction; various kinds of engineering surveys, including the re-establishment of public land surveys; drainage, dams, water supply and irrigation work; subdivision design and development; concrete and steel bridge and culvert construction; and municipal work.

Among his more important mining clients were the Phosphate Mining Company at Nichols, Pierce Phosphate Company at Pierce, Armour and Company at Agricola, the Charleston Mining Company at Fort Meade, and Swift and Company whose mine is located near Bartow—all in the State of Florida. Dams, railways, terminal facilities, and heavy mine buildings were included in his work, as well as a considerable amount of surveying.

With the increase in highway transportation, Mr. Boyd devoted considerable time to highway engineering. In 1924 the Board of County Commissioners of Polk County retained him as county engineer. He made surveys, drew up plans and specifications, and supervised some of the construction, including the solving of many drainage problems. The work amounted to about $10,500,000. His standards were high and, in fact, when ordered to construct work which he felt was of an unsatisfactory quality and, indeed, a public hazard, he promptly resigned from office.

For various Works Progress Administration projects in and around Bartow, Mr. Boyd served as nonrelief supervisor. Considerable paving, as well as design and construction of storm sewers, were included in this work. There was also work on the plan of the Summerlin Institute, at which Mr. Boyd had studied as a youth. Under the same auspices he completed the Civic Center at Bartow.

The bulk of his work as consultant was devoted to surveying projects. While individually these were not extensive—each, nevertheless, involved its own problem. Principally, this phase of his practice consisted in untangling property lines which had become so snarled that the abstractors, the lawyers, and the owners were all in confusion. Re-establishing "obliterated" corners was an important feature of this work. Mr. Boyd was particularly satisfied with his success on such problems, especially when they involved dispute and his work could avoid litigation.

In May, 1900, Mr. Boyd was married in Montezuma, Ga., to Julia Dupree. Of their two sons, only James William Boyd survives him. Mrs. Boyd died in August, 1923. In February, 1945, he was married, a second time, to Mrs. Grace Pemberton in Tampa, Fla. His widow survives him.

Mr. Boyd had a certificate as a professional civil engineer in Florida, and also, one as a professional land surveyor. He became a member of the Tuscan Lodge No. 6 of the F. and A. M. at Bartow on March 16, 1916. At his death he held the position of county surveyor of Polk County.

He died in Tampa on June 11, 1947, and was buried in Bartow on June 13, 1947.

Mr. Boyd was elected an Associate Member of the American Society of Civil Engineers on December 31, 1913, and a Member on October 15, 1923. He became a Life Member in January, 1945.

PAUL LEMON BROCKWAY, M. ASCE[1]

DIED JULY 20, 1945

Paul Lemon Brockway was born in Riverton, Nebr., on May 4, 1879. He was the son of Charles L. and E. S. (Leech) Brockway.

He went from the public schools of his home town to the University of Nebraska in Lincoln, from which he received the degree of Bachelor of Science in Civil Engineering in 1905. In 1927 he received the degree of Civil Engineer from the same school.

In 1905 Mr. Brockway entered the engineering department of the Chicago, Rock Island and Pacific Railway Company as assistant engineer in charge of masonry construction. Two years later, in 1907, he was called to the position of assistant city engineer of Wichita, Kans., and remained in this department until his death. He was made city engineer in 1917.

His success in remaining in an unusually turbulent city government for thirty-eight years demonstrated the sufficiency of the lifelong principle he held as to public service. On entering the city employ, he was familiar with the common belief that such employment was, of necessity, temporary, and that each year added new enemies to one in public service because of the requests one felt compelled to refuse, until a sufficient number of enemies necessitated removal. Mr. Brockway reasoned, "Why not make a friend of each potential enemy?" With expenditure of time and painstaking tact he followed this policy for thirty-eight years and at his death, he was secure in his office with the respect and confidence of all.

During his service Wichita passed from the council to the city manager form of government. Also, the city went through two depressions, an oil boom, and, finally, the war boom which doubled the population in two years. Councilmen, commissioners, and city managers came and went, but after each upheaval the city rallied around this one engineer whose judgment was dependable and whose honesty was unimpeachable.

---

[1] **Memoir prepared by Lloyd B. Smith, M. ASCE.**

To the customary duties of a city engineer—supervising the construction and maintenance of pavements, sewers, and public improvements of a community passing from a country county seat to a city with a population of a quarter of a million—were added the difficult problems of flood protection, which, at times, became very serious. In the unusual flood of 1945 he abandoned his car, waded through a quarter of a mile of floodwater, rode to his office on a high wheel blade grader, and continued to supervise rescue work without changing his clothes. This incident typically demonstrates his faithfulness to duty and his serious attitude toward his responsibilities, and, in this case, tragically so—as he never rallied from its effects.

Exhibiting unusual tact and ability in forming and developing the thoughts of his political employers and the general public on new ideas in government, such as city planning, a city manager, and revisions of outmoded methods, he even changed the point of view of officials who were elected to office on the platform of ousting him from office.

Mr. Brockway engaged in many activities outside the strictly technical functions of his engineering duties. He was secretary of the city planning commission from its inception until his death, in addition to holding membership on the board of directors of the American Society of City Planning Officials, and the City Planning Committee of the Society. A past-president of the American Society for Municipal Improvements, he was also past-president of the Kansas Section of the Society, the Kansas Engineering Society, and the Wichita Engineering Club. Mr. Brockway was secretary of the Kansas Registration Board for Professional Engineers, and took an active part in the American Association for the Advancement of Science.

For thirteen years he was one of the three members of the local membership committee of the Society. His judgment on the qualifications of prospective members was invaluable in helping maintain a high and uniform standard of membership for the Kansas Section. He was the author of numerous articles in technical publications, such as *Engineering News-Record, Public Works, Concrete,* and *American City.*

Mr. Brockway never forgot that he was a professional man and, as such, service was always the predominant motive of his life. He had an unusual combination of the scientific and artistic in one personality. Although his musical talent was only partly developed due to his professional duties, he took part in the musical activities of the city. Much of his leisure was spent in the cultivation of flowers, and he was especially active in Kiwanis and the Boy Scout movement.

During his later years he had six operations in succession. When confined in the hospital, he wrote and later published a small book of poems on hospital life[2] with such captions as "The Surgeon's Saw," "The Closed Door," and "The Transfer Wagon."

In 1904 Mr. Brockway was married to Kate Weed of Bailey, Nebr. Besides his widow, Mr. Brockway is survived by five children: Stella (Mrs. S. E. Omopundro) of Leadville, Colo.; Lawrence Olin Brockway, professor of chemistry at the University of Michigan, in Ann Arbor; Esther Brockway of Independence, Mo.; Lois (Mrs. C. M. Jamison) of De Land, Fla.; and Charles Edward Brockway of Akron, Ohio.

Mr. Brockway was elected a Member of the American Society of Civil Engineers on May 28, 1923.

ORA BUNDY, M. ASCE[1]

Died June 12, 1946

Ora Bundy, the son of Amasa D. and Mary Ann (Hemby) Bundy, was born in Carthage, Ind., on October 14, 1882. After graduation from high school, he obtained employment in Indianapolis, Ind., to earn funds to start college. Three years later he entered Purdue University in Lafayette, Ind., and was graduated in 1908 with the degree of Bachelor of Science in Civil Engineering. While attending the university Mr. Bundy was active in athletics and was a member of the football and track teams.

In June, 1908, he was employed by the Memphis (Tenn.) Bridge Company, and continued in its service until 1912. During this period he was advanced, successively, from draftsman to detailer, designer, and foreman, and spent a year and a half as erection foreman throughout the southern states. From 1913 to December, 1917, Mr. Bundy was associated with the Vincennes (Ind.) Bridge Company; during 1916 and 1917 he was general superintendent of construction operating in eleven states.

In December, 1917, he was commissioned a Captain in the Corps of Engineers, United States Army, and served as constructing quartermaster, Construction Division, in charge of wartime projects in Alabama, Arizona, Maryland, New Jersey, Texas, and Utah until January 1, 1922, when he resigned from the Army with the rank of Major to enter the engineering and contracting business at Ogden, Utah, with Joseph S. Kroft as a partner. At this time he accepted a commission as Major, Corps of Engineers—Reserve, U. S. Army, and was called to active service many times. His last tour of duty was as commanding officer at the Fort Douglas Reserve Officers' Camp, Salt Lake City, Utah, in 1938.

Among the contracts completed by Kroft and Bundy was the forty-two mile Wendover highway across the mud flats and salt beds of western Utah—famous throughout the world for its difficult construction. After Mr. Kroft's withdrawal from the firm in 1925, Major Bundy carried on the business under the name of Ora Bundy, Engineer and Contractor, in Utah, Idaho, Nevada, and Wyoming in the construction of concrete and bituminous paved highways, bridges, and other projects.

---

[2] "Temple of Healing," by Paul Brockway, Wichita, Kans., 1938.

[1] Memoir prepared by R. K. Brown, O. C. Lockhart, and R. R. Mitchell, Members, ASCE.

Major Bundy served as mayor of the City of Ogden from 1930 to 1934, and was appointed a member of the Utah Public Works Advisory Board in 1933. In 1941 he achieved membership on the Utah State Department of Publicity and Industrial Development Commission, becoming its chairman in 1944, and also was president of the National Reclamation Association and the Ogden River Water Users' Association. Major Bundy was a member of the Society of American Military Engineers; the Ogden Chapter, American Association of Engineers (past-president); the Ogden Engineers' Club (past-president); and the Utah Section, later the Intermountain Section of the Society (past-president). In addition, he was a past-president of the Utah Chapter of the Associated General Contractors of America and a past-commander of the Utah department of the American Legion and of the Baker-Merrill Post No. 9 of Ogden. A Thirty-Second Degree Mason, he was a member of George Washington Lodge No. 24 in Ogden; El Monte Commandery No. 2, Knights Templars; Ogden Chapter No. 2, Royal Arch Masons; Ogden Council No. 3, Royal and Select Masters; El Kalah Temple, Nobles of the Mystic Shrine; the Ogden Shrine Club; and the National Sojourners. He was also on the advisers' council, Order of DeMolay, and belonged to the Ogden Lodge No. 719, Benevolent and Protective Order of the Elks; the Weber Club; the Ogden Golf and Country Club; the Chamber of Commerce; the Forty and Eight of the American Legion; and the Pine View Yacht Club.

Possessed with a yearning for leadership in order to carry out his ideas and vigorous opinions, Major Bundy also had the driving force to attain that leadership. In the professional, business, and fraternal associations with which he was affiliated, he took an active part and left his stamp as a positive character. Essentially a man of action, he vigorously prosecuted whatever work he had in hand.

During his term on the Utah State Department of Publicity and Industrial Development Commission, of which he was chairman until his death, he was appointed director of postwar planning by the Hon. Herbert B. Maw, Governor of Utah, and made an extensive survey with the cooperation of others. As a result his department published a comprehensive volume.[2]

Major Bundy wrote:

> "To the Reader:
>
> "The transition to a wartime footing brought great changes to Utah and especially to the narrow strip designated as the Wasatch Front, where approximately two-thirds of the population of the State reside. Another transition, to a peacetime basis, lies ahead.
>
> "The 'After Victory' Report for Utah and the Wasatch Front, enclosed herewith, is the result of a cooperative attempt, under the auspices of the State of Utah, to assemble and make readily available pertinent information about the economy of Utah, past and present, for use in formulating plans for the Utah of the Future.

---

[2] "After Victory, Plans for Utah and the Wasatch Front," Utah State Dept. of Publicity and Industrial Development, June, 1945.

> "The compilers of this Report are well aware that any course of action charted at this time must be subject to change as dictated by changing conditions. Planning must be a continuing effort. It is hoped that the Report may prove to be a logical starting point for further research and may be found useful in reaching decisions that will stabilize the peacetime economy in Utah along planes of high standards. It is hoped further that persons and agencies outside of this State, confronted with problems of readjustment that are fundamentally similar to those of the Wasatch Front, may derive some benefit from the Utah effort."

Major Bundy was a capable administrator, possessing a keen, analytical mind; and he had the welfare and growth of his state at heart, contributing many helpful suggestions for its future. He had a pleasing personality, made friends readily, and had the happy faculty of working constructively with others.

The passing of Major Bundy was a great loss to the community and, in fact, to the state, where men of real character and ability are so needed in the difficult times which lie ahead.

In 1926 Major Bundy was married to Lovina Singleton Lewis. He is survived by his widow; a son and daughter by a former marriage, John A. and Mrs. Dora Goddard; a stepson, Glen Lewis (Major, U. S. Army Air Corps); two brothers, Warren and Earl; two sisters, Mrs. Wilma Eubank and Mrs. Myrtle Mahon; and three grandchildren.

Major Bundy was elected a Member of the American Society of Civil Engineers on April 3, 1922.

ARTHUR GRAY BUTLER, M. ASCE[1]

Died September 7, 1947

Arthur Gray Butler, the son of Thomas J. and Mary Elizabeth (Lee) Butler, was born in Terre Haute, Ind., on September 20, 1888. He was the youngest in a family of four boys. After receiving his early education at the public schools in Terre Haute, he entered Rose Polytechnic Institute there. He was graduated in 1910 with the degree of Bachelor of Science in Civil Engineering and later he received the degree of Civil Engineer.

Shortly after graduation Mr. Butler was employed by the Vandalia Railroad at Logansport, Ind., where he remained until 1912. He became associated with the Henry Bickel Company at Louisville, Ky., in May, 1912, and worked for that firm until 1917. In the capacity of construction superintendent during this period he was in charge of construction of the High Street steam station built for the Louisville Railway Company and supervised the widening of the Louisville-Portland Canal. During World War I he was responsible for constructing the water supply and other camp developments at Camp Taylor in Kentucky.

---

[1] Memoir prepared by Ralph B. Horner and Harrison G. Roby, Members, ASCE.

After the armistice Mr. Butler became general superintendent of construction of the Byllesby Engineering and Management Corporation in charge of work for the Louisville Gas and Electric Company, and retained that position until 1929. This was a period of rapid development in the company's property, and he was directly responsible for a wide diversity of projects, including additions to the Waterside steam station, several large substations, the Bear Grass gas plant, several large gas holders, and many miles of pipe line in eastern Kentucky and in the vicinity of Louisville. Another project which he constructed during this period was the Ohio Falls (Ky.) hydro-electric plant, which has an installation of 80,000 kilowatts. Coincident with this, he was in charge of construction of dam No. 41 in the Ohio River at Louisville, built under contract with the government.

In 1929 Mr. Butler transferred to Pittsburgh, Pa., where he became manager of the Pittsburgh branch of the Byllesby Engineering and Management Corporation in charge of engineering and construction work for the Duquesne Light Company and other subsidiaries of the Philadelphia Company. Early in this association his integrity and straightforward manner won him the respect and loyalty of those with whom he was associated. In 1936, due to changes in organization, Mr. Butler became "Manager of Engineering and Construction" of the Duquesne Light Company. Until his death he was actively engaged in supervising a large number of projects, including two extensions to the James H. Reed power station rated at about 60,000 kilowatts each.

Before completion of the latter projects, the design and construction of a new power station (Frank R. Phillips station) was undertaken. This plant has been operating at about 75,000 kilowatts since it was placed in operation in the fall of 1942. Constructed for the most part during World War II, it is a tribute to Mr. Butler's diligence and energy. In spite of labor shortages, difficulties in obtaining materials, and other wartime obstacles, this plant was completed in time to render useful service for three years of the war, due in large measures to Mr. Butler's tireless efforts to complete the project at the earliest possible date.

In addition to his duties with the companies with which he was affiliated, Mr. Butler took an active part in related engineering activities. From 1938 to 1940 he was chairman of the engineering section of the Pennsylvania Electric Association, and served as a member of the executive committee. He also served in various positions with the Edison Electric Institute of New York, N. Y., being appointed in 1946 to the engineering division general committee.

He was an officer of the Lions Club in Louisville, and handled much organizational and charitable work for this group. After coming to Pittsburgh Mr. Butler became active in the work of the Pittsburgh Association for the Improvement of the Poor, having been appointed a trustee of that organization in 1936. He served as chairman of the men's industrial committee from 1943 to 1946, and was elected vice-president of the association in May, 1947.

One of the projects under his supervision for this organization was the design and construction of a fresh air camp for underpriviledged children. This was constructed about twenty miles from downtown Pittsburgh and provided facilities in the country, including dormitories, mess halls, a swimming pool, a recreation building, and playground equipment to accommodate nearly one hundred and fifty children at a time. Mr. Butler's interest in his development was intense; he spent much effort and many week ends studying plans to obtain the maximum facilities for the limited money which was availible, with the result that the camp provided an exceptionally efficient layout. The dormitories were placed on a wooded hillside, so located as to afford unobstructed views of the recreation facilities and surroundings, and so arranged as to keep removal of trees to a minimum.

Mr. Butler was also affiliated with the Federation of Social Agencies and was a member of the committee on plant and maintenance from 1944. Educational work actively interested him too. He served two terms as an alumni trustee of Rose Polytechnic Institute, and in 1932 was elected to life membership on this board. He proved to be a most valuable member, regular in attendance, sound in his judgment, and active on several important committees—including the building and ground committee, which was engaged in an expansion program for the school at his death.

Mr. Butler was a past-president of the Engineers and Architects Club of Louisville, and was a member of the Engineers Society of Western Pennsylvania. He served as a member of the house committee of the latter organization during the remodeling of their headquarters and club rooms in 1940, and was also chairman of a special committee appointed that same year to study methods of improving the society and of making it of greater benefit to its members.

Golf was his favorite recreation and for many years prior to his death he was a member of the Longue Vue Club in Pittsburgh. He was a member of the University Club and the F. and A. M. While in Louisville Mr. Butler was as elder of the Crescent Hill Presbyterian Church, and on coming to Pittsburgh became an elder in the Third Presbyterian Church in that city. He served for several years as chairman of the building committee, planning extension of the church facilities.

On September 25, 1912, in Terre Haute, Mr. Butler was married to Marie Rhoads. He is survived by his widow; four daughters, Harriet Ingold (Mrs. Thomas S. Stevenson), Anne Spottswood (Mrs. Harry B. Stewart, Jr.), Mary Gray, and Jean Lee; and four grandchildren.

Mr. Butler was elected a Junior of the American Society of Civil Engineers on June 23, 1916; an Associate Member on October 9, 1917; and a Member on April 18, 1927.

BENJAMIN LUCIEN CAMPBELL, M. ASCE[1]

DIED DECEMBER 9, 1944

Benjamin Lucien Campbell was born at Mankato, Minn., on January 24, 1883. He was the son of Benjamin and Carrie (Gillette) Campbell. His father and mother were both of Scotch-Dutch descent. Although most of his life was spent in and around Portland, Ore., his work took him to Boston, Mass., Providence, R. I., San Francisco, Calif., Seward, Alaska, and numerous other points. His elementary and high school training was received in Portland. He attended the Colorado School of Mines at Golden and Rensselaer Polytechnic Institute in Troy, N. Y., each for one year, studying mining engineering.

From 1900 until 1928 he worked as draftsman, surveyman, locator, construction supervisor, and designer on numerous mining and railroad projects throughout the United States and Alaska. In 1928 Mr. Campbell was engaged by the Corps of Engineers, United States Army (Portland District), as assistant engineer. He then served with the War Department continuously (except for a few months in 1933 and 1934), until his retirement in 1944 a few months before his death.

During World War II Mr. Campbell worked with the District Engineer Office in Seattle, Wash., as supervisor of construction of facilities at the Annette Island Air Base in Ketchikan, Alaska. The Signal Corps, U. S. Army, borrowed his services for special work while at this base. His failing health was partly responsible for his return to Portland in 1943.

As an amateur gunsmith Mr. Campbell held a noteworthy reputation; and his principal hobbies were hunting, target shooting, and boating. He was a member of the Society of American Military Engineers and the Professional Engineers of Oregon.

He was married at Kansas City to Sally Brown Skinner of Portland in 1903. He is survived by his widow and three daughters, Mrs. Ida F. Miller, Mrs. Esther Wright, and Mrs. Clarissa Bronn.

Mr. Campbell was elected an Associate Member of the American Society of Civil Engineers on August 17, 1931, and a Member on June 26, 1933.

EDWARD EMERY CARPENTER, M. ASCE[1a]

DIED OCTOBER 26, 1945

The death of Edward Emery Carpenter marked the passing of another of that group of pioneer engineers who had an important part in the building of the West. His engineering career began in the era of railway building and spanned the transition period well into the time when hydroelectric development became of primary importance on the Pacific Coast.

---

[1] Memoir prepared by Stanley E. Sporseen, M. ASCE.

[1a] Memoir prepared by C. E. Blee, M. ASCE.

Descended from pioneering stock, Edward Emery Carpenter, the son of Emery Wilcox and Maryette (Austin) Carpenter, was born in Americus, Kan., on November 12, 1872. Edward was one of a large family, having three brothers and one sister. His parents were living in Wisconsin at the time of their marriage and moved west in a series of migrations, stopping in Ohio, Kansas, Colorado, and other places. Some of the earlier steps were made by horse and wagon. Finally the family arrived in San Luis Obispo, Calif., when Edward was fifteen years old. This became the permanent family home. Edward was graduated from high school at San Luis Obispo and entered Stanford University at Palo Alto, Calif., with the class of 1896, one year after the founding of that institution. He was graduated from the university with the degree of Bachelor of Science in Civil Engineering in June, 1898, having found it necessary to remain out of school for two years in order to earn funds with which to continue his education.

Upon graduation, Mr. Carpenter entered railway construction work as chainman and rodman with the Atchison, Topeka and Santa Fe Railway Company and quickly advanced to the position of assistant engineer on maintenance of way for the Southern Pacific Railroad Company. In 1901 and 1902, he carried out preliminary surveys for a proposed power project on the Stanislaus River in California, one of the earliest major hydroelectric power developments in the United States and one with which he was later to be prominently identified. Returning to railway work, during the next three years he was in charge of difficult railway location assignments in Mexico, northern California, and Nevada. In much of this work he was associated with that master railroad builder, William Hood,[2] M. ASCE, chief engineer of the Southern Pacific Railroad.

In 1905 the Stanislaus power project again became active and Mr. Carpenter joined the organization of Sanderson and Porter, which firm designed and constructed the project. Here the young railway engineer had an opportunity to demonstrate his administrative ability and genius for organizing a construction project for it was a difficult job and had to weather the disruptive effects of the financial panic of 1907. He rose to the position of chief construction engineer and successfully completed the project at the end of four years.

Next Mr. Carpenter worked on the design and construction of the Jordan River power development as chief engineer for Sanderson and Porter. This project was in an almost unexplored section of Vancouver Island, British Columbia, Canada, without access save landing from boats on an open beach. In spite of the difficulties of penetrating into an unmapped territory covered by an exceedingly heavy timber growth and subject to a high rate of rainfall and snowfall, this project was completed in the scheduled time of three years.

---

[2] For memoir, see *Transactions*, ASCE, Vol. 91, p. 1096.

In 1912 Mr. Carpenter became chief of the Department of Civil Engineering on the building of the Panama-Pacific International Exposition at San Francisco, Calif. This was an intricate job involving the placing of dredged fill and the installation of all the utilities, streets, and other facilities that would be required for a large city. The drawing together of these facilities so as to permit the opening of the Exposition on schedule was a constant battle against time and required the closest coordination.

When this assignment was completed Mr. Carpenter was forced to retire from active work for more than a year in order to undergo a serious operation on his eyes. His eyesight had been injured early in his career from running a transit in the bright sun over the snow fields of the high mountain country. This condition became progressively worse until he could scarcely use his eyes at all. The operation only partially corrected the trouble so he was subject to this affliction during most of his active career. It was characteristic of the man that he rose above this handicap and carried on in a normal manner, scarcely ever mentioning the impairment of his vision.

In 1917 he returned to active work as executive engineer and manager of a ship building plant at Raymond, Wash., for Sanderson and Porter. Starting from nothing but a tidelands site, a complete plant was constructed and ten wooden ships were turned out for the United States Shipping Board in a little more than two years' time.

On completion of the ship building job, Mr. Carpenter returned to San Francisco and together with the late Shirley Baker, M. ASCE, a close personal friend and associate since college days at Stanford, formed the engineering firm of Baker and Carpenter. During the next four years this firm carried on a general engineering business including the design and construction of the stadium at Stanford University. Also Mr. Carpenter was a member of the commission responsible for the design and construction of the Memorial Stadium of the University of California, at Berkeley.

In 1923 Mr. Carpenter was called to British Columbia to organize and head a construction department for the British Columbia Electric Railway Company, Limited, the utility serving the lower mainland and Vancouver Island. He continued in this position until retirement, about two years before his death. During these twenty years of service he was responsible for the design and construction of a number of major hydroelectric projects as well as other engineering services for the company.

Mr. Carpenter was a man of large physical stature and of outstanding personality. He conducted himself with a dignity that commanded the respect of his associates and of the contractors, labor representatives, and others with whom he had professional or business dealings. As an engineer and constructor, he had outstanding administrative ability and a genius for organization. In engineering design he never lost sight of fundamental principles. He had a well-earned reputation for sound judgment.

Being a great reader, Mr. Carpenter was well informed on a wide variety of subjects and was a most interesting conversationalist. He had a naturally genial nature which made him many friends. Moreover he was highly regarded and esteemed by those who worked for him and wherever he went on jobs from Mexico to Canada he was always accompanied by former associates from previous jobs.

On April 20, 1904, Mr. Carpenter was married to Aileen Morrison of Chico, Calif., who predeceased him by four years. He is survived by his son, Donald D. Carpenter, of Vancouver, British Columbia.

Mr. Carpenter was elected a Member of the American Society of Civil Engineers on October 1, 1912. He became a Life Member in January, 1943.

JOHN TAYLOR CHAMBERS, M. ASCE[1]

Died April 27, 1947

John Taylor Chambers, son of Henry E. and Ellen (Taylor) Chambers, was born in New Orleans, La., on March 10, 1885. After being graduated from the public high school, he matriculated in the College of Engineering of Tulane University, at New Orleans, and was graduated with the degree of Bachelor of Civil Engineering in the year 1906.

Following graduation, Mr. Chambers completed special study work at Harvard University in Cambridge, Mass. Later, in connection with his professional work, he was given special training in reinforced concrete design at the head office and laboratory of an outstanding fabricator. Although an excellent student, he found time to devote to athletics and other extracurricular activities. He captained football teams in both high school and college. His family connections gave him a position of prominence, but his athletic ability and attractive personality placed him high in popular esteem, and "Jack" Chambers was well and favorably known before he embarked on his professional career.

Mr. Chambers devoted the first five years of his professional life to engineering construction and was employed by the United States Corps of Engineers, and the Sewerage and Water Board of New Orleans on major assignments. In 1911 he formed the Chambers-Morris Company, a civil engineering firm, and this organization was responsible for the structural design of a large number of buildings in the New Orleans area. During the five-year period, from 1915 to 1920, Mr. Chambers operated a 5,000-acre cotton plantation in Chicot County, Arkansas, but his liking for engineering and construction asserted itself and he returned to the field he had selected for his life work. For the succeeding seven years (from 1921 to 1928), first as a partner in the firm of Chambers and Bowers, Contractors, and then as an individual contractor, he completed many contracts and professional engagements in the sewerage, water, and drainage fields.

---

[1] Memoir prepared by a Committee of the Louisiana Section consisting of John Riess, Chairman, and Frank C. Carey, Members, ASCE, and Charles M. Kerr, Assoc. M. ASCE.

Beginning in 1928, as a partner in the firm of Lewis-Chambers, this construction company undertook major levee, canal, and earth moving contracts involving millions of dollars. Lewis-Chambers owned some of the heaviest dirt handling equipment in the south. Limitations and the need for long-range equipment prompted Mr. Chambers to design, build, and later operate the "Chambers bridge," a large, mobile, steel structure powered by diesel engines, over which cable operated cars transported 12 cubic yards of excavated material for 350-foot distances in 50-second cycles. By use of the Chambers bridge many of the largest Mississippi River levees were constructed. These levees contained millions of cubic yards of fill obtained from borrow pits far from the crown of levees. The machine designed by Mr. Chambers was used in building a section of the proposed Atlantic-Gulf Ship Canal across Florida. It has also been profitable and economically used in coal stripping operations.

He was married to Marion Monroe, daughter of the late Judge Frank A. Monroe, distinguished jurist and chief justice of the Louisiana Supreme Court.

Mr. Chambers was elected an Associate Member of the American Society of Civil Engineers on September 3, 1912, and a Member on September 11, 1944. He became a Life Member in January, 1947.

CLEVELAND BEACH COE, M. ASCE[1]

Died November 29, 1945

Cleveland Beach Coe, the son of Charles C. Coe and Anna S. (Beach) Coe, was born in Kenova, W. Va., on March 12, 1893.

He received his elementary education in West Virginia and Tennessee. After being graduated from the Johnson City (Tenn.) High School in 1908, he enrolled in the Sewanee Military Academy. In 1909 he attended Hobart College in Geneva, N. Y., and received the degree of Bachelor of Science in 1914. He also attended the Carnegie Institute of Technology in Pittsburgh, Pa., doing postgraduate work in civil engineering from 1914 to 1916; and, after World War I, the École des Travaux Publics in Paris, France. As both teacher and student he displayed throughout his career a very active interest in engineering education and contributed much to the advancement of the profession. Several articles written by him were published in *Engineering News-Record* and the *Transactions* and *Proceedings* of the Society. He was a member of Sigma Phi and Theta Nu Epsilon fraternities.

Mr. Coe became interested in the civil engineering profession early in life. At the age of sixteen he was employed during the summer vacation as a rodman on railroad construction. His first position after graduation from college in 1916 was as draftsman and inspector for the Mead Paper Company of

---

[1] Memoir prepared by F. W. Webster, M. ASCE, and F. E. Bosland, Assoc. M. ASCE.

Chillicothe, Ohio. He resigned in June, 1917, to enlist in the 323rd Field Artillery, United States Army, as a Second Lieutenant. He participated in the Meuse-Argonne offensive and other active engagements of the 323rd Regiment and remained with the Army of Occupation until June, 1919, when he was relieved from active duty with the rank of First Lieutenant.

He immediately resumed his chosen profession and became associated with the O. K. Morgan engineering firm in Johnson City. In 1923 he became Commissioner of Streets and City Engineer for Johnson City, and in this capacity was responsible for the administration, engineering, and construction of municipal public works. He returned to private practice in 1927, and in addition to his regular practice became special valuation engineer for the Atlantic Coast Line Railroad Company and the Carolina, Clinchfield and Ohio Railway Company.

He continued private practice until October, 1933, when he joined the engineering staff of the Tennessee Valley Authority. His first assignment was to assist in the planning, layout, and construction of the Town of Norris, Tenn. He then was assigned to the relocation of highways in the inundated areas. His ability as a leader and engineer was soon recognized, and he advanced to the position of associate highway engineer, the position he held when he returned to active duty in the United States Army in November, 1940.

His first assignment in World War II was as Adjutant to the Commanding Officer, First Military Area, Knoxville, Tenn., and later he served as Commanding Officer. He attended the Command and General Staff School at Fort Leavenworth, Kans., after which he was assigned to Camp Forrest, Tullahoma, Tenn., and Camp McCain, Grenada, Miss., as Executive Officer. At Camp Butner, N. C., in addition to his duties as Executive Officer, he also served as Commanding Officer of the Redistribution Station. He was stationed at Fort Oglethorpe, Ga., when he received his overseas orders and was transferred to Camp Lee, Va., for processing. Following the Japanese surrender, his unit was deactivated and Colonel Coe, at the time of his death, was under orders to report to Fort Lewis, Wash.

Colonel Coe gave himself wholeheartedly to the service of others. His unceasing devotion to his church, his country, and his fellowmen was an inspiration to all who knew him. A member of the Episcopal Church from his youth, his interest in religious music had wide influence. He wrote a full choral communion service and more than one choir in Eastern Tennessee has sung his "Benedicite." Fittingly, at his burial service, the choir sang his own "Nunc Dimittis." He was a lay reader in the church and an active member of the vestry for many years; he became secretary of the vestry of Saint John's Church in Johnson City.

He gave himself unstintingly to the services of his community. The Community Chest, Rotary Club, Red Cross, Crippled Children, Salvation Army, and Boy Scouts suffered an irreplaceable loss in his passing. All kinds of people were fond of him. There was a quiet strength always manifest in him; Mr. Coe was gentle though manly, possessed of deep wells of human sympathy. What the trainees in the several camps he served and the young engineers with whom he was associated owe to him only they could measure.

Cleve Coe was an active member of the ASCE, serving as Secretary-Treasurer of the Chattanooga Sub-Section of the Tennessee Valley Section in 1939 and he was in charge of the arrangements for the Society's Spring Meeting held in Chattanooga that year. He was active in the American Legion, Reserve Officers' Association, and the Society of American Military Engineers. In spite of his many activities he somehow found time for his three hobbies: Designing and Collecting book plates, stamp collecting, and music.

In Johnson City, in 1924, Colonel Coe was married to Margaret Summers. He is survived by his widow and two sons, Robert and Kenton.

Colonel Coe was elected an Associate Member of the American Society of Civil Engineers on December 13, 1937, and a Member on August 24, 1942.

CLARENCE KNIGHT CONARD, M. ASCE[1]

DIED MAY 16, 1947

Clarence Knight Conard, the son of William and Adelaide (Zimmerman) Conard, was born in Washington, D. C., on March 3, 1876. His father, a native of Philadelphia (Pa.) and a Quaker, was a veteran of the Civil War. At the close of the war William Conard settled in Washington and represented claimants who brought actions against the government for many acts which occurred during the war and in the postwar period. The father later entered the postal service and became chief of the inspector's division of the Colorado district. His mother came of a Virginia family whose sympathies were with the Southern cause during the war.

Mr. Conard's early life was spent in Florida and in Maryland. He attended Brookville Academy in Brookville, Md., and the high school at Ithaca, N. Y. In 1894 he entered Cornell University at Ithaca and majored in civil engineering for two years, when the sudden death of his father caused him to leave the university and seek employment.

During the summer of 1896 he was a rodman with the United States Coast and Geodetic Survey and then engaged in the lumber business in Missouri conducted by his uncle. In April, 1898, when war was declared with Spain he enlisted in the second Missouri Infantry and remained in service until February, 1899, when he was mustered out. At the close of the war he returned to Washington and became a rodman on street railroad construction and a draftsman with the U. S. Coast and Geodetic Survey.

---

[1] Memoir prepared by John C. Riedel, M. ASCE, and Arthur S. Tuttle, Past-President and Hon. M. ASCE.

He left the latter position on July 15, 1899, to join a surveying party in Nicaragua headed by Thaddeus Merriman,[2] M. ASCE, under the general direction of Francis Lee Stuart,[3] Past-President, ASCE, who had been appointed by the newly created Isthmian Canal Commission to survey that part of the Nicaragua Canal route extending eastward from the lake. While engaged in this service, he was advanced from the grade of rodman to that of transitman.

On his return to the United States in August, 1900, he was employed by the Baltimore and Ohio Railroad Company as transitman, and later as resident engineer in the construction of the Quemahoning branch, and on preliminary surveys, with which work Mr. Stuart (a brother-in-law by marriage) was responsibly connected. From August, 1904, to March, 1905, he was a member of the staff of the surveyor general of Arizona. In 1905 Mr. Stuart was appointed chief engineer of the Erie Railroad Company and at once summoned Mr. Conard to join his staff as assistant engineer in charge of construction of forty-five miles of Erie and New Jersey Railroad, costing more than $7,000,000. He was also in charge of the four-track entrance into Jersey City (N. J.) through the Bergen Hill, costing $3,000,000, and of the location and construction of the Elmira, Corning and Waverly Railway, an electric interurban line in New York, costing $800,000.

Mr Conard remained with the Erie Railroad until 1912, when he became construction engineer of the Norfolk Southern Railway Company on the extension of the line from Raleigh to Charlotte in North Carolina. This work involved seventy-five miles of new lines, costing $3,000,000, and was completed in 1914. For the next two years he was assistant engineer of the Southern Railway Company, in charge of the double-track betterment from Orange to Charlottesville in Virginia at a cost of $1,500,000. When this work was completed in 1916, he went to Cuba as chief engineer for railroad surveys and located a line from the mines of the Davison Sulphur and Phosphate Company to sea level.

He was in Cuba during the outbreak of the Cuban revolution in 1917. The United States entered World War I in April, 1917, and on November 9, 1917, he was commissioned as Major in the Ordnance Section, United States Army. He was detailed as constructing quartermaster in charge of building the Raritan Arsenal in Metuchen, N. J., which was to cost $1,400,000. The area comprised nearly four square miles and the work consisted of the erection of barracks for about seven thousand men, a tank assembly layout, railroad facilities, and some two hundred magazines of various sizes. At the river front a dock two thousand feet long was built and the salt meadow area was protected by a dike. He remained in the Army until October 30, 1919, when he was discharged.

Mr. Conard entered the employ of the International Conveyor Corporation, working on the development of coal handling machinery, but found in about a year that the market was dead for new developments. He then worked for the J. G. White Engineering Corporation in 1921 as division engineer on a survey in Nicaragua for a railroad to extend from the east coast to the lake region. The surveys were completed within the year and he returned to the United States. The depression of 1921 caused a cessation of engineering opportunities and for a time he made local surveys in Hempstead, Long Island.

---

[2] For memoir, see *Transactions*, ASCE, Vol. 107, 1942, p. 1799.

[3] *Ibid.*, Vol. 101, 1936, p. 1493.

In 1922 Mr. Conard was appointed railroad engineer on the staff of Arthur S. Tuttle, Past-President and Hon. M. ASCE, who was then chief engineer of the Board of Estimate and Apportionment and in charge of the design and construction of a freight and passenger railroad between Brooklyn and Staten Island, involving a tunnel under the Narrows. Mr. Conard was placed in charge of the surveys, extending from Arthur Kill in Staten Island to Brooklyn, and the final location of the line. The project also included the design of a large classification yard in Staten Island and connections with the Baltimore and Ohio Railroad near Arthur Kill, with the city piers at Stapleton, with the Long Island Railroad at Bay Ridge, and with the Fourth Avenue Rapid Transit System in Brooklyn. His work was well advanced and the city had taken title to the land required for the classification yard and for some of the right of way when the project was discontinued by the New York State Legislature, in 1925, owing to the strong objections which had been raised by some of the trunk line railroads operating in the metropolitan area.

In 1925 and part of 1926 Mr. Conard was construction engineer for the Conard Construction Company of Florence, S. C., on the extension of the Winston-Salem branch of the Piedmont and Northern Railway Company. Local political difficulties caused cessation of the work, however, and Mr. Conard returned to New York. Mr. Tuttle subsequently invited Mr. Conard to return to his staff, and on January 1, 1929, he became railroad engineer on grade crossing eliminations—a position which he held until his retirement in 1943. Many grade crossing eliminations affecting the privately operated rapid transit railroad on Staten Island and the Long Island Railroad in Brooklyn and Queens (the latter work including the entire Jamaica section) were carried out. Mr. Conard represented the chief engineer of the Board of Estimate on all these projects.

Mr. Conard was a man of sterling qualities and of the highest integrity. He had an amazing capacity for work and was precise in his determination. At Cornell University he was an athlete and he retained his athletic prowess until he was well on in his sixties, playing tennis long after men of his years had abandoned such activities. Modest in demeanor, he possessed countless friends; his associates esteemed him most highly. Mr. Conard was a member of the American Railroad Engineering Association.

He is survived by his widow, Rosalie Rives Conard, to whom he was married on May 6, 1903, and by four children: William of Metuchen; Clarence K., Jr., of Clark, N. J.; Elizabeth of New Paltz, N. Y.; and Rosalie (Mrs. Frank R. Switzer) of Baranquilla, Colombia. Also surving are two brothers, Charles Conard, Rear Admiral, U. S. Navy (Retired), and Robert Conard of Guntersville, Ala.

Mr. Conard was elected a Member of the American Society of Civil Engineers on October 5, 1909.

JOHN ROSE WILSON DAVIS, M. ASCE[1]

Died February 19, 1947

John Rose Wilson Davis, oldest of the five children of Thompson and Ann Supplee (Rose) Davis, was born in Phoenixville, Pa., on October 26, 1868. The Davis family came from Wales and settled in America in about 1650, in eastern Pennsylvania. Thompson Davis and his father, Thomas Davis, operated an overshot flour and feed mill on Pickering Creek, a tributary of the Schuylkill River. This mill dated back to Revolutionary days.

On his mother's side, the Rose forbears also came from Wales in about 1680. Part of the early Rose farm occupied the site of the present Pennsylvania Railroad Station at Thirtieth and Chestnut streets in Philadelphia, Pa. Grandfather John Rose, for whom the future engineer was named, had a sword—one of several extant— made in the prerevolutionary woolen mill of the Rose family. This factory was devoted to manufacturing swords for officers of the Colonial Army.

John Davis attended the public schools in Chester County, Pennsylvania, the West Chester State Normal School (from 1883 to 1885), and Schwartz's Academy in Bethlehem, Pa. (in 1886). He started the civil engineering course at Lehigh University in Bethlehem in September, 1886, and, after absence during the second term (in 1889–1890), he re-entered the senior class in September, 1890, being graduated in June, 1891, with the degree of Civil Engineer. In college, he was a serious student, a member of the university tug-of-war team, and active in the Engineering Society.

As he approached his professional career Mr. Davis selected the then prominent field of railroad maintenance. His whole adult life, for about five decades, was spent in this type of work. During the first few years, his employment and employer varied frequently, as he sought to gain basic experience. For a year, in 1891–1892, he was rodman for the New York, Lake Erie and Western Railroad Company at Port Jervis, N. Y., being promoted then to assistant supervisor of track. The following six years, until 1898, he was assistant engineer. He then left this road, to serve for two years as division engineer for the Chicago and Erie Railroad Company at Huntington, Ind. Again, for short periods, he worked for the Chicago and Alton Railroad Company at Bloomington, Ill., and the Erie Railroad Company at Jersey City, N. J.—in both instances as engineer, maintenance of way. In the latter position he was in charge of maintenance engineering from New York, N. Y., to Buffalo, N. Y., and Jamestown, N. Y.

With this excellent training, and with the reputation it provided him, he was prepared for larger responsibility. The opportunity came in 1903 when he was asked to take a similar, but more exacting position with the Great Northern Railway Company in St. Paul, Minn., as engineer, maintenance of way. James J. Hill,[2] F. ASCE, then president of the Great Northern, felt he needed a capable man in charge of maintenance, and so Mr. Davis was selected while employed by the Erie Railroad. It proved to be a wise choice. He remained with this line up to his retirement—thirty-six years later.

---

[1] Memoir prepared by George G. Smart, Edmonds, Wash.

He found the Great Northern had a main line of about two thousand track miles, with six thousand miles of secondary tracks, stretching from the Great Lakes to the Pacific Coast—a vast system on which the maintenance had been sadly neglected. The rail was light, no tie plates were in use, ties were in poor condition, and what little ballast existed was of poor quality. Prior to his arrival it had been the practice for several years to take ties purchased for renewals and use them in construction of new branch lines. It was generally too late to secure additional ties, and the renewal of ties having been neglected resulted in a very poorly maintained line with frequent derailments. This, of course, adversely affected the operation. It was a discouraging outlook for one coming from a well maintained road.

As a granger railroad, finances were a serious problem, especially during those seasons when crops were poor. For many years it was an uphill fight to secure sufficient appropriatons to improve the physical condition of the property. Mr. Davis was a man of great determination, not easily discouraged, and constantly fought for the needed improvements.

Gifted with unusual foresight, he could look into the future and see the problems that would confront the railroad before they became apparent to the average man. It was this ability which made him one of the outstanding maintenance engineers in the United States. His advice and counsel was sought by many of the large railways of the United States. In planning and carrying out his work he built for the future. He could see the constant increase in axle loads which would require a stronger track structure than was then in use. Many of the improvements he pioneered are now standard practice on American railways.

For example, his was the first line west of the Mississippi to use the 6-inch track spike, the large tie plate, the 8-foot 6-inch cross tie, a wider roadbed, and heavier ballast section. He encouraged his employees and the railway supply houses to use their ingenuity in developing and perfecting labor-saving devices and was always willing to test them under service conditions. Under his encouragement the following were developed and set in operation: The self-propelled ditching machine, the self-propelled weed burner, the bolt machine, the adzing machine, and the rail laying machine. These are now standard equipment on all large roads.

His activity in developing an improved tie plate was typical. In about 1920 Mr. Davis saw the need for a much larger plate than was then in vogue anywhere in the United States. The soft wood ties placed in service on the Great Northern Railway soon became badly damaged by rail cutting; and, looking ahead to the increased traffic and heavier equipment of the future, Mr. Davis designed and recommended a plate 9 inches by 11½ inches. The management was aghast at such a large plate, but after months of discussion 120,000 of them were ordered from the steel mills and used with new rail.

---

[2] For memoir, see *Transactions*, ASCE, Vol. LXXXI, December, 1917, p. 1829.

The management, however, soon felt that a smaller plate might be satisfactory, using the same rails. Mr. Davis objected because the plate would be out of balance and would not give satisfactory service. His decision was overruled and a 7½-inch plate instead of a 9-inch plate became standard. During the succeeding five years many millions of these plates were placed in the track. Twelve years later, however, the plates started to give trouble, the breaking off of the ends caused the rail to tip or cant inward, tightening the gage. It became necessary to replace them with a larger plate and the smaller plates were used on less important lines. By the time of Mr. Davis' death, the tie plate he had orginally designed, with slight modifications, had become standard on eighty per cent of the main tracks of American railroads.

With the adoption of heavier rail sections similar problems were experienced. Over a period of many years successful increases of 5 pounds per yard were soon abandoned after a short service proved them too light. Mr. Davis personally designed and secured approval of a section known as G. N. (Great Northern) 110-pound section which was used for many years. In the late nineteen thirties the American Railway Engineering Association (A.R.E.A.) appointed a committee to study and adopt a rail section which could be used as a standard on all American lines, except for a few with very heavy traffic. After about three years of study the committee adopted the present A.R.E.A. 112-pound rail which is nearly identical with the G. N. rail designed by Mr. Davis. The fishing surfaces of both rails are the same. The committee hesitated to adopt any existing pattern, which was the reason for the slight change from the G. N. rail.

Mr. Davis constructed and placed in operation six ballast washing and crushing plants. About seventy per cent of the main line had been ballasted with this material when he retired. There were many other examples throughout his long service, in which his foresight and judgment were finally proved to be correct.

On May 1, 1925, Mr. Davis was promoted to chief engineer and the position of maintenance engineer was left vacant. The management felt that with his experience and ability he could handle both positions, in which estimate they were correct. After his retirement the position of maintenance engineer was again established.

During the period when he held both positions the Great Northern was extended from Bend, Ore., into California, and the main line Cascade Tunnel in Washington, whose length of 7.78 miles made it the longest tunnel in the western hemisphere, was constructed. Many line changes were completed and curve reductions carried out, and the line generally was improved in anticipation of faster train schedules and of the new streamlined "Empire Builder" since placed in service. During the last fifteen years of Mr. Davis' service from ten to eighteen million dollars were expended annually under his direct supervision.

Throughout his staff of some one hundred and sixty officials and seven thousand other employees, there was a feeling of loyalty and respect for his fairness and consideration. Employees were seldom dismissed from the service, but demoted and given a chance to redeem themselves and again work up to better positions. He was ever thoughtful, kind, and considerate of others.

When he retired on January 1, 1940 (at the age of seventy one), his loss was keenly felt by the management and employees, and the American railroads lost an outstanding authority in maintenance engineering.

The condition of the Great Northern, then one of the best maintained roadbeds in the west, was a monument to his ability and steadfastness of purpose. He received numerous tributes to his outstanding service. A particularly revealing one was from Ralph Budd, Hon. M. ASCE, a former president of the Great Northern who, as a result of almost forty years of personal knowledge, wrote in part:

> "No one in my acquaintance has been more continuously devoted to railroad service. * * * You [Mr. Davis] certainly have the right to look back on your thirty-six years with the Great Northern with a great deal of satisfaction. The changes in standards of maintenance, and in the road's ability to serve the territory, have been to a considerable extent due to your ingenuity and efficiency."

Following his retirement in 1940, Mr. Davis lived for a while in Portland, Ore., and finally, in Seattle, Wash. He devoted his time principally to his large grain and livestock ranch in Idaho, which he owned and managed. He always was a great reader and analyst. He never reached the point where he felt he "knew it all," but was still a student of large scope up to the time he passed on. Ever fond of outdoor activities, he enjoyed country life—mainly, farming, walking, and motoring.

On January 10, 1934, in Spokane, Wash., he was married to Ethel Margot Ronald of Seattle. He is survived by his widow. In his name she has established the John R. W. Davis Memorial at Lehigh University. A substantial capital fund is to be held and the income only to be allotted to meritorious and needy students in civil engineering. In this way the idealism that led him to help so many young men in his lifetime is to be maintained in a splendid "living memorial" that will perpetuate his memory through years to come.

Mr. Davis was a charter member of the American Railway Engineering Association and was active in that organization, particularly in the formulation of track standards. He was a member of the Presbyterian Church.

Mr. Davis was elected a Member of the American Society of Civil Engineers on November 4, 1903. He became a Life Member in January, 1938.

ORVILLE VOSBURY DERR, M. ASCE[1]

DIED NOVEMBER 23, 1947

Orville Vosbury Derr, the son of William Lloyd and Lillian (Kies) Derr, was born at Needham, Mass., on April 27, 1885. He attended the public schools and the high school in Elmira, N. Y., and from September, 1901, to June, 1904, studied civil engineering at the Stevens School, Stevens Institute of Technology, in Hoboken, N. J.

---

[1] Memoir prepared by H. F. King, M. ASCE.

From 1904 to 1910 Mr. Derr was an engineering employee of the Baltimore and Ohio and Lehigh Valley railroad companies in a subprofessional capacity; and from 1910 to the end of 1911 he was an engineering employee of the Boston and Albany Railroad Company in a professional capacity.

Early in 1912 he entered the employ of the Erie Railroad Company as resident engineer, a position he retained until April, 1918. He received a furlough from the railroad to enter military service during World War I. Commissioned a Second Lieutenant, Corps of Engineers, in the Railway Transportation Corps, he was honorably discharged from the Army as a Captain at the termination of hostilities, after serving in France with the United States Expeditionary Forces. Subsequent to his discharge from military service, Mr. Derr again entered the employ of the Erie Railroad Company in August, 1919, as resident engineer, and thereafter successfully advanced to general office engineer in 1929, and to valuation engineer in 1931, resigning in August, 1938, to enter private practice.

His practice consisted of consultation on valuation and construction projects, and he retained this practice until late in 1941, when he abandoned it to associate himself with Foley Brothers, Contractors, of Pleasantville, N. Y., as field engineer. He took charge of constructing a graving dock, piers, and warehouses (known as the St. Helena project) being constructed for the United States Navy at Norfolk, Va.

Mr. Derr continued in this capacity until ill health forced him to resign late in December, 1942. After recovering his health early in 1943, he was reemployed by Foley Brothers as field engineer to supervise the Republic Plant extension at Farmingdale, Long Island, N. Y. After the termination of this project, he became associated with the Merritt-Chapman and Scott Corporation. Subsequent to 1945 and early in 1946, at the behest of L. Morris Mitchell, Assoc. M. ASCE, former vice-president of Merritt-Chapman and Scott Corporation, he became a part of L. Mitchell, Incorporated, organized to carry out heavy construction and specializing in foundation work and pulp and paper mills. Mr. Derr continued in this capacity until his death.

While with the Erie Railroad Company he specialized in heavy construction work, and one of his most important achievements was the elimination of railroad grade crossings through Paterson, N. J., an undertaking ordered by the Public Utilities Commission of the State of New Jersey.

Mr. Derr came from a family of engineers starting, to available knowledge, with Isaac Derr, an engineer in Napoleon's army in Egypt, the Derr name appears in Napoleonic records of Egyptian surveys. Isaac came to America later with a brother and founded the Derr family here, and it is said that no Derr male lived a happy life unless he followed the engineering profession. Mr. Derr followed that profession with zest, giving his best to every position he held.

His work was always characterized by accuracy and good judgment. He was a gentleman of very likable personality and made many lasting friendships in business and in his home surroundings.

On Thanksgiving Day (November 29), 1934, at Melrose, Mass., he was married to Ruth Barry. He is survived by his widow, and four sons by a former marriage. Mr. Derr's sons, Thomas, Lewis, William, and Ralph, served in World War II.

Mr. Derr was elected a Member of the American Society of Civil Engineers on May 15, 1939.

RICHARD GRIFFITH DEVELIN, M. ASCE[1]

Died March 22, 1947

Richard Griffith Develin, the son of James and Mary (Aylward) Develin, was born in Philadelphia, Pa., on November 15, 1866. There was a family of eight children, equally divided between boys and girls. One boy died in infancy and Richard was the youngest of the three who survived.

His early education was received in the public schools, and he was graduated from the Philadelphia Central High School in 1884. In October of that year he entered the service of the Pennsylvania Railroad Company as a clerk in the office of the chief engineer. An aptitude for building childish structures was displayed in early boyhood, which no doubt pointed the way to an engineering career. During this clerkship his studies were continued in the evenings, and in 1888 he entered the department of civil engineering of the University of Pennsylvania at Philadelphia, taking a special two-year course in structural engineering.

The years from 1890 to 1894 were stepping stones; he was draftsman and designer with the New Jersey Steel and Iron Company (Trenton, N. J.), with Cofrode and Saylor (Pottstown, Pa.), and with the Pencoyd Bridge Company (Pencoyd, Pa.). In 1896 Mr. Develin returned to the service of the Pennsylvania Railroad Company and was appointed assistant engineer in the Maintenance of Way Department, where in due course he was placed in charge of masonary design, renewals, and plans for general maintenance structures on the lines east of Pittsburgh (Pa.), Erie (Pa.), and Buffalo (N. Y.) to New York (N. Y.). May, 1920, marked his course as assistant engineer of bridges under the chief engineer of the Pennsylvania Railroad lines east of Pittsburgh, Erie, and Buffalo, and in January, 1923, his promotion to assistant engineer of bridges and buildings took place.

To this position, in addition to regular bridge repairs and maintenance, were added the responsibilities of structural design, including foundations and superstructure of shops at Renova (Pa.), Wilmington (Del.), and Trenton, which consisted of locomotive erecting shops, blacksmith shops, roundhouses, planing mills, powerhouses, and other buildings. Other similar projects were under his charge in Philadelphia at Park Shops, Mantua

---

[1] Memoir prepared by Henry J. Sherman, M. ASCE.

Transfer, Gray's Ferry Shops, and the export warehouse for the Girard Point Storage Company, including powerhouses, a roundhouse, two machine shops, a storeroom, and so on. In addition he supervised a locomotive testing plant, a locomotive erecting shop, and a lathe and machine shop at Altoona, Pa.; a powerhouse and car shop at Enola, Pa.; and a steel-concrete coal wharf at Morrisville, Pa. Mr. Develin was honorably retired on pension in March, 1932.

In addition to regular employment with the railroad company, he was consulting engineer for the Philadelphia Suburban Transportation Company, serving suburban Philadelphia in the three thickly settled counties of Delaware, Chester, and Montgomery. In the early 1900's, when electric railway expansion was at its height, this company extended its lines into new areas where considerable bridge and viaduct projects were required. Mr. Develin was in responsible charge of the design and construction of these structures. After completion of these lines he continued with the company as consultant on maintenance and changes until 1942. He was the engineer for the New Hope-Delaware Bridge Company on the design and erection of a steel highway bridge across the Delaware River connecting New Hope, Pa., and Lambertville, N. J. The design of bridges and viaducts for the Cuba Eastern Railway Company were among his activities.

Recognizing early in his career the value of professional group contacts, he joined the Philadelphia Engineers' Club on February 5, 1898, continuing this membership until his death. As a director from 1908 to 1913, he served on many committees during this long membership and was active in promoting and maintaining the club's technical meetings from its early struggling days. Among his memberships was that of the American Railway Engineering Association.

Mr. Develin was deeply interested in the affairs of the Society and was seldom absent from the annual meetings in New York. Through the inspiring leadership of his college friend, Richard L. Humphrey,[2] M. ASCE, who was a member of the Committee on Development, he took an active part in the reorganization of the early 1920's. Following this accomplishment and the establishment of the Philadelphia Section, Mr. Develin was a regular attendant at its meetings, and he participated in its proceedings, exerting his best efforts to increase the membership and growth of the section. Mr. Develin was almost, if not quite, the last of those who had close association with the older group of distinguished Philadelphia engineers, among whom were Lewis M. Haupt,[3] and John C. Trautwine, Jr., Members, ASCE, William H. Brown, and Joseph T. Richards.

Early in life he formed good reading habits and found time to read more widely during his retirement. In addition to keeping abreast of engineering developments, he included the subjects of biography, economics, and to a lesser degree, the better class of fiction. He kept himself well informed on the topics of the day and took great pleasure in discussing them with many friends.

---

[2] For memoir, see *Transactions*, ASCE, Vol. 93, 1929, p. 1839.

[3] *Ibid.*, Vol. 103, 1938, p. 801.

Mr. Develin was an agreeable companion and his ample background of knowledge supplied a full range of interesting conversational subjects. He was small of stature, and in manner mild, dignified, and reserved. Although firm and persistent in his opinions, tolerance of those who did not agree with him was always in evidence; he was keenly aware of his social responsibilties and his charities kept pace with his income.

Mr. Develin was a devout, consistent church member all his adult life, serving as teacher in the church school, as a member of the choir, and as a trustee of the Centenary-Tabernacle Methodist Church of Camden, N. J. Twenty-three years ago, at the time of his moving to Moorestown, N. J., he became identified with the First Presbyterian Church and served many years as a trustee. His recreations were travel, golf, sailing, and music. He made extensive trips throughout the United States and Canada, often combining them with Society meetings. With his wife he made an extensive tour of Europe in their early married life.

On November 3, 1892, Mr. Develin was married to Helen Archer of Camden. They had two children, a son, F. Archer, and a daughter, Dorothy A. (Mrs. Charles P. Walnut). Mrs. Develin died on November 23, 1924. He was married to Mrs. Anna (Sharp) Carson, a childhood friend, on June 26, 1926. She died in March, 1937.

Mr. Develin was elected a Member of the American Society of Civil Engineers on May 1, 1907.

ELBERT HERMON DRESSER, M. ASCE[1]

DIED MAY 21, 1943

Elbert Hermon Dresser, the son of Hermon L. and Phebe (Rhoades) Dresser, was born in Jefferson, Iowa, on October 9, 1878. His parents were of New England ancestry and he followed in the footsteps of his father in choosing the profession of a railway civil engineer.

After graduating from the Jefferson High School, he moved with the rest of the family to Duluth, Minn., in 1896, and spent a year in business school and two years as a clerk for the Zenith Paper Company.

He then found work for about a year with a survey crew of the Duluth, Missabe and Northern Railway, leaving in the fall of 1900 to enter Rensselaer Polytechnic Institute, Troy, N. Y., from which school he was graduated with the degree of Bachelor of Science in Civil Engineering. His summer vacations were spent on survey work for the "Missabe Railway," and after graduation he continued with the same company.

The Mesabi Iron Range was greatly developed during the next few years, and the young engineer was kept busy on a variety of work, as locating engineer, resident engineer, and maintenance engineer, under the supervision of his father then chief engineer of the railway. This rapid expansion was necessary to keep up with the mining industry.

---

[1] Memoir prepared by a Committee of the Duluth Section consisting of Gordon H. Butler and Wayne A. Clark, Members, ASCE.

In August, 1918, he entered the Corps of Engineers of the United States Army, Railway Division, with the rank of Captain, and served until January, 1919.

On return to civilian life, Mr. Dresser spent a few months in the contracting business in Texas and then became manager and later president of a concrete (products) business, with plants in Duluth and St. Paul, Minn. He sold out his interest in December, 1927.

In January, 1928, he returned to the Duluth, Missabe and Northern Railway Company (which, in 1937, became the Duluth, Missabe and Iron Range Railway Company) as chief engineer in charge of all construction and maintenance of way and structures. He held this position at the time of his death. This position had been held by his father up to 1919 when he retired.

Among the professional, business, and social organizations to which he belonged were: The Society of American Military Engineers, United States Army Officers Reserve Corps, American Legion, Sons of the American Revolution, American Railway Engineering Association, Duluth Chamber of Commerce, Kitchi Gammi Club, Northland Country Club, and Chi Phi fraternity.

Mr. Dresser was also a member of Palestine Lodge No. 79, F. and A.M.; Duluth Commandery Knights Templars; and AAD Temple of the Mystic Shrine.

In January, 1911, Mr. Dresser was married to Isabel Meads, who survives him. He is also survived by two sisters, Lulu (Mrs. Ed Barrett) and Ethel (Mrs. Harry Deutschbein).

He was of a kindly disposition with a keen sense of humor and a dislike of sham and pretense. With his strong character and sterling integrity he made friends easily and was intensely loyal to them. He was fond of home life, but one of the greatest pleasures was traveling with his wife.

Mr. Dresser was elected a Member of the American Society of Civil Engineers on September 9, 1919.

CLIFFORD GEORGE DUNNELLS, M. ASCE[1]

Died April 15, 1947

Clifford George Dunnells, the son of Clifford Charles and Susan Emma (Christy) Dunnells, was born on July 8, 1875, at Glenfield, Pa. His father was a pioneer furniture manufacturer in Pittsburgh, Pa., where the family had moved during his boyhood. He attended the public schools and was graduated from the Pittsburgh Central High School in June, 1893. He then entered Lehigh University in Bethlehem, Pa., in September of that same year, and was graduated in June, 1897, with the degree of Bachelor of Science in Civil Engineering.

---

[1] Memoir prepared by a Committee of the Pittsburgh Section, consisting of Eugene N. Hunting, Chairman, Louis P. Blum, and George S. Richardson, Members, ASCE, and C. A. Keelen, Assoc. M. ASCE.

During the summer college vacations he worked for the Pittsburgh Bridge Company and, after graduation, continued working with this company until June, 1900, when he became estimator for the American Bridge Company in Pittsburgh, remaining with that company until March, 1910. After that he entered the office of Charles Bickel, a Pittsburgh architect, as structural engineer. Later in the same year he became a member of the firm of Hunting, Davis and Dunnells, being associated with Eugene N. Hunting, M. ASCE, and Lawrence Davis, in Pittsburgh, engaging in the design and construction of many important industrial buildings and structures throughout the eastern United States. This association continued during his entire lifetime.

With his associates he developed a large practice as consultant on the design of steel and reinforced concrete structures. Simultaneously with his professional practice, Mr. Dunnells became a professor on the faculty of the Carnegie Institute of Technology in Pittsburgh. He began as assistant professor of mathematics and physics in the School of Engineering in 1910, and in 1912 he organized the Department of Building Construction in the School of Applied Industries—the first course of this kind organized in any American college. The work of this school has been a permanent influence on the building construction industry of the Pittsburgh district. In 1918 he became associate professor, and in 1933, professor of civil engineering, continuing in this position until his retirement in 1943. This culminated a noteworthy and continuous teaching career of thirty-three years.

Realizing that a college should give a student more than mere technical education, he urged extracurricular activities. During his entire association with Carnegie "Tech," he was a director of the college Young Men's Christian Association (Y.M.C.A.) and served a term as its president. He was active in community affairs as a member of the Pittsburgh Chamber of Commerce, the Highland Country Club, and the Neighborhood Men's Bible Class, in addition to being a director of the Northside Y.M.C.A. and state director of the Y.M.C.A. He belonged to Trinity Episcopal Church and held many church offices.

Professor Dunnells was a member of Alpha Tau, honorary engineering fraternity. He also served as a member of the Building Code Commission of the State of Pennsylvania. For many years he was an enthusiastic member of the F. and A. M., a member and past-master of McKinley Lodge, No. 318; a past high priest of Shiloh Chapter, No. 257, Royal Arch Masons; a member of the Pennsylvania Consistory, Ancient Accepted Scottish Rite; and a member of Syria Temple, Ancient Arabic Order Nobles of the Mystic Shrine.

He took an active part in the formation of the Pittsburgh Section of the Society in January, 1918, and served as its President from 1935 to 1937. In October, 1936, the convention of the Society was held in Pittsburgh, and his work as President of the section during this convention was outstanding. In 1939 he was appointed to fill a vacancy in the Society's Board of Direction occasioned by the election of Director Roland P. Davis, M. ASCE, as Vice-President, serving until January, 1943. He was enthusiastic in the performance of his duties as a Director, and gave especially fine service to the Committee on Publications, being its efficient chairman for the year 1942.

He was also a member of the Engineers' Society of Western Pennsylvania and the American Concrete Institute, and was a registered engineer in the states of Pennsylvania and Ohio.

He is survived by his widow, Inez E. (Mullian) Dunnells; a son, Clifford, Jr., of Port Washington, N. Y.; four daughters, Dorothy R., Margaret E., Mary S. (Mrs. Theodore J. Lutz, Jr.) of Pittsburgh, and Elizabeth C. (Mrs. Victor Greulach) of Houston, Tex.; a sister, Cora (Mrs. R. H. Harris); and nine grandchildren.

"His works do follow him." A master workman has finished his earthly structure, leaving a record of achievement to serve as an inspiration to the many younger members of the profession whom he has been privileged to guide and assist.

Mr. Dunnells was elected a Junior of the American Society of Civil Engineers on February 4, 1902; an Associate Member on January 2, 1907; and a Member on March 4, 1913. He became a Life Member in January, 1944.

JOHN CONRAD FITTERER, M. ASCE[1]

Died March 12, 1947

John Conrad Fitterer, the son of J. S. and Rosina E. (Moop) Fitterer, was born in Clyde, Ohio, on December 15, 1871. His formal education resulted in acquiring the degrees of Bachelor of Science from Ohio State University at Columbus, in 1898, and Bachelor of Science (Civil Engineering) in 1904, and the degree of Civil Engineer from the University of Colorado at Boulder in 1912. In 1913 and 1914 he secured additional training in mathematics at the University of Chicago, in Chicago, Ill.

He was employed by the American Clay Machinery Company of Bucyrus, Ohio, prior to graduation in 1898, and after receiving his degree, he was in charge of the purchasing department with this firm until 1903. He was with the United States Geological Survey in the summer of 1904 as transitman on water supplies in Montrose, Colo., and later joined the Bureau of Reclamation in various capacities on the La Plata project in New Mexico, Kremmling Reservoir in Colorado, and the North Platte project in Wyoming and Nebraska. His design and construction of reclamation projects secured for him the position of assistant engineer with the Bureau of Reclamation which organization he left in the fall of 1906. From the fall of 1906 until the spring of 1908, he was connected with the Central Colorado Power Company in survey and construction of a hydroelectric plant on the Colorado River. Then, from spring to fall of 1908 he worked with the Orchard Mesa Irrigation Project on triangulation, topography, and line surveys.

---

[1] Memoir prepared by R. L. Downing, M. ASCE.

Mr. Fitterer was professor of civil and irrigation engineering at the University of Wyoming in Laramie from September, 1908, to September, 1928. He was dean of the College of Engineering and head of the Agricultural Experiment Station from 1922 until 1924. During his association with the University of Wyoming, Professor Fitterer developed this department into one of the strongest at the university. From September, 1928, until his death, he was professor of mathematics there; and he became head of the department of mathematics at the Colorado School of Mines in Golden, in 1929.

His record of educational attainment clearly indicates the high standards he believed necessary in each field of work. This concept is further emphasized by his contemporaries, who elected him to outstanding professional societies, in which he took an active part. He was a member of Tau Beta Pi, Phi Kappa Phi, the American Society for Engineering Education, the Mathematical Association of America, the American Mathematical Society, the American Association for the Advancement of Science, the National Education Association, and the Colorado Society of Engineers. Professor Fitterer's counsel was regularly sought by his associates; his interest and competence are further demonstrated by many engagements as consultant, and in various publications.[2,3] These engagements as consultant covered work in water supply, irrigation, drainage, and sewerage.

Professor Fitterer's understanding, friendliness, and wise counsel endeared him to students and associates. His early professional development in reclamation remained a strong interest, and he continued to take an active part in the progress of the west by irrigation, water supply, and drainage. Long and successful experience in engineering education gave him many friends and a wide acquaintanceship in engineering in the Rocky Mountains.

On December 27, 1899, he was married to Lucia E. Kemp of Perrysberg, Ohio, who survives him.

Professor Fitterer was elected an Associate Member of the American Society of Civil Engineers on April 2, 1912, and a Member on October 22, 1924.

JAMES BENTON FRENCH, M. ASCE[1]

Died December 2, 1947

James Benton French was born at New Hartford, N. Y., on September 12, 1863. He was the son of Thomas Paine and Sarah Brooks (Benton) French.

---

[2] "Reclamation by Drainage," by J. C. Fitterer, *Bulletin No. 90,* Wyoming Agri. Experiment Station, Univ. of Wyoming, Laramie, 1911.

[3] "Meteorology for Twenty Years," by J. C. Fitterer and H. G. Knight, *Bulletin No. 100,* Wyoming Agri. Experiment Station, Univ. of Wyoming, Laramie, 1913.

[1] Memoir prepared by Frederick Auryansen, M. ASCE.

His paternal ancestor, William French, emigrated from England to Cambridge, Mass., in 1635, removing to Billerica, Mass., in 1653, where he was the first representative from that town, and also was a Lieutenant of the Militia. In 1850 Thomas Paine French went to California to seek gold, but later returned to central New York, where he became a farmer.

A second paternal ancestor was Roger Gillmore, one of the original settlers of Jaffrey, N. H. As First Lieutenant, then Captain, of the first military company organized in 1775, he served during the Revolutionary War. He was the first Justice of the Peace in Jaffrey, and a delegate to the New Hampshire State Constitutional Convention in 1791.

Sarah Brooks Benton was the daughter of James Benton who came from England. He became a builder at Utica, N. Y., where he constructed important buildings and also served as mayor of that city.

From such sturdy pioneering stock James French inherited the firm, steady purpose and high character which were clearly evident to all who knew him, and which prompted a friend to remark, "You can't say too much that's fine of him."

Mr. French attended the public schools at New Hartford and Utica, and was graduated from Cornell University in Ithaca, N. Y., in 1885, with the degree of Bachelor in Civil Engineering (later changed to Civil Engineer). The following year he held the McGraw fellowship in engineering at Cornell.

For the subject of his graduation thesis he chose the Niagara cantilever bridge, a structure for which Charles Conrad Schneider,[2] Past-President, ASCE (then chief engineer of the Pencoyd (Pa.) Bridge Company), was consulting engineer. Mr. Schneider welcomed the young engineer who sought his counsel and helped him with data and suggestions for his thesis.

When Prof. Charles D. Marx,[3] Past-President and Hon. M. ASCE, recommended Mr. French for the position of rodman on construction of a branch line of the Missouri Pacific Railroad Company in Kansas in 1886, he asked Mr. Schneider's advice, and following it, "went west." The next year he became transitman on location of the Fort Worth and Denver City Railway Company across the Texas Panhandle.

As these field experiences confirmed his preference for bridge engineering, Mr. French then spent a short time as draftsman with the Groton (N. Y.) Bridge Company. At that time various bridge companies were in active competition, each with its special designs, usually tendered on a lump sum basis, so that designs and costs were not readily comparable. The Groton Bridge Company was a small company and not to be classed with the Keystone Bridge Company at Pittsburgh, Pa., the Edgemoor Company in Delaware, the Phoenix Company at Phoenixville, Pa., the Union Company at Athens, Pa., or the Pencoyd Company near Philadelphia, Pa.

An inspector at Groton advised Mr. French to transfer to a larger and better company, so in 1888 he applied at the Pencoyd Company, where he was welcomed by Mr. Schneider, who was not only a good judge of ability, but a world leader in the art of high-grade bridge design and construction. Under such favorable auspices the young assistant engineer was fortunate in being charged with planning original designs and stress sheets, developing details for tracers, and making erection plans, bills of material, and shipping bills—the entire paper work for the bridge. His work at Pencoyd included steel structures for the Reading Terminal at Philadelphia and the arched train shed of the Pennsylvania Railroad at Jersey City, N. J., which was removed in 1922.

---

[2] For memoir, see *Transactions*, ASCE, Vol. 81, 1917, p. 1665.

[3] *Ibid.*, Vol. 105, 1940, p. 1785.

In 1889 Mr. French was granted a leave to join fifty other members of the Society, together with members of the American Society of Mechanical Engineers and the American Institute of Mining and Metallurgical Engineers, on a visit to European countries and the Paris exposition in France. His experiences are delightfully recounted, in concert with three other members of the group, in *Civil Engineering.*[4] In port his first observations were on the Liverpool, England, landing stage, providing for a 30-foot tidal range, and connected to the shore by hinged bridges—prototypes of his later specialty.

By 1892 Mr. French had become so thoroughly experienced in the best structural steel practice of the period that Mr. Schneider recommended him for the position of assistant engineer of the Chesapeake and Ohio Railway at Richmond, Va., where he later served as bridge engineer until 1898. The New York, New Haven and Hartford Railroad Company engaged him in 1899 to design a number of bridges. These were based on E 40 live loading, then considered ample provision for current and future capacity. In 1900 Mr. French was appointed bridge engineer by J. V. Davies,[5] M. ASCE, chief engineer of the West Virginia Short Line, a new railroad (later part of the Baltimore and Ohio Railroad system) built between Clarksburg and New Martinsville, W. Va., to develop a region rich in coal and oil resources. The fifty-four bridges were designed for E 40 loading and all structural steelwork was under Mr. French's direction.

Mr. Davies became chief engineer of the Atlantic Avenue improvement of the Long Island Railroad in 1901, and appointed Mr. French bridge engineer for this important grade crossing elimination through the heart of Brooklyn, N. Y. This work comprised three miles of steel viaducts and two and a half miles of cut-and-cover subway[6] with a complicated subsurface passenger terminal at Flatbush Avenue, Brooklyn, which was later surmounted by an elevated freight terminal with a large angular divergence between the two. With bold assurance Mr. French developed a cellular grillage of heavily reinforced concrete to distribute the column loads of the elevated structure over the roof of the passenger terminal. Although cracked and seriously undermined by a subsequent excavation for a city subway, causing subsurface column settlement up to 6 inches, this reinforced concrete grillage is still carrying the load of a surface freight station and an overhead bottle storage warehouse after forty years of service.

---

[4] "Engineers' Odyssey to Europe—1889," by A. Prescott Folwell, James B. French, Andrew J. Provost, Jr., and T. Kennard Thomson, *Civil Engineering*, January to May, 1945.

[5] For memoir, see *Transactions*, ASCE, Vol. 105, 1940, p. 1826.

[6] *Railroad Gazette*, April 25, 1902.

This and other pioneer work in the design and use of reinforced concrete, when its theory was still being formulated and its practice evolving, was thoroughly justified by the endurance of the structures he built. Another example is the "concrete-steel," 3-inch slab fence surrounding the Sixth Avenue depressed freight yard in Brooklyn, which is still in good condition after forty years of exposure.

From 1904 to 1908 Mr. French was bridge engineer of the Long Island Railroad, continuing to employ conservative yet progressive elements of good construction.

Subsequent to 1908 he was in private practice as consulting engineer for various contractors, as well as for the Erie Railroad, the Lehigh Valley Railroad, the New York Central Railroad, the Long Island Railroad, and the Hudson and Manhattan Railroad Company. His unique work during this period was the development of a special type of screw-suspended transfer bridge,[7] with a "contained apron" operated as a unit with the bridge itself, but free to rise as cars left a float to go shoreward on the main span of the bridge. Seventeen of these bridges have been built for the aforementioned railroads, and one for the Bronx market—all in New York (N. Y.) harbor .

Other notable work under his charge was a series of nineteen railroad and thirty-five highway bridges, and a 3,535-foot-long, four-track reinforced concrete tunnel built for the Bay Ridge improvement of the Long Island Railroad —the important freight link from Bay Ridge, N. Y., over Hell Gate Bridge to New England. The extensive shop and inspection buildings at the Hudson and Manhattan Railroad Company terminal in Jersey City were also designed by him during this period.

Mr. French was a charter member of the American Institute of Consulting Engineers, and a former member of the American Society for Testing Materials, the Cornell Society of Engineers, the Brooklyn Engineers Club, and the New York Railroad Club. He presented papers before the Brooklyn Engineers Club and the Connecticut Society of Civil Engineers, and his discussions of papers are recorded in the *Transactions* of the Society.

Whatever work Mr. French undertook was always completed with the sincere purpose to do better than before, so that his work exemplified not only good practice of the time, but improvement wherever feasible. He was an ardent, skilful practitioner and displayed sound judgment in his chosen profession. His friends cherish the fine example of his exemplary life of service, his dignified bearing, his gracious and cordial friendship, and his unswerving loyalty to principle. He was a devout member of the Protestant Episcopal Church, and one of the founders of the Young Men's Christian Association at Jamaica, N. Y.

In 1891, he was married to Flora Williams of Groton, who died on January 13, 1945; he is survived by two sons, Thomas Morton French, and William Gillmore French, and by a brother, Charles G. French.,

Mr. French was elected a Junior of the American Society of Civil Engineers on February 6, 1889; an Associate Member on February 7, 1894; and a Member on November 2, 1898. He became a Life Member in January, 1929.

---

[7] *Engineering News*, December 28, 1911, p. 770.

PIERCE POWERS FURBER, M. ASCE[1]

DIED DECEMBER 19, 1947

Pierce Powers Furber, the son of Pierce Powers and Kate (Montgomery) Furber, was born in St. Louis, Mo., on October 23, 1885. The family moved to Northfield, Minn., when he was a small boy, and there he received his education in the public schools. Later, he attended the University of Minnesota, in Minneapolis, from which he was graduated in 1908 with the degree of Civil Engineer. While in college he joined Sigma Alpha Epsilon and Theta Tau fraternities.

Mr. Furber's early experience was with the Minneapolis Steel and Machinery Company, and with C. A. P. Turner, M. ASCE, consulting engineer. Working under the guidance of Mr. Turner he received a fundamental training in the art of reinforced concrete, from the standpoint of practical construction, as well as design and detail. A broad field was covered in bridge and building work, setting the pattern for his career in later years.

In 1912 he was assigned to the Chicago (Ill.) office of the C. A. P. Turner Company as district engineer, holding this position for four years. During this time he handled the estimates, designs, and sales promotion of the "mushroom" system, which applied principally to heavy building construction.

In 1917 the advent of World War I, brought Mr. Furber into the Boston (Mass.) office of Stone and Webster, as engineer in the structural division. His work included supervision of the structural designs and specifications for various important buildings at the Rock Island (Ill.) Arsenal—the field and siege building, the boiler house, and various storehouses—and also the designs for the plant buildings of the Westinghouse Lamp Company at Trenton, N. J. During 1918 he was transferred to the American Shipbuilding Corporation at Hog Island, Pa., as assistant to the superintendent of materials in the hull fabrication department.

Early in 1919 when the Hog Island work slowed down, Mr. Furber returned to St. Paul, Minn., as agent for the Paul J. Kalman Company, later the Kalman Steel Company. For the following ten years he continued in the general capacity of manufacturers' representative, estimating, designing, detailing, and selling reinforcing steel, fire-proofing materials, and steel building products. In St. Paul he designed the reinforced concrete buildings for the complete publishing plant of the Louis F. Dow Company; also, for the Butler Brothers Company, he designed and supervised general concrete construction. In 1922 he became district engineer for the Corrugated Bar Company at St. Paul.

In Danville, Va., during 1923–1924, as a partner in the firm of "Wiseman and Furber," Mr. Furber continued to act as manufacturers' representative and distributor for the Berger Manufacturing Company and others. After working for a short time in Washington, D. C., in 1925, he went to Jacksonville, Fla., where he continued as sales representative for about five years.

---

[1] Memoir prepared by Wallace E. Belcher, M. ASCE.

Returning to Philadelphia, Pa., in 1930, Mr. Furber was employed by the United Engineers and Constructors, Incorporated, in the structural division as checker and designer. He was engaged on the Pennsylvania Railroad improvements in Philadelphia; in the design of the manufacturing plant for the Solvay Process Company at Baton Rouge, La., and on the design of the Pittsburgh Plate Glass Company factory in Newark, N. J.

From 1941 to 1943 Mr. Furber was supervising structural designer for Ford, Bacon and Davis, Incorporated, on the project at the Jacobs Aircraft Engine Company in Pottstown, Pa. Working under wartime pressure, his general health became somewhat impaired. However, he had apparently recovered, and for several years prior to his very sudden and untimely death he maintained an office in the Philadelphia Fidelity Building as consulting structural engineer.

On September 5, 1928, Mr. Furber was married to Helen Minium of Jacksonville. He is survived by his widow and by their two daughters, Frances Minium and Mary Montgomery.

Mr. Furber took an active interest in the affairs of the Society, and his community. He had an even, affable disposition, and possessed a ready wit that made and held many friends. He was a member of the Sons of the American Revolution, the Minnesota Historical Society, the American Concrete Institute, and the Republican party. A frequent contributor to technical discussions, he was also the author of several articles.[2, 3]

Mr. Furber was elected a Member of the American Society of Civil Engineers on September 10, 1923.

JAMES EDWIN GIBSON, M. ASCE[1]

DIED OCTOBER 15, 1947

James Edwin Gibson, the son of John Wilson and Mary Elizabeth (Dickinson) Gibson, was born near Pine Bluff, Jefferson County, Ark., on December 19, 1874. He received his early education in his home state and was graduated from the University of Arkansas at Fayetteville with the degree of Bachelor of Mechanical Engineering in 1894.

His engineering activities began in 1895 in the employ of Jay M. Whitham, consulting mechanical engineer, of Philadelphia, Pa. During the fifteen months with Mr. Witham he assisted in more than three hundred boiler and engine efficiency trials and conducted as principal more that one hundred boiler tests.

---

[2] "The St. Croix River Bridge," by Pierce P. Furber, *Engineering News*, December 28, 1911.

[3] "Reinforced Concrete Columns," by Pierce P. Furber, *Proceedings*, A. C. I., Vol. 12, 1916.

[1] Memoir prepared by John Anderson, M. ASCE.

In 1896 he was employed by the Moro Phillips Company in Camden, N. J. (later General Chemical Company) as a mechanical engineer. During this time he designed and rebuilt Glover and Gay Lussac acid towers, and a phosphorus plant.

Although well on his way to becoming a chemical engineer, Mr. Gibson entered the civil engineering field in 1897 when he became an assistant engineer for the American Pipe and Construction Company in Philadelphia. The chief engineer of this company was the very well-known J. W. Ledoux,[2] M. ASCE, of whom Mr. Gibson always spoke with greatest respect and admiration. The American Pipe Company manufactured pipes for water systems but also designed and constructed works and owned and operated some thirty plants throughout the eastern states. During his employment of twenty years with this company, Mr. Gibson's activities covered a wide range. At first he was concerned with the design and construction of a number of hydroelectric and gas plants but eventually his sphere was restricted to the design and construction of water works.

He served in 1897 as manager of the Waukesha (Wis.) water works, supervising the sinking of two 1,800-foot artesian wells and the installation of steam pumping and auxiliary equipment. One of the largest projects on which he was engaged was the development of the water supply system for the main lines of the Pennsylvania Railroad Company between New York, N. Y., and Washington, D. C., Philadelphia, and Pittsburgh, Pa. This work embraced every type of water supply and included more than fifty impounding dams, innumerable steel tanks and standpipes, and more that five hundred miles of pipe. The cost of construction was more than fifteen million dollars. A similar project involved design and construction of three systems for the New York Central and Hudson River Railroad at Buffalo, Rochester, and Syracuse in New York.

In 1903 Mr. Gibson made his first acquaintance with Charleston, S. C., when he was sent there to make studies for the proposed Goose Creek supply for that city. The plant was constructed in 1904 and was at that time unique. A tidal stream was dammed by a low earth dike to impound fresh water upstream and keep out salt water downstream. Climatic conditions at times caused the evaporation from the surface of the reservoir to reach twenty million gallons daily—about four times the city's requirements.

When the Charlestown Light and Water Company which owned and operated this water works system was purchased by the City of Charleston, Mr. Gibson was in charge of the operation of the plant. He accepted the city's offer of the position of manager and engineer of the water works, retaining the position until his death.

Mr. Gibson was a leader in the use of cement lining for cast iron pipes. In cooperation with the American Cast Iron Pipe Company in Birmingham, Ala., he contributed greatly to the development of this method of protecting cast iron piping. Experimental lines installed in Charleston in 1922 were so successful that no other type of lining has been used there since. In the summer of 1939 a number of mains totaling two and a half miles were lined in place by the Tate method. At the time of his death a research project on the effectiveness of this cement lining was under way on a line supplying the City of Summerville, S. C., which draws water from the Charleston supply.

---

[2] For memoir, see *Transactions*, ASCE, Vol. 104, 1939, p. 1956.

What is regarded as Mr. Gibson's outstanding achievement was the conception and construction of the twenty-three mile Edisto-Goose Creek tunnel aqueduct for Charleston. This is an unlined tunnel constructed through a marl formation at an average of thirty feet below sea level for its entire length. The boldness of this project can be realized when it is remembered that the area was visited by a violent earthquake in 1886. However, no signs of faulting were discovered in the tunneling operations.

Another noteworthy engineering accomplishment was the extension of the water supply system of Charleston to the west side of the Ashley River, involving the construction of a subaqueous supply main on the bottom of the river. At the time of his death Mr. Gibson had practically completed an entirely new treatment and pumping plant.

With his death there passed from the City of Charleston an outstanding water works engineer and a citizen of the highest standards. A man of greatest integrity, he was a rugged individual who never failed to exert himself for the best interests of the community he served for thirty years. His efficient administration of the Charleston municipal water works made it an exception to the usual gloomy story of public operation of utilities.

It may be said of him, as it was said of Christopher Wren: "If you wish to see his monument, look around you." The Charleston water works, with which Mr. Gibson was associated in one way or another for more than forty years, although not so easily seen as the great architect's cathedral, is nonetheless concrete evidence of a great achievement.

Mr. Gibson was a pioneer in spirit, boldly undertaking new and untried engineering feats, but he was not by any means a romantic adventurer. All his ventures were carefully planned. He took nothing on faith, but practiced St. Paul's admonition: "Prove all things." It was said among his associates that he knew all the answers but invariably checked them with his slide rule, an instrument so long and so well used that hardly a figure or a graduation remained on it.

He served as a contact member for The Citadel Student Chapter of the Society from 1937 until 1942. He took a very active interest in the work of the chapter, being a frequent vistior at meetings and always ready with helpful advice and suggestions.

His alma mater honored him in June, 1941, by conferring on him the honorary degree of Doctor of Science. In the same year the American Water Works Association bestowed on him the George W. Fuller Memorial Award. The citation accompanying this award reads:

> "For his distinguished leadership in water supply construction in North America; for his effective administration of the water supply system of the city of Charleston; and for his worthy contribution to the activities of the Association."

Mr. Gibson was an honorary member and past-president of the American Water Works Association; a member of the New England Water Works Association; a member of the American Society of Mechanical Engineers; and a charter member and past-president of the South Carolina Society of Engineers. He was also a past-president of the Charleston Rotary Club and a Royal Arch Mason.

In 1903 he was married to Florence A. Blair in Waukesha. He is survived by his widow; two children, Edwin Emmons Gibson and Henrietta Blair Gibson; and one grandchild.

Mr. Gibson was elected a Member of the American Society of Civil Engineers on September 6, 1910.

EDWARD GODFREY, M. ASCE[1]

DIED JUNE 25, 1947

Edward Godfrey was born in Pittsburgh, Pa., on March 7, 1871. His parents were James and Eliza (Ferguson) Godfrey.

When a young man of seventeen his first interest in engineering developed in the employ of G. W. G. Ferris and Company in 1889 as a draftsman on the detailing of the famous ferris wheel operated at the Chicago World's Fair in the early eighteen nineties. This interest culminated in his being graduated as a civil engineer from the University of Pittsburgh (then the Western University of Pennsylvania) in that city in 1893. Following graduation, Mr. Godfrey joined the organization of engineers known as the Robert W. Hunt Company and was continuously associated with this company until his death, except for two years early in the nineteen hundreds when he taught structural engineering at the University of Pittsburgh.

In his active work for the Robert W. Hunt Company, Engineers, which continued until his untimely death as a result of an automobile accident, he had interesting, diversified, and unique experience. This extended from mill, shop, and erection experience on structural steel for bridges and buildings to the testing of erected structures; the checking of designs of important and unique structures built over the period of his fifty-four years with the Robert W. Hunt Company; the examination and correction of structures deemed unsafe to the public; and also the investigation of failures in steel and reinforced concrete structures.

His life and experience extended over a period in which there were marked developments in engineering design for structures of steel and reinforced concrete. It was only natural that in his direct and continuous contacts—from checking the design of others through to following up failures—he early recognized and saw certain weaknesses in engineering theories of design, particularly in the early stages of design development.

---

[1] Memoir prepared by H. H. Morgan, Vice-Pres., and Chf. Engr., Robert W. Hunt Co., Pittsburgh, Pa.

Threfore, it is not surprising that in his personal writings and in his participation in engineering discussion, he was quite prone to bring out the weaknesses of contemporary practices for correction. He did, however, feel sincerely that no early changes could be effected unless he was to take the position of an iconoclast, so that his discussions and his writings manifest a spirit which was truly not a personal characteristic. However, it has since been generally recognized that a great number of his criticisms of those practices in design were sound and his improvements have become an integral part of present engineering practice. Many of his recommendations regarding the design of reinforced concrete and dam construction are illustrations.

Mr. Godfrey was a lover of nature, a lover of children, and a profoundly religious man. He was an elder of the Point Breeze Presbyterian Church in Pittsburgh, conducting Bible classes and nature studies for children. In the early days of the Westinghouse radio station KDKA, he was "Uncle Ed" on a children's program, telling bedtime stories.

He was an artist and one of the most lovable and sincere persons one could ever know. He is one of the few men who gave to charity a fixed percentage of his annual gross income throughout his business life.

Edward Godfrey was an engineer and a man revered and beloved by all who knew him. His contributions in literature to the advancement of his profession are recognized and appreciated.

On September 12, 1894, Mr. Godfrey was married to Clarinda Clyde Marks in Washington, D. C. He is survived by a son, Norman F. Godfrey.

Mr. Godfrey was elected a Member of the American Society of Civil Engineers on October 5, 1909. He became a Life Member in January, 1942.

ALFRED COOKMAN GREGORY, M. ASCE[1]

DIED NOVEMBER 30, 1946

Alfred Cookman Gregory, the son of Benjamin and Louise (Parshall) Gregory, was born in New York, N. Y., on March 26, 1872. He studied at the public schools in New York City and in Newark, N. J., and at the Model School, in Trenton, N. J. He then attended Rutgers College in New Brunswick, N. J., where he was a member of Delta Kappa Epsilon fraternity. After completing a four-year course he received the degree of Bachelor of Science in 1896.

Immediately after graduation he accepted employment with the Lehigh Valley Railroad Company, spending the summer as rodman. Following a few months as transitman with the county engineer of Mercer County, New Jersey, he spent his first term of service with the City of Trenton as transitman and draftsman, from February, 1897, to October, 1900. Then, after a brief intermission of a few months as assistant supervisor for the Philadelphia and Reading Railway Company on the Bethlehem (Pa.) branch, he returned to the employ of the City of Trenton as transitman for a year and a half. Again, a few months intervened during which he worked for Mercer County, this time as assistant county engineer on the design and construction of county roads and bridges.

---

[1] Memoir prepared by N. A. K. Bugbee, Assoc. M. ASCE.

With Mr. Gregory's third employment by the City of Trenton, beginning in April, 1903, the association became permanent—he remained until his death. During the first eight years, until 1911, his position was that of engineer of sewers, in charge of the design and construction of sewerage and drainage systems. Beginning in 1911, the position was broadened to that of engineer, sewers and water, with the additional responsibility for planning and building a water filtration plant and a sewage disposal plant. A public campaign for the adoption of a modern sewage disposal system was under way, and Mr. Gregory had the satisfaction of seeing the adoption of his own plan for this project, and also of supervising its construction. Other work under his charge included the city crematory and the important concrete retaining wall along the Delaware River and Assumpink Creek.

In 1935 Mr. Gregory was appointed chief engineer and director of the department of public works for the City of Trenton, in which capacity he served for the terms, 1935–1936 and 1937–1938. After 1938, he was again sewer and water engineer. These changes corresponded with modifications in the form of city government. Mr. Gregory had the distinction of serving under three types of administration. When he first entered the city employ, it was under the control of a common council, with appointment subject to a commissioner. In 1935 the council manager type of government was inaugurated. Finally, in 1939, the city commission organization was adopted.

One of Mr. Gregory's outstanding interests and contributions was in the Engineers Club of Trenton. Indeed, he received most of the credit for its very existence. During its planning and actual formation, he was most active, agreeing to take the responsibility for the organization; and, as a result of his enthusiasm and hard work, the club was not only inaugurated, but also successfully administered. Recognizing its indebtedness to him, the club adopted a Resolution on the occasion of his death. Mr. Gregory, the Resolution stated:

> "* * * affectionately known by all of us as 'Pope' was very prominent in the formation of the Engineers Club of Trenton in 1914, and served as its second President in 1915; and was the first member of the Club to be awarded a Life Membership in 1942, and was one of the Club's most enthusiastic members."

The Resolution also paid tribute to his "kindly personality, his cooperation and his advice."

Mr. Gregory was also a member of Trenton Lodge No. 5, F. and A. M., Crescent Temple, and the Sons of the American Revolution. In Trinity Cathedral (Episcopalian), of which he was a member, he sang in the choir for forty years. He was a member of the American Geophysical Union and the Rutgers Club of Trenton.

On October 14, 1905, in Trenton, he was married to Ethel Wimpenny, who survives him.

Mr. Gregory was elected an Associate Member of the American Society of Civil Engineers on June 5, 1907, and a Member on November 12, 1913.

HEINRICH EDUARD GRUNER, M. ASCE[1]

Died November 28, 1947

Heinrich Eduard Gruner was born in Basel, Switzerland, on February 8, 1873. He acquired the preparatory education for a technical college course in his home town. His father, a noted civil engineer, enjoyed an extended practice in Switzerland and southern Germany, and this may have been a deciding factor in the son's following in his father's footsteps.

After an interim sojourn as apprentice in a mechanical factory in England, Mr. Gruner entered the Swiss Federal Technical University in Zurich, from which institution he was graduated in 1897 as a civil engineer.

Two years of practical engineering in the office of Herman Laub,[2] M. ASCE, who specialized in structural steel design of bridges and factory buildings in Pittsburgh, Pa., gave the young engineer a broadened outlook on life and on his chosen profession.

At that time Mr. Gruner's father had begun preliminary studies for a hydroelectric power development of 70,000 horsepower on the Rhine River at Laufenburg above Basel. After the elder Gruner died, this enterprise was supervised and carried to completion by his son from 1899 to 1918.

The successful consummation of this project opened the way for the establishment of an engineering office in Basel for the consultation and the direction of projects of major magnitude. From 1913 to 1918 he was engaged on the construction of the Fala hydroelectric power plant (40,000 horsepower) in Austria (later Jugoslavia), and in 1918 he supervised the design and construction of the hydroelectric plant (24,000 horsepower) at Broc, Switzerland, involving the then novel construction of a slender arch dam, 185 feet in height. From 1919 to 1924 Mr. Gruner directed the construction of the Chancy-Pougny power plant (43,000 horsepower) on the River Rhône, and, from 1929 to 1936, the design and construction of the Albruck-Doggern power plant (100,000 horsepower) on the River Rhine. When large earth dams were constructed for the latter project, he established a laboratory for soil mechanics which later led to the installation of a similar laboratory at the Federal Technical University at Zurich.

---

[1] Memoir prepared by A. L. Sonderegger, M. ASCE.

[2] For memoir, see *Transactions*, ASCE, Vol. LXXXIII, 1919–1920, p. 2263.

The Albruck-Doggern plant built across the border of Switzerland and Germany is generally credited as a work of major magnitude in power development.

These projects extended Mr. Gruner's reputation as an authority on hydroelectric development throughout the European continent and resulted in his being called for consultation by the governments of Austria, Czechoslovakia, Hungary, and Poland, as well as by industries and banks in Spain, France, Belgium, and Norway. He spent considerable time as consultant on irrigation works for the governments of Iran and Turkey, where, thanks to his diplomatic accomplishments and political acumen, he succeeded in bridging the contrasting European and Oriental ideologies and business methods to obtain the confidence of his clients.

Mr. Gruner followed with steady interest the scientific investigation of hydraulic problems, and with other engineers, he was instrumental in establishing, against considerable opposition by older colleagues, the now famous hydraulic laboratory at the Federal Technical University. This institution in Zurich offers consultations and experimental research to private engineers who are confronted with hydraulic problems.

A professorship offered him at Zurich was declined because he preferred the practical application of his vast experience to that of teaching. In recognition of his outstanding accomplishments in hydraulic engineering his alma mater conferred upon him the title of Doctor of Technical Science, *honoris causa*.

For many years Dr. Gruner was the chairman of the Swiss Commission on High Dams and the Swiss delegate to the Commission Internationale des Grands Barrages. In addition to the Society, he was also a member of the Swiss, English, and Austrian civil engineering societies.

In 1904 the young engineer was married to his cousin, Jenny Kern, and thus was initiated a happy home and a life companionship. The cheerful, buoyant nature of the young woman and her deep, religious convictions were an ever-animating support in the life of the husband whose exacting business called for frequent and often protracted absences in almost every part of Europe.

This marriage was blessed with three children who grew up to mature years during their parents' lifetimes. The sons, Eduard (M. ASCE) and Georg, following their father's profession, took over the business at his death. A daughter, Irène (Mrs. Rudolph Gally) and the sons survive.

In the summer of 1947 the writer had the pleasure of visiting the country estate of Dr. Gruner in Switzerland after more than forty years of separation. The friendship, developed during college days, continued through life despite the long separation. Although at the time Dr. Gruner was still grieving over the passing of his wife in 1945, his untiring energy and old enthusiasm for great pending works were in evidence as much as ever. A project of major proportions and importance, the development of hydroelectric power on the Aswan Dam on the Nile, was then nearing the successful consummation of planning and design. For many years Dr. Gruner had been consultant for the government of Egypt and a member of a board charged with the second raising of this dam (1929). This gigantic project undertaken in collaboration with two English firms formed an appropriate finis to a colorful life.

Personal responsibility always rested heavily on a man like Dr. Gruner who fought not only the vicissitudes of nature, but also the opposition and the rivalries which always seem to arise with large and international enterprise. It is in the nature of such a career that success is contingent upon undaunted courage, uprightness, moderation, and persistency in all phases of the work. These qualities combined with technical ability of the highest order carried him to success. His was the satisfaction of having accomplished works of lasting benefit to mankind. To quote the poet, Robin Atthill, Dr. Gruner was "* * * the whole man, compass of whose being swings past magnetic fields to his true North."

The poet Alexander Reid wrote:

> "It is not our business to be successful—
> What is important is only to be faithful
> To the inner vision."

The simple truth implied in these lines was strickingly borne out in the career of Dr. Gruner. His was a life of uprightness of purpose and of persistent application of his technical knowledge to the best of his strength and ability; and so it followed, as the night the day, that all those other things were added unto him—success and honor, a happy home life, and the reasonable enjoyment of the modern comforts of living to very mature years.

Dr. Gruner was elected a Member of the American Society of Civil Engineers on June 1, 1925.

JULIUS REED HALL, M. ASCE[1]

Died March 26, 1947

Julius Reed Hall was born in Elmira, N. Y., on September 18, 1880. He was the son of Robert A. and Augusta (Pratt) Hall.

His engineering career began in 1899, four years before he entered college. From July, 1899, to September, 1903, he was employed as a draftsman and detailer of machinery by such well-known designers, builders, and manufacturers as the Barnett and Record Company of Chicago, Ill.; the Skillen and Richards Manufacturing Company of Chicago; the Sterling Company of Barberton, Ohio; the Brown Hoisting Machinery Company of Cleveland, Ohio; and the American Bridge Company (plants at Elmira, Ohio, Athens, Pa., and Chicago). In several of these engagements he was entrusted with designing machines as well as detailing work.

During the two years from September, 1903, to September, 1905, Mr. Hall was enrolled as a special student in bridge engineering at Lehigh University at Bethlehem, Pa. He then became associated with his brother, Charles F. Hall, in Chicago, designing and manufacturing special machinery, until March, 1906. This was followed by detailing and designing steel mill buildings and coal-loading machinery during 1906 and 1907. The following two years he was treasurer and assistant manager of the Hall Machine Company in Chicago, designing and developing special machinery.

---

[1] Memoir prepared by Theodore L. Condron, M. ASCE.

In 1909, after a few months as detailer and designer of structural steel for buildings for the Purdy and Henderson Company of Chicago, he was engaged by the Strauss Bascule Bridge Company there as designer, then as chief draftsman, and, from 1912 to 1915, as designing engineer. At the Strauss Company he designed or supervised the preparation of specifications and estimates, approved the shop drawings, and directed the erection of more than one hundred bascule bridges. Included among these structures were: A 336 ft double-leaf simple span for the Canadian Pacific Railroad Company; four single-leaf railroad bridges with spans of from 118 ft to 235 ft; another bridge with a double deck; a bridge with four tracks; two direct lift bridges for the United States government of 210-ft span; and another of 248-ft span for the City of Chicago. Also, these designs included bridges for the Russian, Canadian, Egyptian, and Norwegian governments.

During World War I he was commissioned a Captain, and from 1916 to 1918 he served in the armed forces of the United States (515th Engineers) in charge of building locomotive shops and warehouses at Gieves, France. Following the close of World War I in 1919, Mr. Hall returned to the United States.

During 1920 he was employed by the City of Chicago as engineer of bridge design, and from 1921 to 1923 the Chicago Sanitary District employed him as principal assistant engineer in charge of the design and construction of all sewage work. From 1923 through 1929 he was superintendent for John Griffiths and Son Company, general contractors for the Northside sewage treatment plant, the Grant Park Stadium, and the Civic Opera House—all in Chicago. For the next three years Mr. Hall was general superintendent of construction for the same company on buildings, utilities, and roads for the Century of Progress Exposition at Chicago, and in 1935 he was superintendent on the building of the cofferdam and pile substructure of the Mississippi River lock No. 26, at Alton, Ill. While on this project estimates for several other Public Works Administration (P.W.A.) projects were prepared by him.

Mr. Hall was engaged from 1935 through 1937 by the Village of Oak Park, Ill., as engineer in charge of the design and construction of a $2,000,000 intercepting sewer, having a maximum diameter of 9 ft. and a length of 20 miles. At the completion of that work, the designs for several swimming pools, as well as an intercepting sewer for Decatur, Ill., were made by him. Also, he served from 1939 through 1942 as examining engineer (P.W.A.), at Chicago, under the assistant regional director, Federal Works Agency at Chicago, examining and making recommendations on applications for federal grants for engineering projects. From 1942 to 1944, Mr. Hall was advisory engineer on purchasing engineering equipment for United States Army Engineers and United States Navy Sea Bees. In addition, he completed reports on plants for the Reconstruction Finance Corporation. From January, 1947, until his death in the following March, he was associate engineer of Consoer, Townsend and Associates, Consulting Engineers in Chicago.

He was married to Rita Sabin in 1906, who died on March 25, 1925, and to Mary Owen on October 21, 1928. He is survived by his widow; a son, Norman Reed; and a daughter, Priscilla Jean (Mrs. Chester Demus).

A man of extreme modesty, Mr. Hall was devoted to his family, active in church and community affairs, and a conscientious engineer. His home was in Oak Park.

Mr. Hall was elected a Junior of the American Society of Civil Engineers on June 6, 1905; an Associate Member on December 3, 1912; and a Member on January 18, 1916.

LESLIE STANDISH HALL, M. ASCE[1]

DIED JULY 5, 1947

Leslie Standish Hall, the son of the late Leslie Chapman and Bertha York (O'Neill) Hall, was born on February 1, 1892, in Corning, N. Y. He was a descendant of Myles Standish and ordinarily used the name of L. Standish Hall.

His early education was received in Albany, N. Y., and Boston, Mass., where he completed high school. He was graduated from the Massachusetts Institute of Technology in Cambridge, in June, 1914, with a degree of Bachelor of Science in Civil Engineering (hydraulic option).

On completion of his formal education Mr. Hall was appointed a civil engineer student with the United States Office of Public Roads and was promoted to junior highway engineer on surveys and supervision of the construction of roads in California, Virginia, and Maine. After this assignment, he was engaged on survey, design, and drafting work, including nearly a year on master plans for an industrial village for the Delco Company, south of Dayton (Ohio), and he spent six months as structural steel draftsman for the United States Navy in the Bureau of Yards and Docks.

He became associated with H. L. Haehl, M. ASCE, of the consulting engineering firm of Duryea, Haehl and Gilman of San Francisco, Calif., in November, 1917. Mr. Hall spent most of the next seven years with this organization on the investigation of irrigation projects, including the study of stream flow and location of storage reservoir sites, the design of diversion dams, and the location and design of canals. Other work involved duty of water measurement and the investigation of ground-water conditions in the Sacramento-San Joaquin delta in California and other areas. He also made a study of the application of statistical methods of probability to hydrology and geological studies of fault lines and earthquake hazards in the San Francisco Bay

---

[1] Memoir prepared by J. S. Longwell, M. ASCE.

(California) area. Mr. Hall spent short periods during these seven years with various irrigation districts and consulting engineers, performing work along lines similar to his assignments with Mr. Haehl.

In July, 1924, Mr. Hall entered the service of the then newly organized East Bay Municipal Utility District of Oakland (Calif.); under the direction of Arthur Powell Davis,[2] Past-President, ASCE, he served as assistant engineer on the preliminary studies and the final plans for the $39,000,000 Mokelumne River water supply project. In 1926 he was advanced to chief hydrographer in charge of extensive investigations, that involved water rights litigation, to determine the effect of the district's diversion from the Mokelumne River on the water supply in the Mokelumne River Basin. He played a very important part in litigation concerning water rights on the Mokelumne River, both in preparing engineering data for court hearings, and in giving expert testimony in court.

Mr. Hall was made hydraulic engineer in 1936, and later, principal hydraulic engineer, a position he held at his death. During this period he served under J. S. Longwell, M. ASCE, chief engineer and general manager of the district. In addition to the water rights and ground-water investigations, his professional work with the district included stream gaging and hydrographic surveys; hydrology; river regulation and flood control; hydraulic research; land use analyses; municipal water supplies; erosion and reservoir sedimentation control; and operation studies of dams, spillways, and hydraulic structures, together with water supply and sewage disposal studies.

During the two years prior to his death he directed long-range planning in connection with the growth of the area served by the district, both for water supply and sewage disposal—the latter in connection with the newly formed Special District No. 1. These studies required exhaustive investigations on population increases, probable water consumption, and sewage quantities. His reports on these surveys were comprehensive and were generally considered outstanding achievements.

At an early stage in life, Mr. Hall developed a keen interest in hydraulics and enlarged this interest to the point where he was a well-recognized authority in the hydraulic engineering field. He was an excellent mathematician, and a lifelong student, devoting considerable time and effort after graduation to university extension courses in engineering, business administration, and related fields which would be of value to him in his life's work.

Mr. Hall did considerable original work on the design of spillways and the flow of water through pipes and orifices. He contributed greatly to the field of hydraulics through published writings in the *Transactions* of the Society, both original reports and discussions. One paper[3] resulted in his being awarded the Collingwood Prize for Juniors. In 1943 another paper[4] brought him the award of the Karl Emil Hilgard Prize for hydraulics. He was invited to attend the Third Hydraulics Conference at the University of Iowa

---

[2] For memoir, see *Transactions*, ASCE, Vol. 100, 1935, p. 1582.

[3] "Probable Variation of Yearly Runoff as Determined by California Streams," by L. Standish Hall, *Transactions*, ASCE, Vol. LXXXIV, 1921, p. 191.

[4] "Open Channel Flow at High Velocities," by L. Standish Hall, in "Entrainment of Air in Flowing Water: A Symposium," *ibid.*, Vol. 108, 1943, p. 1394.

(Iowa City) in June, 1946, at which time he presented a paper entitled, "The Influence of Air Entrainment on Flow in Steep Chutes."

Mr. Hall served as consultant on hydraulic problems for several organizations. His work was always of a high order as he possessed unusual aptitudes for the analysis of engineering data. He rendered useful and valuable service to the organization with which he was connected and to the engineering profession as a whole. Mr. Hall was extremely interested in the professional advancement of engineers, serving on the Committee on Professional Standards of the San Francisco Section of the Society.

On October 30, 1923, he was married to Geraldine Adeline Bohannon. He is survived by his widow and their three children, Ruth Josselyn (Mrs. John Paul Juhl), Emily Patricia, and John Standish.

Mr. Hall was a member of the American Water Works Association, the American Geophysical Union, the American Philatelic Society, the American Academy of Political Science, the American Geographical Society, the East Bay Engineers' Club, the Athens Athletic Club, the Society of Mayflower Descendants, the Scottish-Rite bodies, and the Rockridge Masonic Lodge. His religious affiliation was Episcopalian.

Mr. Hall was elected a Junior of the American Society of Civil Engineers on May 13, 1918; an Associate Member on January 17, 1921; and a Member on November 10, 1941.

BORDEN BAKER HARRIS, M. ASCE[1]

DIED OCTOBER 26, 1947

Borden Baker Harris, the son of Edward Enoch and Lydia (Baker) Harris, was born in Vernon County, Wisconsin, on September 20, 1875, at the farm of his maternal grandfather, Alvin Baker.

He received his early education in the public schools of Quincy, Ill., and Evansville, Ind., being graduated from the high school at Evansville. He completed three years of liberal arts training at Chaddock College in Quincy, in addition to a one-year business course at Monmouth Business College in Monmouth, Ill., and three years of civil engineering at the University of Illinois in Urbana.

From 1903 to 1906 he was assistant to the chief engineer of the Midland Valley Railroad Company, in Fort Smith, Ark., remaining with the company from the commencement of location surveys to the completion of construction. In 1906 he became office engineer in the firm of J. G. White and Company, Incorporated, Engineers and Contractors, in St. Louis, Mo. Mr. Harris was placed in charge of the preparation of plans, estimates, and reports for the proposed construction of lines of the Philippine Railway Company on the islands of Panay, Negros, and Cebu; he was soon sent to the Philippines as assistant chief engineer in direct charge of the engineering organization and construction forces engaged in building the Philippine railway. He remained in the Philippine Islands, carrying on this work until 1912.

---

[1] Memoir prepared by Ray Baker Harris, Washington, D. C.

From 1912 until 1928 Mr. Harris was associated with Col. Frederic A. Molitor,[2] M. ASCE, in New York, N. Y., in private practice as a consulting engineer. During this period he was engaged in making out project reports including the following: The Kansas City, Mexico and Orient Railway Company of Texas (the Texas Mexican Railway Company); the railway situation in the New England States; the West Side Terminal in the Borough of Manhattan (New York), for the Board of Estimate and Apportionment; the Brazil Railways, for the receiver, covering the physical condition of properties, additions, and betterments required, future traffic trends, and operation methods; the proposed extension of the South Carolina and Western Railway Company; the possibility of constructing new lines for the Delaware and Hudson Company; the operation and physical condition of the Chicago, Milwaukee, St. Paul and Pacific Railroad Company; and the National Railways of Mexico for the Committee of Foreign Bondholders. After making his reports for the Brazil Railways Mr. Harris was retained by that company to analyze and report on monthly operating results.

In 1929 he was placed in full charge of the affairs and operation of street railway and bus companies owned by the United Coach Company in New York, as manager of the railway and bus department of the Utility Management Corporation (formerly the J. G. White Management Corporation). He was named vice-president, director, and member of the executive committee.

As a result of the widespread changes in policy and personnel made by the Securities and Exchange Commission in the utility management field, Mr. Harris retired from the Utility Management Corporation in 1941 and resumed practice as a consulting engineer. He did considerable survey and special report work on transportation properties owned by H. E. Salzberg and Company in Maine (Lewistown-Auburn Transit Company, Portland Coach Company, and others) and on the British Columbia Electric Railway Company properties at Vancouver, B. C. He also contributed numerous articles to publications in the transportation field.

He died suddenly in New York of acute coronary thrombosis on October 26, 1947. His death occasioned many expressions of respect and admiration from professional associates who deplored the loss of his wide experience and progressive thinking in the engineering and transportation fields. All spoke of his personal qualities of dignity, courtesy, and consideration for others. Funeral services were held from his home in Washington, D. C., and interment was in the family plot at Cedar Hill Cemetery there.

Mr. Harris is survived by his widow, Mary Allen Harris, to whom he was married at Hull, Ill., on September 21, 1898. He is also survived by a son, Ray Baker Harris, a daughter, Eleanor Harris—all of Washington, and by a brother, Phil B. Harris, M. ASCE, of Los Angeles, Calif.

His memberships included those in the American Institute of Consulting

---

[2] For memoir, see *Transactions*, ASCE, Vol. 103, 1938, p. 1838.

4

Engineers and the American Railway Engineering Association. He belonged to the Lotus Club, Metropolitan Club, Railroad-Machinery Club, and the National Republican Club, all of New York.

Mr. Harris was elected an Associate Member of the American Society of Civil Engineers on April 3, 1907, and a Member on March 13, 1917. He became a Life Member in January, 1942.

HERBRAND HARVEY, M. ASCE[1]

DIED FEBRUARY 22, 1947

Herbrand Harvey (Alexander St. John Herbrand Harvey) was born at Bellary, India, on October 10, 1872, of a Scotch family of merchants, statesmen, and diplomats. His father was an East India merchant.

At about six years of age, he was sent to Great Britain to be educated, and obtained his preparatory education at Marlborough School. He was then an apprentice for five years (1889 to 1894) with A. and W. Smith and Company, Eglington Engine Works, Glasgow, Scotland, and in 1896 received a Certificate of Engineering Proficiency for the civil engineering course at Glasgow University.

During the next four years (1896 to 1900) Mr. Harvey was engaged on surveying in British Honduras and on gold and silver mining and railroad construction in the State of Oaxaca, Mexico. In 1900 and 1901 he worked as draftsman for the American Bridge Company at St. Louis, Mo., and for D. H. Burnham and Company in Chicago, Ill., and was employed on construction work with the Hillman Land and Iron Company in Kentucky.

In 1902 and 1903, he was superintendent of construction on the Whitehall Building and the Fourteenth Street Store in New York, N. Y., for the George A. Fuller Company. The following year, he went to Guam for Milliken Brothers of New York as superintendent of construction of the Commercial Pacific Cable Company's station there.

Mr. Harvey then, in 1905, became associated with S. Pearson and Son, Limited, of London, England. It was an association that, but for an interval of a year and a half, lasted eighteen years. At first he was in charge of construction of a section of the Thuántepec National Railway, including the Atlantic terminal, and also the complete construction of the Port of Mexico at Coatzacoalcos, Mexico. This project included the construction of jetties at the mouth of the river; dredging the channel across the bar; building extensive steel pile wharves and wharf sheds and warehouses; the erection of a power plant; and the installation of wharf cranes. This was a busy port until after the building of the Panama Canal.

On the completion of this job, Mr. Harvey was sent to Belém, Pará,

---

[1] Memoir prepared by George H. Bayles, M. ASCE.

Brazil, where he had charge of the construction of the Port of Pará. This consisted of building a bulkhead wall a mile and a half long of massive concrete blocks, a broad wharf with several railroad tracks the length of the wall, and eight large wharf sheds, in addition to dredging the basin and channel—a job that was completed in 1912. Both of these ports stand today—the one practically abandoned, the other, the busy port of entry to the whole Amazon Basin—as monuments to the thoroughness and fidelity of their construction.

When the latter was finished, Mr. Harvey went to New York as a director of S. Pearson and Son, Incorporated, of New York. There he directed the work of the New York office and had personal charge of contract 20 B on the New York State Barge Canal.

During the first three years of World War I, he was occupied with munitions for the British Navy and in inspecting small arms manufacture for Great Britain in America. This work having been reduced to routine, he went to Chuquicamata, Chile, in the latter part of 1917, for the Chile Exploration Company as resident engineer on construction of large extensions to the then existing plant. Through a staff of seventy, he directed the work of twenty-five hundred men. The project included, besides building extensions to crushing, screening, leaching, dechloridizing, and electrolyzing plants, the construction of a large steel frame machine shop, a reinforced concrete foundry building, houses for thirteen thousand people, sewerage and sewage disposal for two towns, concrete water storage tanks, and a ninety-four kilometer steel pipe line across the world's driest desert. In addition, he made a study and estimate of cost of a proposed hydroelectric development in the near-by mountains.

Shortly after the end of World War I, the construction program at Chuquicamata was abandoned, and Mr. Harvey returned to England where he became resident engineer on the erection of steel mills for the Appleby Iron and Steel Company, near Lincoln. In early 1920, he rejoined S. Pearson and Son, Limited, and was sent to Mexico as general manager of the Minatitlán Refinery of The Anglo-Mexican Oil Company, where work on an extensive enlargement of the refinery was about to begin. After two years' work, the failure of the company's production in the Tampico field caused the curtailment of the program and Mr. Harvey was sent to India for the Whitehall Petroleum Company on oil exploration. After spending two years in India, he severed his connection with the Pearson firm and spent the next year mainly in designing and constructing a home for himself and Mrs. Harvey on the highest ridge of The Downs, midway between London and Portsmouth.

During 1925 and 1926, Mr. Harvey directed, for Balfour and Beatty, the construction of the Lochaber Project of the British Aluminum Company, near Fort William, Scotland, consisting of a sixteen-foot tunnel eighty thousand feet long, a steamship pier, a connecting railway, and auxiliary works. In 1927, he joined old Pearson friends as director of Sir Ernest Moir and Company, remaining until the death of Sir Ernest in 1929.

He then became general manager of the Port of La Guaira in Venezuela, and continued managing and improving the property until he negotiated its sale to the Venezuelan government in 1938. He returned later, however, and prepared an estimate of the cost of further extensive improvements.

Before leaving La Guaira, he became a director of the Manaos Harbor Corporation, a Booth and Company property, a thousand miles up the Amazon River, and was their consulting engineer, as well—a connection he maintained until his death.

When the Germans damaged the harbor works at Scapa Flow in Scotland, early in World War II, he was called on to assist in the reconstruction. The location of his home, unsurpassed for beauty in England, now proved a handicap. The almost constant flight of war planes, British, American, and German, over it eventually impaired Mrs. Harvey's health and he was compelled to give up the work at Scapa Flow and, after 1942, devoted almost his entire time to her.

At times during his career, he found the opportunity to make surveys for the harbors of Buenaventura and Cartagena in Colombia; to make preliminary studies for the port work at Valparaiso, Chile, and Fortaleza Ceará, Brazil; to study and estimate engineering works in Mesopotamia; and to prepare the estimate for a bidder on raising the Aswan Dam in Egypt.

Mr. Harvey was a man of profound learning and broad culture. He was thoroughly grounded in the principles of engineering and had an unsurpassed capacity for detail and thoroughness. He possessed a quick perception and keen insight and was an infallible judge of character. His capacity for friendship was unlimited. He enjoyed the society of his friends who, in turn, admired his vigorous personality and clear intellect. He was six feet tall and of slender build, but was nevertheless an active participant in athletic sports. In his fifties he still bowled respectably at cricket and played above average tennis.

Having worked in ten or more countries and traveled in many others, he had a lively interest in world politics and finance and for years had kept up an active correspondence on these subjects with friends throughout the world. Besides his loss to the engineering profession, these friends and younger men to whom his life was an inspiration will miss him sadly.

While working in Mexico, Mr. Harvey met Bessie Kingsland of St. Louis, daughter of Lawrence D. Kingsland, manufacturer of farm and saw mill machinery. In 1895 she had been the second Veiled Prophet Queen of St. Louis. He gave up his work in Mexico and went to St. Louis where he got a job. They were married in 1903 and thereafter she was his companion on every job until the last one at Scapa Flow.

While working with his gardener and in his usual good health and spirits, Mr. Harvey suffered a heart attack and died within a few minutes. His wife survived him only five days. They had no children.

He was a member of the Institute of Civil Engineers, a Fellow of the Royal Geographical Society, and belonged to the Devonshire Club of London.

Mr. Harvey was elected a Member of the American Society of Civil Engineers on April 2, 1913.

PHILIP WALTER HENRY, M. ASCE[1]

DIED NOVEMBER 7, 1947

Philip Walter Henry was active in his chosen profession from the time he started working on a surveying party at the youthful age of sixteen, until a very few weeks before his death at the notable age of eighty three. These sixty-seven years of action and service were far beyond those allotted to the average man. He obtained his engineering education through sacrifice, and was honored by his alma mater with the positions of trustee and vice-president for many years. As a consulting engineer Mr. Henry was deeply conscious of his professional obligations and was delegated to serve the American Institute of Consulting Engineers, both as secretary and as president. As he served his fellow engineers, he was honored by them. His devotion to them was returned in full by their affection and respect.

The account of his life as an engineer, builder, and manager of construction projects throughout the United States, South America, Spain, and China presents a colorful and romantic picture. The details given here are to be accepted as Philip Henry's autobiography, presented in the modest style so familiar to those who knew him.

This outstanding engineer was born at Scranton, Pa., on March 24, 1864. He was the son of Eugene T. and Emma E. (Walter) Henry. In 1865 his father moved to Oxford, N. J., where he became general manager of the Oxford Iron Company in the erection and operation of a blast furnace, rolling mill, and nail factory. As the discovery of the cheaper iron ores of the Lake Superior region brought about the discontinuance of much of the iron industry of northern New Jersey, this circumstance (combined with the long illness of his father, who died in 1883) made it necessary for young Henry to go to work at sixteen years of age. Fortunately, he obtained employment in the engineer corps of the Delaware, Lackawanna and Western Railroad Company where he remained for three years on location and construction in western New York State. Through the kindness of a friend of his father, Mr. Henry was able to enter Rensselaer Polytechnic Institute (R. P. I.) in Troy, N. Y., and he was graduated in 1887.

Then his connection began with the Barber Asphalt Paving Company which was rapidly expanding under the business genius of Amzi L. Barber. Serving first at Buffalo, N. Y., and then at St. Joseph, Mo., Omaha, Nebr., and New York, N. Y., as superintendent, he became vice-president and general manager in 1894, and he was responsible for the cost and quality of pavements laid in some sixty cities—ranging from Boston, Mass., to Denver, Colo., and from St. Paul, Minn., to New Orleans, La. In 1893 he made surveys for loading asphalt on piers and tramways from Pitch Lake in Trinidad, British West Indies, and started construction the next year. In 1894 he sent the organization to lay the first asphalt pavements in Buenos Aires, Argentina.

---

[1] **Memoir prepared by Enoch R. Needles and Howard C. Baird, Members, ASCE.**

In 1902, owing to change in ownership, Mr. Henry left the Barber Company and became vice-president of the Medina Quarry Company in Albion, N. Y., having charge of operation and sales. From 1904 to 1909 he was again associated with Mr. Barber in the development of oil and asphalt properties in Mexico and Venezuela. The company in Mexico of which Mr. Henry was vice-president had oil and asphalt leases to the extent of 80,000 acres in the State of Vera Cruz (60 miles south of Tampico) for which development he made investigations. These leases, among which were those of "Juan Casiano" and "Cerro Azul," were sold in 1907 to the Huasteca Petroleum Company (Doheny), forming the basis for the great oil production of that company; one well alone produced 100,000 barrels of oil per day for a number of years.

In Venezuela the company of which Mr. Henry was vice-president purchased asphalt from the Venezuelan Government from the so-called Bermudez Lake, shipped it to New York, and refined and sold it to paving contractors throughout the United States. Owing to the previous seizure of this property by the Venezuelan Government from another American company for alleged failure to live up to its concession, litigation ensued as well as diplomatic correspondence between the Venezuelan and American governments. Since the company with which Mr. Henry was associated had the support of President Castro of Venezuela, it did a profitable business until 1909 when both President Castro and the company were ousted by the revolution that placed Mr. Gomez in the presidency for twenty-six years. Mr. Henry regarded this as the most entertaining and exciting experience of his entire business life. It led to his writing an article which appeared in the *North American Review*.[2]

While Mr. Henry was busy with his Mexican and Venezuelan affairs, he was also president of the South American Construction Company (from 1906 to 1910) during which time it built 125 miles of railroad in Bolivia from Viacha to Oruro on the plateau 12,000 ft above sea level. The road was constructed at a cost 5% under that estimated by the chief engineer of the Bolivia Railway Company, for which it was built. This was due to his selection of two competent general managers—the late F. C. Hitchcock, M. ASCE, who, upon his retirement (because of the altitude) was succeeded by J. P. Hallihan,[3] M. ASCE. Both engineers later had distinguished careers.

In 1908 Mr. Henry was appointed by the Hon. Charles E. Hughes, then Governor of New York State (later Chief Justice of the United States) to represent the state at the First International Roads Congress called by the French government to meet in Paris. Here he presented a paper which was reprinted in *The Kings Highway*,[4] with the following editorial comment: "The author Mr. P. W. Henry of New York, is to be congratulated on his prophetic vision."

---

[2] "Has the United States Repudiated International Arbitration?" by Philip Henry, *North American Review*, December, 1907.

[3] For memoir, see *Transactions*, ASCE, Vol. 109, 1944, p. 1483.

[4] "The Future Road—The Wearing Surface," by Philip Henry, *The Kings Highway*, April, July, and October, London, 1938.

In addition to the discussions, excursions were made in buses and motor cars from Paris to Versailles; in carriages from Melun to Fontainebleau; and in automobiles from Nice through the Maritime Alps to l'Authion, 7,000 feet above sea level, returning via Mentone—a distance of ninety miles. At these places the members of the Congress were entertained by the city officials—in Paris there was a reception at the City Hall and a special performance of the Comédie Française; at Versailles, a luncheon in the Orangerie of Louis XIV in the palace; at Fontainebleau, a reception in the Salon of Henry II in the palace; at Nice, a dinner by the mayor; at Peira Cava, a mountain resort 5,000 feet above sea level, a luncheon; and, at Monte Carlo, a luncheon by the Mayor of Monaco and a special concert in the theater. Mr. Henry considered this the most delightful business trip he ever took.

In 1910, in company with Mr. Hallihan, he made a reconnaissance by automobile of 700 miles of a proposed railroad in Spain (Bilbao to Madrid and thence to Valencia), detailed surveys having been made by Spanish engineers. On this trip they were accompanied by representatives of the National City Bank and of the financial house, Hallgarten and Company, both of New York. They met the Prime Minister, Señor Canalejas, who greatly desired to have American engineers and contractors develop the public works of his country. Unfortunately, the assassination of Señor Canalejas a year later discouraged the American bankers, who had been favorably disposed toward this project, but who had delayed committing themselves.

From 1909 to 1917 Mr. Henry was president of the Central Railroad of Haiti, in the construction of the wharf at Port-au-Prince and in the operation of railroads and public utilities in that country. During this period he also maintained an office as consulting engineer and was employed for three years by the Southern Pacific Railroad Company, in relation to the management of its oil properties in California, Texas, and Mexico.

In 1915, in company with a vice-president of the National City Bank, Mr. Henry made a trip to Brazil, Argentina, and Bolivia, with special reference to the railroad situation in those countries. He made a special investigation of the properties of the Antofagasta (Chile) and Bolivia Railway Company, Limited, the purchase of which was being considered by a certain American financial group.

Mr. Henry was vice-president of the American International Corporation from 1916 to 1923; he would pass on the economic soundness of many engineering projects submitted to that corporation for financing from all over the world. From 1917 to 1923 he was vice-president of the Siems-Carey Railway and Canal Company (a subsidiary of the American International Corporation) and was responsible for the selection of engineers to make railroad and canal surveys in China to the extent of more than $2,000,000. As a result he was made an honorary member of the Association of Chinese and American Engineers, of which the late J. A. L. Waddell, M. ASCE, John F. Stevens,[5] Past-President, ASCE, and John R. Freeman,[6] Past-President, ASCE, were also made honorary members in recognition of their work in China. During this period he had much to do with the International Products Company (another subsidiary) which erected packing and quebracho plants in Paraguay and a packing plant in Colombia.

---

[5] For memoir, see *Transactions*, ASCE, Vol. 109, 1944, p. 1440.

[6] *Ibid.*, Vol. 98, 1933, p. 1471.

From 1933 to 1935 Mr. Henry was a member of the technical board of review composed of fifteen prominent engineers, which passed on the economic soundness of engineering and construction projects presented for financing to the Public Works Administration in Washington, D. C. In 1936 he made an investigation of the transportation system of Buenos Aires for a New York investment trust which had a large investment in overdue notes of the American built subway.

Although Mr. Henry's experience was largely that of an executive, his consulting practice took him into various parts of the world and into diverse fields. As an executive or consultant, his work in engineering or construction embraced Argentina, Bolivia, Chile, China, Colombia, Haiti, Mexico, Paraguay, Peru, Spain, Trinidad, and Venezuela. In the United States it was concerned with oil in California, Colorado, Texas, and Wyoming; coal in Tennessee and Wyoming; railroads and utilities in various states; and the low temperature process for artificially curing alfalfa and other forage crops in the eastern part of the United States, Canada, and England.

As Mr. Henry's activities had taken him to various Spanish speaking countries, he undertook, at the age of forty-nine years, the study of Spanish, mostly at home with grammar, dictionary, and phonograph. He became so interested in the philosophy of Calderón de la Barca, noted Spanish dramatist of the seventeenth century, that he put into English several of his dramas and "autos sacramentales" (morality plays). Selected scenes from his translation of "La Vida es Sueno" were staged by the R. P. I. Players in March, 1939.

From 1942 to 1945 Mr. Henry was chairman (representing the Society) of the Joint Committee on Inter-American Engineering Cooperation formed by the five societies (representing the civil, mining, mechanical, electrical, and chemical engineering fields) which worked in cooperation with the Inter-American Development Commission of which Nelson A. Rockefeller was chairman.

Mr. Henry early became interested in outside activities. Shortly after coming to New York in 1892 he became a member of the Committee of Management for the Young Men's Christian Association (23d Street branch) of which he was chairman from 1916 to 1921. He was a trustee of the United States Savings Bank from 1916 until its merger with the Dry Dock Savings Institution in 1933. From 1918 he had been a member of the Council of the American Geographical Society, serving as vice-president from 1920 to 1941, and as chairman from 1922 to 1939. He was a charter member in 1912 of the Pan-American Society and served on its council from 1919 to 1944. He served the New York City Mission Society as a director in 1922, and as a vice-president from 1927 to 1944.

In 1911 Mr. Henry became a trustee of Rensselaer Polytechnic Institute and in 1938 was vice-president. His memberships included those in Sigma Xi; the American Institute of Mining and Metallurgical Engineers; and the American Institute of Consulting Engineers (a charter member from 1911), serving as secretary from 1924 to 1947, and as president in 1940. He was a fellow of the Royal Geographical Society. Early in his career he joined the Engineer's, University, Union League, and Century clubs of New York, and for a time was a member of the Cosmos Club in Washington, D. C., and the Saturn Club in Buffalo.

In politics Mr. Henry was a Republican, and his religious affiliation was Presbyterian. He served as an elder in the Madison Avenue Presbyterian Church of New York and in the Scarborough (N. Y.) Presbyterian Church. As his mother's family was Moravian, he was baptized in the Moravian Church (Nazareth, Pa., which was his mother's home before her marriage).

Throughout his long and successful career Mr. Henry was both technical and professional. In addition, he had a knowledge of administration and finance, that made him a capable business executive. His quiet and circumscribed boyhood was followed by wide travels into many parts of the world, from which he derived the fullest benefits and inspiration. A practical, realistic, and productive engineer and executive, he was happy in his more esthetic attainments as a reader, writer, student, and linguist. Mr. Henry was an idealist (as so many engineers are) and, since he was born into a religious atmosphere, he continued to be an active churchman throughout his life; he was very forceful but, at the same time, a true gentleman.

He was a man who had friends and treasured them; quiet and modest, he was strong and unswerving in support of his principles. He was gentle and receptive, but his loyalty and determination made him a valuable ally in a chosen cause.

The American Institute of Consulting Engineers had an unusual place in the heart and life of Philip Henry, particularly in his later years. The integrity and advancement of the institute were almost an obsession with him as he gave to its affairs the attention and devotion which are beyond price. In the most minute details he was the servant of the institute and its members and his only compensation was in seeing the institute prosper and in receiving the heartfelt thanks and deep affection of those with whom he worked.

The engineering profession provided a means for the exercise of Philip Henry's particular talents which could hardly have been excelled. His engineering education was followed by engineering experience which fitted him in excellent manner for management, administration, and consultation pertaining to engineering projects. His engineering success was combined with business success, and his service to his fellow man as a professional engineer was merely typical of his service through his church, charities, and societies. He honored his profession as it honored him.

Mr. Henry was married on January 22, 1906, to Clover Cox of Chicago, Ill., and from 1918 they resided at Scarborough. Mrs. Henry died on January 12, 1946. He is survived by a daughter, Clover (Mrs. David Graham), and three grandsons, one of whom was named for him.

Mr. Henry was elected an Associate Member of the American Society of Civil Engineers on January 3, 1894, and a Member on November 2, 1908. He became a Life Member in January, 1929.

CHESTER JAMES HOGUE, M. ASCE[1]

Died November 4, 1946

Chester James Hogue, the son of Harvey A. and Sarah L. (Abrams) Hogue, was born in Portland, Ore., on September 19, 1875. During 1895 and 1896 he had private instruction in drafting and architecture, and then entered Massachusetts Institute of Technology in Boston, as a special student, completing courses in architecture and engineering in 1899.

Mr. Hogue began his professional career in Boston with the firm of Wheelwright and Haven, Architects, in 1899. In 1901 he worked on the construction of the building for the Massachusetts Horticultural Society; then, he entered the employ of the Eastern Expanded Metal Company where he became construction engineer. He joined the Concrete Engineering Company in 1908, in a similar capacity. At this time, Mr. Hogue aided in the development of reinforced concrete construction and had charge of the design and construction of many large structures, utilizing this combination of construction materials, which was then new.

In 1913 he left Boston to return to his native city, Portland, to form a partnership with Edward Foulkes, practicing architecture under the firm name of Foulkes and Hogue. Many of the fine homes of this young and lusty city were conceived on their drafting tables, in addition to the Oregon Building for the Panama Pacific International Exposition at San Francisco, Calif., in 1915.

In 1916 Mr. Hogue became manager of the Portland office of the West Coast Lumbermen's Association and, except for a period during World War I, he remained with this organization until his retirement in January, 1946. During World War I (1917–1918), his services were loaned to the Bethlehem Shipbuilding Corporation to construct shipyards in the San Francisco Bay area.

The completion of the Panama Canal opened markets along the Atlantic seaboard for lumber manufactured in the Pacific northwest. Therefore, he was transferred to New York, N. Y., in 1919, where he effectively introduced west coast species of lumber for construction uses and proved their value. In 1926 he returned to the west to take charge of Trade Promotion and Technical Service with headquarters in Longview, Wash. In 1928 the organization moved to Seattle, Wash. Although his headquarters and his home were nominally in Seattle, his interests and his work carried him to all parts of the United States, to Japan, and to South America. In 1939, he dropped the trade promotion work and devoted all his time to his principal interest, timber engineering—remaining in charge of technical service until retirement. His work became his hobby, and Mr. Hogue was engaged in writing a textbook on timber engineering at his death.

---

[1] Memoir prepared by William Thornton King May, Assoc. M., ASCE.

Constant research in wood technology and in its mechanical properties made Mr. Hogue a well-known authority on the uses of this material during his thirty years with the lumber industry. He was active in the development of the American Lumber Standards which instrumented the grading of lumber for specific use requirements. All important improvements in the use of timber as an engineering material were advanced by his fund of knowledge and his cooperation. He prepared a handbook on the structural uses of west coast species, and wrote many papers and publications on the technical uses of lumber. He always found the time to serve on many technical committees and to aid in solving the numerous problems set forth by professional and nonprofessional men alike.

On January 16, 1901, Chester Hogue was married to Marian L. Chase of Boston. He is survived by his widow; two daughters, Dorothy (Mrs. T. A. Clarridge) and Margaret (Mrs. J. F. Malony); and two grandsons, Arthur Preston Clarridge, II, and Chester Hogue Clarridge.

He was a member of the American Society for Testing Materials, the American Railway Engineering Association, the American Wood Preservers' Association, and the American Petroleum Institute. He was also a member of the F. and A. M.

Mr. Hogue had an exceptional capacity for work, and he was accomplished in every field which attracted his interest. His unfailing friendliness won him many friends throughout the world. A man extremely well versed in his lines of endeavor, his opinions were constantly sought by experts, and he never was too busy to devote time to younger men. Personal integrity distinguished him throughout his life.

Mr. Hogue was elected an Associate Member of the American Society of Civil Engineers on December 6, 1905, and a Member on June 6, 1911.

HENRY MACY JONES, M. ASCE[1]

DIED JULY 18, 1947

Henry Macy Jones was born on April 17, 1892, in Los Angeles, Calif. His father, Elmer E. Jones, came to Los Angeles from Delaware, Ohio; his mother, Mary (Macy) Jones, was graduated from the old Hanna College on Temple Street in Los Angeles. This was an exclusive finishing school in the early days of the city. Her parents, originally from Kansas, owned one of the early orange ranches in Riverside, Calif.

---

[1] Memoir prepared by Donald M. Baker, M. ASCE.

At the time when Macy Jones was born, Los Angeles was a small town with a population of about 60,000 and Los Angeles County had a population of little more than 100,000. Before his death he saw the city grow to a size of 1,800,000 and the county reach a population of 3,750,000, and he knew that he had a definite part in this development.

Mr. Jones received his early schooling in Los Angeles, attending the old Hoover Street Grammar School when it was first opened, and he was graduated from the Los Angeles Polytechnic High School in 1913. He enrolled in the civil engineering course at Stanford University (Stanford University, Calif.) immediately afterward, and in the spring of 1917, a few months before graduation, he left Stanford to take his first engineering job as assistant on the construction of the Copco Dam for the California-Oregon Power Company in southern Oregon. He returned to Stanford in the summer of 1918 and received the degree of Bachelor of Arts in Civil Engineering on October 4, 1918.

After entering military service in August, 1917, Mr. Jones rose, successively, through the ranks of Private, Corporal, Sergeant, and Second Lieutenant. His regiment, the 154th United States Engineers, was waiting to embark overseas in New York, N. Y., on Armistice Day in 1918.

Mr. Jones entered the office of the county surveyor in Los Angeles County on February 26, 1919, as a junior draftsman, and he passed, successively, through the positions of senior draftsman and chief draftsman before being appointed office engineer and deputy county surveyor on July 1, 1925. This period saw an extremely rapid growth in the population of Los Angeles County. Subdivisions (literally in the thousands) were filed with the county surveyor's office for approval and checking, large numbers of roads were built involving acquisition of rights of way, and many plans and specifications were prepared for municipal improvements—all this work coming under Mr. Jones' supervision as office engineer.

He resigned to accept an appointment as chief engineer of the Harold G. Ferguson Engineering Corporation in November, 1928, and held this position until February, 1931. During this time the company laid out several imposing residential subdivisions along the Roosevelt Highway in the famous Malibu Ranch on the coast, north of Santa Monica, Calif., making plans and supervising the construction of improvements on a number of these subdivisions. It also laid out and constructed improvements for Canoga Estates, south of Ventura Boulevard, in Los Angeles (in the San Fernando Valley). He resigned as chief engineer of the Ferguson Engineering Corporation in March, 1931, and from this time until August, 1932, he was engaged in private practice as a consulting engineer in Los Angeles.

In September, 1932, he was appointed secretary to the California State Board of Registration for Civil Engineers, and in June, 1935, he became executive secretary for the board. The California board had been created during 1929 by the legislature, and under the "grandfather clause provision" of the Civil Engineers Registration Act, more than 5,100 certificates of registration were issued. Following the issuance of these certificates, few applicants for registration were considered for several years, as engineering employment (because of the economic depression of that period) was not very active. This opportunity enabled the board to organize itself in order to sell the idea of registration—not only to the civil engineering profession, but also to the employers of engineers, to numerous city and county civil service commissions in particular, and to the state civil service organization.

This work during the formative period of the state board was carried on under Mr. Jones' direction, and the best testimony of its effectiveness lies in the fact that by the middle of the 1930–1940 decade civil engineering registration in California was strongly supported by the profession (including the Society), and also by both public and private employers of engineers. One of the most important accomplishments of this period was to sell the idea to public agencies that civil engineers taking civil service examinations for jobs which involved engineering responsibility should, as a prerequisite to taking such examinations, be registered. As all administration of regulatory statutes exposes certain weaknesses in the original statute, various amendments were made to the act to overcome them.

In the late 1930's applicants for examinations increased and the examinations became more standardized. The board worked in close cooperation with the four sections of the Society in California, and much of this excellent cooperation was the result of Mr. Jones' endeavors. He was active on numerous committees in the Los Angeles Section and was Secretary during 1931–1932, Vice-President in 1935, and President in 1939.

Because of ill health, Mr. Jones took a leave of absence from his position with the board on August 15, 1945, and moved to Balboa, Calif., where he resided until his death. He resigned as executive secretary on October 3, 1945.

While at Stanford Mr. Jones met Elizabeth Mitchell Judah, to whom he was married on December 5, 1921. It was a very happy marriage as they were ideal companions, and her solicitous care of him during his last illness was a tribute to their attachment to each other. He is survived by his widow and a sister Mariam N. Jones of Los Angeles.

Mr. Jones was elected an Associate Member of the American Society of Civil Engineers on October 15, 1923, and a Member on June 26, 1933.

MARION REED KAYS, M. ASCE[1]

Died March 13, 1946

Marion Reed Kays, the eldest child of Emory and Emma (Track) Kays, was born on January 22, 1881, at Tonica, Ill. His parents and their four children moved to Phoenix, Ariz., in 1886, and by 1889 were operating a small dairy ranch four miles outside that city.

---

[1] Memoir prepared by H. J. Morrison, Assoc. M. ASCE, and Lynn Perry, M. ASCE.

The energy of Marion, together with that of his two younger brothers, provided the essential help in feeding, milking, irrigating, and performing the myriad of ranch chores while they were attending the rural grammer school and city high school. The dairy prospered and after a few years Emory Kays became manager of a cooperative creamery. After graduation from the Phoenix High School in June, 1899, Marion was employed at the creamery, which provided his first business experience.

Mr. Kays registered in the mining engineering department at the University of Arizona, in Tucson, in the autumn of 1901. He studied there only one year and the following September he entered the alma mater of both his father (Class of 1879) and mother (Class of 1881), the University of Illinois, in Urbana. He was graduated from the civil engineering department in June, 1906. His formal education was interrupted and his graduation delayed by an engagement with the United States Reclamation Service from June, 1903, to August, 1904. During this period he was engaged on location surveys for the Roosevelt Dam in Arizona and its connecting roads and canals. He spent the following summer vacation on similar work on the North Platte project in Wyoming. These engagements with the U. S. Reclamation Service provided him with some practical engineering experience and a close acquaintance with able engineering personnel; it also fired his imagination and stimulated his sense of accomplishment, leading him directly to the field to which he devoted the productive years of his profession.

After receiving the degree of Bachelor of Science in Civil Engineering Mr. Kays was immediately employed by the U. S. Reclamation Service as assistant engineer and remained with that service for three years. During this time he engaged in surveying, designing, constructing, and operating irrigation works on the North Platte project in Wyoming and Nebraska, and the Salt River project in Arizona. He resigned in November, 1909, to join his father in business; within a few months, however, he joined some of his former associates and returned to reclamation work—this time with the Idaho Irrigation Company.

On this project he served as assistant construction superintendent for J. G. White and Company with headquarters at Richfield, Idaho, from February 1, 1910, to November 30, 1911, when he became vice-president and general manager of the Idaho Irrigation Company. He served the company in this capacity for more than eight years, during which time he was responsible for the business management of the irrigation project and "allied features"; the collection and disbursement of all funds; public relations; incidental construction; all maintenance and operation, including the reservoir and canal system; repairs and improvements; precipitation and stream flow measurement and records; and water supply between project lands and decreed rights in cooperation with the state authorities. Mr. Kays conducted duty-of water studies and demonstrations; negotiations over water supply and project acreage, frequently leading to litigation; and the grouting of dam foundations. In addition, he promoted land ownership and use. The "allied features" included laying out two town sites, together with the construction of their public buildings, hotels, deep wells, pumping stations, water distribution systems, an electric light and power plant, miles of telephone network, and two 160-acre fully improved demonstration farms.

In August, 1920, Mr. Kays became associated with the Beckman and Linden Engineering Corporation of San Francisco, Calif., in charge of its Phoenix (Ariz.) office. He acted as project engineer for the Paradise-Verde Irrigation District (later known as the Verde River Irrigation District) in Arizona, with headquarters in the Heard Building in Phoenix. He was in charge of the analyses and estimates of available water and the area which it would serve, and the design of dams, reservoirs, diversion works, and canals. A part of his attention was devoted to the Carrick and Mangham, and to the Maricopa and Queen Creek projects in Arizona. In July, 1923, he became special field engineer for the Salt River Valley Water Users' Association. Some features of the Verde River district were: Measurements on canal water loss, studies to determine the quantity of water that could be saved by lining the canals and the additional area it would serve, and the remodeling and enlargement of the power plant at the Roosevelt Dam.

He was appointed manager of the Lake Worth Drainage District in Florida, in January, 1924, upon the suggestion of those who had participated in financing the Idaho Irrigation Company previously. This appointment brought to the drainage district a thorough, practical knowledge of the principles of the artificial control of water for agriculture; the relation between soil moisture and plant growth; the construction, operation, and maintenance of reclamation works; the financial and economical aspects of such developments; and an appreciation of the responsibilities of public agencies engaged in such service—accumulated in Mr. Kays' seventeen years experience in the west.

The Lake Worth Drainage District originally provided an area with a system of canals and laterals to dispose of excess water, and control works to prevent overdrainage and to maintain proper ground-water levels. At the time Mr. Kays became manager the canals were from two to six years old and had not been systematically or adequately maintained. The system of canals was incomplete, and the control works were insufficient; there were no pumps. As manager, he was retained to salvage a failing enterprise, which required construction and personnel, use of land and collection of assessments, and the payment of the interest on bonds and the reduction of debt. As manager of the district, he was also chief engineer, secretary, and treasurer.

After a rehabilitation policy was approved by the bondholders, a program of improvement and maintenance was adopted and successfully carried out. Small-scale land development was encouraged at first which led to larger projects. To advance agricultural development, many houses were built. Finally, the improved growing conditions aroused interest and attracted capital. The collapse of land values in 1925, the hurricanes of 1926 and 1928, bank failures in 1927, the crash of 1929, and ensuing years of depression left their scar. They disrupted business and presented unexpected problems which Mr. Kays faced with characteristic ability and patience. He never lost sight of his duty to build a successful and profitable agricultural system on a foundation of water control suited to local conditions. His program had been laid out and he adhered to its fundamental provisions tenaciously. Methods and practices suitable to local conditions were devised and demonstrated. Irrigation by ground-water control was perfected and developed into large-scale use.

In 1938 he relinquished the management of the district and prepared an exhaustive and detailed report dealing with the engineering phases of the development, making recommendations for future construction and operation, all of which were based on the principles which had been established and in harmony with the long-time program which he had adopted fifteen years before.

Mr. Kays was appointed chief engineer of the West Palm Beach Housing Authority in Florida in August, 1939. He was in charge of construction of the 246-unit "Dunbar Village," a low rent housing project (colored), and the 122-unit "Southridge" project (white), with an administration building for each. The buildings were of concrete block construction, completely equipped with electricity, gas, water, sanitary sewers, and storm water drainage, and they were landscaped when they were ready for occupancy.

Following this project, in 1941 he served as resident manager of the "Morrison Homes," a 150-unit defense development of which the West Palm Beach Housing Authority was managing agent. The tenants were the military and civilian personnel at Morrison Field, an Army air base near West Palm Beach. He became construction engineer of District No. IV of the Federal Public Housing Authority in Atlanta, Ga., in September, 1942, a position which he held at the time of his death.

Active in religious affairs as well as in technical, business, and civic matters, Mr. Kays was a leader in every undertaking with which he was associated. He was a trustworthy friend, respected and beloved by all who were privileged to know him. He will be remembered for his intelligent and enthusiastic handling of the drainage problems in the Everglades of Florida and for his tenacity in striving to accomplish the long-time objective. He was a builder and developer; in the wake of his activity were houses and green fields.

Mr. Kays was a charter member and the first president of the Palm Beach County Engineers' Club; secretary of the Association of Florida Drainage Districts; past-president of the Palm Beach County Chapter of the American Red Cross, and chairman of its Preparedness and Relief Committee; president for five years of the Palm Beach County Council of the Boy Scouts of America, serving as its representative on the National Council; past-president of the Miami Section of the Society; director and vice-president of the West Palm Beach Chamber of Commerce and the West Palm Beach Zoning Board; and a member of the Rotary Club and of the Presbyterian Church there.

He was married to Alice Adeline Grier of Phoenix. Surviving are his widow; a son, Marion Reed Kays, Jr., of West Palm Beach; and a daughter, Mrs. Arnold G. Asp, of Fair Lawn, N. J.

Mr. Kays was elected an Associate Member of the American Society of Civil Engineers on December 6, 1910, and a Member on October 10, 1927. He became a Life Member in January, 1945.

JOSEPH GRANT KELLEY, M. ASCE[1]

DIED NOVEMBER 14, 1941

Joseph Grant Kelley was born in Orono, Me., on August 14, 1864. He was the son of Jones Shaw Kelley and Harriet (Brown) Kelley. Mr. Kelley attended the University of Maine at Orono and received a degree in civil engineering in 1884.

Upon his graduation, he engaged in surveying and lumbering activities in the State of Maine. In 1886 hearing the call of the West, he went to work for the Shasta County (California) engineer. Later he was employed by the Shasta Lumber Company locating timber lands and constructing timber flumes. Mr. Kelley went back to Maine in 1887 and until 1892 was engaged on engineering and surveying at Bar Harbor, Me., and other towns along the eastern seaboard.

However, in 1892 he returned to the West undertaking both contracting and engineering work in Eugene, Ore. In 1896 he was employed by the Portland (Ore.) District of the United States Engineers, remaining with that organization for about four years during which time he was engaged on river and harbor surveys and construction and on the construction of fortifications at the mouth of the Columbia River. After leaving the United States Engineers, Mr. Kelley was superintendent of the Port of Portland for two years.

During the remainder of his professional life, he was primarily engaged in private practice—the location and the construction of irrigation canals and flumes; the installation of pumping plants and small hydroelectric power developments; and the investigation and the evaluation of hydroelectric power sites, water rights, and water systems in Oregon and Washington.

In 1886 Mr. Kelley was married to Cora Oliver of Redding, Calif. Two sons, M. S. Kelley and J. G. Kelley, Jr., survive him.

Mr. Kelley was elected a Member of the American Society of Civil Engineers on October 7, 1914. He became a Life Member in January, 1939.

---

[1] Memoir prepared from information on file at Society headquarters.

JOHN EDWARD KEMP, M. ASCE[1]

DIED JANUARY 19, 1946

John Edward Kemp, the oldest child of John McMillan Kemp and Elizabeth (Atteridge) Kemp, was born in Chicago, Ill., on January 14, 1878. His maternal grandfather, Thomas Atteridge, came from Northern England to Skibberee, County Cork, Ireland, where he was married to Mary Cole, a pretty Irish girl, and together in the early part of the nineteenth century, they came to the United States. Thomas Atteridge purchased land from the government and built his home, in what later became Lake Forest, Ill., a suburb of Chicago.

In later years, the family had among its most precious possessions, the papers signed by Martin Van Buren and by James K. Polk, former Presidents of the United States, transferring these grants of government land to Thomas Atteridge, whose descendants still live in or near Lake Forest.

The paternal side of the Kemp family was of English and Scotch extraction. John E. Kemp's father was employed as chief telegrapher of the Western Union Telegraph Company at the Union Station in Chicago.

When John Edward was seven years old, his father bought seven acres of the old Atteridge homestead, where his wife had lived as a girl, and built a comfortable home on it. The family then moved from Chicago to Lake Forest, since the elder Kemp felt a city was "no place to raise a family." There, with a younger brother and three sisters, John E. Kemp grew to young manhood.

He attended Lake Forest Academy, after completing grade school, being graduated as valedictorian of his class, in 1895. He then entered Lake Forest College, where he finished his four-year course in 1899. He continued his education in civil engineering at the University of Illinois, in Urbana, being graduated in 1901 with the degree of Bachelor of Science in Civil Engineering. In 1908 he received the degree of Civil Engineer. At the University of Illinois, Mr. Kemp ranked in the top four of his class and was elected to membership in the honorary engineering fraternity of Tau Beta Pi. He was also a member of the social fraternity of Phi Delta Theta.

During his vacation time, while not attending school, he served in 1898 as stadiaman for the United States Engineering Corps on the Hudson River. In 1901 and 1902, he was rodman and instrumentman for the Chicago and North Western Railway then being built in Iowa.

From 1902 to 1904 Mr. Kemp was city engineer, superintendent of the Water Department, and plumbing inspector for Kewanee, Ill. As city engineer, he designed and constructed pavements, ten miles of sidewalks (concrete), six miles of six-inch and eight-inch water main and half completed the installation of a $30,000 sanitary sewer system. In March, 1912, he was made secretary of the City Water Commission to recommend a new water works system for Kewanee. He resigned as city engineer to become civil engineer at the Kewanee Works National Tube Company which position he held from 1904 to 1917. Later this plant was sold to the Walworth Company.

---

[1] Memoir prepared by Pauline King Kemp, Kewanee, Ill.

In July, 1917, Mr. Kemp enlisted in the Engineering Reserve Corps of the United States Army, being commissioned a Captain, and, on September 4, 1917, was ordered to Fort Leavenworth, Kans., for officers' training camp. In November, 1917, he sailed, with the first 100,000 Americans, for France, where he was in the United States Ordnance Department as Construction Engineer for Ordnance Depots (built for repairing the munitions of World War I). He made a record for speedy construction of this type of building and was placed in charge of ordnance depot construction at various places in France. Since he spoke French fluently, he often did the necessary purchasing in France of army building materials such as concrete, sand, etc., which were not furnished by the United States Army. He spent a little more than one year in France, returning to his home in Kewanee on December 18, 1918.

In January, 1919, Mr. Kemp was again employed by the Walworth Company—as superintendent of maintenance—and served in that capacity until 1931. During this period, he designed and constructed a new steel skelp mill building, 120 feet by 230 feet with a 60-foot crane to replace the old structure (without interrupting mill operations); a steel and brick pipe storage building 80 feet by 150 feet with an 80-foot crane; a three-story reinforced concrete pattern storage building 50 feet by 110 feet; and a modern boilerhouse. He was in charge of all sewer, water-main, tracks, oil-storage, fire protection systems; boiler installations; furnace construction; and buildings in the Walworth plant.

From 1932, until his retirement on January 1, 1946, Mr. Kemp was chief engineer and director of training. He organized the training department of the Walworth Company, laying out the three-year course offered to the employees, assigning the instructors, writing a number of the textbooks used, and teaching several hours each week.

Mr Kemp was most proud of his work during World War II, and it was doubtless the cause of his death. Great responsibility, the rush of the work, which had to be completed quickly, and the lack of skilled labor made his load a heavy one. For a long time he had had heart trouble but his sincere patriotism would not allow him to rest while his country had war work for him to do. He designed and erected two shell-plant buildings, one 80 feet by 320 feet and one 60 feet by 177 feet where anti-aircraft shells for World War II were made.

On January 17, 1942, he was appointed resident engineer "to inspect, approve and expedite all items of labor, material, construction and engineering details in connection with the construction of a plant for the Navy at Washington Park, Illinois," near East St. Louis, Ill. This plant consisted of five main buildings; a light finishing building, 120 feet by 400 feet; a heavy finishing building, 86.8 feet by 400 feet; a steel foundry, 86.8 feet by 400 feet; a pattern shop, 50 feet by 125 feet; and an office building, 56 feet by 160 feet. These buildings were of steel, concrete, and brick.

Eight months after Mr. Kemp went to Washington Park, he turned the buildings over to the new manager for operation. Walworth valves, fittings, etc., were made here for the new ships built by the United States Navy. This was a big job and Mr. Kemp gave seven days each week to its accomplishment.

After living at Collinsville, Ill., a town near this Navy plant, for eight months, Mr. Kemp returned to the Walworth Company in Kewanee, where more construction work, necessitated by the war, awaited him. At the Kewanee plant, he designed and erected a steel iron foundry extension and cupola addition, 81.6 feet by 163 feet; an extension to the steel valve finishing department; a new plant hospital; and an extension to the main office, increasing it to three times its original size.

Mr. Kemp was a charter member of the American Legion and helped organize Kewanee Post No. 31, of which he was commander. He was active in Masonic circles and assisted in forming the Commandery of Knights Templar in Kewanee, serving as its second commander. He belonged to the Illinois American Foundrymen's Association, the Rotary Club, Midland Country Club, and Lake Calhoun Association. Among his other interests was his Boy Scout Troop, one of the largest and most active in the city, of which he was scoutmaster for over twenty-five years, receiving the national honor award of the Silver Beaver. He organized and taught a Sunday School Class of ex-service men of both World War I and World War II for more than a quarter of a century, called "The Yanks Class," in the Presbyterian Church, where he was an elder, and an active worker.

He was an outdoor man and enjoyed golf, fishing, bird hikes, and nature study. He loved to write poetry as a pastime and published for his friends, a clever collection of his poems, entitled "Kemp Kodaks." Each January, he sent out an original poem for the New Year.

On September 4, 1906, John Edward Kemp was married to Pauline Harris King, daughter of Theodore F. King and Ida (Harris) King, at their home, "The Maples," near Prescott, Iowa. To this union, were born two sons, John Theodore Kemp, a practicing physician in Michigan City, Ind., and Harris Atteridge Kemp, an architect in Dallas, Tex. Mr. Kemp is survived by his widow; his two sons; and four grandchildren, John Edward, Jr., David Anders, Barbara Ann, and Peter Alan Kemp.

Mr. Kemp was elected an Associate Member of the American Society of Civil Engineers on April 1, 1908, and a Member, on May 28, 1912. He became a Life Member in January, 1943.

RICHARD SMITH LEA, M. ASCE[1]

Died November 11, 1945

Richard Smith Lea was born on April 11, 1866, in Victoria, Prince Edward Island, Canada. An original settler of this community, a Lincolnshire (England) yeoman and an ancestor of Mr. Lea, landed on Prince Edward Island early in the 1800's, intent on making a new home for himself in the colony. Accompanied by his wife, several married sons and daughters, and their children, he settled on a farm in Victoria, on the Northumberland Straits shore, a few miles west of Charlottetown.

In due course Richard Lea was sent to the local school and on to Prince of Wales College in Charlottetown, then and now the island's best preparatory school. Because he showed more than ordinary ability in mathematics and the physical sciences, he decided, rather against his family's wishes, to study mechanical engineering at McGill University in Montreal, Que. From this institution he received the degree of Bachelor of Science with distinction in 1890. Postgraduate work at Cambridge University in Cambridge, England, and active practice in Boston, Mass., with Joseph P. Frizell,[2] M. ASCE, and Marshall M. Tidd,[3] M. ASCE, followed. In 1893 his alma mater, McGill University, granted him the degree of Master of Engineering and added him to her staff.

By 1903, with ten years of additional experience accumulated by working summers and in his spare time with Freeman C. Coffin,[4] M. ASCE, he had become assistant professor of civil engineering. In 1899 he was appointed consultant to the newly formed Provincial Board of Health of Quebec, and in 1900 formed a partnership with Mr. Coffin for handling the latter's Canadian commissions which were rather numerous at the time. In 1903 Mr. Lea established the independent consulting practice he was to follow for the rest of his life.

Up to this time his experience had been mostly in the municipal field—a field in which his major interest always lay—although later he was to be recognized as an expert in the broader aspects of hydraulic power development and in engineering legal matters. However, he made not infrequent excursions into other fields; he was a good example of the versatility which is now less common.

Detailed mention of the many commissions he executed during the forty odd years of his active practice is not possible here; most of the larger Canadian cities and corporations and many smaller ones retained him at one time or another. He made friends readily and his talents were evident enough to secure many return engagements. Some of the work he considered his more important included: The preliminary investigations for the hydroelectric plants on the Winnipeg River for the City of Winnipeg, Man.; the design and construction of extensive improvements to the water supply and power system of Medicine Hat, Alberta; the repair of serious damage to Winnipeg's ninety-mile Shoal Lake aqueduct, caused by unforeseen foundation conditions; the planning in broad outline of the Greater Vancouver (B. C.) sewerage system; the rehabilitation of the water distribution system of Ottawa, Ont.; similar work on a much larger scale for the City of Montreal; the Bassano Dam across the Bow River in Alberta; a group of earthquake resistant buildings in Kingston, Jamaica, B. W. I.; many of the studies on which the proposed St. Lawrence Waterway is based; and, finally, the many arbitration and legal cases in which he appeared as an engineering adviser.

---

[1] Memoir prepared by R. DeL. French, M. ASCE.

[2] For memoir, see *Transactions*, ASCE, Vol. LXXIII, September, 1911, p. 501.

[3] *Ibid.*, Vol. XXXVII, June, 1897, p. 568.

[4] *Ibid.*, Vol. LVIII, June, 1907, p. 532.

Fortunately, Mr. Lea's practice was always large enough so that he could choose his clients with a great deal of freedom. Although he had a strong sense of professional responsibility, he was quite aware of the field in which he excelled, and never hesitated to advise a client to take his work elsewhere, if he thought it would be to the client's advantage. Naturally, this trait made him popular with his less well-established confrères, who looked upon him as a sort of fatherly adviser.

Beginning in 1911 the writer was in his employ for some years, seeing him not infrequently after leaving his employ, and thus writes from personal knowledge in noting that he concealed a kindly personality under a gruff exterior. Being of the most straightforward character, Mr. Lea found it hard to think ill of anyone, but woe to those who imposed upon him. The writer recalls with relish the occasion on which he skillfully and ruthlessly flayed a younger engineer who had been using Mr. Lea's advice, freely and cheerfully given, to impress one of his own clients.

Mr. Lea did not marry until his consulting practice was well established, which perhaps accounts for his love of travel. It was his custom to visit England and the Continent every three or four years, partly for pleasure, no doubt, but mainly to see what was going on in engineering circles. It was on one of these trips in 1914 that he was interned in Austria, disappearing completely for many months. He was ultimately released and made his way to Switzerland, crossing France and the Channel to England before finally returning to Canada. His experiences provided the material for many stories, all of which reflected credit upon Austria and its people.

He retired from active practice in 1931, after admitting his younger brother, William S. Lea,[5] M. ASCE, to partnership some years before. His death is regretted by his many professional acquaintances in Canada, the United States, and abroad. He was one of the few remaining "old time" engineers.

Mr. Lea was a member of the Institution of Civil Engineers; the Engineering Institute of Canada, from which he had received both the Gzowski and Julian C. Smith medals, and had served on the council; and the Association of Consulting Engineers in England. He also belonged to the University Club of Montreal and the Royal Montreal Golf Club.

---

[5] For memoir, see *Transactions*, ASCE, Vol. 113, 1948, p. 1493.

Brought up in the Methodist faith yet never a church member, after that church lost its identity in the United Church of Canada in 1925, he attended both Anglican and Presbyterian churches as opportunity offered and remembered both the "Auld Kirk" of Charlottetown and the United Church at Victoria generously in his will.

In 1914 Mr. Lea was married to Mrs. Wanda Schador. He is survived by his widow and two sisters, Mrs. Edward Boswell and Mrs. J. T. Windsor.

Mr. Lea was elected a Member of the American Society of Civil Engineers on July 1, 1908.

WILLIAM SCHURMAN LEA, M. ASCE[1]

DIED JULY 5, 1947

William Schurman Lea was born on May 10, 1877, at Victoria, Prince Edward Island, Canada. His parents were William Charles and Annie Lea. He was educated at Prince of Wales College at Charlottetown, Prince Edward Island, and at McGill University in Montreal, Que., Canada, from which he was graduated in 1908, a Medallist with the degree of Bachelor of Science.

His engineering experience, both before and after graduation, extended over a period of forty-five years. His advice was sought by municipalities in every province of Canada, as well as by provincial and dominion governments and industrial organizations, including many pulp and paper firms and power companies. He appeared as an expert before the International Joint Commission on questions of pollution of boundary waters, of water storage, and of runoff control. In large measure he was responsible for design and construction of large-scale sewerage and sewage treatment projects for Montreal, Winnipeg, Man., and Vancouver, B. C.; of water supply systems or extensions in these cities and in scores of others; and of hydroelectric developments in many parts of Canada.

Mr. Lea's early engineering experience was had with two leading sanitary engineers in Boston, Mass., the late Freeman C. Coffin, and the late Frank A. Barbour, M. ASCE. While working with Mr. Barbour, he was engineer in charge of construction for the Fredericton Filter Plant in New Brunswick, the first modern type filter plant built in Canada. He then operated this plant for six months. From 1908 to 1911 he lectured in municipal engineering and hydraulics at McGill University. In 1911–1912 he was engineer in charge of the Department of Water Works and Bridges for the City of Vancouver, and in 1913 he joined his brother, R. S. Lea,[2] M. ASCE, in the partnership of R. S. and W. S. Lea. Together they were consulted on many outstanding projects, among the more outstanding of which were the Vancouver and Districts Joint Sewerage and Drainage project; the Slave Falls Hydroelectric Plant for the City of Winnipeg; the Manitoba Power Company's 200,000 horsepower hydroelectric plant on the Winnipeg River; the Canadian Niagara Company's plant for the Ontario Government; the Hydroelectric Power Commission of Ontario's Queenston-Chippewa development and the proposed St. Lawrence Power and Navigation Development; the Nesbitt Thompson plant of about 100,000 horsepower on the Back River at Montreal; Price Brothers Kenogami Dam and later their paper mill dam; dams built by the Quebec Streams Commission; a dam for the Spanish River Pulp and Paper Company and several of their small storage dams; development of the aqueduct power for the City of Montreal, and many other large projects.

---

[1] Memoir prepared by J. Long, Secy., William S. Lea, Cons. Engr., Montreal, Que., Canada.

[2] For memoir, see *Transactions*, ASCE, Vol. 113, 1948, p. 1491.

The partnership continued until the retirement of R. S. Lea in 1930. Since that time, William S. Lea had been retained by municipalities in every province of Canada and by the following large corporations: The Laurentide Pulp and Paper, Canadian International Paper, Building Products, Brown Corporation, Spanish River Pulp and Paper, St. Maurice Paper, Lake of the Woods Milling, Aluminum Company of Canada, Limited, as well as the Hydroelectric Power Commission of Ontario, Quebec Streams Commission, Quebec Public Service Commission, and others.

Mr. Lea acted as technical expert for the City of Winnipeg and the Winnipeg Electric Railway Company on a number of occasions. For nine years he was a member of the Montreal Sewers Commission and consulting engineer on its twenty million-dollar trunk sewer construction program and at his death was retained by the City of Montreal on the design of the St. Pierre River Sewage Pumping Plant. He was consulting engineer to the City of Winnipeg for the design and construction of trunk sewers and a 25,000,000 gallons per day sewage treatment plant for the Greater Winnipeg Sanitary District—a four million dollar project. As a member of the commission appointed by the Quebec Government he investigated and reported on the contract between the City of Montreal and the Montreal Tramways Company and on the general question of transportation for the Greater Montreal District. He was also consultant to the cities of Windsor, Ont.; Brantford, Ont.; Saint John, N. B.; Charlottetown; Fredericton; and many others in connection with water, sewage, or electrical power problems.

Mr. Lea was a member of the American Concrete Institute committee appointed many years ago to draw up standard specifications for plain and reinforced concrete sewers. As a result of the findings of this committee, reinforced concrete came to be more widely used for large sewers and other similar structures.

As a student, Mr. Lea played on the McGill intercollegiate football team, and throughout his life maintained a very keen interest in sports. He was an active supporter of all Montreal and McGill University teams, and well known to the players, coaches, managers, and sports editors in that city.

In 1927, Mr. Lea was married to Mildred Florence Parkins who survives him.

He was a member of the Engineering Institute of Canada, the Corporation of Professional Engineers of Quebec, the Association of Professional Engineers of Manitoba, the American Water Works Association, the McGill Graduates Society, the University Club of Montreal, and the Senneville Golf Club.

Mr. Lea was elected a Member of the American Society of Civil Engineers on January 14, 1918.

GEORGE ALLEN LEE, M. ASCE[1]

DIED JULY 13, 1945

George Allen Lee was born in Dunedin, New Zealand, on July 1, 1881. From late pioneering days, his father, William Lee, had been prominently identified with the commercial and industrial development of Otago Province, especially in the mining of coal and scheelite and shale oil and in the dredging, sluicing, and reef mining of gold. George, his eldest son, was thus brought up in close contact with engineering problems.

At the age of eighteen, after having received part of his schooling in England, he commenced an apprenticeship with a Dunedin firm of mechanical engineers. This firm had been prominent in the design and construction of mining dredges (at that time a development pioneered in New Zealand and later perfected in California and elsewhere).

After a three-year apprenticeship, he became a cadet in the office of C. L. Watt, consulting mechanical engineer, and at the same time began night classs in the Otago School of Art and in the Otago University School of Mines at Dunedin.

His first responsible appointment was as mechanical engineer and surveyor with the Kaitangata Coal Mining Company. Later he was assistant to J. Blair Mason, engineer to the Otago Harbour Board, who had an extensive consulting practice with various colonial organizations. Subsequently, after some years as assistant engineer for the Otago Harbour Board and as engineer for the Bluff and Napier harbour boards, Mr. Lee, in 1918 entered partnership with Mr. Mason, and, during the following fourteen years, this firm became prominently identified with the harbor engineering of the Dominion, constructing various projects or reporting on possible developments in the majority of the ports. The most extensive engagements were on the remodeling of Otago Harbour and on the construction of the breakwater port of New Plymouth.

On the retirement of Mr. Mason, the partnership of Mandeno, Lee and Brown was formed. As a member of this firm, one of Mr. Lee's major responsibilities was the selection of a site and the design of the new breakwater harbor near the Town of Mackay, the port for the principal sugar-producing district of Queensland, Australia. Completed during World War II, this new harbor introduced profound changes in the transport system of Queensland. The breakwaters exceed two miles in combined length and are nearly seventy feet in total height at the entrance.

---

[1] Memoir prepared by Lloyd Mandeno, Cons. Engr., Mandeno, Lee & Brown, Auckland, New Zealand.

Mr. Lee came into the profession "the hard way" so that the natural skill of his hands became fully developed. It was ever his pleasure to utilize this dexterity, and he applied it in a wide variety of ways. For years he added to the admirable collection of insects commenced in boyhood, and he achieved high standards in the art and technical excellence of his photography. Although so proficient himself, it was his delight to have those less skilled gain experience in the mechanical and wood-working workshop at his home.

In 1911 he was married to the sweetheart of his boyhood, Fan McAdam, by whom, with his two daughters, Mary (Mrs. John S. Stronge) and Frances Betty (Mrs. James R. Steer), he is survived.

Mr. Lee was elected a Member of the American Society of Civil Engineers on December 31, 1913.

CARLOS LOBO, M. ASCE[1]

DIED SEPTEMBER 29, 1946

Carlos Lobo was born in Curacao, N. W. I., on September 22, 1873. He received his prepartory school education in Caracas, Venezuela.

Having decided to become a civil engineer, he came to New York, N. Y., in 1889 and in the fall of that year entered New York University (New York) as a member of the class of 1893. After completing the civil engineering course, which at that time was the only engineering course offered at the university, he received the degrees of Bachelor of Science (1893) and Civil Engineer (1894), after completing the required postgraduate work.

In October, 1894, Mr. Lobo joined the engineering staff of the Bureau of Water Supply, Department of City Works, Brooklyn, N. Y. He worked mainly on the distribution system, and in 1911 he was placed in full charge of the operation, maintenance, and repairs of that system for the then Borough of Brooklyn. This was part of the work of the Department of Water Supply, Gas and Electricity which was responsible for the entire water system of the City of New York. In October, 1914, Mr. Lobo was appointed borough engineer supervising all the water supply work for the Borough of Brooklyn under the Department of Water Supply, Gas and Electricity.

Resigning in September, 1919, he entered the Kelvin Engineering Company of which his brother, Gustavo Lobo, an electrical engineer, was president and treasurer. This company manufactured and exported machinery mainly for South American countries. In January, 1932, Mr. Lobo became vice-president and director of the company. He retired from active duty in January, 1936.

---

[1] Memoir prepared by William W. Brush, M. ASCE.

Mr. Lobo and the writer were classmates and fellow workers until he left the New York City Water Department in 1919.. He was a man who, as an engineer, citizen, and friend, lived up to his high ideals and was outstanding in his loyalty to the organizations and persons with whom he was connected. His kindness and generosity were also noteworthy.

Following his graduation from the university he was the perennial secretary of his class and one who could always be depended upon to be present at alumni meetings.

In June, 1904, Mr. Lobo was married to Edith Eder, who was born of American parents in Colombia and whose family had very extensive sugar interests in that country. He was a devoted husband and father and was constantly with his wife during the long period of illness prior to her death in 1942. He is survived by a daughter, Mrs. Clarita L. Collins, and a son, Walter E., who is a chemical engineer and a graduate of Massachusetts Institute of Technology in Cambridge.

Mr. Lobo was elected an Associate Member of the American Society of Civil Engineers on May 3, 1905, and a Member on March 14, 1916. He became a Life Member in January, 1940.

JAMES BUTTON CLYDE LOCKWOOD, M. ASCE[1]

DIED DECEMBER 2, 1945

James Button Clyde Lockwood was born on May 27, 1863, in Marshalltown, Iowa. He was the son of Edwin Wallace and Phoebe Anne (Putnam) Lockwood who were of Scotch and English descent.

Mr. Lockwood attended Iowa State Agricultural College at Ames, from which he was graduated in 1885 with the degree of Bachelor of Science in Civil Engineering. Immediately thereafter, he began his professional career as draftsman and estimator with the Chicago Forge and Bolt Company. Two years later he severed this connection and joined the Berlin Iron Bridge Company as a designer.

In 1889 Mr. Lockwood moved to Seattle, Wash., where as engineer and manager he opened a branch office of the San Francisco Bridge Company. During the following ten years he actively directed the operations of this branch, engaging in a wide variety of construction projects. They included the design and erection of a bridge across the Columbia River for the Spokane Falls and Northern Railway, the building of two steamers for the Boston and Alaska Transportation Company, and the execution of various dredging contracts throughout the Pacific Northwest. In 1898, without change of ownership or management, the operations of the company were transferred to the newly organized Puget Sound Dredging Company, with Mr. Lockwood serving as president and chief engineer. Shortly thereafter, this company purchased the Bowers hydraulic dredges *Anaconda* and *Python,* which were immediately redesigned and rebuilt. These two dredges were among the earliest hydraulic machines built in the United States. Under his supervision, much dredging and bridge building was undertaken and successfully completed.

---

[1] Memoir prepared by H. W. McCurdy, M. ASCE.

Mr. Lockwood moved temporarily to Washington, D. C., in 1900. There he designed and had charge of the construction of a hydraulic dredge for the Atlantic Gulf and Pacific Company, an affiliated organization. On completion of this work in 1902, he returned to Seattle, severed his connection with the Puget Sound Dredging Company, and entered a private consulting practice.

His unusual experience in the design and construction of dredges was becoming widely and favorably known, and he was soon engaged by the Port of Portland, Ore., to design and supervise the construction of two hydraulic dredges. They were the *Portland* and the *Columbia,* twenty-inch and thirty-inch dredges, respectively.

In 1903 Mr. Lockwood moved to Portland and opened an office as consulting engineer, specializing in the design and construction of dredges and bridges. The Port of Portland continued to make use of his services in the construction of its first floating dry dock, which was of ten thousand-ton capacity; the hydraulic filling of a lowland area in central East Portland; and the filling of terminal yards and wharf sites. In addition, he was the city's representative in connection with the Spokane, Portland and Seattle Railway bridge across the Columbia River.

He purchased a substantial interest in the Drummond Lighterage Company of Seattle in 1907, and later became its president. Simultaneously, he retained his interest and position in the company with his wide consulting practice until 1923, when he disposed of his ownership stock.

Mr. Lockwood entered the full-time employ of the Port of Portland as its chief executive and engineer in 1908. He continued in this capaciy until late in 1909, when he was retained as a consultant. In 1911 he designed and supervised the construction of the port's hydraulic dredge, *Willamette,* and, in 1916, its hydraulic dredge, *Tualatin,* both of which were thirty-inch dredges and among the largest and most powerful in the United States at that time. In 1913 he designed two large hydraulic dredges for the United States Engineer office in Portland, and, a year later, the hydraulic dredge, *Duwamish,* for the Duwamish Waterway Commission in Seattle.

Mr. Lockwood returned to Seattle in 1919 and continued his consulting practice. He undertook much work for the Puget Sound Bridge and Dredging Company, the successor to the Puget Sound Dredging Company of which he had been president. Among his many activities for this company was the design and supervision of construction of the hydraulic dredges, *Seattle* and *Olympia;* the electrification of the hydraulic dredge, *Tacoma;* the rebuilding of the dipper dredge, *Puget Sound;* and the design and supervision of construction of two dump scows, the hydraulic dredge, *T. B. Noble,* at Mabelle, Tex., and the hydraulic dredge, *Seattle,* at Garza, Tex.

The General Construction Company of Seattle engaged Mr. Lockwood in 1933 to rebuild and modernize its floating plant, including the redesign and rebuilding of the hydraulic dredges, *Washington* and *Missouri.* During this period he was simultaneously managing his many investments, which included extensive mining interests in Alaska.

He returned to the employ of the Puget Sound Bridge and Dredging Company in 1944 and designed a new hydraulic dredge to be built at the cessation of World War II. Throughout his life he had taken an interest in the operations of this company; for many years, for sentimental reasons, he kept his name on its payrolls at one dollar a year.

As a young man Mr. Lockwood was married to Mabel Dana. After her death and on September 5, 1906, he was married to Mary Reynolds who, with four children, survives him. The children are James Button Clyde, Jr.; William Putnam; Lucile (Mrs. Kenneth Chisholm); and Marie (Mrs. John R. Martin).

Mr. Lockwood was an energetic, capable, and friendly gentleman. He was a pioneer in the dredging industry, and during the sixty years of his business life he was a leader in his profession, probably designing more dredging plants than any other individual. His good health and vigor were retained until shortly before his death.

Mr. Lockwood was elected a Member of the American Society of Civil Engineers on March 6, 1901.

CHARLES HARVEY MACCULLOCH, M. ASCE[1]

DIED JUNE 24, 1947

Charles Harvey MacCulloch, the son of Charles and Emma (Harvey) MacCulloch, was born in Albany, N. Y., on February 23, 1877. He attended the public schools in Albany, graduating from the Albany High School; he then pursued an electrical engineering course at Union College, in Schenectady, N. Y., from which he was graduated in 1900 with the degree of Electrical Engineer.

Mr. MacCulloch began his engineering career in the employ of the state engineer and surveyor of New York State as a rodman in 1900. He then was engaged in surveying for the Erie Barge Canal and in highway work until 1905, at which time he became an instructor in the engineering department of Union College.

In 1907 he returned to the state engineer's office in charge of barge canal construction work and in 1909 was made responsible for the preparation of plans and specifications for electrical equipment in connection with the operation of canal locks, powerhouses, substations, and gasoline engine generating stations. He also designed electrical equipment for canal terminals and the state-owned grain elevators at Oswego, N. Y., and Gowanus Bay, New York City, together with electrical equipment for the 8,000-horsepower generating plants at Crescent and Vischer Ferry in New York.

---

[1] **Memoir prepared by H. O. Schermerhorn, M. ASCE.**

Ill health forced Mr. MacCulloch to relinquish temporarily his strenuous duties in 1926. After nearly a year's rest, he became associated with the firm of Whitman, Requardt and Smith in the preparation of plans for a new water supply system for the City of Albany.

He again returned to the employ of the New York State Department of Public Works in 1931, and from that time was engaged on all kinds of electrical design and estimate work connected with the department's activities. On January 31, 1947, Mr. MacCulloch retired after forty-one years of state service.

In 1916 Mr. MacCulloch was married to Angeline Emery. Besides his widow he is survived by a son, Charles MacCulloch, and a daughter, Elizabeth (Mrs. R. H. Mattox), all of whom are Albany residents.

He was a member of the Society of Professional Engineers, the Psi Upsilon fraternity, and the Calvary Methodist Church of Albany; for many years he was actively engaged in Boy Scout activities in the Albany area.

Mr. MacCulloch was beloved by all who had the good fortune to know him. He was devoted not only to his family and to his profession, but was loyal to all his friends and acquaintances as well. He enjoyed the respect of all who knew him, and his passing is mourned by a host of friends.

Mr. MacCulloch was elected a Member of the American Society of Civil Engineers on April 1, 1914.

GEORGE ALFRED MANEY, M. ASCE[1]

DIED MAY 10, 1947

George Alfred Maney was born in Minneapolis, Minn., on December 9, 1888. He was the son of Thomas A. and Ella (Hallum) Maney. His education was received in the public schools of Minneapolis, and at the University of Minnesota (Minneapolis), from which institution he was graduated with the degree of Civil Engineer in 1911.

In 1912 he was appointed to a research fellowship in theoretical and applied mechanics at the University of Illinois in Urbana, receiving the degree of Master of Science in 1914. As a student, Mr. Maney early displayed the marked originality of thought and deep interest in research which were to become such outstanding characteristics of his later professional career.

Mr. Maney began his long career in teaching as instructor in structural engineering at the University of Minnesota in 1914. He was promoted to assistant professor in 1921, and to associate professor in 1926. His work at the University of Minnesota was interrupted by World War I, during which, while on leave from the university, he performed outstanding services—first as special analyst from the Forest Products Laboratory investigating stresses in aeroplane propellers, and later as engineer in charge of the analyses and design of the frame sections of concrete ships for the Emergency Fleet Corporation, United States Shipping Board. In the summer of 1918 he enlisted in the United States Army rising to the rank of Second Lieutenant before his discharge at the end of the war.

---

[1] Memoir prepared by M. B. Lagaard, M. ASCE, with the cooperation of J. I. Parcel, L. T. Wyly, and Alvin S. Cutler, Members, ASCE.

In 1927 he accepted the appointment as professor of civil engineering at Northwestern University in Evanston, Ill., becoming administrative chairman of the School of Engineering in 1936. He contributed largely to the remarkable growth of the school, and, when the new technological institute was established in 1939, he became chairman of the Department of Civil Engineering, a position he held until his death.

Professor Maney's career in engineering practice was as interesting as his teaching career. In 1919 he designed a 10,000-ton floating and lifting dry dock for the United States Navy in competition with twelve other structural engineers. His design won the competition, but, because of the passing of the war emergency, the project never was built.

He served as special consultant to L. R. Whitson, architect, in 1920 on the design of the eighteen-story First Mortgage Building in El Paso, Tex. In 1923 he again acted as consultant to Mr. Whitson on the design and construction of the $4,000,000 Santa Fe Terminal Building in Dallas, Tex. At the time of its construction this twenty-one story structure was the tallest reinforced concrete building in the United States. Acting as consulting engineer to the Minneapolis Bridge Company, Professor Maney prepared the design for the $750,000 continuous truss highway bridge over the Mississippi River at Savanna, Ill., in 1931.

In 1939 he undertook the extraordinarily difficult and unusual assignment of designing an eight-story building to be placed on top of an existing building (which had already been increased from four to six stories), and which had to be constructed while the building underneath was occupied, and without making any use of the original foundation; Mr. Whitson was the architect. Professor Maney organized a group of engineers in 1933, with L. T. Wyly, M. ASCE, as principal associate, known as G. A. Maney and Associates. This group promoted and designed an $8,500,000 highway bridge over the Mississippi River connecting Missouri Avenue in East St. Louis (Ill.) with Franklin Avenue in St. Louis (Mo.). Construction of this project was begun after his death under the direction of this group as joint engineers and with C. P. Hazelet, M. ASCE, and Ingolf Erdal, as executive officials.

His contributions to research in the field of structural engineering were many and varied. His first publication[2] was an outstanding contribution and has become widely known in the profession as "Maney's Method." In 1915 Professor Maney made perhaps his greatest contribution to the theory of structures, and one which was to win for him international recognition when he presented to the profession the slope-deflection method.[3] Although the slope-deflection equation had been developed by O. Mohr in 1892 and applied extensively to the analysis of secondary stresses by European engineeers, no other use of the principle appears to have been made and it was practically unknown to American engineers.

---

[2] "Relations Between Deformation and Deflection in Reinforced Concrete Beams," by G. A. Maney, *Proceedings*, A.S.T.M., 1914.

The Maney presentation included: (1) A method of successive approximations for the secondary stress problem; (2) a rational solution of the wind stress problem of a building frame; and (3) a general solution for the rigid frame with loads applied between the joints. This presentation of the slope-deflection method was thus the first of the so-called "modern" methods of rigid frame analysis. It opened the door for the remarkable development which has since occurred in this field. It is worthy of note that Professor Maney developed the slope-deflection theory independently, without any knowledge of prior work, while a research fellow at the University of Illinois in 1913–1914.

Also, in 1915, he published, in collaboration with W. M. Wilson, M. ASCE, the first successful attempt to analyze rationally and in numerical terms a large building frame.[4] This treatment became a classic in the field. In 1922 Professor Maney, in collaboration with John I. Parcel, M. ASCE, published the results of an analytical and experimental study of the secondary stresses in a 512-foot riveted railway bridge.[5] This investigation was the first to reveal and explain the serious discrepancy which exists between actual secondary stresses occurring in practice in a large bridge and the computed stresses obtained by the conventional analysis.

In 1926 Professor Maney published in collaboration with John I. Parcel a textbook on indeterminate structures, which achieved a place as one of the standard treatises on the subject.[6] During the period from 1929 to 1931, Professor Maney published an important series of articles on shrinkage and plastic flow in concrete in the magazine *Engineering and Contracting.* These studies, which he continued for the next ten years, brought out the fact that shrinkage was predominant in producing compressive stresses in the reinforcing steel and in causing increased deflections in beams and slabs under working loads. The results were published in 1940[7] in a joint article with M. B. Lagaard, M. ASCE, and again in 1941.[8]

He prepared in 1932 (in collaboration with John E. Goldberg, Assoc. M. ASCE) a remarkable simplified approximate solution of building frames by the slope-deflection method.[9] This paper presented for the first time the one and one-half cycle procedure. In 1936 Profeson Maney received the Wason Medal from the American Concrete Institute for his paper on "The Analysis of the Multiple Rigid Frame Bridge by the Slope-Deflection Method."

---

[3] "Secondary Stresses and Other Problems in Rigid Frames: A New Method of Solution," by G. A. Maney, *Studies in Engineering No. 1,* Univ. of Minnesota, Minneapolis, Minn., 1915.

[4] "Wind Stresses in Steel Frames of Office Buildings," by G. A. Maney and W. M. Wilson, *Bulletin No. 80,* Univ. of Illinois Eng. Experiment Station, Urbana, Ill., 1915.

[5] "The Secondary Stresses in the Kenova Bridge," by G. A. Maney and J. I. Parcel. *Studies in Engineering No. 4,* Univ. of Minnesota, Minneapolis, Minn., 1922.

[6] "Elementary Treatise on Statically Indeterminate Stresses," by G. A. Maney and J. I. Parcel, John Wiley & Sons, Inc., New York, N. Y., 2d Ed., 1936.

[7] "Stress Increases in Compressive Steel Under Constant Load Caused by Shrinkage," by G. A. Maney and M. B. Lagaard, *Proceedings,* A.C.I., June, 1940.

[8] "Studies in Engineering, No. 1," by G. A. Maney and M. B. Lagaard, Northwestern Technological Inst., Evanston, Ill., 1941.

[9] "Simplified Methods for the Analysis of Multiple Joint Rigid Frames," by G. A. Maney and John E. Goldberg, *Northwestern University Bulletin,* Vol. XXXIII, No. 7, October 7, 1932.

During the World War II emergency Professor Maney undertook a comprehensive series of investigations for the War Production Board including: (*a*) A new type of pre-cast reinforced concrete arch for airplane hangers; (*b*) a reinforced concrete barge with a continuous metal seal for watertightness; (*c*) the use of wood fiber as a filler for cement; (*d*) pre-stressed spirally reinforced concrete columns (with M. B. Lagaard); and (*e*) an independent rail-supporting reinforced concrete railroad tie (with H. A. LaRoy). He also carried on (with L. T. Wyly) an extensive investigation of riveted joints for the National Advisory Committee for Aeronautics.[10]

The failure of the Tacoma Narrows Bridge in Washington aroused Professor Maney's interest in the problem of preventing the vibration of suspension bridges under wind loads. His solution of the problem was to introduce diagonal cable members under dead load tension; laboratory models which he built indicated that a ninety per cent reduction in the vibration was effected. The results of his study were presented in addresses to the Western Society of Engineers in 1941 and aroused very wide interest.[11] During the last two years of his life Professor Maney's principal research interest was concentrated on an investigation with L. T. Wyly of the fatigue of riveted and bolted joints.[12] He was one of the men who made up the original group which developed into the Rivet and Bolt Research Foundation.

In 1946 Professor Maney suffered a very serious illness from which he never entirely recovered. He died at the Evanston Hospital, in Evanston, on May 10, 1947.

He possessed one of the most brilliant and original minds in the profession and during his entire professional life he devoted himself to research and development in his technical field with a zeal and single-mindedness rarely equaled. His untimely death when at the peak of his career was a tragic loss to the profession. On the personal side, his wide range of interests, keen sense of humor, and capacity for good fellowship won him a large circle of friends.

Professor Maney was married on April 7, 1920, to Mabelle O. Draxton. His widow and two children, Thomas D. and Elizabeth K., survive him.

He was a member of Tau Beta Pi, Sigma Xi, the American Railway Engineering Association, the American Concrete Institute, the Western Society of Engineers, the American Society for Engineering Education, the American Association of University Professors, and the University Club of Evanston.

Professor Maney was elected a Member of the American Society of Civil Engineers on March 5, 1928.

---

[10] "Impact Properties at Different Temperatures of Flush-Riveted Joints for Aircraft Manufactured by Various Riveting Methods," by G. A. Maney and L. T. Wyly, *Advance Restricted Report AAR No. 5H28*, National Advisory Committee for Aeronautics, Washington, D. C., September, 1945.

[11] "Nature, Causes and Prevention of Resonant Vibrations in Suspension Bridges," *Journal*, Western Soc. of Engrs., March 25, 1941.

[12] "Fatigue Strength of Flush-Riveted Joints for Aircraft Manufactured by Various Riveting Methods," by G. A. Maney and L. T. Wyly, *Advance Restricted Report AAR No. 5H28*, National Advisory Committee for Aeronautics, Washington, D. C., December, 1945.

WAYNE DICKSON MAXWELL, M. ASCE[1]

DIED FEBRUARY 2, 1947

Wayne Dickson Maxwell was born in Prescott, Iowa, on August 18, 1881. He was the son of Benjamin N. and Lucy (Simmons) Maxwell.

He attended the public schools at Prescott, and Des Moines (Iowa) and was graduated from East High School in Des Moines, in 1899. Mr. Maxwell received the degree of Bachelor of Civil Engineering in 1906, from Iowa State College in Ames. He earned the professional degree of Civil Engineer from the same school in 1911, after five years of engineering work; during the last three years of this period, he was office engineer in the engineering department of the City of Des Moines, where he remained until 1913. His outstanding piece of work was designing the Walnut Street Bridge over the Des Moines River, and the supervision of its construction. This bridge, which bears his name on a bronze plate, is one of the main downtown bridges, and it is in constant use.

During the period from 1913 to 1925, Mr. Maxwell was county engineer of Sac County, Iowa, and was in direct charge of construction and maintenance on bridges, drainage structures, and one hundred and fifty miles of county and primary road. From 1920 to 1925, he managed much of the road administration of Sac County; and in the capacity of county drainage engineer, he was active in reclaiming marshlands for agricultural purposes, an activity of high priority during World War I.

Following his administration as county engineer of Sac County, he was made secretary of the Associated Pennsylvania Constructors (State Constructors' Association, State Branch of the Associated General Contractors of America). This position was most interesting to Mr. Maxwell since in his office management he continually handled road contractors' problems. It was a subject with which he was well qualified to deal because of his broad experience and technical training in such matters. He held this position until December, 1933.

Mr. Maxwell moved to Greeley, Colo., in December, 1933. He was connected with many projects in the Rocky Mountain region, on which he served as superintendent of construction for structural steel projects, concrete foundations, and the sewage treatment plant in Windsor, Colo. He was representative for Greeley, a city of eighteen thousand persons, on construction of the sewage treatment plant there.

In the early years of World War II, he engaged in defense work; first, as utilities engineer on the new unit constructed at Fort Francis E. Warren in Cheyenne, Wyo., and later in the same capacity at the Denver (Colo.) ordnance plant in Denver. During his residence in Greeley, Mr. Maxwell was also engaged in consulting and general engineering work, and he was always keenly interested in civic affairs, doing much to further sound engineering development.

---

[1] Memoir prepared by Carl A. Gould, Regional Engr., U. S. Forest Service, Denver, Colo.

His hobby was reading literature of varied subject matter—fiction, history, travel, and technical studies. He was a member of the United Presbyterian Church, and attended services regularly as long as his health permitted.

In 1910, Mr. Maxwell was married to Elizabeth Taylor in Chicago, Ill. He is survived by his widow and a daughter, Jean Elinor.

Mr. Maxwell was elected an Associate Member of the American Society of Civil Engineers on November 12, 1913, and a Member on December 4, 1922.

THOMAS MERO McCLURE, M. ASCE[1]

Died November 5, 1946

Thomas Mero McClure, the son of John T. and Clara (Webster) McClure, was born on July 21, 1895, at Beaver City, Nebr. The family moved to Roswell, N. Mex., in 1908, where they established a permanent home. His father was a leading lawyer of that city, holding the office of district judge for many years.

Mr. McClure completed his preliminary education in Roswell and attended the New Mexico Military Institute there; he was graduated in 1914. He entered Iowa State College in Ames as an engineering student and later transferred to the University of Kansas at Lawrence. His education was interrupted in 1917 when he entered the army as a Second Lieutenant and served with the 91st and 89th divisions; he spent a year in France with the 342d Machine Gun Battalion, engaging in the Meuse-Argonne and St. Mihiel offensives.

After his discharge, he went to work for the Saint Louis and San Francisco Railroad Company in Texas as an instrumentman and chief of party. In 1922 he was appointed project engineer for the State Highway Department of New Mexico. Returning to Texas in 1926, he worked for five years with the Missouri Pacific Railroad Company as assistant division engineer. From 1931 to 1933 he again worked with the New Mexico highway department as location engineer and assistant district engineer.

In 1925 the State of New Mexico entered into a cooperative agreement with the United States Geological Survey to make a comprehensive investigation of the Roswell artesian basin, whose water supply for irrigation uses was seriously threatened by overdeveloped and extensive losses from the uncontrolled flows of leaking, abandoned wells. During 1925 and 1926 Mr. McClure participated in the basin study and it was during this time that he first became actively interested in the conservation and utilization of ground waters.

---

[1] Memoir prepared by a Committee of the New Mexico Section consisting of John H. Bliss, M. ASCE, Chairman, Berkeley Johnson, M. ASCE, and Rufus H. Carter, Jr., Assoc. M. ASCE.

On September 7, 1933, Mr. McClure was appointed state engineer of New Mexico and served continuously in that position until his death. As state engineer he had jurisdiction of the public waters of the state; in this capacity he proved himself a thorough and capable administrator. He spent considerable time in the administration of the state underground water law which became effective in 1931. His pioneer work in the administration of this law and the promulgation of rules and regulations for the equitable application of water right principles in this relatively new and untried field stands as a monument to his keen insight and thorough understanding of the problems involved. The New Mexico underground water code has become a standard which several western states have consulted liberally in drafting their ground-water laws.

In 1936 he served as president of the Association of Western State Engineers. Becoming an active member of the association in 1933, Mr. McClure worked consistently toward a better understanding between states and a coordinated approach to the solution of their mutual water problems. In 1934 and 1935 he served as vice-president for New Mexico in the National Rivers and Harbors Congress. He was secretary of the State Planning Board and a member of the State Flood Commission as well as the National Reclamation Association. Mr. McClure was the first secretary of the State Board of Registration for Professional Engineers and Land Surveyors, a position he held at his death.

The legislature created in 1935 the Interstate Stream Commission of which Mr. McClure, as state engineer, was ex officio secretary. The commission was formed for the purpose of negotiating interstate stream compacts, matching congressional appropriations for cooperative investigations, instituting legal proceedings in interstate stream controversies, and doing everything necessary in the conservation and development of these waters. In connection with his duties as secretary of the commission Mr. McClure took an active part in the supreme court suit brought by the State of Texas against the State of New Mexico over the use of the waters of the Rio Grande within the Middle Rio Grande Conservancy District. This suit was not settled in the courts, but led directly to negotiations between the states of Colorado, New Mexico, and Texas, looking toward an equitable allocation of the waters of the Rio Grande among the three states. Mr. McClure served as New Mexico's commissioner on these negotiations which culminated in the drafting and final adoption of the Rio Grande Compact on May 31, 1939.

He served as commissioner on the Costilla Creek Compact negotiations, a compact agreement being approved by the states of Colorado and New Mexico, which was signed in 1946 by the Hon. Harry S. Truman, President of the United States. Mr. McClure was also commissioner for New Mexico on the Pecos River negotiations, which were interrupted by his untimely death. As secretary of the Interstate Stream Commission he was intimately connected with the Colorado River problems and assisted in the negotiations leading to the Boulder Canyon Adjustment Act, as well as the drafting of the treaty with Mexico. He was appointed by the Hon. John J. Dempsey, Governor of New Mexico, in August, 1946, to represent the state in discussions looking toward an equitable allocation of the waters of the Colorado River Basin above Lee Ferry (Ariz.) among the upper basin states.

During the last days of October, 1946, Mr. McClure was attending a four-day series of basin meetings of the Upper Colorado River Compact Commission. The party had driven to Farmington, N. Mex., the afternoon of November 1, where the last of the meetings was to be held the following day. Mr. McClure suffered a heart attack that evening and succumbed the morning of November 5.

As a past-president and prominent member of the New Mexico Section of the Society, Mr. McClure was being sponsored as a candidate for Director of the national organization at his death. He was one of the leading figures in the local and state organizations of the American Legion and the Forty and Eight, holding a number of posts in those organizations: Post Commander of the Montoya y Montoya, Post No. 1, in 1942; Grande Chef de Gare of the state organization of the Forty and Eight (1944–1945); and a member of the National Executive Committee in 1946. He was a member of the Theta Xi fraternity. In addition to these activities Mr. McClure found time to make stamps his hobby, specializing in United States issues; a stranger needed no better introduction than an expressed interest in philately.

Mr. McClure possessed a high degree of ability as an engineer and was a loyal, faithful advocate of protecting and conserving the rights of New Mexico in its waters. He possessed the character, courage, integrity, and tenacity of purpose which inspired respect and confidence in all with whom he came in contact. He has long been regarded as one of the outstanding engineers and reclamation authorities in the west. He has been truly a faithful servant of the people of New Mexico.

The following editorial, appearing in the November 6, 1946, edition of the *Santa Fe New Mexican,* aptly describes the feelings he inspired among his co-workers:

> "With the death of Thomas M. McClure, New Mexico has lost an able public servant and many a person has lost a valued friend. Tom McClure was a man of courage, integrity and intelligence. He would never hedge on a question; everyone always knew precisely where he stood and why. He was a respected antagonist in an argument, but never let a mere argument interfere with a friendship.
>
> "His State Engineer's office was remarkably free from politics, because Tom McClure placed the welfare of the state before that of his party or himself. Even the knowledge of his precarious health for several years could not diminish his energy and his interest in his work. He died with his boots on, doing the work he loved and knew best. There are never many men in the world like Tom McClure. He will be missed."

Mr. McClure was married on May 22, 1926, in Santa Fe, to Margaret L. Murphy of Waterloo, Iowa. He is survived by his widow, a daughter, Margaret Jean, and a sister, Gretchen, of Long Beach, Calif.

Mr. McClure was elected an Associate Member of the American Society of Civil Engineers on May 13, 1935, and a Member on November 14, 1938.

ROBERT LEE McCORMICK, M. ASCE[1]

Died April 10, 1947

Robert Lee McCormick, the son of Leander C. and Katherine (Gwynne) McCormick, was born at Charlestown, Clark County, Ind., on January 12, 1867. His boyhood was spent on a small Indiana farm. After he was graduated from the local high school, he entered Franklin College at Franklin, Ind., where his interest in mathematics was awakened.

Before completing work for a degree at Franklin, he transferred first to Indiana University at Bloomington, and then to Rose Polytechnic Institute in Terre Haute, Ind., to study civil engineering. At Rose Mr. McCormick came under the influence of Malverd A. Howe,[2] M. ASCE, and developed a major interest in structural engineering. He was graduated in June, 1891, with the highest academic record in his class, for which he received the Hemingway Medal.

On being graduated, Mr. McCormick accepted an appointment to the staff at Rose as instructor in mathematics and civil engineering. He remained at Rose for forty-six years, being, successively, instructor and assistant professor of mathematics and civil engineering and, finally, professor of civil engineering. He retired in 1937 at the age of seventy. During his long term at Rose, he contributed to the development of the character of many outstanding engineers and left a lasting impression on those who passed through his classes as students. He was popular with the Rose alumni as well; this was evidenced by the fact that they elected him president of the alumni association for several terms.

Professor McCormick engaged in numerous professional undertakings in addition to his teaching. At the turn of this century he assisted Professor Howe in the design and construction of the Wabash Avenue Bridge across the Wabash River at Terre Haute. For many years, too, he engaged in mine surveying for the numerous coal mine operations in the vicinity of Terre Haute. After his retirement, he moved to Brazil, Ind., and served as county surveyor for several terms. He had refused to run again for this office shortly before his death.

He was a member of several professional and scientific societies; among these were the American Railway Engineering Association, the American Association of University Professors, the American Society for Engineering Education, the American Association of Professional Engineers, the Indiana Engineering Society, and the Tau Beta Pi fraternity. Also, he was a member of the Student Chapter Committee of the Society in 1933 and a member of the first Indiana Board of Registration for Professional Engineers and Land Surveyors in 1921, serving as its first chairman. Professor McCormick was a member of the Terre Haute Lodge No. 86 of the Benevolent and Protective Order of Elks until his retirement from Rose in 1937, after which he transferred his membership to the Brazil lodge.

---

[1] Memoir prepared by R. E. Hutchins, M. ASCE.

[2] For memoir, see *Transactions*, ASCE, Vol. 106, 1941, p. 1599.

He was married to Floy Maguire of Kansas on December 24, 1894. His widow and two sons, Bruce R. and Francis, survive him.

Professor McCormick was elected a Member of the American Society of Civil Engineers on December 14, 1925.

WILLIAM NAYLOR McDONALD, M. ASCE[1]

Died January 6, 1947

William Naylor McDonald was born at Cool Springs, Clarke County, Va., on May 27, 1868. He was a nephew of the late Hunter McDonald, Past-President and Hon. M. ASCE, who was for many years chief engineer of the Nashville, Chattanooga and St. Louis Railway in Nashville, Tenn. As a young man, William McDonald had considerable experience in railroad work in and near Nashville.

His first experience, when he was in his "teens," was office work in Louisville, Ky., in 1887. For a year, in 1888, he was with the maintenance of way department of the Louisville and Nashville Railroad Company at Hopkinsville, Ky. Feeling the need of further training, he spent a year as a student at the University of Virginia in Charlottesville, after which he was reemployed by the railroad company as a draftsman and inspector of masonry. For a few months in 1890 he was assistant engineer on bridge masonry for the Pine Mountain Iron and Coal Company in Pineville, Ky. Beginning in May, 1890, and for eight years, he served on the engineering staff of the Nashville, Chattanooga and St. Louis Railway on surveying and construction work which included depots and yards with shop buildings. Much of the work was in or near the Clay Street yard in Nashville.

Responding to the patriotic call for service during the Spanish-American War, he joined the Second Regiment, Louisiana Volunteer Infantry, in June, 1898. Within five months he was promoted to Second Lieutenant, in which capacity he served until he was mustered out in April, 1899. After an introduction to Cuba, he spent the next few years in the government engineering service on the island. Until the end of 1902 he was assistant engineer and superintendent in the Department of Streets at Havana. This government service placed him in charge of all street work, including some surveys and the establishment of street grades. He supervised the building of the celebrated Malecon sea wall and drive. Under his plans and specifications about $7,000,000 worth of work was laid out—$4,000,000 worth of which was finished. The new paving system was started after the United States government relinquished control.

---

[1] **Memoir prepared by Robert M. Angas, M. ASCE.**

Mr. McDonald then returned to Nashville and in 1903 and 1904 served as chief engineer of the Tennessee Central Railroad Company. Feeling the urge to undertake construction work as a business, however, he left Nashville, and spent the remainder of his life in Florida. For twenty-one years, from 1905 to 1925, inclusive, he was continuously in the contracting business, except for a period of service in World War I. Prior to the war this work consisted largely of erection of concrete and reinforced concrete buildings, bridges, and dams. His war service comprised the latter half of 1918, following receipt of a commission as Captain of Engineers in the United States Army. Continuing his contracting work during the years from 1919 to 1925, he was engaged largely on construction of concrete highways and bridges at a total cost of about $2,000,000.

The development of real estate in Florida then attracted Mr. McDonald's attention. During 1926 and 1927, he was chief engineer for the D. P. Davis Properties—including Davis Shores in St. Augustine, Fla. The work consisted of design and construction at an estimated cost of $3,000,000. Then for two years, 1928–1929, he was city manager of St. Augustine. In 1930–1931 he served as engineer in charge of drainage work for J. L. Ketterlinus. His next engagement was at the close of 1933 and during most of 1934 when he was engineer in charge of the design and construction of the sea wall at Jacksonville Beach, Fla. This was a Works Progress Administration project, costing approximately $130,000, and he designed and constructed a reinforced concrete sheet pile bulkhead to protect the waterfront.

With the approach of ill health, Mr. McDonald went into virtual retirement. He purchased a small farm near the village of Hogan, Fla., where he devoted his time to raising Satsuma oranges and gladiolus bulbs.

During his residence in Nashville, Mr. McDonald was married to Emma Hicks on December 10, 1900. They had one daughter, Catherine, who was born in October, 1901; she died on December 6, 1927. Mrs. McDonald died in 1946 and from that time until his death Mr. McDonald lived with his sister, Mrs. Nannie Eustaphieve, in Jacksonville. He is survived by Mrs. Eustaphieve and three other sisters—Mrs. Bayard Stevens and Catherine Powell McDonald, both of Washington, D. C., and Mrs. W. E. Suddath, of Warrenburg, Mo.; also, by a brother, Hugh M. McDonald, of Gloversville, N. Y.

Besides the Society, Mr. McDonald was a member of the Society of American Military Engineers and the Florida Engineering Society. He also was a member of St. Marks Episcopal Church in Jacksonville.

Mr. McDonald was elected a Member of the American Society of Civil Engineers on September 3, 1902.

BENJAMIN McKEEN, M. ASCE[1]

DIED DECEMBER 16, 1947

Benjamin McKeen was born at Terre Haute, Ind., on Jauary 29, 1864. He was the son of William Riley and Ann (Crawford) McKeen.

His father was a prominent financier and railroad executive, who was head of the McKeen National Bank and directed the affairs of the Vandalia Railroad Company before and after it became a part of the Pennsylvania Railroad System.

Mr. McKeen enrolled at Worcester Polytechnic Institute (Worcester, Mass.) and when Rose Polytechnic Institute (Terre Haute) opened its doors, he entered as a junior and completed his technical education; he was graduated with a degree in mechanical engineering in 1885, with the first class.

In September, 1885, he entered railroad service as draftsman in the office of the superintendent of motive power and machinery of the Terre Haute and Indianapolis Railroad Company (later included in the Pennsylvania System). He later served as rodman in the engineering department and in April, 1886, was appointed resident engineer on the work of cutting down the Rockville grade on the Logansport (Ind.) division of that railroad. On January 1, 1887, he was appointed engineer, maintenance of way, of the Logansport division, and in 1889 was chief engineer on completion of the Indiana and Lake Michigan Railroad Company lines, in addition to being engineer, maintenance of way, of the Logansport division.

On August 1, 1892, Mr. McKeen was transferred to the Peoria (Ill.) division as engineer, maintenance of way, and, on January 5, 1894, he was advanced to the position of superintendent of that division.

Mr. McKeen was appointed superintendent of the main line division of the Vandalia Railroad on June 10, 1901, and, on April 1, 1902, he became superintendent of the Chicago (Ill.) terminal division of the Pennsylvania Lines West of Pittsburgh (Pa.). This transferred his residence to Chicago. He was further advanced on December 15, 1903, to the position of general manager of the Vandalia line with an office at St. Louis, Mo. Ten years later he was appointed general manager of the Pennsylvania Lines West of Pittsburgh, and was vice-president of this railroad from 1917 to 1920 and vice-president of the southwestern region from 1920 to June 1, 1925, when he was made a vice-president of the Pennsylvania Railroad System. He retired as vice-president on February 1, 1934, on reaching the age of seventy.

All railroad officers and employees who served with Mr. McKeen held the highest respect and esteem for him. He was a man who, even under the most trying conditions, always maintained his equanimity, and was an inspiration to all the men who served under him.

Mr. McKeen was the first president of the alumni association of Rose Polytechnic Institute. He was a director of the Mississippi Valley Trust

---

[1] Memoir prepared by W. S. Hanley, Assoc. M. ASCE.

Company and was a member of the St. Louis Country Club and the Noonday Club. He belonged to the Episcopal Church, and in politics was a Republican.

On October 20, 1891, he was married to Anna Strong, daughter of Joseph Strong of Terre Haute. Surviving are an only daughter, Mary Josephine (Mrs. Neal Creighton) of La Jolla, Calif.; a grandson, Thomas Niedringhaus of St. Louis, who made his home with Mr. McKeen; and a granddaughter, Mrs. Mary Kaiser, and a nephew, William R. McKeen—both of Terre Haute.

Mr. McKeen was elected a Member of the American Society of Civil Engineers on November 6, 1895. He became a Life Member in January, 1930.

JOHN THOMAS LAMAR McNEW, M. ASCE[1]

Died December 21, 1946

John Thomas Lamar McNew, known affectionately as "Mac" by his many close friends and respectfully as "Mr. Mac" by all his students, was called to his final reward as he neared the peak of his profession as an educator. When stricken with his last illness, he was filling the newly created office of Vice-President of Engineering at the Agricultural and Mechanical College of Texas at College Station.

He was born in Belcherville, Tex., on January 20, 1895. He was the only son and the older of two children of Edgar O. and Sarah E. (Taylor) McNew. John Thomas Lamar and his sister, Vera, each elected to become teachers, a profession in which both parents served.

The McNew ancestors were Scots, descendants of that part of the ancient clan of McNew which had migrated from Scotland to Northern Ireland several generations before the colonial McNews settled in the United States in Washington County, Virginia. The McNew family was well established in Virginia by 1879 and was called "Scotch-Irish" by the other residents of the area, a term used to indicate that its members were Scots from Ireland instead of people of mixed Scotch and Irish heritage.

Edgar O. McNew had a long career as an educator in Texas. His professional life included many years as a classroom teacher in several public school systems in the state and twenty-three years as superintendent of the Independent School District of Mineral Wells, Tex. He loved children and was in turn beloved by all his pupils. He taught for the love of teaching and appreciated his responsibility in guiding young lives. Young Lamar, as John Thomas Lamar was called in his youth, was his father's most ardent admirer and he adopted as his own the wise and canny philosophy of life and rules of personal conduct so ably taught and exemplified by his dad.

Lamar received his high school education and training in Mineral Wells. He was graduated from high school in 1912 and in the fall of that year entered the University of Texas at Austin where he enrolled for the civil engineering course. He spent two school years as a student; then he decided that he wanted to work. During the fourteen months spent in the service of the Wells Fargo Express Company as an express trainman his desire to become a civil engineer crystallized and he acquired the ambition to attend Texas A. and M. With this in mind he announced his wish to become a student again and his father told him that he might return to college if he could save fifty dollars in the very short time remaining before the opening of the fall semester. Lamar's determination plus his native Scotch thrift and business acumen in a quick sale of some of his wearing apparel combined to produce the stipulated amount, so he matriculated in the class of 1918 at A. and M.

---

[1] Memoir prepared by O. H. Koch and J. H. Bringhurst, Members, ASCE.

Mr. McNew spent the two years prior to the entry of the United States into World War I in school and worked at his profession during the summer months. His education was interrupted when he was drafted into the Army in October, 1917. However, he returned to A. and M. after his discharge in 1919 and received the degree of Bachelor of Science in Civil Engineering in 1920, and that of Master of Science in Civil Engineering in 1926. The Iowa State College of Agriculture and Mechanical Arts at Ames conferred upon him the degree of Civil Engineer in 1925.

One of his professors on the civil engineering faculty at A. and M. wrote the following description of him, as a college student:

> "I had Lamar as a student in practically every class I taught. I learned to hold him in very high esteem for his sincerity, his enthusiasm as a student, depth of thinking, high personal and professional idealism, deep sense of fairness, inherent friendliness, intellectual curiosity and well balanced sense of humor.
>
> "Mac was a quiet young man who evidently believed in thinking through a proposition before explaining his opinion.
>
> "In short, J. T. L. McNew was the kind of student that a teacher wishes he might have in every class. He was always responsive to intellectual stimulation, cheerful in attitude, vigorous in effort, analytical in thought, kind in sprit and gentlemanly in conduct."

Mr. McNew's first actual contact with engineering work in the field occurred during the summer vacation of 1917, at which time he accepted a position with the Strawn Coal Mining Company of Strawn, Tex., as engineer in charge of surveying, mapping, and layout for two mines.

When he was inducted into the United States Army in the fall of 1917, he spent seven months at Camp Travis in San Antonio, Tex., before he was admitted to the Fourth Officer Candidates' Training School at Camp Lee in Petersburg, Va. He was commissioned a Second Lieutenant on May 25, 1918, and was assigned to Company A, 113th Engineers at Camp Shelby in Hattiesburg, Miss. In September, 1918, Lieutenant McNew was sent overseas and was placed in charge of road construction and maintenance in France until the armistice. After hostilities ceased, he was charged with the operation and maintenance of 540 miles of highway in his division area in the Army of Occupation in Germany.

On his honorable discharge from the armed service in July, 1919, he accepted a position as foreman on construction of concrete road paving and structures. In September, 1919, he resigned this position in order to return to college for his last year's work toward his undergraduate degree.

During the Christmas vacation of his junior year at Texas A. and M. Mr. McNew met and fell in love with Edna E. Murphy who was teaching at Graford, Tex., a town near Mineral Wells. This romance had developed to the engagement stage, duly sealed with the solitaire, by the time he resumed his civilian status in 1919. On May 27, 1920, the day after commencement, the two were married and went to Beaumont, Tex., where Mr. McNew had accepted his first assignment as a graduate engineer, a position with the City of Beaumont as assistant city engineer on paving work.

In the fall of 1920 Mr. McNew joined the staff of the civil engineering department of his alma mater, beginning his career as an educator with the title of instructor. On December 1, 1921, he was appointed assistant professor in civil engineering; on September 1, 1924, he was made associate professor in civil engineering; on September 1, 1925, he was named professor of highway engineering; and on June 1, 1940, Professor McNew, after four months as acting head, became head of the civil engineering department. He held this title until February 1, 1945, when he was appointed to the newly created position and title of Vice-President of Engineering. On July 12, 1945, he was given, in addition to his other duties, the work as director of Engineering Extension Service.

During his period of service on the faculty at A. and M., Professor McNew had many opportunities for intimate contact with his students. His humane and sincere personality instilled genuine confidence in his students and attracted them to him for consultation and advice. "Mr. Mac" was the students' real friend who sought and found many ways to help those who were helping themselves secure their education. He gave them jobs in his testing laboratory, in his professional practices, and in his home. One of the very needy boys, who used to earn money taking care of the two McNew children while the parents were out in the evenings, avowed that the stipend paid for his services was negligible compared to the cost of replenishing the larder after he and "Mr. Mac" had finished their customary raid, and that "Mr. Mac" insisted that the position of "sitter" carried with it unlimited access to and enjoyment of anything eatable that could be found.

Professor McNew was primarily interested in highway construction and was a recognized authority on this subject. He realized the importance of the teacher having personal contact with the actual work in the field. He used most of his summer vacations in getting actual experience with the various phases of highway engineering construction. He spent one summer as assistant city engineer for the City of Beaumont, another summer with the county engineer of Dallas County, Texas, and other vacation periods with contractors on highway bridge and highway pavement projects throughout the state. He was district engineer for Works Project Administration

(W.P.A) of the Dallas District and state field engineer for the W.P.A. at San Antonio during parts of 1935 and 1936.

His record in Society activities reveals a genuine interest and active participation from the very beginning. In less than three years after he became a corporate member of the Society he was elected Secretary of the Texas Section. He served in this post with great credit for a period of ten years. During this time he was the spark plug of the Texas Section. His interest and aggressiveness is given credit for the unusual growth and progress of the section during that period. One of the important institutions of the Texas Section is the regular publication, *The Texas Engineer,* which was developed and edited by Secretary McNew. It became, under his regime, a real house organ with genuine personality typical of the Texas Section. His tremendous interest in the affairs of the section and his efficient method of looking after the details and arrangements of the semiannual meetings were in a great measure responsible for the continued interest of the members in the Texas Section. In recognition of his excellent services and because he had the proven qualifications of leadership, he was elected President of the Texas Section in 1938. In 1941 he was chosen to represent District No. 15 as the National Director and then was returned to the National Board in 1946 as Vice-President of Zone No. 4.

He served as a member of the Society's committees on Juniors and Local Sections and, while Vice-President, as Chairman of the Committee on Engineering Education. This last committee undertook an aggressive program working toward a plan of sponsoring the young engineering student from the beginning of his college work on through his entry into the profession. The enthusiasm, the possibilities, and the objectives of the committee's program are a reflection of his keen, sympathetic interest in the student engineers. He wrote two technical bulletins [2, 3] and miscellaneous discussions in *Proceedings* and *Transactions.*

In civic affairs Professor McNew was quite active in the work of building his home city and community. He took the lead in getting the City of College Station incorporated; he was a member of the City Planning Committee, and served for three years as city engineer. In negotiating with the city commission for this professional assignment, he suggested and received an annual fee of fifty cents claiming that his code of professional practice demanded that he receive compensation.

When the United States entered World War II officers were badly needed and the cadet corps at A. and M. was called upon to supply officer material. Many of the faculty also entered the armed forces, on leave of absence from the college. Professor McNew, although quite busy in the essential work of conducting the war training program with a reduced faculty, knew he had experience and ability in paving and roadwork that was demanded at the

---

[2] "A Study of the Fluxing of the Bitumen Contained in Texas Limestone Rock Asphalt," by J. T. L. McNew, *Technical Bulletin No. 31,* Eng. Experiment Station, Agri. and Mech. College of Texas, College Station, Tex., 1926.

[3] "Proceedings of the Fourth Annual Short Course in Highway Engineering," by J. T. L. McNew, *Technical Bulletin No. 36,* Eng. Experiment Station, Agri. and Mech. College of Texas, College Station, Tex., 1928.

front, so he presented himself to the Officer Procurement Division of the U. S. Army as a candidate for a commissioned overseas assignment. He was commissioned a Major in the Corps of Engineers in the summer of 1943 and sent first to Ft. Belvoir, Virginia, and then to Orlando, Fla., for training before going overseas.

Major McNew was forty-eight years old at this time. He had served in one world war and to him there was no glamour in the wearing of a uniform and brass buttons; no spirit of adventure called him from an honored and respected position as head of the civil engineering department and from a happy home life with a devoted wife and two fine children to seek the hardships and rigors of warfare on foreign soil. His was the deepest sense of patriotism that sent him, smiling, by plane from Houston, Tex., to the China-Burma-India (CBI) front on October 25, 1943. His wife, no less valiant a soldier than he, waved him farewell in the same assumed spirit of cheerfulness, knowing full well that long lonely months, or years, or maybe eternity, might elapse before they should meet again.

How well he served his country may be understood from the words of a fellow officer and former student:

> "I had served with Mr. Mac at Belvoir, Richmond, and again at Orlando, Florida. On my way overseas I was sitting on a bench outside the ATC operations office at Accra on the Gold Coast of Africa, when a plane landed and Mr. Mac was the first man off the plane. On arriving in India, we found ourselves together again and I was in close contact with him until his return home. General Stuart Godfrey [4 M. ASCE] (who has since been killed) leaned on the Colonel for nearly every decision that he made. Colonel McNew had accomplished wonders before his return to College."

On February 1, 1945, Professor McNew returned to A. and M. from active army duty with the rank of Lieutenant Colonel which commission he had won in the CBI area planning and supervising roads and air bases, among which were some of the first B-29 bases in China and Burma. He was immediately given the duties and title of Vice-President of Engineering, a position just created.

The teaching of good citizenship and the work of character building among his students were emphasized on every appropriate occasion, as is shown in his summary of a few of the responsibilities of an educated citizen given in Vice-President McNew's commencement address to the graduating class at A. and M. on May 25, 1945. In this category he included:

> "1. A continuing acquaintance with the current events of government.
> "2. An obligation to express a considered opinion on matters of public concern.
> "3. An obligation to vote at elections—National, State, County, City, and School.
> "4. An obligation to serve the public with your talents in an ethical way.
> "5 An obligation to make your technological skill contribute to the better living of your fellow man.

---

[4] For memoir, see *Transactions*, ASCE, Vol. 112, 1947, p. 1449.

> "6. An obligation to be an example as an all around honest, straightforward and useful part of your community and to take from it less than you contribute."

Professor McNew was a great lover of the outdoors, and he enjoyed and participated in most of the outdoor sports. He earned the coveted athletic "T" on the track team at A. and M. and in 1917 broke the state record in javelin throwing. He was also an ardent golfer, hunter, and fisherman. One of his most prized trophies was the mounted head of a marvelous specimen of white tail buck which he killed in the brush country of south Texas while hunting on the ranch of one of his college students. This buck was so large that he had to send his student host back to the ranch house for a team while he built a fire and guarded his kill for many hours.

"Mac" was a keen observer of nature and possessed a marked literary ability in describing the scenes and people encountered in his travels. The descriptions in his letters from the CBI theater of the customs and modes of living of the natives and their construction methods and equipment show not only an unusual capacity for the understanding of their problems, crude analyses, and solutions, but also a mastery of the art of writing seldom equaled by an engineer and surpassed only by the great authors.

His life work represented many and varied fields of activity. Whatever the work at hand he did it with enthusiasm and with marked success. He shone equally as a student, a teacher, a consultant, a soldier, a Society member and officer, a citizen, and finally, a husband and father. The profession needs more men like Professor McNew. In Society affairs and especially in its Texas activities he will be sorely missed.

He is survived by his widow and by two children, Elizabeth (Mrs. Don Dale Little) and J. T. L. McNew, Jr.

Professor McNew was elected an Associate Member of the American Society of Civil Engineers on February 25, 1924, and a Member on October 7, 1929.

RUDOLPH PHILIP MILLER, M. ASCE[1]

DIED JANUARY 18, 1947

At this time of uncurbed condemnation of building codes, it is significantly pertinent to record the accomplishments of a long and faithful professional career devoted to the improvement and administration of progressive and equitable building laws. With the death of Rudolph P. Miller the building industry and the profession lost an outstanding engineer who gave his time and ability unreservedly to the promotion of fire-safe and economical building. As a staunch advocate of freedom in the application of new developments and methods of construction, without sacrifice of safety to life and property, he won the fitting title of "dean of building codes." His unswerving conviction of right and his resolute will power, which were tempered by an innate modesty and charm of manner, won for him the admiration and the esteem of his associates. His integrity and professional accomplishments well deserved the high honors which were conferred upon him by the public and the profession.

---

[1] **Memoir prepared by George E. Strehan, M. ASCE.**

Rudolph Philip Miller was born in New York, N. Y., on July 5, 1866, the seventh of thirteen children. His parents were Anton William and Matilda (Schmahl) Miller. The door of the hospitable Miller home on Madison Street was always open, and it gave entry to the wholesome family surroundings in which he grew to manhood. A love and an appreciation of music were fostered in the frequent home gatherings of family and friends who spent evenings in good cheer and fellowship. This early training in art was supplemented by a deep interest in the natural sciences, which subsequently led to an application, with four other friends, for membership and eventual admission into the Agassiz Association as Chapter No. 87.

After initial private schooling, at ten years of age he entered Public School No. 34 from which he was graduated in 1880. In 1885 he received the degree of Bachelor of Science from the College of the City of New York. His postgraduate studies were pursued in the School of Mines at Columbia University in New York, leading to the degree of Civil Engineer in 1888, and later to the honorary degree of Master of Science.

After graduation Mr. Miller was engaged for six years in railroad construction and operation. First, he was connected with the Richmond and Danville Railroad (later the Southern Railway Company) as assistant engineer and supervisor of track, laying out and superintending the erection of bridges, culverts, station buildings, and shops, including the replacement of fourteen Howe trusses with new steel bridges. Later, he became supervisor in charge of maintenance of one hundred and nineteen miles of main line and three terminal railroad yards. Subsequently, he served as engineer on the location and construction of the Montauk Extension of the Long Island Railroad Company. In 1895 he undertook public service work for the City of New York, successively, holding the positions of special inspector, plan examiner, and assistant engineer in the department of buildings; and from 1901 to 1905 he held the post of chief engineer, supervising methods of construction and planning and directing tests for new materials, in addition to developing new forms of building construction.

As chief engineer of the department of buildings, Mr. Miller collaborated with Prof. Ira H. Woolson of the Columbia University School of Engineering, undertaking an extensive investigation of fireproof building construction. The test standards and practices developed by them in the first decade of the century established the basis for fire-resistive requirements and procedures which later became standard practice. This work led, in 1912, to the printing of a report and graphical tabulation of test results of various types of fireproof floor systems.[2] Those records are an important and rare contribution to the literature concerning fire-resistive materials and assemblies for building construction. In 1905 he also promulgated for New York the first rules governing reinforced concrete construction which established the basis for design of that material and which became a model for similar regulations throughout the country.

---

[2] *Proceedings*, International Soc. of Testing Materials, Sixth Cong., New York, N. Y., 1912.

Beginning in 1905 Mr. Miller engaged in private practice, designing and supervising all types of building construction with special emphasis on the early development of reinforced concrete structures. He was also consulted as an expert in analyzing safety of buildings from the standpoint of fire hazard, unusual stress conditions, and adequacy of exit facilities; in the investigation of structural and fire failures; in development and testing of new materials and forms of construction; in arbitration of disputes in the building industry; and in expert testimony in court procedures. In 1907 he became a member of the General Society of Mechanics and Tradesmen of the City of New York, giving unreservedly of his time to the educational program of that institution; he served on the finance, library, and pension committees, and subsequently became president of the society in 1920.

In 1910 George McAneny, President of the Borough of Manhattan, appointed Mr. Miller superintendent of the Bureau of Buildings of the Borough of Manhattan, an office he held until March, 1914. In addition to his official duties much of his time was devoted to the formulation of progressive building legislation, the development of material standards for the American Society for Testing Materials (ASTM) of Philadelphia, Pa., and in work for the National Fire Prevention Association in the improvement of building codes and practices to insure structural, fire, and sanitary safety of buildings.

He relinquished the position of building superintendent in 1914 to accept a commission with the Board of Aldermen for the revision of the building code of New York. Two years of intensive work and close cooperation with all branches of the building industry accomplished the adoption of a new building code, which remained in force until 1938. Mr. Miller's code succeeded where thirteen prior attempts at a cost of $200,000 to the city had failed. This code was acclaimed and is still recognized as a model building regulation noted for clarity, simplicity, and efficiency. On February 8, 1916, he was tendered a testimonial dinner by the joint committee on city departments of New York and was presented with an engraved scroll and watch, certifying the honor, esteem, and respect in which he was held by the city administrative officials and representatives of the building industry.

When a group of building officials having jurisdiction over the administration of building laws in various localities throughout the United States met in New York in May, 1914, to discuss their common problems, Mr. Miller served as temporary chairman of the conference. He proposed and became the founder of the Building Officials Conference of America which was subsequently incorporated under the laws of the State of Connecticut. The objective of that organization was to strengthen the principles and practices underlying laws and ordinances for public safety and health through effective building regulations and their administration, to improve the status and civic standing of the building official, and to enable that official to do his work as it should be done. To Mr. Miller is ascribed the success of this pioneering group, which he guided through ten successive terms as president of the conference. On his retirement as president, he was elected an honorary member of the conference. At its annual convention in Detroit, Mich., in 1939, the twenty-fifth year of its existence, the conference, dedicated that meeting to "Mr. Rudolph P. Miller as a token of respect, esteem and appreciation for the invaluable counsel and guidance which he so generously and unselfishly contributed." To him is due the interest and enthusiasm which kept the organization alive through its formative years and which later culminated in the expanded scope of the Building Officials Foundation for the promulgation of a national basic building code.

In July, 1916, Mr. Miller was appointed chairman of the newly created Board of Standards and Appeals by the Hon. John Purroy Mitchell, then Mayor of New York, with the task of organizing and establishing the board's methods of procedure. The functions of that board involved the interpretation of all statutes governing building construction in New York, including the use and occupancy of buildings when difference of opinion arose between applicants and administrative officials concerning building permits, the formulation of rules to interpret and supplement the legal provisions of the building code and other statutes, the granting of variations of the provisions of the zoning ordinances, and the testing and approval of new materials. He laid the groundwork so efficiently that there has been no essential change in procedure to this day, and these practices have been followed as a guide by similar boards throughout the United States in the testing of building materials and in the administration and application of zoning laws.

In his book,[3] John P. Comer gives the following appraisal of Mr. Miller's accomplishments as chairman of the Board of Standards and Appeals:

> "The chairmanship went to Rudolph P. Miller, the able professionally-trained engineer, brought up in the Manhattan building bureau, unselfish in sharing his broad knowledge of building conditions with official superiors and laymen alike, experienced as head of the most important local building agency, and a nonpartisan servant of the people through the application of scientific principles."

This tribute is further amplified by the 1947 chairman of the Board of Standards and Appeals in its official bulletin.[4]

During World War I, Mr. Miller acted as chairman of a draft board in New York resigning to accept the appointment as consultant to the United States Housing Corporation of the War Department in Washington, D. C. In the latter capacity, he was called upon to negotiate with legislative and administrative authorities of municipalities throughout the United States for temporary suspension of building code restrictions in the interest of conservation of critical construction materials required for the war effort. With the end of hostilities, he re-entered private practice.

---

[3] "New York City Building Control—1800–1941," by John Preston Comer, Columbia Univ. Press, New York, N. Y., 1942.

[4] Bulletin of Board of Standards and Appeals-City of New York, January 28, 1947, Vol. 32, No. 4, p. 81.

In 1920, Borough President Henry Curran again appointed Mr. Miller superintendent of buildings of the Borough of Manhattan of New York, and on completion of his second incumbency, he resumed private practice in general consultation. Much of his work was devoted to the preparation and study of building codes, zoning ordinances, and fire prevention laws. New building codes were written by him for Rochester, Niagara Falls, and Pelham Manor, in New York; Stamford, Conn.; Cumberland, Md.; and Richmond, Va.; as well as the original draft of the model code for the State of Connecticut. Prentice White, administrator of the Connecticut State Housing Authority, in connection with the adoption of the state model code in 1946, writes: "It would have been very difficult if not impossible to have compiled the code without the very fine work that he did for our Authority." Mr. Miller also was the author of the 1931 Recommended Building Code of the National Board of Fire Underwriters (New York). He served in a consulting capacity on the revision of building codes for Springfield, Mass.; Ossining, Bronxville, and Scarsdale, in New York; and prepared many other municipal and state building regulations.

From 1919 until 1928 Mr. Miller represented the engineering profession on the National Board of Jurisdictional Awards in the Building Industry, serving as chairman for five years. The function of that board was to adjudicate disputes arising between unions operating in the building industry. The board, by its decisions, is accredited with untold savings in time and money through the prevention of strikes. Ernest John Russell, the architect, said:

> "Rudolph P. Miller was a worthy representative of the engineering profession on the Board. It was a feeling of confidence to me and a great pleasure to be fortunate enough to serve with him for about four years."

Ferdinand John Conrad Dresser, M. ASCE, who also served with him throughout the entire life of the National Board of Jurisdictional Awards wrote:

> "I soon became aware that because of his experience in public affairs and their relation to the building industry, Mr. Miller had acquired an understanding and a grasp of the problems in the building industry such as few men in industry possessed. This service was always given without any thought of himself and with the only consideration that these problems must be reconciled for the benefit and common good of the industry as a whole."

Herbert Hoover, Hon. M. ASCE, former President of the United States, who was then Secretary of Commerce, appointed Mr. Miller a member of the Building Code Committee of the United States Department of Commerce in 1921. He served in this capacity throughout the life of the committee and was active in urging the continuance of the work under the procedure of the American Standards Association when the committee was discontinued in 1934 and the work was transferred to the Building Code Correlating Committee of that organization. Mr. Miller represented the Society in that body;

and in 1935 he was elected chairman of the committee, a position which he held until his resignation in 1944.

Because of his unfailing interest in civic improvements, his acknowledged judgment in all matters pertaining to building laws, and his willingness to share the results of his experience, Mr. Miller's services were continually in demand and they were freely donated, without thought of pecuniary reward, to committee work of technical organizations and to the solution of all building and planning problems in the public interest. In this connection he was a member of the gypsum and other committees of the ASTM throughout his professional life. Many of the standard test specifications governing fire-resistive constructions were prepared and adopted by Committee C5 (later E5 of the ASTM) under his chairmanship. This activity was one in which he took great pride and which resulted in a widely recognized standard.[5]

In 1914 he was a member of the advisory committee for New York on the proposed recodification of the labor law, then recommended by the New York State Factory Investigating Commission. In 1919 Mr. Miller occupied the post of chairman of the Advisory Committee on Mercantile Establishments at the request of Frances Perkins, former Secretary of Labor, then Commissioner of the New York State Labor Department. From 1930 to 1940 he served on the labor departments Advisory Committee on Fireproof Construction and Fire-Resisting Materials for the Division of Industrial Codes, acting as the representative of the ASTM.

Mr. Miller in 1923 become a member of the committee on seasonal operation in the construction industries by appointment of Herbert Hoover, who was Secretary of Commerce when that committee was formed, in conjunction with the president's conference on unemployment, to study the causes of, and remedies for, ungainful seasonal employment. In 1924 he was chairman of the committee on better building of the American Construction Council by appointment of the late Franklin D. Roosevelt, and ten years afterward was elected a member of the board of governors of that body. Secretary of Commerce Robert P. Lamont appointed him in 1931 to the committee on construction of the president's conference on home building and home ownership. In 1941, the seventy-fifth anniversary of the National Board of Fire Underwriters, Mr. Miller received the award of the New York Safety Council with the citation, "For forty-five years of service in the cause of fire prevention through intelligent engineering."

Associated with numerous building industry organizations, he took an active part in the deliberations of their technical committees. Among them he served on the committee to draft specifications for fireproofing of structural steel of the American Institute of Steel Construction; the Planning Committee of the National Conference on Construction; The Fire Waste Conservation Committee of the National Federation of Construction Industries; the Charter Revision Committee and the Safety Committee of the New York Building Congress; and as chairman of the Building Laws Committee of the City of New York on the Proposed Uniform Certificate of Occupancy.

---

[5] **"Standard Methods of Fire Tests of Building Construction and Materials," A.S.T.M.** Designation C-14-41, revised 1941.

Numerous articles and papers by Mr. Miller were contributed to the technical press and he was associate editor of several handbooks.[6, 7] He frequently lectured on building codes and their application before technical gatherings and as a member of the staff of the extension school of New York University (New York).

Mr. Miller's membership in the Society dated from 1889 and in recognition of his long association the Society made him a Life Member in 1931. He was indentified with its Steel Column Committee, Reinforced Concrete Committee, and the Committee on Docks, Wharves, and Piers; he represented the Society on the Standards Council of the American Standards Association, and was chairman of the Committee on Professional Relations of the Metropolitan Section of the Society in 1924.

He was a charter member of the American Institute of Consulting Engineers (New York) and served a term as vice-president. At his death, he was an honorary member of the National Fire Protection Association, having served as vice-president from 1922 to 1924, and as president from 1924 to 1926. He helped to shape the policies of that organization as a member of the executive committee from 1913 through 1922, as president of the New York chapter of the association for three years, and as a member of the committees on building construction, city planning, zoning, and safety to life, at various times.

In 1892, Mr. Miller was married to Agnes Robinson Nelson of Stafford, Va., and through that alliance he became interested in the history of colonial Virginia and the families that played important roles in the life of those pioneer times. His genealogical studies and research in colonial history proved a source of unending pleasure and interest to him throughout life. He is survived by three children, Rudolph Nelson Miller (village engineer of Bronxville), Mary Moncure Miller, and Tillie Miller of White Plains, N. Y.

As a public official, Mr. Miller will long be remembered for his freedom from undue influence of any kind in the performance of duty. Stanley Macomber, M. ASCE, then president of the Steel Joist Institute (Washington, D. C.), in a letter written a few years ago, said, "You always stood out in my mind as the best example of a building official that we had in the United States."

A practicing builder on chance meeting remarked to him, "You were hard-boiled, but everybody got a square deal." Ernest John Russell, past-president of the American Institute of Architects (Washington, D. C.), who served with Mr. Miller on the National Board for Jurisdictional Award in the Building Industry, stated, in tribute,

> "His knowledge and experience in building problems earned the respect of the contending parties who appeared before the Board; and his opinions were of great value in formulating the decisions that were rendered by the Board."

---

[6] "Architects' and Builders' Handbook," by Thomas Nolan, Editor in Chief, 16th Ed., John Wiley & Sons, Inc., New York, N. Y.

[7] "American Civil Engineers' Pocketbook," by Mansfield Merriman, Editor in Chief, 1st Ed., John Wiley & Sons, Inc., New York, N. Y.

In bestowing the honorary degree of Master of Science in 1914, Dr. Nicholas Murray Butler of Columbia University cited Mr. Miller as "exercising a strong influence in drafting and codifying building laws." It is by such estimates that he will always be remembered as an engineer and a public servant.

Rudolph P. Miller was modest and unassuming by nature, a man who traveled quietly through life serving the community through his profession without thought of financial return or public recognition. A man of moderate means, he contributed liberally to a variety of charitable enterprises. His record of public and professional accomplishment is an enviable one.

There is no more fitting tribute to Mr. Miller than the following quotation from Mr. Comer's book:

> "If an objective appraisal of building officialdom could be presented in terms of a hero, Mr. Miller would certainly be that man, for he served in one capacity or another for over thirty years and introduced practices that tended to increase the efficiency of the bureau of buildings in its administration of the laws and regulations."

Mr. Miller was elected a Junior of the American Society of Civil Engineers on January 2, 1890; an Associate Member on April 7, 1897; and a Member on January 5, 1904. He became a Life Member in January, 1931.

CHARLES ADAM MIXER, M. ASCE[1]

DIED OCTOBER 24, 1947

Charles Adam Mixer, the son of Alfred A. and Martha (Hinkle) Mixer, was born in Hamilton, Ohio, on June 9, 1859.

Mr. Mixer obtained his first engineering job in 1883 as assistant to the city engineer of Hamilton, engaging in municipal and hydraulic work. He next turned to the field of transportation which included work on both steam and street railways. From 1883 to 1886 he was affiliated with various railroad companies as assistant engineer on location and construction. During the period from 1886 to 1888 he was employed as principal assistant engineer on the design and construction of cable roads in Cincinnati, Ohio, and in Denver, Colo., and from 1889 to 1890 he was engaged in the same capacity during the construction of the Providence (R. I.) Cable Trainway. In 1890–1891 Mr. Mixer served as assistant to the city engineer and the commissioner of public works in Providence.

With this preliminary training, he was prepared for what was to become his life work. In 1892 he was employed by the Power Company and Rumford Falls (Me.) Light and Water Company and with these companies he spent his succeeding, active professional life—covering almost half a century.

---

[1] Memoir prepared by P. W. Thomas, Engr., Rumford Falls Power Co., Rumford, Me.

After a ten-year service as resident engineer, he was advanced to the position of engineer in 1902. In 1902 the Rumford Falls Realty Company commenced a housing development, part of which is known as "Strathglass Park" and Mr. Mixer was very active in the layout and construction of these properties. In November, 1919, his title was chief engineer.

As of March, 1931, he was retired from active duty but continued his association as consulting engineer. This unusually long connection with a single organization was commemorated on the occasion of his eightieth birthday (1939) when he was the recipient of a bronze plaque signalizing his forty-seven years with the Rumford Falls Power Company.

During this long period of service Mr. Mixer had a major share in the considerable engineering, civic, and industrial progress in Rumford. He completed the accepted plans for developing Rumford Falls, an important source of power on the Androscoggin River. His subsequent work included the making of surveys and working plans for the construction, operation, and maintenance of the dams, canals, water supply system, pumping station, power station, electrical systems, and paper mills. In all instances, he supervised the construction. He was also instrumental in developing public schools, libraries, churches, and parks, particularly Chisholm Park.

One of his engineering interests had more than local importance. He originated a method of estimating the water content of snow cover, which made possible long-range forecasting of water storage probabilities and flood hazards.

In Rumford Mr. Mixer was prominent in religious and community activities. He was a charter member of the Rumford Cosmos Club, deacon of the First Baptist Church, organist for the church for forty years, and one of a committee of three to lay the cornerstone of the new church. He also served as park commissioner of Rumford. Among the national organizations he was an associate (later life member) of the American Institute of Electrical Engineers and a member of the New England Water Works Association as well as of the Franklin Institute (Philadelphia, Pa.).

After a lingering illness, Mr. Mixer died in a nursing home on October 24, 1947. He was buried from the First Baptist Church of Rumford and interred at Sunnyside Cemetery in East Rumford.

He was married twice. His first wife, Nellie M. Naughton, died in 1934. On September 24, 1942, he was married to Emma Richardson, who survives him. Also surviving are two sons, Charles N., of Livermore Falls, Me., and Alfred L., of Belmont, Mass.; two daughters, Martha L., of Newark, N. J., and Alice (Mrs. Richard Rouillard), of North Hinsdale, N. H.; five grandchildren; and two great grandchildren.

Mr. Mixer was elected a Member of the American Society of Civil Engineers on November 1, 1893. He became a Life Member in January, 1928.

ROBERT REGNAULT NACE, M. ASCE[1]

Died May 19, 1947

Robert Regnault Nace was born in Philadelphia, Pa., on August 27, 1882. His parents were Joseph R. and Eliza (Davis) Nace. He was graduated from the Northeast Manual Training School in Philadelphia in June, 1900, supplementing his studies with work at Drexel Institute of Technology, also in Philadelphia. Throughout his studies his evident interest in railroad work brought him to the early attention of his instructors and to his initial employment as rodman in the Jersey City (N. J.) office of the Pennsylvania Railroad Company. This was to form an association which lasted until his death.

His work and promotions followed the usual pattern of supervisory maintenance on the lines east of Pittsburgh, Pa., until 1914, when he was appointed supervising pilot to head and develop the field and office work of the valuation department. In 1917 his assignment was that of special duty work on the South Philadelphia improvements, a major grade crossing elimination project; and in 1919 Mr. Nace was appointed assistant engineer in the general office at Philadelphia. With the system reorganization in 1920, Mr. Nace was made engineer, maintenance of way, on the Eastern Ohio Division, later entering the transportation department as superintendent on the Schuylkill and Indianapolis divisions. He returned to the maintenance of way department as chief engineer, maintenance of way, of the New York Zone, in 1928, holding that position until his death in 1947.

Mr. Nace was a member of the Society, the Engineers' Club of Philadelphia, the Society of American Military Engineers, the Roadmasters and Maintenance of Way Association, the Metropolitan Roamers, and the Miles River Yacht Club. He held a commission as Lieutenant Commander in the United States Naval Reserve. He was a member of the F. and A. M., and an enthusiastic sportsman, spending his brief vacations boating and fishing with extended trips on the water.

To those who worked with him, the factual data of his life leave much unsaid. "Bob" Nace was an able engineer and an outstanding leader with a personality both forceful and pleasing. A tireless worker, his approach to the many problems in his professional life of forty-six years showed a prophetic insight into the field of modern railroad practice. He pioneered in roadway drainage of major proportions, in mechanized work equipment, and in heavy rail that anticipated the later higher speeds and loadings.

He was forceful in speech with a dry wit often expressed in unusual terms that will long be remembered. Even more, his capacity for enduring friendships left his memory stamped indelibly on those who were his friends. Throughout a full life in his chosen field, he always found time to be sympathetic and helpful, and his passing leaves many who will remember him always.

---

[1] **Memoir prepared by Henry D. Stowe, Assoc. M. ASCE.**

On June 15, 1909, in Philadelphia, he was married to Elsie May Porr. He is survived by his widow; a daughter, Mildred R. (Mrs. William G. Hart); and two sons, Robert Porr, and Charles Derick.

Mr. Nace was elected a Member of the American Society of Civil Engineers on July 6, 1920.

CHARLES EDWARD PARRY, M. ASCE[1]

DIED JULY 7, 1946

Charles Edward Parry was born in Newcastle upon Tyne, Northumberland County, England, on February 4, 1877. His parents were Henry and Sarah Jane (Newbold) Parry. He was educated in England at the Portsmouth Grammer School (Portsmouth), Eton College (Eton), and Cambridge University (Cambridge), being graduated from the latter with the degrees of Master of Arts and Bachelor of Science in 1897.

In 1898 he became associated with the business of his father, a consulting engineer in Newcastle upon Tyne. At the outbreak of the Boer War, Mr. Parry entered the British Army as a First Lieutenant; he served throughout the war, receiving the South African War Medal. During Mr. Parry's army service his father died and his practice was closed. Mr. Parry then left for Canada where he entered the service of the Northwest Mounted Police.

Beginning in 1906 he was employed by the Minneapolis, Saint Paul and Sault Sainte Marie Railway Company in Minneapolis as assistant chief draftsman in the office of the superintendent of motive power and served in that capacity until 1912. In that year he was engaged by the Minnesota and Ontario Power Company as assistant chief draftsman on the plans of a 110-ton capacity newsprint paper mill for Fort Frances, Ont., Canada. At intervals he also acted as construction engineer for the company.

In 1913 he accepted the position of city engineer of Fort Frances, with the privilege of conducting a private practice. While he was city engineer, he designed, complete in all details, an eight-room schoolhouse and sanitary sewers and designed and built a 2,300-volt substation for the city power supply.

On the outbreak of World War I in 1914 he was commissioned in the Canadian Army and was quartermaster of the garrison barracks. He applied to England for active service in the British Royal Engineers and was duly commissioned as a First Lieutenant. In the First Battle of the Marne he was gassed, recovered in three weeks, and received a field commission as Captain. Gassed again in the Marne sector and invalided out of service, he returned to America in 1915, taking up residence in West Palm Beach, Fla.

---

[1] Memoir prepared by Charles H. Parry, Philadelphia, Pa.

In the latter part of that year, he was employed by Palm Beach County as resident engineer on the construction of eleven miles of penetration asphalt roadbed on the Dixie highway. Upon the completion of this work he was employed by the Pebble Phosphate mines at Brewster, Fla., designing and constructing phosphate washers and storage bins and making monthly computations of output for one year. In 1917 Mr. Parry was chief draftsman for the Tampa Dock Company on the building of four Ferris-type wooden ships.

Following the war, in 1918 he was offered the position of assistant city engineer in the City of Allentown, Pa., which he accepted and held for three years. During this period, Mr. Parry designed and wrote the specifications for a $40,000 public comfort station, a $300,000 sewage disposal plant, a steel highway bridge of four spans, concrete roadways, and street crossings.

Because of a change in administration, he resigned in 1920 and joined the Municipal Disposal Company of Philadelphia, Pa., as chief engineer in charge of construction and design. The work involved the completion of construction of the Philipsburg, N. J., and the Allentown sewage treatment works, and making complete designs and writing specifications for a large amount of prospective work. This company specialized in sewage disposal plants but failed two years later. He then entered the service of the Public Service Commission of the State of Pennsylvania and, as transit engineer, assisted in the valuations of the Philadelphia Suburban Gas and Electric Company and also those of the Philadelphia Rapid Transit Company. Six months later the Municipal Disposal Company was reorganized and refinanced under the name of the Direct Oxidation Corporation and Mr. Parry was recalled as chief engineer. This company constructed, or redesigned for efficiency, sewage disposal plants in all parts of the United States before being dissolved in 1927.

For the next twelve years he was again employed by the Public Service Commission as street railway engineer, being dismissed in 1939 because of a reduction in budget and personnel. After a lapse in employment he became chief engineer for the Pennsylvania Utility Consumers Service. His work consisted of the acquisition of privately-owned utilities and their purchase by municipalities. Rate studies were also made with a view to reductions for the consumers. This company became defunct in 1940. In the latter part of 1940 Mr. Parry was employed by the Public Works Department of the Philadelphia Navy Yard as assistant engineer checking contracts.

In the early part of 1941 he was recalled by the Public Utilities Commission (formerly the Public Service Commission) and employed as a railroad engineer in the Bureau of Engineering on highway and railroad crossings. In 1944 he was made chief gas engineer on matters pertaining to natural and manufactured gas utilities, and in 1946 was given complete charge of the investigation of all informal complaints to the Public Utilities Commission against public utilities (gas, electric, water, etc.) in the State of Pennsylvania, continuing in this capacity until his death on July 7, 1946.

Mr. Parry's life was a full one because of his many different types of employment all over the world, his experiences in two wars, and his service on the Northwest Mounted Police of Canada. Incidently, it has been stated that some of Mr. Parry's experiences were used by James Oliver Curwood in his book "The Sunset Trail."

He belonged to the American Association of Engineers, American Concrete Institute, Engineers Society of Pennsylvania, and the Society of American Military Engineers. He was a Thirty-Second Degree Mason in the Trimble Lodge of New Jersey. He was a member of the Episcopalian Church. Mr. Parry became a citizen of the United States in Philadelphia in 1921.

His two brothers served in the British Navy: His brother, Harry, on the Chief Surgeon's staff and his other brother, Reginald, served on the flagship as an electrical engineer. Both were knighted by the King of England and decorated for bravery in battle.

On September 3, 1903, he was married to Carrie Leonora Kaiser in the Church of the Holy Trinity in Toronto, Ont., Canada. Mr. Parry lost his wife on July 6, 1946, after she was operated on for gangrene poisoning of the leg. He suffered a heart attack forty-six hours later due to grief. He is survived by his only son, Charles H. Parry, of Philadelphia. Mr. Parry was highly respected and loved by the friends and co-workers he left behind, some of whom felt his passing almost as much as his son. He and his wife were cremated and laid to rest in the Whitemarsh Cemetery in Whitemarsh, Pa.

Mr. Parry was elected a Member of the American Society of Civil Engineers on May 19, 1924.

HARRY NELSON PHARR, M. ASCE[1]

Died November 3, 1947

Harry Nelson Pharr, the son of Capt. Henry Newton and Stella Houghton (Scott) Pharr, was born in LaGrange, Lee County, Ark., on May 8, 1875. His father, of Scotch-Irish ancestry, was a Presbyterian, originally from the Rocky River settlement of North Carolina; he moved to Shelby County, Tennessee, in 1838, and built his home near Raleigh (Tenn.), which was then the county seat of Shelby County. Following the termination of the Civil War, in which he took an active part in the Confederate Army, Captain Pharr moved to LaGrange, where he acquired considerable farm property. At LaGrange, three sons and one daughter were born.

Harry Pharr, his second child, attended the Lee High School at LaGrange, and the University of Arkansas at Fayetteville, from which he was graduated in 1893 with the degree of Bachelor of Science in Civil Engineering. In 1900, after several years of experience as assistant engineer and then chief engineer of the St. Francis (Ark.) Levee District, he did research work and submitted a thesis on the "Theory, Practice and Economy of Embanking Lands of the

---

[1] Memoir prepared by Herbert S. Gladfelter, M. ASCE.

Mississippi Valley Against Floods," for which his alma mater conferred on him the degree of Master of Civil Engineering.

He grew up in an engineering environment and it was natural for him to devote his life to this work—especially to flood control projects. After the Civil War, his father, a civil engineer, resumed his work and, after engaging in railroad location and construction until 1883, became interested in the reclamation of the overflowed lands and swamplands of his adopted state—Arkansas. In 1893, he initiated the organization of the St. Francis Levee District of Arkansas and Missouri and served as chief engineer of the levee boards of both states until his death in 1897. This man pioneered the work in the organization of the levee district and in the development of the lands within it. The northeastern part of Arkansas, known as the Arkansas Delta, was at that time largely overflowed land and swampland, sparsely settled with few roads; it is now one of the most productive agricultural sections of the United States, and contains numerous cities with populations ranging from 2,000 to 10,000.

Harry Pharr's first engineering employment after being graduated from the University of Arkansas was as assistant engineer under his father. In 1897, following his father's death, he was made chief engineer of the St. Francis Levee District of Arkansas.

In 1907 he resigned as chief engineer to engage in the private practice of general engineering, and for eight years was engaged as chief engineer of the newly organized Crawford County (Arkansas) Levee District in locating and constructing a levee along the north bank of the Arkansas River. This was followed by surveys and reports on flood control for the Obion River in Tennessee, the Kansas River in Missouri and Kansas, the Wabash River in Indiana, and the White River in Arkansas. In addition to flood control work, he also acted as consultant for numerous counties, corporations, and individuals in the design and construction of highway bridges over the St. Francis River. He was also consultant on flood control and drainage problems for the Mobile and Ohio Railroad Company; the Nashville, Chattanooga and St. Louis Railway Company in Tennessee; the Big Four Railroad Company in Indiana; for various municipalities in the design and construction of streets, sidewalks, sewers, and layout of subdivisions.

In 1915, following the great floods of 1912 and 1913, when the existing levee system proved inadequate in protecting the St. Francis Basin, he again was appointed chief engineer of the St. Francis Levee District, and during the next two years planned and constructed extensive enlargements of the levee system. Two years later Mr. Pharr left the levee district to engage in private business, largely devoted to the improvement and development of a large tract of land he owned in Lee County, Arkansas.

He returned again to the St. Francis Levee District in 1918 as chief engineer, remaining in that position until 1935. During this period, in cooperation with the federal government, he planned and constructed extensive works of levee enlargement and strengthening of the entire levee system within the district. Also he engaged in the relocation and rebuilding of part of the levee. During this time he was responsible to a large extent for the interpretation of the laws and rules for assessing benefits to land and property resulting from the construction of levees.

In June, 1935, he resigned as chief engineer of the St. Francis Levee District to accept the appointment by the Hon. Franklin D. Roosevelt, then President of the United States, as member of the Mississippi River Commission on which he served as a valued member until his death in November, 1947. The Mississippi River Commission has charge of the design and construction of flood control projects, including levees, revetments, dikes, spillways, floodways, cutoffs, and outlets on the Mississippi River from Cairo, Ill., to the Gulf of Mexico and, in addition, of dredging for the improvement of navigation.

In accepting his resignation as chief engineer of the St. Francis Levee District, the board of directors of that district adopted the following resolution:

> "In the course of worldly affairs it so happens that men are called upon to perform a vast and intricate multitude of duties, some in humble pursuits, others in pursuits more exalted. Men qualify for higher position by the quality of effort put forth in lower position. Thus, there comes promotion for those to whom promotion is due;
>
> "And Whereas, Harry N. Pharr, for a long number of years, spanning a good portion of his active life, has occupied by the free will, choice and selection of this Board, elected over and over again by its membership, the office of its Chief Engineer, and, as such has linked his name and fame eternally with the successful prosecution of the gigantic work for which the St. Francis Levee District was created and now exists; having throughout the years borne himself in the highest traditions of his profession, a true Knight of Old, buttressed in the bright and shining armor of honest resolve and consecrated purpose; and
>
> "Whereas, the President of the United States, Honorable Franklin D. Roosevelt, conscious of the importance of choosing for membership on the Mississippi River Commission talent of the highest order, has tendered appointment to that body to Mr. Pharr, who has accepted, and has resigned his position as Chief Engineer;
>
> "NOW, THEREFORE, BE IT RESOLVED, by the Board of Directors of St. Francis Levee District assembled in annual meeting at the domicile of the same, West Memphis, Arkansas, that we express to Mr. Pharr our late Chief Engineer, and now member of the Mississippi River Commission, our profound regret occasioned by his departure from us, and along with the same express our gratification that his ability and other qualifications have received this high recognition; that the Levee Board has undoubtedly sustained a great loss which we should be unwilling and loath to take were it not so patently to his great advantage, and also, to the advantage of the whole country which has so large an interest in the development to a state of nearer completion of the public works program proceeding on the Mississippi River and its tributaries under the supervision and control of the Commission of which he is now a member.
>
> "BE IT FURTHER RESOLVED, that we express our unbounded appreciation of Mr. Pharr's services; of his character as a man and true gentleman; of his uncompromising fidelity to principle and his fearless discharge of duty; of his professional skill, which is of high order, and commend him to his future associates in the responsible field of public duty to which he has been called by our honored Chief Executive."

Following Mr. Pharr's death in November, 1947, Maj.-Gen. R. W. Crawford, United States Army, President of the Mississippi River Commission, in announcing the death of a member of the Commission, stated:

> "Mr. Pharr was appointed to the Commission in 1935. Connected in earlier years with the St. Francis Levee District of Arkansas, his knowledge of flood control in the Alluvial Valley became a valuable asset to the Commission. As a member he was actively connected with all phases of the Commission's work, and his close and industrious application, his judgment and his understanding were of great assistance to the Commission in its deliberations.
>
> "He will be missed both as a valued professional associate and a most cordial and pleasant fellow member."

Mr. Pharr was a member of the Board of Trustees of the University of Arkansas from 1900 until 1902; a Colonel and chief of engineers of the Arkansas National Guard from 1907 until 1908; past-president of the Memphis Engineering Society, which was disbanded about 1900 and later revived in the Engineers' Club of Memphis of which he was a charter member; and he was active in the formation of the Mid-South Section of the Society of which he was a charter member. In addition, he also belonged to the University Club of Memphis.

The demands of his work precluded Mr. Pharr's indulgence in hobbies; however, in later life he did devote much of his spare time, in cooperation with the director of the proposed William R. Moore School of Technology in Memphis, to the planning, construction, and fitting out of this industrial trade school for boys. This school was built and completed in 1938 from a trust fund established in 1912 by the late William R. Moore. Mr. Pharr was appointed a trustee of the then-proposed school in 1936 and took a great deal of interest in its building in 1938, and in its development and operation until his death.

He was a member of the Idlewild Presbyterian Church of Memphis, taking an active part in its operation as a member of the Board of Elders. Always an enthusiastic worker in the church, the spirit of the church permeated his home and his work. He was known as an engineer of the highest integrity, soft spoken, but firm in his convictions—one who believed in doing his work thoroughly and well.

About 1900, Mr. Pharr transferred his residence from Arkansas to Tennessee, establishing a home in Memphis. In 1908 he was married to Creola Strong, and they had one son, Samuel Strong, who later became an attorney-at-law in Memphis. Mr. Pharr was married a second time in 1923 to Rose Scott Coleman of Uniontown, Ala., who survives him; they had two children, Kitty Grey (Mrs. Thomas R. Long) of Uniontown, and Stella Scott, who also survive.

Mr. Pharr was elected an Associate Member of the American Society of Civil Engineers on October 3, 1900, and a Member on June 6, 1905. He became a Life Member in January, 1935.

HENRY CYRUS PORTER, M. ASCE[1]

DIED MAY 31, 1946

Henry Cyrus Porter was born in Rockdale, Tex., on September 3, 1886. His parents were Louis Henry and Mary Catherine (Hamblen) Porter.

In 1904 he was graduated from the Rockdale High School and later attended the Engineering Department of the University of Texas, at Austin, from 1906 to 1912. At the university he was a member of the Phi Gamma Delta fraternity. Because of insecure finances he was unable to attend college continuously; he would work for a year or two and then go to school until his funds were exhausted. During his college career he took selected courses in electrical and civil engineering, preparing himself for municipal administration work, or what is now called the position of city manager.

From October, 1909, to October, 1910, Mr. Porter worked as chainman, draftsman, and computer for the Fred A. Jones Engineering Company on surveys for railroad and interurban railways in Oklahoma and Texas. He was resident engineer for the Fred A. Jones Engineering Company, from October, 1910, to October, 1911, on the construction of White Rock Dam for the City of Dallas, Tex.

On leaving the University of Texas in June, 1912, he was office engineer and assistant to the chief engineer of the San Benito and Rio Grande Valley Railway Company. In July, 1913, he became city engineer of Kingsville, Tex., and was in charge of the design and construction of a water system, sanitary sewage system, and street improvements.

Mr. Porter resigned this position in May, 1917, to attend the first Officers' Training Camp at Leon Springs, Tex. In August, 1917, he was commissioned a First Lieutenant in the Corps of Engineers, United States Army, and was assigned to the 315th Engineers, 90th Division, which was activated in September, 1917. His first duty was that of training troops. In February, 1918, he was made regimental supply officer and was promoted to Captain in June, 1918. He served overseas in France and Germany from June, 1918, to August, 1919, as regimental supply officer and regimental adjutant, and was in charge of the operations of the railroads in the 90th Division area while the division was part of the Army of Occupation in Germany. He was discharged in August, 1919, and immediately returned to Kingsville, where he was appointed county engineer of Kleberg County, responsible for the design and construction of a highway system.

In June, 1922, Mr. Porter resigned to become a division engineer for the Texas Highway Department, and he served in divisions No. 3, No. 4, and No. 9, supervising construction and maintenance of state highways. Then, in January, 1926, he became field engineer for Nagle, Witt, Rollins, and Gilchrist, Consulting Engineers, in charge of highway construction. The following year, in November, 1927, he was appointed resident engineer for the

---

[1] Memoir prepared by John A. Focht, M. ASCE.

Texas Highway Department in Goliad and Victoria counties, where he served until December, 1929. Then, he moved to Austin and was in charge of the Division of Specifications in the Austin office until May, 1930. During this time he advocated many improvements in the designs of highways, including the wide, flat type of cross section which is used very widely today.

One of his important publications at this time was a manual on roadway construction.[2] He became special resident engineer in McLennan and adjoining counties, in May, 1930, designing and constructing highways. In June, 1932, he returned to Austin as engineer of soils and research, which position he held until his death.

Mr. Porter did much research and wrote extensively on many of his findings. He contributed materially to the development of the cotton mat for curing concrete pavements, and was connected very closely with the Guadalupe County (Texas) Research Project, observing it over a long period of years; he compiled four large volumes of data collected on this project, from which he drew many conclusions which have been incorporated beneficially in design and construction of highways in Texas. He also developed the use of an electrical resistivity instrument to locate base materials.

In the last four or five years of his life Mr. Porter devoted his attention to soil foundations for pavements, and he wrote widely on the results of these findings. He was insistent that more attention should be given to drainage, soil selection, densification, and stabilization of the soil foundation for highways. His publications are found in the *Texas Highway Department Information Exchange,* the *Texas Parade,* the *Texas Engineer,* the *Manufacturers' Record, Roads and Streets, Engineering News-Record;* also, *Proceedings* of the National Research Council, *Proceedings* of the International Conference on Soil Mechanics, and *Foundation Engineering* of Harvard University, at Cambridge, Mass.

The culmination of his research and publications is found in a series of eleven bulletins.[3] It was his burning desire to present this information, along with many of his other findings, in a textbook on soil mechanics to be used in engineering colleges. He had prepared much of this manuscript, and had collected many cuts and illustrations, when a fatal illness prevented its completion.

Mr. Porter was quiet and cheerful—a friendly, unassuming individual who never sought publicity. He was most industrious, staying at the office or laboratory early and late; even his spare time was devoted to initiating new experiments and approaches. Tireless and meticulous in every detail, he did not object to the great precision required to eliminate all but one variable, or to check his results by other approaches. His office was always kept in neat order, and at his passing his personal affairs were complete, with his books ready to be closed. Mr. Porter was most enthusiastic about his work, and he was always happy to have an opportunity to discuss new developments with interested colleagues. When given an office in the penthouse of the Highway Building in Austin, he was delighted, for here he could work without interruption by the casual visitor; only those who had business with him would climb the extra flight of stairs. He welcomed the opportunity of passing on the information he had collected to others.

---

[2] "Manual for Surveys, Plans and Specifications for Construction of Roadways," by Henry C. Porter, January, 1930.

[3] "Roadway and Runway Soil Mechanics Data," by Henry C. Porter, Texas Eng. Improvement Experiment Station, College Station, Tex., 1942.

Mr. Porter was a registered professional engineer in the State of Texas, a member of the Texas Society of Professional Engineers, and the Church of Christ, and was a Thirty-Second Degree Mason.

On September 4, 1920, Mr. Porter was married to Grace Ellen Bludworth. He is survived by his widow and one son, Louis Henry Porter.

Mr. Porter was elected an Associate Member of the American Society of Civil Engineers on November 21, 1921, and a Member on October 1, 1926.

DE WITT PAWLING PUGH, M. ASCE[1]

DIED FEBRUARY 27, 1947

De Witt Pawling Pugh was born at King of Prussia, Pa., on April 19, 1872. He was the son of John Ramsey and Emily (Knight) Pugh. His early education was received in the vicinity of Philadelphia, Pa., and he was graduated from Ohio Northern University at Ada, in 1894, with the degree of Bachelor of Civil Engineering.

In the fall of 1894 Mr. Pugh joined the engineer corps of the Borough of Norristown, Pennsylvania, after which he became affiliated with the Pennsylvania Railroad Company. For this company he specialized in reconnaissance, location, construction, foundations, elimination of grade crossings, and estimates on economic railroad work. In the words of an associate, he was an outstanding locating engineer, who had a natural eye for line and grade. His life's work was spent in the service of the Pennsylvania Railroad Company.

He was a member of the following orders: Charity Lodge No. 190, Norristown Chapter No. 190, Hutchinson Commandery No. 32, the Scottish Rite, and the Nobles of the Mystic Shrine—all of Pennsylvania. Also, he was a member of the vestry of Washington Memorial Chapel at Valley Forge, Pa., from 1923 until 1946. During his association with the patriotic shrine, he was interested in augmenting the finances and activities of the chapel and especially in developing the churchyard, which flourished under his leadership.

His ancestors were Episcopalians and because of this early association with the church, he cherished a close relationship with the Episcopal Academy —a preparatory school for boys in Overbrook, Pa., where four of his sons were graduated. He was vitally interested in its development during his life.

Mr. Pugh, of Welsh descent, was an American of outstanding integrity, ingenuousness, and sincerity of purpose. He died on February 27, 1947, and was buried in the Washington Memorial Chapel churchyard.

---

[1] Memoir prepared by Mrs. De Witt Pawling Pugh, Wayne, Pa.

On October 7, 1908, he was married to Anna Elizabeth Ramsey. He is survived by his widow and three sons, Lt. Col. John R. Pugh (Fort Leavenworth, Kans.), Edward Pugh (Baltimore, Md.), and Capt. Theodore R. Pugh (Fort Riley, Kans.).

Mr. Pugh was elected a Member of the American Society of Civil Engineers on May 6, 1908. He became a Life Member in January, 1943.

REINALDO RAMIREZ, M. ASCE[1]

DIED JANUARY 19, 1947

Reinaldo Ramirez, the son of Severiano and Isolina (Marini) Ramirez, was born at Cabo Rojo, Puerto Rico, on December 28, 1892. A few years later this island of the Greater Antilles became a possession of the United States as a consequence of the Spanish-American War. He was, therefore, born a citizen of Spain, but, like most Puerto Ricans, became an American citizen in 1917 when Congress passed the Jones Act.

Mr. Ramirez received his early education in the public schools of the island. Later he attended the Baltimore Polytechnic Institute at Baltimore, Md., and was graduated in 1912. He then entered Cornell University at Ithaca, N. Y., from which he received the degree of Civil Engineer in 1916. His record as an outstanding student won the admiration of his Latin American classmates, friends, and students, some of whom offered him attractive positions to work in Cuba and South America.

He preferred to work for his native land, however, and returned to Puerto Rico to become assistant engineer and, later, chief assistant engineer for the South Puerto Rico Sugar Company, the largest cane sugar corporation in the island and, at the time, one of the largest in the world. His work was of varied character, covering land surveys, mapping, and the design and construction of irrigation works, which included pumping stations to supply cane plantations, houses, warehouses, piers, and the miscellaneous type of construction which is ordinarily required by a large sugar company. On the death of his father, having been appointed by the court administrator of the Ramirez estate, he was obliged to leave this position to take care of family business. At the same time he was able to engage in some engineering work.

In June, 1921, he was appointed assistant engineer in the newly organized Isabela Irrigation Service, a government institution created for the purpose of designing, constructing, and operating an irrigation system to water 15,000 acres in the northwest part of the island, and, incidentally, to develop hydroelectric power. At first he was in charge of surveys and subsoil investigations for an earth dam that was to have a maximum height of 123 feet and a 38,000 acre-foot reservoir, and later he assisted in the design of the dam and appurtenant structures, as well as of other features of the project. When construction work was started early in 1924, he was appointed assistant resident engineer.

---

[1] Memoir prepared by Rafael A. González, M. ASCE.

Mr. Ramirez resigned that position in January, 1926, to become chief assistant engineer in the Maintenance of Ways and Structures Department of the American Railroad Company, a corporation owning most of the public railroads on the island. Besides maintenance, his work involved design and construction of bridges, buildings, and miscellaneous structures. In May, 1943, he was promoted to chief engineer of that department and held this position until his death.

Mr. Ramirez was always interested in, and devoted much time to, the activities of the engineering profession. He was one of the founders of the Puerto Rico Section of the Society and was its first secretary-treasurer, a position he held from 1929 until 1935, when he was elected president. He was also a member of the local society, Sociedad de Ingenieros de Puerto Rico, of which he was president in 1940. Frequently, he was a member or chairman of committees dealing with some problem or phase of the engineering profession. His memberships also included the social club, Casino de Puerto Rico, and the Cabo Rojo Lodge of the F. and A.M.

The principal attributes of Mr. Ramirez were his integrity, modesty, and industriousness. The last is demonstrated by the fact that, while engaged with the American Railroad Company, he found time to do some private engineering work, which included contracting in the building industry. He possessed winning manners which earned him willing cooperators in all his undertakings. High moral principles always guided his actions. He was an excellent husband and father, a reliable friend, and a faithful citizen.

On January 18, 1919, Mr. Ramirez was married to Eulalia Irizarry Ramirez of Yauco, Puerto Rico. He is survived by his widow; three sons—Reinaldo Elliot, a graduate (Bachelor of Science in Agriculture) of Southwestern Louisiana Institute of Liberal and Technical Learning at Lafayette; Rene Octavio, an architect, graduate of Cornell University; and the youngest, Hernan Rolando; and two grandchildren.

Mr. Ramirez was elected an Associate Member of the American Society of Civil Engineers on September 10, 1923 and a Member on March 27, 1933.

ALLAN APPLETON ROBBINS, M. ASCE[1]

DIED FEBRUARY 17, 1947

Allan Appleton Robbins, the son of George Appleton and Susan (Morris) Robbins, was born at Scarsdale, N. Y., August 25, 1872. His early education was received at Eversons and Halsey's School, and he was graduated with the degree of Bachelor of Philosophy from the Sheffield Scientific School, Yale University, at New Haven, Conn., in 1893.

---

[1] Memoir prepared by R. T. Betts, M. ASCE.

Mr. Robbins did not enter an active engineering career until the autumn of 1896 at which time he became a draftsman for William Barclay Parsons,[2] Hon. M. ASCE, in connection with rapid transit work. Later, in the autumn of 1897, he joined the late F. Stuart Williamson, M. ASCE, as engineer in charge of piling, foundation, and masonry on the Manhattan Valley viaduct (extension of Riverside Drive, New York, N. Y.). This assignment was interrupted in the summer of 1898 by his service in Puerto Rico, as a First Lieutenant with the First United States Volunteer Engineers. On his discharge from the Army, he returned to Mr. Williamson and served with him until 1900.

It was during his work on the viaduct that he met John W. Ripley,[3] M. ASCE, who also was employed by Mr. Williamson as engineer in charge of the steelwork. From this friendship came the formation of The Robbins-Ripley Company in New York, in October, 1909.

From 1899 to 1901 Mr. Robbins was assistant general manager and director during the erection and operation of the plant of the Pinkney Mining Company at Pinkney, Tenn. He was resident engineer from May, 1901, to May, 1904, with the Rapid Transit Subway Construction Company, in charge of the fourth division, including the terminal yard and shops, 148th to 150th streets, New York, and the construction of substations No. 16 and No. 18 for the Interborough Rapid Transit Company. From May, 1904, to October, 1909, Mr. Robbins served as resident engineer, then as principal assistant engineer, with the New York and Long Island Railroad Company for a proposed tunnel under the East River (later used as a part of New York's subway system).

Mr. Robbins was president of The Robbins-Ripley Company, from the time of its organization in October, 1909, until his death in February, 1947. During his incumbency as president, The Robbins-Ripley Company rebuilt considerable stretches of the New York Dock Company's waterfront in Brooklyn, N. Y., involving the removal of approximately ten old piers, and the building of ten new piers, as well as rebuilding approximately 3,000 linear feet of bulkhead. Other projects include the waterfront improvements for Carl Fisher in his development of Montauk Point, Long Island, N. Y.; the rebuilding of pier No. 54 in the North River, after its destruction by fire in May, 1932, for the Board of Fire Underwriters, and the Cunard Steamship Company; and the design and construction of the Eastern Steamship Lines terminal at piers No. 25 and No. 26 (North River).

On his mother's side (Susan Morris Robbins), he was a descendant of Lewis Morris, first governor of New Jersey as a separate province (from 1738 to 1746), for whom Morristown, N. J., was named. He also was related to Robert Hunter Morris, one-time mayor of New York, and to Gouveneur Morris, signer of the Declaration of Independence.

He was a member of the Society of Colonial Wars, and of the University, Piping Rock, and Turf and Field clubs.

---

[2] For memoir, see *Transactions*, ASCE, Vol. 98, 1933, p. 1485.
[3] *Ibid.*, Vol. 109, 1944, p. 1517.

Mr. Robbins was elected an Associate Member of the American Society of Civil Engineers on October 2, 1901, and a Member on September 6, 1904.

WILLIAM JOSEPH ROWLAND, M. ASCE[1]

Died May 21, 1947

William Joseph Rowland, the son of William Joseph and Bobbie (Roberts) Rowland, was born in Vicksburg, Miss., on September 29, 1902. His education tion was received at the Saint Aloysius High School (Vicksburg), the Georgia Military Academy (Millidgeville, Ga.), the University of Alabama (Tuscaloosa), and the United States Military Academy (West Point, N. Y.). The death of his father forced his resignation from West Point after completing only two years study.

The early years of Mr. Rowland's professional life (from 1922 to 1931) were spent with various lumber companies in the south supervising plant construction and mechanical installations. When the United States Waterways Experiment Station was established at Vicksburg in 1931, Mr. Rowland was placed in charge of the soils testing section as chief technician; he served in that capacity until 1938, when he was placed in charge of all field operations of the soils division of the experiment station. He also served as consultant to various engineer districts on problems relative to field explorations and the installations of new soils laboratories.

Mr. Rowland was in the Reserve Officers' Corps, United States Army, and in 1942 was called to active duty with the rank of Captain in the Corps of Engineers. He served on the construction of the Alaska Highway until 1945 when he was discharged for disability. After his discharge, Mr. Rowland operated his own engineering and construction business in Vicksburg and handled a number of construction projects, including the widening of the east approach to the Vicksburg Bridge.

He was a member of the Engineers' Club of Vicksburg, the American Legion, and the F. and A. M. Mr. Rowland was of sterling character and strong personality He was always true to his ideals and to his friends, and his sound judgment and will for work commanded the respect of his associates and fellow workers.

He is survived by his mother, Mrs. Bobbie E. Rowland; three sisters, Juliette Rowland, Mrs. W. J. Melsheimer, and Mrs. R. B. Hutchinson; and several nieces and nephews.

Mr. Rowland was elected an Associate Member of the American Society of Civil Engineers on March 14, 1938, and a Member on January 11, 1943.

---

[1] Memoir prepared by W. L. Wells, Assoc. M. ASCE.

JOSHUA STEWART SAWYER, M. ASCE[1]

DIED APRIL 6, 1947

Joshua Stewart Sawyer was born in Montrose, Colo., to the former Mary Edlen Peters of Cincinnati, Ohio, and Edward Bertrand Sawyer, of Hyde Park, Vt., on March 20, 1883. His father, a civil engineer, had emigrated to the northwest, where he engaged in mining and railroad construction, and the son's indoctrination into the profession began at an early age. Some of Mr. Sawyer's earliest recollections were of construction sites and survey camps in the great plains and mountains of the northwest, of pack horses, and of fishing and hunting—experiences that bred in him a love of the outdoors which remained with him and colored his entire life.

He attended the Colorado School of Mines, in Golden, working during the summer periods with his father and, in 1910, was appointed assistant city engineer of the City of Medford, Ore. There followed various work as locating engineer on railroads and highways and a term as city engineer of Brandon, Ore. These experiences perhaps assisted in determining the particular field in which he was most interested, for in 1917 he joined the Oregon State Highway Department as locating engineer. From that time on Mr. Sawyer was identified with highway design and construction and with the development of materials.

After about six years of service as division engineer in charge of Division No. 2 of the Oregon Highway Department, Mr. Sawyer joined the Shell Oil Company in San Francisco, Calif., in 1929 as manager of the Asphalt Department and in that capacity organized and directed the department. In 1932 he came to New York, N. Y., to organize the Asphalt Department on the East Coast for the Shell Oil Company and remained as head of the department until his death.

Interested in highway construction and, in particular, in asphalt construction, he assisted in developing asphaltic materials for that purpose, and was well known for his work with emulsified asphalts.

An ardent fisherman and hunter, he spent most of his leisure in the field, and was particularly skilful in building fishing rods and gear. With a huge capacity for friendship, he had the respect and affection of all who knew him.

Mr. Sawyer was married to Minne P. McIntyre. He is survived by his widow and their two children, Irma (Mrs. W. C. Wells) and Joshua Stewart Sawyer, II, an engineer with the Oregon State Highway Department.

Mr. Sawyer was elected a Member of the American Society of Civil Engineers on January 13, 1930.

---

[1] Memoir prepared by James P. Williams, Assoc. M. ASCE.

FRANCIS JOSEPH SEERY, M. ASCE[1]

Died July 27, 1947

Francis Joseph Seery, the son of Thomas H. and Mary (Seery) Seery, was born on May 24, 1874, in Waterbury, Conn. Thomas Seery was a mill superintendent and a grandson of Gregory Delacy Seery, the leader of the United Irishmen in 1798.

Francis Seery received his secondary education in Waterbury High School, and then went to work. For seven years, from 1893 to 1900, he served his home city as engineering assistant on construction for additional water supply including Wigwam Dam. In 1900 he was transitman for the Isthmian Canal Commission exploring possible canal routes in Darien, Panama.

With this background of practical experience he decided to continue his education. In 1901, at the age of twenty seven, he entered Tufts College in Medford, Mass., and in 1905 received the degree of Bachelor of Science in Civil Engineering. During his summer vacations he was employed by the cities of Waterbury and Fulton, N. Y., and also by the United States Geological Survey on topographic work in Kentucky.

After being graduated from Tufts he served, during the summers from 1905 to 1908, as assistant engineer in Department D of the New York State Engineers' Office on construction of the Barge Canal.

In September, 1905, Mr. Seery was appointed instructor in hydraulic engineering in what was then the College of Civil Engineering at Cornell University in Ithaca, N. Y. This was the beginning of a teaching career at this institution which continued until his retirement as professor emeritus of civil engineering in 1942. In 1907 he was promoted to the grade of assistant professor, and in 1918 to a full professorship of hydraulic engineering, serving in the latter position for twenty-four years.

Professor Seery organized and taught courses in water supply, hydraulic construction, hydraulic engineering, water power (devoted to characteristics, hydrology, and operating conditions), water power and pumping plants (a design course), and conservancy and reclamation problems.

He was not idle during the summers of this long teaching career. Besides spending much time on very careful preparation of the practical engineering problems (on which he later had his students present reports), he undertook numerous jobs for which his assistance was requested. He prepared a report on hydroelectric development on Geneganslet Creek, and Oswegatchie River in New York in 1909 and also a design of the Potters Falls Dam on Six Mile Creek in Ithaca. In 1910 he made a valuation appraisal of the water plant at Lyons, N. Y., and in 1913, a similar appraisal for Watertown, N. Y.

With the participation of the United States in World War I, during the summer of 1917, Professor Seery was superintendent of water supply and sewers at Camp Dix, N. J.; and, during the summer of 1918, he served as hydraulic engineer for the United States Geological Survey assigned to investigate and make a confidential report to the Capital Issues Commission, United States War Board, on a proposed water plant at Watertown.

---

[1] Memoir prepared by a Committee of the Ithaca Section consisting of Charles L. Walker and Ernest W. Schoder, Members, ASCE.

His activities also included: A design survey and estimate for extensions to the water supply of Moravia, N. Y. (in 1919); a report on five proposed water power developments on Jacques Cartier River in Quebec, Canada, for the Donnaconna Paper Company (in 1921); a report on proposed alterations of Hoffman Hill Reservoir in Elmira, N. Y. (in 1922); a report to Congress as a member of the board of consulting engineers appointed to review the "Jadwin Plan" for flood protection of the Mississippi River (in 1925–1926); a report on the sufficiency of Hemlock Lake for the water supply of Rochester, N. Y. (in 1935); testimony as expert witness on hydraulics for the State of New York on the Montezuma, Oneida Lake, and Ley Creek cases (in 1936); a report on the extension of water works for Hammondsport, N. Y. (in 1937); work as expert on hydraulic meters for the New York State attorney general's department on water supply in Rochester (in 1937–1938); and expert witness in the Harrington Farms case (in 1938).

Professor Seery served the City of Ithaca as a member of the Board of Public Works for six years, from 1919 to 1925, being vice-chairman for the last three years; also, from 1943 to 1946, he was a member of the Ithaca Civil Service Commission.

The varied practical experience of Professor Seery in the field of hydraulic engineering had convinced him of the value of such experience to the teacher and to the student of engineering. Design problems of considerable scope, length, and suggestiveness were made a part of the courses which he taught. The educational worth of these problems is attested by the fact that almost all his students valued them highly, preserving their reports when returned at the end of the course, and frequently benefiting by them in their professional careers after graduation. By spending much time and energy in the composition of these problems Professor Seery asserted his conviction of the basic truth in the words of Aristotle: "For the things we have to learn before we can do them, we learn by doing them." This feature of the teaching methods of Professor Seery has become incorporated in other fields in the curriculum of the School of Civil Engineering at Cornell. His work thus will extend an influence toward the better training of students in the future.

Since he always had available in his memory much interesting and supporting extraneous material gleaned from his own practical experience and from extensive reading, his classes were far from dull. He made much direct use of current engineering literature, both to fortify the textbook and to orient the student toward the actual living world of engineering. He did not, however, neglect basic theory in developing the student's grasp of subject matter. In his work as a teacher he stressed conception, design, construction, and maintenance as elements in the structure of an engineering education and as activities that would be required of the engineer in his varied experience after graduation.

Professor Seery was married to Elnora C. McElligot of Waterbury, in 1909. He is survived by three daughters: Miriam, with whom he resided in Dover, Del., after leaving Ithaca; Virginia, of New York; and Francesca (Mrs. J. Theodore Chamberlain), of Tarrytown, N. Y.; and a grandson, Teddy Chamberlain.

Professor Seery was elected an Associate Member of the American Society of Civil Engineers on April 3, 1907, and a Member on June 6, 1921. He became a Life Member in January, 1942.

VALENTINE BERNARD SIEMS, M. ASCE[1]

DIED FEBRUARY 15, 1946

Valentine Bernard Siems, the son of A. G. John and Anna K. B. (Kohlhepp) Siems, was born in Baltimore, Md., on October 30, 1887. After being graduated from a Baltimore public school in 1903, he attended and was graduated from the following schools: The Maryland Institute (Baltimore) in 1906; the Baltimore Polytechnic Institute (Baltimore) in 1907; and Milton University (Baltimore), receiving the degree of Bachelor of Science in 1909. In 1941 Milton University honored him with the degree of Doctor of Engineering.

Subsequently, he was awarded the degree of Master of Science from Washington College (Chestertown, Md.) in 1919. Later he studied business administration and finance at the Alexander Hamilton Institute (New York, N. Y.) from 1919 to 1922 and law at the University of Maryland (Baltimore) during the same period. In 1923 he received the degree of Civil Engineer from Gettysburg College (Gettysburg, Pa.).

In March, 1909, Mr. Siems became a member of the engineering staff of the Baltimore city water department and was engaged in design and construction of various parts of that system, including two years as resident engineer on the construction of the Montebello filtration plant in that city. In 1918–1919 he was supervising and construction engineer in the construction division of the United States Army for the water systems at Camp Meade and Camp Franklin in Maryland. He was also assistant water engineer for the Baltimore water department (from 1918 to 1923) and president of the water board and water engineer for Baltimore (from 1923 to 1927).

Mr. Siems became interested in the North American Waterworks Corporation in New York, which was one of the large holding companies in 1926, and served as vice-president, general manager, and director until 1928. In addition, he was president and director of some of the subsidiaries of this corporation. For the next two years he was vice-president and chief engineer of the National Waterworks Corporation of New York and also president and director of subsidiary companies.

---

[1] Memoir prepared by William W. Brush, M. ASCE.

In 1931 Mr. Siems formed V. Bernard Siems and Company, an organization of consulting engineers specializing in management work and at the same time had an interest in the firm of George H. Siems and Company, contractors.

Mr. Siems was the author of numerous articles on the operation and management of utilities and of reports on water works for municipalities. After leaving the Baltimore water department he became interested in financial as well as engineering work and served as trustee of the St. James Savings Bank in Baltimore, and as director of the North Bond Street Permanent Building and Savings Society also in Baltimore. He was a member of the American Water Works Association and the American Association of Engineers.

For a number of years before his death, Mr. Siems gave a great deal of time to church activities and was an honorary trustee of the Bethany Lutheran Church in Elmhurst, N. Y., and of the Evangelical Lutheran Church of Our Savior in Baltimore.

On June 6, 1911, he was married to Mary Fens, who, together with their son, V. Bernard, Jr., Jun. ASCE, survives him.

Mr. Siems was elected an Associate Member of the American Society of Civil Engineers on November 25, 1919, and a Member on December 3, 1926.

SIDNEY MONTEFIORE SIESEL, M. ASCE[1]

Died September 15, 1946

Sidney Montefiore Siesel was born in Macon, Ga., on August 20, 1889. He was the son of Jacob and Clementine (Wolf) Siesel, who were natives of Germany and had been married in Macon. In 1891 the family moved to Pittsburgh, Pa., where they continued to live until Mr. and Mrs. Siesel's passing in 1906 and 1933, respectively.

In acquiring an education, Mr. Siesel attended the public schools of Pittsburgh, and in due time entered the Carnegie Institute of Technology there. He was graduated in 1908 with the degree of Bachelor of Science. Following his graduation he studied reinforced concrete for thirteen months at the Technical School in Charlottenburg, Germany. In 1932 he was awarded the degree of Civil Engineer from the Carnegie Institute of Technology after submitting a thesis on reinforced concrete piling.

Mr. Siesel became affiliated with the Pittsburgh Testing Laboratory in 1910 as inspector and served in this capacity on various classes of work in Pittsburgh and Milwaukee, Wis., for a period of two years. This was followed by a position in Pittsburgh where he became associated with James L. Stewart as construction engineer, estimator, and expediter, continuing until 1914. At that time he returned to Milwaukee as president of the O. H. Bossert Company, Contractors, with headquarters in Milwaukee, and continued as chief executive of this company for four years.

---

[1] Memoir prepared by Lebrecht J. Klug, Assoc. M. ASCE.

In September, 1918, he entered business as S. M. Siesel, Contractor, and continued to operate in this way until 1922 when he incorporated under the name of S. M. Siesel Company. Later it was changed to Siesel Construction Company, of Milwaukee and Pittsburgh, with Mr. Siesel as president and general manager. He was very active in this capacity until December, 1945, when illness prevented further work and resulted in his death in 1946.

Mr. Siesel was widely known, both professionally and socially. He was an active member of the Engineers' Society of Milwaukee, the Wisconsin Society of Professional Engineers, the Associated General Contractors of America, and the Master Builders Association of Pittsburgh.

In addition, he served as a member of the Milwaukee Building Code Committee and as president of the Milwaukee Employers' Building Trades Association, which he was instrumental in founding. He was influential in bringing about the enactment of the Tehan Slum Clearance Law by the Wisconsin State Legislature in 1945. This law granted to municipalities the power to condemn and claim land for slum clearance and to eliminate blighted areas.

He was a Thirty-Second Degree Mason and maintained memberships in the Milwaukee Association of Commerce, the Milwaukee Athletic Club, the Elks Club, the Woodmount Country Club, and the Westmoreland Country Club in Pittsburgh. He also served as treasurer of the Milwaukee Urban League and as president of the Mount Sinai Hospital Association of Milwaukee.

The work of Mr. Siesel was characterized by an ability, determination, and thoroughness which commanded the respect and praise of clients, fellow engineers, and contractors. Among the many engineering and construction accomplishments in which he was engaged during his lifetime, the following are particularly deserving of mention:

The Milwaukee County Court House, perhaps the largest and most beautiful building in Milwaukee County, the Mariner Tower, the Shorecrest Hotel, and the Mount Sinai Hospital—all in Milwaukee; the German Evangelical Protestant Smithfield Church and the Allegheny County Office Building in Pittsburgh; the Scottish Rite Cathedral in Newcastle, Pa.; the Department of Justice Building foundations in Washington, D. C.; and the twenty-seven-story Exchange National Bank Building in Tulsa, Okla.; and the Robert E. Lee Bridge in Richmond, Va.

During World War II, under the administration of the Defense Plant Corporation, Mr. Siesel's work was most outstanding in the construction of the Allis-Chalmers supercharger plants No. 1 and No. 2 in West Allis, Wis., the addition to the Ford Motor Company plant in Milwaukee, and the new forge shop for the Continental Can Company in McKees Rocks, Pa.

In 1944 he was chosen as a member of a special committee by the Society to complete an exhaustive study and to prepare a report on classifications and salaries for engineering employment in the Milwaukee area. His diligence and untiring efforts while serving on this committee were invaluable as he was intimately familiar with the local situation, and, at the same time, was able to maintain the objective point of view which, in itself, was a noteworthy achievement. The results of the study and the subsequent report as prepared by this committee were valuable contributions to the engineering profession.

On June 24, 1915, Mr. Siesel was married, in Pittsburgh, to Dorothea Jacobs, a native of England. Shortly thereafter he moved his family to Milwaukee. He is survived by his widow and three daughters, Barbara, Ruth Peggy, and Claude Sally.

Mr. Siesel was elected a Member of the American Society of Civil Engineers on February 24, 1931.

JOSEPH EMORY SIRRINE, M. ASCE[1]

Died August 7, 1947

Joseph Emory Sirrine was born in Americus, Ga., on December 9, 1872. He was the son of George William and Sarah Euodias (Rylander) Sirrine. Both his father and grandfather were Confederate soldiers, although the family was originally from Connecticut.

The family of George William Sirrine moved to Greenville, S. C., in 1876, and Joseph was educated in the Greenville Military Institute (1883–1886); then, he received the degree of Bachelor of Science from Furman University in Greenville in 1890. Mr. Sirrine received the degree of Honorary Mechanical Engineer from Clemson College (Clemson, S. C.), in 1928, and a degree of Doctor of Laws from the Presbyterian College (Clinton, S. C.), in 1941.

Mr. Sirrine engaged in private practice as a civil engineer from 1890 to 1895. He then worked (from 1895 to 1902) with Lockwood, Greene and Company of Boston, Mass., and from 1898 to 1902 served as southern manager for that company. In 1902 he went into business for himself and in 1921 formed a partnership with eight of his associates in the name of J. E. Sirrine and Company, Engineers. The business now continues under the name of J. E. Sirrine Company with eleven partners.

The scope of the company's work is illustrated by its activities during World War II. Among other projects were included the erection of four shipyards for the United States Maritime Commission, two major camps for the War Department, together with many smaller projects for the War Department, such as air bases, ammunition depots, and so on. The firm also did major construction work for the Charleston (S. C.) navy yard and the Parris Island (S. C.) marine barracks. Also, for the Defense Plant Corporation they built aluminum plants, and two plants for the manufacture of rayon tire cord, as well as doing a good deal of work for the Reconstruction Finance Corporation.

Mr. Sirrine belonged to the following organizations: The American Society of Mechanical Engineers; the American Institute of Electrical Engineers; the Newcomen Society of England; Greenville Elks Lodge, No 858 (honorary member); the A. F. and A. M. Recovery Lodge, No. 31; the Knights Templar; and Hejaz Temple. He was also a life trustee of Clemson College, past-president of the Greenville Chamber of Commerce, head of the Greenville Community Chest, past-president of the Young Men's Christian Association, and past-president of the Greenville Rotary Club (one of the international organization's few honorary members). Mr. Sirrine was an Episcopalian, being a former member of the vestry of Christ Episcopal Church in Greenville. In politics he was an independent Democrat. His clubs were the Greenville Country Club, the Poinsett Club, the Cotillion Club, the Biltmore Forest Country Club, and the Merchants Club of New York.

---

[1] Memoir prepared by H. L. Hagerman, M. ASCE.

He was chairman of the board of the Brandon Corporation, vice-president of four South Carolina textile concerns, and a director of nineteen other textile companies throughout the southeast. He was also a director of the First National Bank of Greenville, the Liberty Life Insurance Company, and the *Greenville News-Piedmont* (morning and afternoon papers in Greenville).

Mr. Sirrine was a member of the Cotton Mill Industry Advisory Committee for the War Production Board during World War II. During World War I he was chairman of the Price Fixing Committee for Cotton Textiles.

The following words speak so well of Mr. Sirrine, as the world knew him, that they can well be repeated herein:

> "The value of his counsel and assistance so freely given, and the value of his sound and experienced judgment can be visualized but it cannot be measured.
>
> "His ability in solving difficult problems and in clarifying difficult situations was one of his many great characteristics.
>
> "His willingness to give of his time, his thought and his substance for the benefit of others was known by all who knew him, and by many who only knew of him.
>
> "His was a life based upon studious practical learning, sound thinking, fair dealing, honest ambition for others as for himself, faith in honest undertakings, and a belief in his fellow men. He was tolerant with men's failures and intolerant only with dishonesty and greed.
>
> "In his heart there was love of thousands and hate for no one. His life, his vision and his encouragement have been an inspiration to an untold number of people."

His death marked the close of a colorful career.

On November 8, 1898, Mr. Sirrine was married to Jane Pinckney Henry of Greenville. She died on July 31, 1938. They had no children.

Mr. Sirrine was elected a Junior of the American Society of Civil Engineers on June 20, 1893; an Associate Member on May 4, 1898; and a Member on May 6, 1903.

LEONARD SEWALL SMITH, M. ASCE[1]

DIED JULY 1, 1947

Leonard Sewall Smith was born in East Troy, Wis., on February 14, 1864. His father, Charles Warren Smith, a merchant, who had migrated from Vermont to Wisconsin in 1840, was married to Mary Buell Edwards. Their family consisted of four daughters and two sons.

Young Leonard matriculated at the University of Wisconsin in Madison, entering from the Whitewater Normal School (Whitewater, Wis.) as a special freshman in 1886. In college he was a member of Beta Theta Pi fraternity and of the college band. The university conferred on him the degree of Bachelor of Science in Civil Engineering in 1890.

His first job as a young engineer was with the Union Pacific Railroad Company at Omaha, Nebr., where he worked as a draftsman in the office of the chief engineer from June to December, 1890. In the spring of 1891, he went to upper Michigan to do some topographic surveying for Prof. Charles R. Van Hise (later president) of the University of Wisconsin. He continued there until the spring of 1892 when he began work as transitman on the international boundary survey.

Mr. Smith began his teaching career as an instructor in engineering at the University of Wisconsin in January, 1894. He taught surveying and established a reputation as an effective teacher in that field. When the "good roads" movement began to attract wide attention, he introduced in 1905, the university's first course in roads and payments. This course expanded and, in 1925, became the course in highway engineering. As interest in housing and city planning developed in the more advanced nations, Professor Smith introduced a course in city planning in 1915 that proved popular. In 1920, Professor Smith was appointed professor of highway engineering and city planning, which title he held until his retirement.

Professor Smith was not an academic recluse; he felt the urge to put his professional knowledge to practical application. As a young teacher of surveying, he frequently pulled on the high boots that distinguished the young engineer of the period, and took to the field to lay out a new subdivision for the growing little city of Madison. He designed and platted many of its well-known sections. These contacts led to business connections that made him director for both the Royal Securities Company and the Madison Realty Company. From 1897 to 1905, Professor Smith was hydrographer for the Wisconsin Geological and Natural History Survey and from 1901 to 1905 was engineer for the Wisconsin Levee Commission. During the summer of 1918 he worked for the United States Housing Corporation of the Department of Labor.

In the early part of the century the records show that a number of the members of the engineering faculty engaged in local political activity. Professor Smith was one of this group and served as alderman from the Fifth Ward in 1905–1906. He introduced the ordinance that established the famous "Half-mile Dry Zone" around the university and created a desert that existed until the repeal of the prohibition amendment. After he had been appointed professor of city planning, Professor Smith was called upon by various Wisconsin cities to assist with the preparation of city plans and zoning ordinances—Appleton, Eau Claire, Madison, Manitowoc, Oshkosh, Racine, Waukesha, and Wausau all benefiting from his professional assistance and advice.

---

[1] Memoir prepared by Ray S. Owen, Prof. Civ. Eng., Univ. of Wisconsin, Madison, Wis., based on resolutions adopted by the faculty of the Univ. of Wisconsin, dated October 6, 1947.

A paper on surveying was the result of his experience as surveyman on the international boundary survey between the United States and Mexico.[2] This paper, written in 1895, was accepted as a thesis for the professional degree of Civil Engineer and was also printed as a United States Senate document. A "Field Manual of Engineering Instruments" appeared in 1900. A hydrographic map of Lake Mendota and the adjacent topography, published by the Wisconsin Geological and Natural History Survey, was based upon surveys made by civil engineering students under his direction, and, in 1908, the same survey published Professor Smith's report on the water powers of Wisconsin. In 1910, he revised an earlier textbook[3] by J. B. Johnson,[4] M. ASCE.

Professor Smith made a number of trips abroad. In 1912–1913, he took his family on a journey through England, France, Germany, and Switzerland and in 1920, he was an American delegate to the World Congress on Housing and Town Planning held in London, England. Accompanied by Mrs. Smith, in 1923, he attended the Fourth International Good Roads Congress in Seville, Spain, where he presented a paper on the "The Relation of Modern Traffic to the Planning of Streets." Following the meeting, he and Mrs. Smith toured Spain, France, Italy, Norway, and Sweden. He spent the month of October, 1926, in Japan, inspecting the work of reconstruction that followed the great earthquake of September 1, 1923. The Japanese government extended many courtesies to him and he lectured at the universities at Osaka, Kobe, and Tokyo.

The Engineering Society of Wisconsin was one of Professor Smith's hobbies. He served the society as secretary from 1915 to 1926, during which time the membership quadrupled and the society resumed and continued publication of its *Proceedings.* At a joint meeting of the society and the Technical Club of Madison, on February 18, 1927, he was presented with a watch in recognition of his eleven years of service.

He retired from teaching in February, 1929, and went to California, where some of his children had established themselves. He joined the Los Angeles City Planning Association and engaged actively in city planning. In addition to other work, he assisted with the preparation of plans for San Clemente and laid out that subdivision. He also made plans for National City and for Hollywood Rivera. (All those communities are in Southern California.)

---

[2] "An Experimental Study of Field Methods Which Will Insure to Stadia Measurements Greatly Increased Accuracy," by Leonard S. Smith, *Bulletin No. 5,* Univ. of Wisconsin, Madison, Wis., 1895.

[3] "The Theory and Practice of Surveying Designed for the Use of Surveyors and Engineers Generally, But Especially for the Use of Students in Engineering," by J. B. Johnson, 17 Ed., rewritten by Leonard S. Smith, John Wiley & Sons, Inc., New York, N. Y., 1910.

[4] For memoir, see *Transactions,* ASCE, Vol. LI, December, 1903, p. 454.

For the few years prior to his death Professor and Mrs. Smith lived in Stockton, Calif., at the home of their daughter, Ruth. To the last, he maintained his interest in educational activities, attending classes in German and other subjects, and occasionally lecturing to the students of the College of the Pacific (Stockton) on city planning.

His death removed one of the most colorful of the early teachers who created the College of Engineering at the University of Wisconsin. Through a long career, he retained, undimmed by the passing years, an energy that drove him from one accomplishment to another, and an enthusiasm that made all work seem worthy of his best effort. He was devoted to his family, of which he had reason to be proud, and was a good citizen who took his duties seriously, both in private life and in public office. Professor Smith was a conscientious and effective teacher. There developed through the years a "Lenny Smith" tradition as he was enthusiastic and colorful; it is held in fond remembrance by hundreds of engineers who, as students, sat in his classes and worked in the field under his guidance. From their contacts with him, they acquired, along with professional competence, a fund of anecdotes that will, for many years, keep alive his memory.

Besides serving the Engineering Society of Wisconsin, Professor Smith was a member of the American Institute of City Planning, the American Civic Association, the City Planning Commission of Madison, the Los Angeles City Planning Association, the University Club of Madison, and the Los Angeles City Club.

In 1893, Leonard Smith was married to Lucy Fowler Austin in his home town of East Troy. In the course of time, the family circle was increased to six by the arrival of four children. He is survived by his children: Maxwell Austin, dean at the University of Chattanooga (Chatanooga, Tenn.); Helen (Mrs. Robert Postgate) of Maracaibo, Venezuela; Ruth, professor of French at the College of the Pacific; and Lucy (Mrs. Alfred E. Ellerby) of Lander, Wyo. There are seven grandchildren. Two sisters, Grace (Mrs. White Jones), of Madison, and Marie (Mrs. E. S. Park), of Fond du Lac, Wis., also survive.

Professor Smith was elected an Associate Member of the American Society of Civil Engineers on June 2, 1897, and a Member on February 28, 1911. He became a Life Member in January, 1932.

JESSE BAKER SNOW, M. ASCE[1]

DIED JUNE 16, 1947

Jesse Baker Snow was born on May 26, 1868, at Nantucket, Mass., on the Island of the same name off the south coast of Cape Cod. He was one of five children born to Charles E. and Emily Jane (Carpenter) Snow.

---

[1] Memoir prepared by Louis E. Robbe, M. ASCE.

An ancestor landed at Plymouth, Mass., in 1623, from the historic *Mayflower* on one of her trips from England and in 1823 his paternal grandfather moved from Cape Cod to Nantucket where he later became the father of fourteen children. Emily Carpenter's grandfather was a Macy, of the family which started the department store of R. H. Macy and Company in New York, N. Y. Both sides of Mr. Snow's family had sons in the early wars of colonial days and in the later wars of the United States. Two of his grandsons served in World War II.

His father was in the packet trade, operating a schooner for many years and later became a Nantucket merchant. It was in this environment that Jesse Snow grew up, knowing something about transportation by sea and observing some of the lack of proper transportation by land. Perhaps he was influenced by the then popular era of street railway construction in choosing engineering as a career. He passed through the common and high schools at Nantucket and was graduated from Union College in Schenectady, N. Y., with the degree of Civil Engineer in 1889.

Immediately after graduation Mr. Snow held for four months a field position on surveys and cost estimates for sewer and drainage works at Goshen, N. Y., and at Long Island City, N. Y., after which he returned to his home town and entered private practice. There he laid out and built the first horse drawn streetcar line and made plans for extending it seven miles to Siasconset. He also assisted with plans and layouts for a sanitary system for Nantucket which then (1890) was only partly constructed. It is interesting to note that thirty-eight years later another engineering firm made a new design which was almost identical with Mr. Snow's, showing that sound principles and good judgment produce engineering designs which are unaffected by years.

In 1891 he followed Horace Greeley's advice and went west as far as West Superior, Wis., where he engaged in the private practice of land surveying and general engineering. About a year later he contracted typhoid fever and was forced to go home to recuperate. When restored to health he went to North Tonawanda, N. Y., engaging in the construction of street railways and land subdivisions.

After a year of this work Mr. Snow formed a partnership with another engineer, T. W. Barrally, in 1893 and together they engaged in municipal engineering. As city engineers for Tonawanda and North Tonawanda they designed and constructed sewers; brick and macadam pavements; extensions to the water supply and pumping plant, including new intake piers in the Niagara River; and bridges and culverts—construction totaling about $550,000. They also made assessors' maps, railroad property valuations, and constructed eight miles of electric street railway. After five years he terminated this partnership and became resident engineer on the constructions of water works and sewers in five towns in New York, Pennsylvania, and Ohio, totaling $100,000. Shortly afterward he built ten miles of single-track steam railroad for the Jamestown, Chautauqua and Lake Erie Railroad Company ($350,000) at Westfield, N. Y.

In 1902, after gaining thirteen years of field experience, Mr. Snow was ready to accept a more responsible assignment. He became chief engineer, first, of the Jersey Central Traction Company and, later, of the West Chester, Kennett and Wilmington Electric Railway (in Pennsylvania and Delaware). He secured the rights of way and the various local franchises and constructed about forty miles of street railway from South Amboy to Red Bank, in New Jersey, from Avondale, Pa., to Brandywine Springs, Del.

After two years of this work Mr. Snow decided to broaden his construction experience by working for a contractor. Thus, he became a superintendent on railroad construction at Occoquan, Va., and on water works improvements at Bellaire and Portsmouth, Ohio. Five months later he returned to engineering, undertaking surveys and estimates for a power development project at Walling, Tenn., for the Tennessee Power Company and, later, made surveys in New York City.

In November, 1905, he started on work which he learned to like so well that it became his specialty—compressed air subaqueous river tunneling. At first he was in a minor position on the engineering staff of the Pennsylvania Tunnel and Terminal Railroad Company of which Alfred Noble,[2] Past-President, ASCE, was chief engineer. At that time the company was constructing the four tunnels under the East River (New York) to be used by the Long Island Railroad. He was quickly promoted to first assistant to the resident engineer in charge of the two southerly tunnels, and was placed in charge of the alinement and inspection forces. Later, this was extended to include the cross-town tunnels extending from the East River back to the Pennsylvania Station on Manhattan Island.

On completion of this work in 1910 and because of his experience with construction under compressed air and water works construction, he was sent to Hauser Lake Dam, in Montana, by the Foundation Company. On this project he was engineer and assistant superintendent on the foundation work and also on the design and construction of the caissons and cofferdams, two of which had failed prior to his going there. After eight months his work was completed and he returned to New York where he was busy for about three years on various projects. These included a proposed large hydroelectric development near Port Jervis, N. Y. ($40,000,000), for which he was field engineer making surveys and reports for Canada Syndicate, Limited, of Montreal. He also prepared reports and cost estimates on other proposed hydroelectric developments for Charles Cooke, consulting engineer.

In 1914 he succeeded in returning to work of his primary interest, becoming resident engineer on the construction of compressed air subaqueous tunnels for the New York Public Service Commission. At that time the commission was undertaking a great subway construction program in New York with four pairs of tunnels planned under the East River. Mr. Snow was, simultaneously, in charge, on the Manhattan side, for both the Old Slip-Clark Street ($7,300,000) and the Whitehall Street-Montague Street ($6,300,000) tunnels for about one year; and then he supervised the Fourteenth Street-North Seventh Street tunnels ($6,800,000) for about three years. Here he came in contact with Clifford M. Holland,[3] M. ASCE, then in charge of all the commission's tunneling operations.

---

[2] For memoir, see *Transactions*, ASCE, Vol. LXXIX, 1915, p. 1352.

In 1919 he and Mr. Holland joined the New York, New Jersey (Interstate) Tunnel Commission, then undertaking the construction of the now famous Holland vehicular tunnels under the Hudson River. He became principal assistant to Mr. Holland, who was chief engineer and greatly assisted in the organization of the force and in the actual construction of the two tunnels.

In 1921 combined railroad and rapid transit tunnels were proposed under the Narrows between Staten Island and Brooklyn by the Board of Estimate and Apportionment of New York City, of which Arthur S. Tuttle, Past-President and Hon. M. ASCE, was then chief engineer. Mr. Snow, now gaining prestige and renown in the construction of subaqueous tunnels, was thus the logical selection as tunnel engineer. He set up an organization; two steel pneumatic caisson shafts were sunk to grade; and he had the designs well advanced for the entire project ($60,000,000) and was advertising for bids for the construction of the two tunnels when the work was stopped in 1925 by the state legislature.

Mr. Snow was then appointed division engineer in the Board of Transportation of New York City (successor to the Public Service Commission), in charge of the subaqueous tunnels to be built in another great expansion of subway construction. Under his direction the following rapid transit tunnels were built under the East River using compressed air: Fifty-third Street, Manhattan-Nott Avenue, Queens (two tunnels); Fulton Street, Manhattan-Cranberry Street, Brooklyn (two tunnels); and Rutgers Street, Manhattan-Jay Street, Brooklyn (two tunnels). In addition, three tunnels were built from 157th Street in Manhattan under the Harlem River to the Bronx and two tunnels, under Newtown Creek in Queens. The contracts for this work totaled about $71,000,000.

In December, 1933, Mr. Snow became acting chief engineer and, in February, 1934, chief engineer of the Board of Transportation. During his incumbency the Sixth Avenue (Manhattan) subway ($50,000,000) was designed and constructed. This was the most intricate and most difficult subway yet built because of the presence, at the time of construction, of the Sixth Avenue elevated railroad and the street surface transportation throughout its length and the Hudson and Manhattan Railroad tunnels and stations along part of its length, as well as the several subsurface transit facilities crossing Sixth Avenue. Between Thirty-second and Thirty-fifth streets are the Long Island Railroad tunnels, the Hudson and Manhattan Railroad station, and the Brooklyn-Manhattan Transit (B.M.T.) subway and station. It was necessary to remove and rebuild the Hudson and Manhattan Railroad station and to alter the other conflicting structures to provide sufficient clearance to build the new subway and station in that area.

---

[3] For memoir, see *Transactions*, ASCE, Vol. 89, 1926, p. 1625.

The Fulton Street subway line in Brooklyn was extended ($25,000,000) and the Hillside Avenue connection in Queens ($7,500,000) was built. In addition, nearly nine hundred new subway cars ($36,000,000) were designed and added to the operation equipment of the subway besides making an extension to the subway to reach the World's Fair grounds, in Flushing Meadows, in Queens ($2,000,000). The existing elevated railroads became outmoded and the Sixth Avenue line, the Ninth Avenue line, and a large part of the Second Avenue line (all in Manhattan) and the Fulton Street line (in Brooklyn) were removed.

In June, 1940, the B. M. T., the Interborough Rapid Transit (I. R. T.), and the Eighth Avenue (Independent System) subway lines were consolidated and Mr. Snow then became chief engineer of the entire system. The consolidation greatly enlarged his duties and responsibilities, and about doubled the engineering employees under his authority.

His plans for future subways and extensions were extensive; among these were the Second Avenue (Manhattan) line estimated at $400,000,000 and the Concourse (Bronx) line extension at an estimated cost of $30,000,000, as well as many others of lesser importance.

Because of his knowledge of compressed air tunnel construction, the New York City Tunnel Authority arranged for his services as consulting engineer on the Queens-Midtown East River vehicular tunnels, then being constructed. This was in addition to his duties with the Board of Transportation. He served in this dual capacity from 1936 to 1939 and again from 1940 to 1943 when the Brooklyn-Battery tunnel was started. (In 1933 he presented a paper bofore the Metropolitan Section of the Society on the "Fulton Street-Cranberry Street East River Tunnels.")

After reaching the mandatory retirement age of seventy the Board of Estimate, on four occasions, extended his employment by the city. However, on May 31, 1945, he finally retired after a period of fifty-six years of engineering activity including twenty-nine years service with the city. His extensive experience and great ability brought him to the forefront in his profession and he was regarded as the outstanding compressed air tunnel engineer and one of the foremost subway engineers in the United States. He had an active part in the construction of twenty-three rapid transit, railroad, and vehicular tunnels of a total of forty one constructed along the periphery of Manhattan, besides the two additional rapid transit tunnels under Newtown Creek in Queens. No other engineer has had such a direct hand in the design and construction of so many subaqueous tunnels by the use of compressed air.

Through his ingenuity the junctioning of the river shields in earth, instead of rock, was first accomplished on the Fulton Street tunnels. He also raised the river bottom of the East River, between Welfare Island and Manhattan, sufficiently to embed the two Fifty-third Street tunnels and provide them with earth and riprap cover for protection. Improvements in the design of the cast-iron tunnel rings by the insertion of a third and intermediate flange lengthwise (which provided greater stiffness and thus enabled wider tunnel rings to be used) were also credited to Mr. Snow. In addition to great ability, Mr. Snow had a keen mind, a very retentive memory, and was attentive to details. He was noted for his generosity and his loyalty to all those who served under him. Outstanding was his great capacity for hard labor, his only hobby was taking work home at night.

Civil engineering fitted him well for he loved the life that it gave, and it brought him happiness and satisfaction, not only through the details of under-river tunnel and subway construction but also in the making of estimates of quantities and costs. He was very retiring by nature and refrained from making many friends; his love of home and of family was his main comfort in life. Mr. Snow's great contribution to the life in which he lived was to provide better, faster, and more extensive transportation for the people of New York City.

Because of his renown as a tunnel and construction engineer Mr. Snow was honored by his alma mater, Union College, with the degree of Doctor of Science in 1930. He was a member of the Railroad Club and the Engineers' Club and was a licensed professional engineer in New York State. In college he had been a member of Delta Upsilon fraternity and of the honorary society of Sigma Xi.

Honoring his memory at his funeral were many of the leading engineers and representatives of large contracting organizations besides many intimate friends and former employees.

He was first married to Eleanor Curtis Harman of Schenectady. To this happy union, two children were born, Annette W. (Mrs. Earl Allington) and Charles G. Snow. His wife died a few years after they were married, but he maintained his home and kept his little family together. In 1903 he was married to May Purdy whom he had met while working in New Jersey. They enjoyed a very happy companionship for many years, broken only by her death a few months after his retirement. They had one son, Edgar P. Snow. Surviving him are the three children, seven grandchildren, and one great-grandchild.

Mr. Snow was elected a Member of the American Society of Civil Engineers on October 2, 1907. He became a Life Member in January, 1939.

ISAAC JOSHUA STANDER, M. ASCE[1]

DIED FEBRUARY 13, 1947

Isaac Joshua Stander was born on January 26, 1872, in Riga, Latvia. His parents were Lazarus Joseph Stander, a gifted musician with a beautiful lyric tenor voice, and Rose (Soloway) Stander. His father died at the age of forty-four as the result of an accident when "Ike" was sixteen years old. Three years later he came to the United States, the "land of promise."

---

[1] Memoir prepared by William Goldsmith, M. ASCE.

In the "land of promise" he followed the usual Horatio Alger pattern. Within one year he had prepared himself by his own efforts for entrance into the School of Engineering of Columbia University in New York, N. Y., in 1893. Mr. Stander belonged to the "Forty-Niners," a group of men who studied when Columbia University held sessions at Forty-Ninth Street and Madison Avenue. In 1897 he was graduated from Columbia University with the degree of Civil Engineer, and immediately entered the engineering field as a draftsman for the City of New York and later for the Purdy and Henderson Company. For sixteen years (from 1900) he specialized in the design and construction of docks, bulkheads, piers, and all phases of water front construction for the Department of Docks of New York. The ten years that followed found him busy as a contracting engineer on water front, building, and underpinning work also in New York.

From 1925 to 1935, when subway construction was active in that city, Mr. Stander became, successively, chief engineer for the Hyman and Goodman Company, chief engineer for the Clementi Contracting Company, and consultant on subway work for other large contractors. In 1935 he went back to his "first love"—water front construction—remaining in this type of work until his death; and, successively, he was associated with the Triborough Bridge Authority, superintending the construction of the East River and Harlem River drives, and with the president of the Borough of Manhattan on the design and construction of the lower section of the East River Drive. In 1942 Mr. Stander was retired on a pension by the City of New York. However, he was not content to be idle, and he became a private consultant, specializing in New York water front construction for local contractors.

His splendid accomplishments in engineering do not tell the complete story of his well-filled life. He was a devoted husband to his wife, Lena Smith, one of his childhood friends, to whom he was married on November 15, 1892. Mrs. Stander died on January 27, 1936, after forty-four years of happy married life. He was a cherished and respected father of three daughters and one son: Thamara, an outstanding physician; Elizabeth (Mrs. Harold Pimblett) of Alvinston, Ont., Canada; Ruth (Mrs. Meyer M. Harris), who was married to a physician; and Joseph L., who followed in his father's footsteps in construction work. There are also four grandchildren of whom Ike was very proud. In March, 1939, he was married to Mrs. Ray W. Sundelson who survives him.

However, Ike's home duties did not prevent him from being of both financial and professional assistance to his many friends. No matter who requested aid, Mr. Stander would always offer a helping hand. The writer can vouch for this wonderful trait in Ike, for it was through Ike's efforts that his first engineering position was received, after graduation from school. That fine deed to a comparative stranger has never been forgotten.

One of Mr. Stander's outstanding characteristics was his unshakable belief that a man's word was as good as his bond, and he persisted in that behavior pattern as long as he lived. Possessed with an excellent sense of humor, it carried him over rough experiences and showed in his good nature. His zest for living remained undiminished, even when his physical condition limited his activities to a minimum. His plans were always cheerful and hopeful. During his last days he was strongly advised against climbing stairs, yet, each morning at the nursing home in Florida where he passed his last days, he climbed the stairs to make the rounds of those who were bedridden—brightening them by reading the newspapers, telling humorous stories, and generally spreading cheer. It was especially fortunate for those who loved him that his last and only illness was of short duration.

It can be said of Isaac Stander, an outstanding engineer, that his principal characteristic was a sense of human values. He recognized that these values were more important than material accomplishments and, when he passed on, his friends and associates realized and appreciated that the world was better, both spiritually and materially, for his having "passed this way."

Mr. Stander was elected a Member of the American Society of Civil Engineers on October 10, 1921. He became a Life Member in January, 1946.

HOMER REED STANFORD, M. ASCE[1]

Died April 7, 1947

Homer Reed Stanford was born on June 26, 1865, in Alton, Ill. His parents were Homer Stanford of Alton and Evalyn Clark of Boston, Mass.

In 1820 Joshua Reed Stanford, whose ancestors came from England to Massachusetts in about 1650, faced the hardships of the westward migration of that time with his wife Elizabeth. They settled in Griggsville, Ill., and later moved to Alton, where a son, Homer Stanford (father of Homer Reed Stanford), was born in 1835. The present city hall of Alton is located on the site of the Stanford family homestead.

Young Stanford attended the local public schools, being graduated from the Alton High School in 1881. He then took a course in the St. Louis (Mo.) Manual Training School and in 1884 entered Washington University in St. Louis. At the university his scholarship won him election to the Sigma Xi society and he was graduated in 1888 with the degree of Bachelor of Engineering, magna cum laude.

During the ten years following graduation he had varied engineering experiences. These included surveys for the design of sewer and irrigation works, shop inspection and erection supervision of steel bridges, supervision of a metallurgical laboratory, and superintendence of a bridge company.

In 1898 the war with Spain created a demand for a material enlargement of the shore facilities of the United States Navy. The existing strength of the Civil Engineer Corps of the Navy was insufficient to handle the expanded requirements and a considerable increase in the corps was authorized by Congress. Additional officers were selected from applicants whose records of education and experience were satisfactory and who attained a high mark in a comprehensive examination. Mr. Stanford met these requirements and in May, 1898, was commissioned a Civil Engineer in the Navy with the rank of Lieutenant (jg).

---

[1] Memoir prepared by DeWitt C. Webb, M. ASCE.

After a short period of indoctrination at the Bureau of Yards and Docks of the Navy in Washington, D. C., he was ordered to the navy yard at Boston. From there he was sent to the Philippines where he made subsurface investigations and prepared plans and specifications for a large coaling plant. Returning to the United States he was assigned, successively, to the navy yards at Mare Island, Calif., Pensacola, Fla., and Philadelphia, Pa., and was promoted, first, to the rank of Lieutenant and then to that of Lieutenant Commander. His efficient and economical handling of day labor construction work at Philadelphia was noted by the Secretary of the Navy. Doubtless this was a material factor in bringing about his appointment, in January, 1912, as chief of the Bureau of Yards and Docks with the rank of Rear Admiral.

As chief of the bureau Admiral Stanford was responsible for the solution of many complex engineering problems, the most difficult of which was the completion, after an initial failure, of a large masonry dry dock at Pearl Harbor, Hawaii. This work was fully described in a paper[2] by Admiral Stanford, and in 1917 he received the Society's James Laurie Prize for this paper.

After the completion of his four-year term as chief of the Bureau of Yards and Docks he was assigned to duty at the Philadelphia Navy Yard and, later, at the Boston Navy Yard. While in Boston the State of Massachusetts was constructing the large Commonwealth dry dock in South Boston. Admiral Stanford was designated naval representative on this work, advising in its design and construction to insure that, when completed, the dry dock would be suitable for use by naval vessels.

After leaving Boston Admiral Stanford was for several years a special adviser to the chief of the Bureau of Yards and Docks. Among other tasks during this period he was ordered to Manila, Philippine Islands, to investigate and report on the location of a large floating dry dock. He was also a member of a board charged with the settlement of claims for damages to fishing rights in Naval waters at Pearl Harbor.

On June 26, 1929, Admiral Stanford was placed on the retired list, in accordance with law, at the age of sixty four. The occasion was marked by a dinner given by his associates in Washington and vicinity. After his retirement he continued to live in Washington, and, later, in Chevy Chase, Md.

Admiral Stanford was awarded the Navy Cross for his work in World War I, and also received the Spanish American War and Philippine Insurrection medals. In 1937 he was awarded the honorary degree of Doctor of Science by Rensselaer Polytechnic Institute in Troy, N. Y. He was a member of the Metropolitan and Chevy Chase clubs, and of the Boston Society of Civil Engineers, the Franklin Institute, and the Newcomen Society. He was an expert contract bridge player and an enthusiastic golfer. Weather permitting, a round of golf was a vital part of his daily routine up to the commencement of his last illness.

---

[2] "Pearl Harbor Dry Dock," by Homer Reed Stanford, *Transactions*, ASCE, Vol. LXXX, December, 1916, p. 223.

Admiral Stanford loved his family, his friends, and his work. He had a keen business sense and was energetic and impulsive. True to Navy tradition he was critical of delays, errors, and inefficiency.

In 1896 he was married to Maidie Mason, the daughter of Carrington and Maria Bodie Mason, of Memphis, Tenn. Mrs. Stanford died on November 10, 1923. Admiral Stanford is survived by a son, Carrington Stanford of Wayne, Me., and a daughter, Elizabeth (the wife of Rear Admiral Earl LeRoy Sackett, U. S. Navy, Retired).

Funeral services were held at the Fort Myer, Va., chapel and interment was in Arlington Cemetery in Virginia with the honors accorded to his rank.

Admiral Stanford was elected a Junior of the American Society of Civil Engineers on October 1, 1890; an Associate Member on November 7, 1894; and a Member on September 4, 1901. He became a Life Member in January, 1929.

HENRY ARTHUR VAN ALSTYNE, M. ASCE[1]

DIED JANUARY 23, 1947

Henry Arthur Van Alstyne was born at North Chatham, N. Y., on October 9, 1869. He attended the Nassau (N. Y.) Academy and Marshall Seminary at Easton, N. Y. He then worked about three years at the Schenectady (N. Y.) plant of the General Electric Company to earn money to pay for a college education. He was graduated from Union College at Schenectady in the class of 1893 with the degree of Civil Engineer. In college he was a member of the Psi Upsilon fraternity and in his senior year was elected a member of the scientific society of the Sigma Xi.

After graduation Mr. Van Alstyne worked about a year as assistant city engineer of Schenectady and another year constructing a system of sewers in the Village of Fort Plain, N. Y. He then spent about three years, 1894 to 1897, as leveler and assistant engineer in the Rochester (N. Y.) office of the state engineer and surveyor. He left this office for about two years to do special work for other employers and re-entered state service in 1899 as first assistant engineer in the Eastern Division office. In that same year he was promoted to resident engineer, and in 1901 division engineer, Eastern Division.

In 1903 the people of the State of New York had approved the Barge Canal Law appropriating $101,000,000 for the construction of the Barge Canal. An Advisory Board of Consulting Engineers was appointed to pass upon the design of the canal. The late Edward A. Bond, M. ASCE, resigned as state engineer to accept the chairmanship of that board. Benjamin B.

---

[1] Memoir prepared by Frederick W. Newton, Assoc. M. ASCE.

Odell, Jr., then Governor of New York State, appointed Mr. Van Alstyne to succeed Mr. Bond as state engineer and surveyor. In November of that year Mr. Van Alstyne was elected to the office for a full term. Eleven years after graduation from college, it became his job to create the organization which was to design and supervise the building of the Barge Canal. He was in charge of one of the greatest engineering projects of all time.

That was not all. The state engineer was also responsible for the highway work in the state. When Mr. Van Alstyne entered office there were no state highways, and in the preceding seven years, from 1898 to 1904, the state had appropriated only $3,273,000 as its share of the cost of building and maintaining "state-aid" roads. Mr. Van Alstyne was not satisfied; he worked and lectured in support of the "good-roads" movement, and for greater participation by the state. In 1905 the people amended the constitution to authorize the state to borrow $50,000,000 for highway work, and in 1906 the state spent more than $5,000,000 on its roads—all designed and supervised by the state engineer's office.

Road building in 1905 was not the science that it later became. Mr. Van Alstyne explained in a lecture given at Cornell University at Ithaca, N. Y., in May, 1905, that under the law he had a choice of a telford or a macadam or a gravel surface and he might in a proper case use brick or a steel rail track. He ordered a series of experiments with the use of road oil and he particularly recommended the use of oils with an asphalt base. Such experiments had already been conducted in California, and Mr. Van Alstyne remarked particularly that Nassau County had taken the lead in similar experiments in the east. The use of concrete in road work was not considered—concrete being mentioned only as a substitute for stone masonry in structures. However, the State Testing Laboratory reported in both 1904 and 1905 that Portland cement had not yet reached the required standards and that many of the samples submitted were rejected. There was so much public distrust of the durability of concrete structures that a special committee was appointed to determine whether concrete was sufficiently reliable to be used in the structures for the Barge Canal.

In his search for suitable materials for road surfaces Mr. Van Alstyne in 1904 retained Prof. William H. Burr,[2] M. ASCE, to test all the rocks of the state for resistance to abrasion and for a stone dust which would make a suitable binder for water-bound macadam. One of the recommendations in Mr. Van Alstyne's report for 1904 is interesting in that it undoubtedly represents the first gun in an ever-continuing battle—the assessment of an automobile license tax, the proceeds to be used exclusively for the maintenance of roads.

The Barge Canal Law, as approved by the voters, contemplated the use of one thousand-ton barges about twenty-seven feet wide by one hundred and fifty feet long with ten-foot draft. The law specified that the locks should be twenty-eight feet wide and eleven feet deep. Mr. Van Alstyne insisted that the locks should be at least as large as those on the Canadian canals—namely, forty-five feet wide by fourteen feet deep. He succeeded in obtaining authority to increase the width of the locks to forty-five feet, and the depth to twelve feet.

---

[2] For memoir, see *Transactions*, ASCE, Vol. 100, 1935, p.1617.

The use of the canal, particularly during World War I and World War II, demonstrated the advantages of the larger locks, and after the canal was completed to the twelve-foot depth large sums of money were expended in deepening it to the fourteen feet which Mr. Van Alstyne had originally recommended.

Why was so young a man given so much responsibility? The answer is that Mr. Van Alstyne had already attracted attention to himself for his initiative and ability. He was in charge of some work on grade crossings over the Erie Railroad on what later became route No. 17 through Suffern and Tuxedo, N. Y., and to the north. Instead of building bridges he relocated the highway to eliminate the bridges. Edward H. Harriman, the railroad builder, and the owner of the land from the Hudson River to the New Jersey state line (which his widow gave to New York State and which later became part of Bear Mountain Park), was so impressed that he personally urged Mr. Van Alstyne's promotion. Mr. Harriman's support had two results: (1) A personal friendship and business association which lasted throughout Mr. Harriman's life; and (2) an order issued by Mr. Van Alstyne when he became state engineer, and which became a part of the permanent policy of the state engineer's office that, in rebuilding a highway, the engineer should change the route whenever such change would materially reduce the cost of future maintenance.

After he became state engineer, Mr. Van Alstyne found that progress in the improvement of highways was impeded by the many, and sometimes conflicting, laws under which highways were built and maintained. He urged amendments and finally, in his report for 1906, he urged that all existing laws relating to highways be repealed and that there should be one act, to be known as the "Highway Law." This recommendation was adopted in 1909 when the "Highway Law" was enacted.

Mr. Van Alstyne was personally responsible for another innovation which is of particular interest to contractors. The memory of the legal battles and the charges of fraud in connection with the so-called "$9,000,000 Improvement" of the canals was still fresh in the minds of the people. Most of that difficulty had resulted from differences of opinion as to the classification of excavated materials. To avoid a repetition Mr. Van Alstyne introduced two new features in state construction work. He required a single bid at a single bid price for "unclassified excavation" including all materials of every nature. However, to make this system practical and to secure bids which would not be excessive, he directed that very complete borings be made and that the record of borings, including samples, should be shown to all bidders. There were no substantial charges of fraud or misconduct in the entire construction of the Barge Canal. If there is any one person who deserves credit for this accomplishment it is the man who built up the organization and who established the policies under which the canal was constructed.

In 1906 Mr. Van Alstyne ran for re-election against Frederick Skene, a Democrat and also the candidate of the William Randolph Hearst Independent League. Charles Evans Hughes was elected Governor of New York State on the Republican ticket. Mr. Van Alstyne ran well ahead of the rest of the Republican ticket but all except the Governor lost to the combination of the Democrats and the Independent League.

Early in 1907 Mr. Van Alstyne was retained as consulting engineer by the Atlantic Gulf and Pacific Company, contractors building a section of the Champlain Canal. In the following year Mr. Harriman organized the Acme Engineering and Contracting Company. He intended that this should be a construction company to build the railroads necessary to complete his transcontinental road from Boston, Mass., to Seattle, Wash. He made Mr. Van Alstyne president and directed that the company take Barge Canal and highway contracts to train the organization for its future railroad work. That company operated under Mr. Van Alstyne's direction from 1908 to 1925 and built several sections of the Barge Canal and of the new state highway system.

During the period from 1925 until Mr. Van Alstyne's retirement in 1935, he was president and chief engineer of the Mohawk Dredge and Dock Company and the New York State Dredging Corporation, and was in charge of the construction of many sections of the Barge Canal, the deepening of the Hudson River below Albany (N. Y.), dredging in Boston harbor, and work for the New York Central and Delaware and Hudson railroads.

An illness from which he never fully recovered forced Mr. Van Alstyne's early retirement from active engineering work. He maintained a very keen interest in public affairs. He read a great deal, particularly history and biography, and was a great lover of music. He maintained contact with many of the younger engineers who had started their professional work under him. He was intensely proud of their achievements and always ready to advise and to give material aid to those who brought their problems to him.

On October 11, 1899, Mr. Van Alstyne was married to Bertha Stone Neher of Nassau. Mrs. Van Alstyne and Mr. Van Alstyne's two sisters, Mabel V. A. (Mrs. F. D. Marsh) and Edith V. A. (Mrs. C. J. Sheldon, Jr.), survive him.

Mr. Van Alstyne was elected an Associate Member of the American Society of Civil Engineers on June 6, 1900, and a Member on December 3, 1907. He became a Life Member in January, 1935.

ALBION NOYES VAN VLECK, M. ASCE[1]

DIED AUGUST 22, 1945

Albion Noyes Van Vleck, the son of Eugene A. and Florence M. (Noyes) Van Vleck, was born on June 6, 1886, at Sligo, Md. His home was just outside the District of Columbia. He attended both grade and high schools in Washington, D. C., being graduated from Central High School in June, 1904. Mr. Van Vleck won a scholarship to Lehigh University in Bethlehem, Pa., and, after completing the regular four-year course at Lehigh, he was graduated in June, 1908, with the degree of Civil Engineer. While at college he composed (and won first prize for) a football song which to this day is always played at its games. In later years he again won first prize for an alumni song.

For a few years, until 1911, he was employed in a number of structural steel offices, including those of the J. B. and J. M. Cornell Company; Post and McCord, Incorporated; Milliken Brothers, Incorporated; the Baltimore Bridge Company; and the American Bridge Company. This work was a fine preparation for his career in structural engineering, giving him experience as detailer, checker, and squad boss. During the following two years Mr. Van Vleck held the position of assistant chief engineer for F. M. Andrews, an architect of New York, N. Y. Beginning in April, 1913, and for four years, he served as chief engineer and construction manager for Guy Lowell, an architect with offices in New York and Boston, Mass.

On the entry of the United States into World War I, he offered his services to the Army and received a commission as Captain. While commanding Battery C, Sixtieth Artillery, Coast Artillery Corps, he participated in the St. Mihiel and Meuse-Argonne offensives in France in 1918. He came through with no injuries but was gassed. This left him with a bronchial weakness which, although it did not incapacitate him, made him susceptible to bronchial ailments. His military services were recognized by promotion, and he received his discharge in January, 1919, as a Major.

Following this Army experience, he held two positions briefly—first, as district plant engineer and secretary of the District Claims Board of the United States Shipping Board for the Great Lakes District, centering in Cleveland, Ohio. After a year of this work he served for two years as assistant general manager of the Saginaw Shipbuilding Company in Saginaw, Mich. From Michigan he moved to Pittsburgh, where he served in an engineering capacity for the Dravo Contracting Corporation for about a year. Mr. Van Vleck's next employment lasted for almost six years, when he returned to New York as chief engineer and construction manager for J. E. R. Carpenter, an architect and builder.

With all this experience and background, he felt qualified to enter the structural engineering field as a principal. In November, 1927, he formed a partnership with the writer under the firm name of Hurlbut and Van Vleck and opened an office as consulting engineer in New York. The firm had an office in the Architects Building at 101 Park Avenue, and later moved to 370 Lexington Avenue, a building on which the firm had executed the engineering design.

---

[1] **Memoir prepared by Charles C. Hurlbut, M. ASCE.**

Although not as colorful, perhaps, as some of his other assignments, this chapter in Mr. Van Vleck's life was one of the most productive of his career. The firm of Hurlbut and Van Vleck not only took over the structural work in connection with Mr. Carpenter's office, but also carried on a general consulting practice in structural engineering—specializing in buildings and foundations in steel and reinforced concrete. Among other assignments from Mr. Carpenter's firm were the fifty-three-story Lincoln Building on Forty-second Street; the Madison Avenue-Fifty-second Street Building (twenty-seven stories); and the forty-five-story office building at 515 Madison Avenue—all in New York. Notable structures were also designed for other clients, including the Marine Hospital at Stapleton, Staten Island, in New York; the National Democratic Convention Hall at Houston, Tex.; and the building at 370 Lexington Avenue in New York (twenty-seven stories). For the International Telephone and Telegraph Company the firm designed an addition to its New York building (twenty-seven stories), as well as the headquarters building in Bucharest, Rumania. The latter building subsequently went through a severe earthquake without a single crack.

In spite of these notable accomplishments, the path of this consulting engineering office, with the onset of the depression, was not an easy one. After struggling against insuperable conditions, the partners decided to give up a losing venture. The office was closed and both took positions with the Public Works Administration. Mr. Van Vleck's work was in the regional office in New York under M. E. Gilmore, M. ASCE, regional director. He continued as senior engineer in charge of work in the State of New York for more than four years, from November, 1933, to January, 1938. In this office he had charge of more than $250,000,000 worth of major projects, such as buildings, bridges, sanitary systems, grade crossing eliminations, docks, highways, hospitals, and schools.

Accepting appointment with the New York City administration, he served for almost two years, until November, 1939, as first deputy commissioner of the Department of Housing and Buildings. Here his wide technical experience served him in good stead, and he maintained his excellent reputation among engineers and architects. He rendered a particularly fine service in organizing six city departments into one combined department under the new city charter. Under his executive charge were 1,100 employees of the department, including engineers and inspectors.

In dealing with the political phases of this position, Mr. Van Vleck did not feel at home; and, as a result, he returned shortly to his former activity as consulting engineer, maintaining an office in New York, from November, 1939, to June, 1941. Among other commissions, he was retained as consultant by the Mortgage Conference of New York, having charge of inspection and rating of all apartment buildings under erection in New York and Bronx counties in New York City. Then, for a year, until April, 1942, he was employed as senior construction engineer (civilian) in the Army Quartermaster Corps, engaged in engineering work covering design and construction of cantonments, and ordnance and chemical warfare plants for World War II. A major assignment in this connection was the construction of Camp Wheeler, Ga.

All his life, Mr. Van Vleck had the gift of making friends. This faculty, plus his long-standing interest in matters of the Masonic Order, led to his next and final business association. Always an ardent member of the F. and A. M., he had held many offices and trusts of responsibility in that fraternal organization. Starting in April, 1942, he devoted the remaining years of his life to looking after the interests of Masons in the government services. His title was Executive Director of the Grand Lodge Committee on Military and Naval Service (New York State) and his duties covered such varied activities as the establishment and continuance of contact with more than thirty-five thousand men serving in the Armed Forces; the establishment and operation of twelve service centers for service men in the State of New York, in total catering to about seven thousand men per week; hospital visitations; and extension of aid to wounded service men. The last function required special emphasis on bridging the financial gap between discharge from the service and reestablishment in civilian life. This social service work, with its emphasis on the welfare of veterans, made a fitting conclusion to his lifetime of service.

Mr. Van Vleck's Masonic activities included the following offices: Past Master, Roome Lodge No. 746, New York; Past Master (First Master), Lodge of the United Services No. 1118, New York; Past Assistant Grand Lecturer, New York; Past High Priest, Jerusalem Chapter No. 8, Royal Arch Masons, New York; Past Commander, Coeur de Lion Commandery No. 23, Knights Templar, New York; Past Master of the United Services Lodge, and also of the Manhattan Chapter of the National Sojourners. He was also a member of the Structural Engineers Society of New York, of which he was a past president; the New York Society of Professional Engineers; the Society of American Military Engineers; the Architectural League of New York (an Associate Member); and the Holland Society of New York.

Possessed of a warmhearted personality, Mr. Van Vleck made a host of loyal friends. Among his recreations, he was an accomplished pianist, organist, and composer (an odd accomplishment for an engineer) and found pleasure in many outdoor sports. In particular, he was adept at hunting, fishing, and golf. His greatest pleasure in later years was gardening and his yard was quite the showplace of the neighborhood. Roses of all types and rare species were his pride.

On August 24, 1907, he was married to Anna L. Magee in Bethlehem, Pa. He is survived by his widow; two daughters, Florence (Mrs. O. Johnson, Jr.) and Anna (Mrs. F. H. Gibbs); three sisters; one grandson, Gerald M. Gibbs; and two granddaughters, Carole A. Gibbs and Anne Marie Johnson.

Mr. Van Vleck was elected a Member of the American Society of Civil Engineers on August 27, 1928.

WILLIAM BERNARD WARD, M. ASCE[1]

DIED JUNE 6, 1947

When the sand of time runs out on a man like "Bill" Ward, in its wake follow reminiscences that reach beyond the mere technicalities of an engineer's life—not just the slide rule, the tee square, and the structures created, but the intimate bits of his life remembered by those who were associated with him. The closer the associations, the finer and the more enduring are those memories.

William Bernard Ward, the son of John and Sarah (Curtiss) Ward, was born in Cincinnati, Ohio, on December 16, 1890. Educated in the public and parochial schools, Mr. Ward began his career in architecture and structural engineering in 1906 as a cub in the office of Samuel Hannaford and Sons, Architects. The priest in charge of young Ward's education must have foreseen in him the superb draftsman and clear thinking engineer of the future when he recommended to Harvey Hannaford that the boy, Bill, be taken in charge by that firm for mutual benefits. During this engagement of some six years, and in the following three years with Krucker and Deckbar, Architects (Los Angeles, Calif.), periods of study in various technical school and college courses built up his engineering foundation.

Returning to Cincinnati in 1915, a short engagement with Garber and Woodward, Architects, and a few months with Tietig and Lee, Architects, intervened before Mr. Ward entered military service. Enlisting on June 27, 1918, he was assigned to the 309th Engineers, United States Army, and was sent to France on base hospital construction and road maintenance. Returning to the United States as a Master Engineer, Junior Grade, of Headquarters Company, he was discharged from the service on July 12, 1919, having taken advantage meanwhile of the opportunity to attend lectures at the Sorbonne in Paris.

Rejoining Tietig and Lee as a structural engineer in 1919, he remained in charge of that department, and was superintendent of construction until 1942, except for two short engagements in Los Angeles. In 1943 Mr. Ward established his own office for the practice of structural engineering, continuing in the wide range of school, office, hospital, and industrial buildings which had been his interest in years past, and with the same simplicity, directness, and orderliness of solution which had formerly characterized his work.

Representative of Mr. Ward's operations are the following buildings, for which he designed the structures, often having a part in the architectural planning and in the supervision of construction: The Cincinnati Chemical Works building, the William S. Merrell Company Pharmaceutical Products plant, the J. and F. Schroth Packing Company buildings, the Knox Presbyterian Church, the Christ Hospital and Nurses' Home, the Parkside Hotel, the Atlas Bank Building, the City Garage and Testing Lane, Shillito's Garage at 7th and Elm Streets—all in Cincinnati—the Medical Arts Building (Memphis, Tenn.), and the Mengel Company war plant (Louisville, Ky.).

---

[1] Memoir prepared by F. W. Morrill and Hunter W. Hanly, Members, ASCE.

Mr. Ward was a modest and retiring man with a keen sense of humor, and it was natural that he should be an avid reader of select current literature. He played a good game of bridge and was proficient at golf. For many years, he was a member of the Advent Canoe Club at Tower Hill in Cincinnati, and, in 1912, he was roller skating champion of Ohio. His family treasures the many trophies Mr. Ward won in tournaments at the Hyde Park Tennis Club in Cincinnati.

He continued his interest in Army associations by holding membership in the Bentley Post of the American Legion and its Voiture of the Forty and Eight. Mr. Ward was an early member of the Cincinnati Architectural Society, an organization of draftsmen and younger architects for social and cultural purposes. Later, he conducted classes in engineering for the club. He was an officer of the Cincinnati section of the Architects' Society of Ohio, and he was admitted to corporate membership in the American Institute of Architects in 1943. In addition, he had been a member of the structural provisions subcommittee of the Building Code Revision Committee for the City of Cincinnati for several years.

In 1927 Mr. Ward was married to Marie Oberschmidt; they had two children, William B., II, and Sally Jo, who survive one of Cincinnati's most amiable and able men in the architectural and engineering professions.

Mr. Ward was elected an Associate Member of the American Society of Civil Engineers on December 15, 1924, and a Member on June 30, 1935.

WILLIAM GODFREY WEGNER, M. ASCE[1]

DIED AUGUST 25, 1947

William Godfrey Wegner, the son of Edward J. Wegner and Emelie (Schmidt) Wegner, was born on October 9, 1897, in New York, N. Y., but his family moved to a German community in the Polish part of Russia before he was a year old. The family lived there until William was about nine years old, when they returned to the United States and took up residence at Niagara Falls, N. Y. He developed a liking for mechanical things and educated himself for the engineering profession by attending night schools and studying with the International Correspondence School.

Mr. Wegner worked as a tracer and draftsman at the United States Light and Heat Company and also at the Titanium Bronze Company, both in Niagara Falls, for a short period. The family then moved to Tacoma, Wash., where he spent the remainder of his life.

---

[1] Memoir prepared by John S. Abel, M. ASCE.

On January 28, 1920, he entered the employ of the Weyerhaeuser Timber Company at Tacoma. With the exception of a few months at Snoqualmie Falls (Wash.) in 1920, he spent twenty-seven years continuously in the service of the Tacoma engineering office of the Weyerhaeuser Timber Company. During this time he worked on plans and improvements for all the manufacturing branches of the company. Starting as a draftsman, he advanced to the position of designing engineer for the company, and contributed much to the progress of new developments in the mills.

His memberships included those of the Zion Lutheran Church, the Society, and the Tacoma Engineers' Club.

He was married to Pearl V. Wilson on April 22, 1924; they had two children—a son, Harvey Edward (born on August 12, 1925) and a daughter, Martha Ann (born on October 4, 1932). He died from a heart ailment, and is survived by his widow and two children, in addition to one brother and four sisters.

Mr. Wegner was elected a Member of the American Society of Civil Engineers on August 13, 1945.

GEORGE GROVER WICKLINE, M. ASCE[1]

DIED NOVEMBER 27, 1943

George Grover Wickline was born on January 5, 1883, near Stephenville, in Erath County, Tex. He was the son of Lloyd Edward and Arie Elizabeth (Ramage) Wickline.

He came of pioneer stock. His father was born in Monroe County, West Virginia. When he was six years old, the family moved to Miami County, Kansas, and later, as a young man, he moved to Erath County, where he became a rancher and a schoolteacher, and where he lived until his death in 1926. His mother, the daughter of Josiah and Margaret Ramage, was born in McCool, Miss. The family moved to Texas in 1873 and settled in Erath County, and there Elizabeth Ramage was married to Lloyd Edward Wickline in 1880.

George Wickline attended the public schools of Stephenville and John Tarlton College at Stephenville. In 1901 he entered the University of Texas in Austin, from which he was graduated in 1904 with the degree of Civil Engineer.

After graduation, Mr. Wickline's first engineering job was that of instrumentman and bridge inspector in the construction of the St. Louis, Brownsville and Mexico Railroad between Corpus Christi, Tex., and Bay City, Tex. Then he was, successively, engineer for the North Texas Traction Company and assistant engineer of Dallas County, Texas, in the expenditure of a large bond issue on the first county-wide highway improvement program in north Texas.

---

[1] Memoir prepared by Banks McLaurin, Assoc. M. ASCE.

On the completion of the highway building program in Dallas County, Mr. Wickline took the advice of Horace Greeley and went west to seek adventure and fortune in California. He obtained employment as assistant engineer in the preliminary location survey, in the preliminary plans, and in the estimates for the first great aqueduct to be built in Southern California; this was known as the Owens River Aqueduct, which was to bring water from the region of Owens Lake on the western edge of Death Valley, to the City of Los Angeles. For the early years of the twentieth century this was a great engineering project and attracted world-wide attention. Mr. Wickline spent three years in the mountains, the desert, and Los Angeles on this great engineering project.

After three years in the west, Mr. Wickline returned to his native state, Texas, working for the El Paso and Southwestern Railroad Company at El Paso, and for the engineering department of that city. He then returned to Dallas in 1910 and was employed by Dallas County as bridge engineer for three years. From 1912 to 1914 he was engineer of bridges and buildings for the Southern Engineering Construction Company in Dallas, later a part of the Texas Electric Railway Company.

After completing this work, he was again employed by Dallas County as engineer in the preparation of the designs and as supervising engineer in the construction of the Commerce Street viaduct. Years later, in 1936, this viaduct was replaced by the Triple Underpass, which was also designed and constructed under Mr. Wickline's supervision as bridge engineer of the Texas Highway Department. When the viaduct was completed, he was employed by the City of Dallas as bridge engineer.

In January, 1918, Mr. Wickline accepted the position of bridge engineer of the Texas Highway Department. This department was created by legislative action in 1917, and Mr. Wickline was its first bridge engineer. He held this position until his death, with the exception of a three-year period (from 1936 to 1939) when he was given a leave of absence to serve as supervising engineer for construction of the Neches River bridge near Port Arthur, Tex.

The Neches River bridge spans the Neches River on the highway between Port Arthur, and Orange, Tex. It has long approaches rising on a 5 per cent grade from the level marshes on both sides of the river to a 680-foot main cantilever span affording 176-foot vertical clearance for ocean-going ships. The over-all length is 7,815 feet. The cost of the project was $2,750,000, and was provided by a Public Works Administration grant, a Texas Highway Department allotment, and a local bond issue of Jefferson County, Texas.

From the minutes of the Texas Highway Commission of December 16, 1943, the following is quoted:

> "Mr. Wickline has served as bridge engineer for the Texas Highway Department since 1918, and in point of service was the oldest member of the engineering staff. The outstanding accomplishment of his long, successful career was the Neches River Bridge between Port Arthur and

Orange, the largest bridge ever built by the Department. This, and many other bridges throughout the state, designed and constructed under his supervision, will stand as a lasting monument to his ability."

Mr. Wickline's pastor, Dr. Edmund Heinsohn, said of him:

> "Men are known by their works. Ye shall know them by their fruits. The work of George Wickline has been characterized by faith, intelligence and action. The great bridges in this state conceived in the faith of George Wickline were born out of his knowledge and understanding, and under his guiding hand were made into concrete actuality. No wonder he had a quiet and an abiding joy in the work of his faith, his mind and his hand."

Mr. Wickline was a registered professional engineer of the State of Texas. He was a member of the Society of American Military Engineers, the Society of Professional Engineers, the American Road Builders' Association, and the American Association of Highway Officials. Mr. Wickline was also a member of the Bridge Committee of the American Association of Highway Officials, and as such he took part in the development of the specifications and standards of the organization.

Death came to Mr. Wickline while en route from Austin to Chicago, Ill., to attend a meeting of the American Association of Highway Officials (A.A.H.O.). In resolutions passed by the Committee on Bridges and Structures of the A.A.H.O. are these words:

> "It is significant that the end came as he was en route to Chicago to attend the meeting. He was a charter member of our Committee. For years we have worked together. There was ever wisdom in his counsel, warmth in the depth of his human sympathy and understanding. Truly down the years we learned to love him greatly."

Mr. Wickline's activities extended beyond the engineering field. He was a member of Austin Lodge No. 12, A. F. and A. M., in Austin, and the Rotary Club of Port Arthur. In religion, he belonged to the University Methodist Church at Austin, serving very efficiently on the Board of Stewards of this church. Mr. Wickline's hours of relaxation were devoted to photography, to collecting historical books, and to gathering the folklore of Texas.

In 1913 Mr. Wickline was married to Hattie Belle Coffman, daughter of William Amos and Margaret (Barbour) Coffman, at Dallas. He is survived by his widow; one daughter, Joyce (Mrs. Robert Darling); and two granddaughters, Ruth and Joyce Darling.

Of modest and dignified bearing, Mr. Wickline was quiet and considerate, as well as painstaking, logical, and thorough. His opinions were formed slowly, but surely and soundly and he was a conscientious and an untiring worker. Mr. Wickline's death removed a fine Christian gentleman, a valuable citizen, and a most capable engineer; no better epitaph can be written of his life than two words: "Well done."

Mr Wickline was elected an Associate Member of the American Society of Civil Engineers on September 3, 1912, and a Member on July 6, 1920.

ARTHUR JACOB WIDMER, M. ASCE[1]

DIED APRIL 30, 1947

Arthur Jacob Widmer was born in Newark, N. J., on April 1, 1883, of Swiss ancestry. He was the son of Gustavus H. and Mina L. (Luerich) Widmer.

He attended the public school of the small town of Port Richmond, Staten Island, which became the Borough of Richmond in New York, N. Y., where he spent his boyhood. Later he studied at the Westerleigh Collegiate Institute in an adjoining small town on Staten Island. He entered Cornell University in Ithaca, N. Y., in 1900. For two years, partly between college terms and on special leave from the university, he was a mechanic on machine output studies in several important eastern manufacturing plants. During the period from 1904 to 1907 he was engaged in Detroit, Mich., as research designer on adding machine improvements; as designer with the Solvay Process Company on coke ovens, coke briquetting plants, ammonia and soda ash processes; and as designer of the then unfamiliar reinforced concrete construction. Later he was appointed manager of the rib bar department of the Trussed Concrete Steel Company.

In 1907 Mr. Widmer was transferred to St. Louis, Mo., for the purpose of opening an area office and managing the business of selling the rib bar product of the Trussed Concrete Steel Company. During this period of infancy of the reinforced concrete construction industry, it was necessary, not only to sell the reinforcing steel products but to design structures using the rib bar, prove the economy of its use, and supervise the installation of the product. Many times full-size loading tests were performed under his supervision to prove the adequacy of the design using the patented rib bar. During the subsequent seven years, Arthur Widmer acted as consulting engineer on numerous reinforced concrete and steel structures.

In 1914 he established a separate office in St. Louis for the independent practice of mechanical and civil engineering and early in 1917 formed the Widmer Engineering Company of which he served as president until his death. In 1921 he formed and headed Arthur J. Widmer and Associates, Incorporated, an architectural firm. He was one of the very few persons licensed in the State of Missouri to practice both engineering and architecture.

On leave from the Widmer Engineering Company during World War I, he was commissioned a Major and assigned to the Engineering Branch of the Construction Division of the United States Army in various positions, terminating as Chief of the Requirements Branch of the Construction Division. Here, in connection with the large construction program of the Army, he was responsible for appropriation estimates, preparation of data for Congress and other agencies, liaison with the general staff, project esimates, authorizations, and allotments of funds. In the Army service, he demonstrated his ability in organization and administration and in the planning of equipment and personnel for production.

---

[1] Memoir prepared by Elmer H. Oechsle, Pres., Widmer Eng. Co., St. Louis, Mo.

Again on leave in 1940, Arthur Widmer, then a Colonel of Engineers in the Army Reserve, served as a special assistant on organization and procedures to the chief of the new Construction Division which was being developed to carry on the enormous building program of the Army for World War II.

Arthur J. Widmer had been responsibly connected with the design of more than one thousand structures of an extremely varied nature. The construction of a large part of these structures was also carried out under his supervision. His activities embraced innumerable types of civil and mechanical engineering projects, spreading largely throughout the United States and extending abroad as far as China. Many of the works successfully designed by him are outstanding for their size and boldness of conception. Included was the reinforced concrete, twenty-one-story Railway Exchange Building in St. Louis, approximately three hundred feet square, which, when constructed in 1910, involved the largest single usage of reinforced concrete for a building. As a pioneer in reinforced concrete construction, he developed, and assisted and directed others in the development of, many outstanding features of modern building construction. He developed and patented a pre-cast type of tunnel design and a method of construction suitable for inland streams subject to extreme floods and for shifting river bottoms which prevented the use of ordinary tunnel construction practices.

Arthur Widmer obtained more than a score of patents in such varied fields as construction methods and materials, tunnels, and processes relating to fine particles of matter. His familiarity with patents and his knowledge of mechanical devices and processes led to his being called upon for expert testimony in many important patent litigation proceedings. He made a hobby of research in connection with finely divided particles of matter, both microscopic and chemical. He had spent more than five thousand hours in physiochemistry research of this nature and as a result formed and temporarily headed a company to market certain finely processed particles in association with an alkaline chemical compound.

Colonel Widmer was prominent in military activities in the Army Reserve Corps and commanded various engineer reserve regiments. He energetically advocated preparedness; he participated in and led numerous patriotic movements and was also vigorously opposed to the many pacifist and unpreparedness movements which sprang up from time to time. He was an active member of the Reserve Officers Association and the local post of the American Legion.

Arthur J. Widmer was a member of the American Society of Mechanical Engineers, both the Missouri and National Society of Professional Engineers, the Society of American Military Engineers, and the American Concrete Institute. Other organizations of which he was a member included the Engineers' Club of St. Louis, the St. Louis County Chamber of Commerce, the Academy of Science of St. Louis, the Cornell Club of St. Louis, and the Missouri Athletic Club of St. Louis. During the past he held various offices in his clubs and societies and actively served on committee work.

In addition to his professional activities, Arthur Widmer gave generously of his time and energy to local civic affairs. He was energetic in his efficacy for the planning of a large city along new lines—initiating the movement to overcome the barrier in the St. Louis metropolitan area formed by the Mississippi River between the city proper and the localities on the east bank. He maintained that numerous crossings for vehicular traffic would, in effect, overcome the obstacle, and that the harbor of St. Louis would be a tremendous asset with increasing value in the future if no more overhead bridges were built and all future crossings were tunnels.

On November 20, 1906, Arthur J. Widmer was married to Lavina Roberts, daughter of George H. and Margaret (Lindsey) Roberts of Staten Island. In addition to his widow, he is survived by a son, Arthur J. Widmer, Jr., and by five daughters, Harriet Lindsey (Mrs. Bertram R. Kellaway), Mary Elizabeth (Mrs. John Hochreiter), Janet May, Alice Lavina, and Margaret Helen.

No more fitting tribute to Arthur J. Widmer can be expressed than that given by the Rev. Alphonse M. Schwitalla, S.J., in his "Tribute of Affection" as follows:

> "I have sought the phrase that will say in brief what I have found in Arthur Widmer's words and acts as most expressive of his true self. To me, he was a man of 'tempered strength.' His mind was strong because he was clear and sure in his knowledge, both human and technical. His emotions and feelings were strong because he was definite and penetrating in his appreciation of values. His will was strong because he was determined and firm in the pursuit of the objectives he had set before himself to attain. And that strength made him a man of achievement. A builder, a soldier, an organizer, a leader; a guide to others but also a cooperator; a man of stone and steel but also a man of dreams and visions; in all his traits and works there stands out the quality of strength as his characteristic imprint upon all he planned and performed, upon all he valued and loved, upon all he hoped for and ambitioned."

Mr. Widmer was elected a Member of the American Society of Civil Engineers on January 19, 1920.

CLEMENT CLARENCE WILLIAMS, M. ASCE[1]

DIED FEBRUARY 20, 1947

Clement Clarence Williams was born in Bryant, Ill., on February 21, 1882. His mother, Martha Ann (Davis) Williams, was a woman of unusual resourcefulness, managing with poise and dignity her family of nine and the simple home that was maintained on very moderate means. Clement's father, Isaac Greenbury Williams, was the operator of a small mill, and as a youth Clement Williams was interested and able in helping his father repair and maintain the mill equipment.

---

[1] Memoir prepared by Bruce G. Johnston, M. ASCE, assisted by R. K. Laros, Pres., R. K. Laros Silk Co., Bethlehem, Pa., and Mrs. C. C. Williams.

In his early childhood, the family lived on a farm a mile and a half from Banner, Ill., where the district school was located. Written notes of his earliest recollections on the long walks to and from school show a strong love of natural beauty. The old dirt road led part way over rolling hills, then along the bed of a winding creek, where in early spring green willows, wild flowers, and ferns in profusion made every turn of the road a cause for wonderment. In the summer the creek became a swimming hole, and in the winter, he and the children from the neighboring farms tried out their ice skates on its frozen surface.

Clement earned most of the funds to pay his way at the Canton High School and then attended the Southern Iowa Normal School at Bloomfield, Iowa, where he earned all his expenses from odd jobs at rates of from 10 cents to 25 cents per hour. At the normal school, where his brother was an instructor of mathematics, he too showed marked ability in this subject, and after being awarded the degree of Bachelor of Science in Natural Science became a teacher in the county school. His ambition for more practical application of his talents led him, in the fall of 1903, to enter the University of Illinois, at Urbana, where he made an excellent scholastic record. After receiving the degree of Bachelor of Science in Civil Engineering, in 1907, his career for a time became twofold; during the three months of each summer he gained experience as a design engineer, and during the remaining nine months he was a teacher of engineering—initially, in 1907, becoming instructor of civil engineering at the University of Colorado in Boulder.

In his summer work, Mr. Williams began, in 1906, as chainman and rodman for the Delaware, Lackawanna and Western Railroad Company at Scranton, Pa., continuing his railroad experience during successive summers, from 1910 to 1917, as checker, and later designer, for the Chicago (Ill.) office of the Chicago, Milwaukee and St. Paul Railroad Company.

In 1910, at the age of twenty-eight, Mr. Williams was promoted to the post of assistant professor at the University of Colorado, and during the same year he was married to Grace Josephine Black, who died six years later. His summer work for the railroads was a factor that led the University of Kansas, at Lawrence, to offer him, in 1914, a post as professor of railway engineering. In 1919, he was made head of the Civil Engineering Department of the same institution. During World War I he took leave of absence from the University of Kansas to be supervising engineer for the War Department on the construction of explosives plants near Nitro, W. Va.

In 1921, at the age of thirty-nine, he was married to Ora Louella Webb, and in the fall of the following year left the University of Kansas to become professor of civil engineering and head of that department at the University of Illinois. In the fall of 1926 he became dean of the College of Engineering at the State University of Iowa at Iowa City. While there, in 1934, he was honored by election to the presidency of the Society for Promotion of Engineering Education (later the American Society for Engineering Education).

In 1935 Dean Williams was chosen president of Lehigh University, at Bethlehem, Pa. Great educational advances had been made at Lehigh during the ten years prior to 1932. These had come nearly to a standstill during the depression years; stringent economies were necessary to keep the university from operating "in the red." Nevertheless, under the able guidance of President Williams, Lehigh University continued on the progressive course that had been charted. Whereas many other schools were forced to stand still, or lose ground, during the years between 1935 and the advent of World War II, Lehigh increased its undergraduate and graduate enrollment by 37 per cent and 57 per cent, respectively, and the endowment was increased by 40 per cent. New buildings, additions, and improvements costing more than a million dollars were effected during these depression years.

While at Lehigh, President Williams continued his broad interest in engineering education matters. He was active in the Pennsylvania Association of College Presidents and was a member of several commissions which made educational and administrative surveys of other institutions.

President Williams received honorary doctorate degrees from Lafayette College at Easton, Pa.; Northeastern University at Boston, Mass.; Bucknell University at Lewisburg, Pa.; the Hahnemann Medical College at Philadelphia, Pa.; the Case School of Applied Science at Cleveland, Ohio; Muhlenberg College at Allentown, Pa.; Moravian College and Theological Seminary at Bethlehem; and Rutgers University at New Brunswick, N. J.

Dr. and Mrs. Williams created in the President's House, at Lehigh, an atmosphere of warmth and friendliness and soon had a large circle of friends both within the university and in the community, into which Dr. Williams quickly extended his interests. He became an honorary member of the Rotary Club in 1936 and was a regular and interested attendant. His talks to the Rotary were examples of his ability to inject a rich good humor into the most profound subject. He was elected to the Board of Trustees of St. Luke's Hospital, became a director of the Union Bank and Trust Company, and was active in the work of the Chamber of Commerce, Community Chest, the American Red Cross, as well as being on the Executive Committee of the Bach Choir. His counsel and good judgment were extremely valuable in all these services and he gave thoughtful and painstaking attention to every problem presented to him.

Dr. Williams was a devout Christian of irreproachable character and had a high sense of devotion to duty. He was kindly, friendly, possessed of unusual administrative ability, high talent, a penetrating mind, and a well-rounded character. Both he and Mrs. Williams will for many years be affectionately recalled in Bethlehem.

In 1944 he retired from the presidency at Lehigh and moved to Madison, Wis., where he continued in practice as an engineering administrative consultant. At Madison he was a member of the University West End Club which, after his death, prepared a memorial, reading in part as follows:

> "In the death of Dr. Clement Clarence Williams the University West End Club lost one of its distinguished members. He became a member of the organization shortly after he and his family moved to Madison in 1944. A man of sterling character and great personal charm, he possessed the capacity not only of making friends, but of being a loyal friend himself. His sense of humor was keen and he could emphasize a point under discussion with an apt quotation, story, or anecdote. His kindness and tolerance for another's point of view coupled with his own reasoned conclusions made him a character to be loved, admired and respected.
>
> "All those who have known Dr. Williams either for a few years or for many, can attest to an enrichment of life which they have enjoyed because of his warm personality, his breadth of knowledge, his wisdom, his cultured taste, his honesty, and his sincerity."

Dr. Williams was the author of several books on masonry structures and foundations, a book on the engineering career,[2] and many bulletins and articles.

In addition to the organizations previously mentioned, he was a Thirty-Second Degree Mason, a member of Acacia, the Triangle Club, the University Club of Bethlehem, and the Manufacturers' and Bankers' Club of Philadelphia. He was a member of the American Railway Engineering Association, the Pennsylvania Academy of Science, Tau Beta Pi, Sigma Xi, Sigma Tau, the American Association for Advancement of Science, and the Academy of Political Science.

Dr. Williams is survived by his widow and their three children, Ora Louise (Mrs. Charles H. Hoffman), Ellen Webb, and Clement Webb.

Dr. Williams was elected a Junior of the American Society of Civil Engineers on June 1, 1909; an Associate Member on October 1, 1912; and a Member on October 8, 1918.

ALBERT WILLIAM ZESIGER, M. ASCE[1]

DIED JANUARY 9, 1948

Albert William Zesiger, the son of John and Elizabeth (Baumgartner) Zesiger, was born on April 11, 1868, on a farm in Switzer Township, Monroe County, Ohio. The farm was near the Village of Powhatan Point, where he passed his childhood and early manhood.

After receiving his high school education in Powhatan Point, he attended Earlham College in Richmond, Ind., and in 1896 was graduated with the degree of Bachelor of Science from Case School of Applied Science (later the

---

[2] "Building an Engineering Career," by Clement C. Williams, McGraw-Hill Book Co., Inc., New York, N. Y., 1946.

[1] Memoir prepared by Virgil A. Eberly, Assoc. M. ASCE, with the cooperation of W. A. Stinchcomb and W. E. Blaser, Members, ASCE, Mrs. Zesiger, and Martha Christiansen.

Case Institute of Technology) in Cleveland, Ohio. In 1901 he was awarded the degree of Civil Engineer from Case.

Mr. Zesiger went to Cleveland in 1893. When he retired in 1945, after seventeen years as chief engineer of the Metropolitan Park Board of Cleveland, he went to the home he had established in Oberlin, Ohio.

Shortly after his graduation, Mr. Zesiger became bridge engineer of parks in Cleveland, and for three years, in the mid 1920's, was chief engineer of bridges in the State Highway Department in Columbus, Ohio. In 1925 he was married to Erna Knapp of Columbus.

The record of his employment, from 1896 until his retirement in 1945, is as follows:

1896–1898—instrumentman, draftsman, and designer, in the county surveyor's office in Cleveland;

1898–1901—superintendent on the construction of intercepting sewers in Cleveland;

1901–1902—engineer associated with an architect;

1902–1903—engineer in the mills of the Jones and Laughlin Steel Company;

1903–1908—engineer in charge of bridges for the Department of Parks in Cleveland;

1908–1910—assistant engineer of the Evers Engineering Company in Cleveland;

1910–1912—engineer of construction in charge of checking plans structurally and of reinforced concrete construction for the Department of Buildings in Cleveland;

1912–1921—bridge engineer in charge of the design and construction of the Detroit-Superior Bridge (approximate cost $6,000,000) and the Brooklyn-Brighton Bridge (approximate cost $650,000) in the county surveyor's office in Cleveland;

1922—engineer for Newhall Company, contractors, and assistant engineer in the water works department in Cleveland;

1923–1925—chief engineer of bridges in the State Highway Department at Columbus; and

1925–1945—chief engineer of the Metropolitan Park Board in Cleveland.

Following Mr. Zesiger's retirement in 1945 he served as consultant on the design of several large bridges for the Metropolitan Park Board in Cleveland. He also coached a number of junior engineers.

"Zes" was a keen mathematician. Those who worked in collaboration with him express the same thought—they have never known a better one. He had a keen sense of logic and was able to develop formulas for the solution of difficult problems. Mr. Zesiger developed the first wholly comprehensive pile driving formula; a formula for punching any material or varve; formulas for the holding capacity of anchors; and designs for cantilever retaining walls.

Mr. Zesiger is properly given much credit for the design of the Detroit-Superior Bridge in Cleveland, since there were so many changes made in the structure after 1912 that it was necessary to redesign practically the entire structure after he took charge of the office that year.

One of the first reinforced concrete bridges that Mr. Zesiger designed was a flat slab skew bridge on the Rockefeller Parkway crossing Doan Creek in Cleveland, just to the north of Superior Avenue. In the light of later developments, this would not be considered much of an accomplishment, but in 1905 it was unusual and excited considerable interest, since it was a definite departure from previous practice. The next notable bridge that he designed as park engineer was the Washington Park viaduct in Cleveland, consisting of a series of reinforced concrete arches, carrying Independence Road in Washington Park.

Mr. Zesiger deserves full credit for the design of the Brooklyn-Brighton Bridge in Cleveland. This too was a series of reinforced concrete spans, carrying West Twenty-fifth Street across the Big Creek Valley from what was then the Brooklyn area to the South Brooklyn area.

He designed and superintended a reinforced arch bridge under the Wheeling and Lake Erie Railway at Brookside Park in Cleveland; a reinforced concrete arch at Edgewater Park in Cleveland; a three-hinged reinforced concrete arch at Greenville, Ohio; and many other bridges, foundations, and retaining walls.

One of the unusual structures which Mr. Zesiger designed was a boulder-faced, three-hinged arch of concrete across Big Creek in Brookside Park in Cleveland. At the time this bridge was built it was the flattest three-hinged arch in existence. Shortly thereafter, another three-hinged, boulder-faced, highway bridge was built following his design in Brookside Park across Big Creek. This bridge now carries the traffic into the area where the Bear Dens and Deer Park are located.

"Zes," besides being a man of good judgment, was a good friend to have. He was absolutely unselfish, and could be counted on to help anyone in trouble, regardless of his position. Quoting W. E. Blaser, M. ASCE, who was chief draftsman under Mr. Zesiger, from 1912–1921: "He was one of the best friends I ever had."

Having struggled for his own education, and having made his way unaided through the county school, summer school, Earlham College, and finally Case School, he knew from experience the hardships that make people sympathetic. Once he achieved his goal and was in a position to help others similarly handicapped, he gave most generously of his counsel and his cash to many who came within the scope of his wide acquaintance. Among relatives, he helped numerous nieces and nephews, sisters and brothers, even grandnieces and grandnephews with financial and other problems. An unfailing source of strength and assistance, he never turned away the stranger at his gate without a spiritual and monetary gift. Truly he has left the world richer than he found it.

His ready friendship for children always established a great confidence in them. In particular, he (together with Mrs. Zesiger) assumed charge of a little boy whose father had died.

He was identified with Sigma Xi society, the City Club of Cleveland, and with the Methodist Church. His hobbies were bowling and gunning.

Surviving are his widow; a sister, Mrs. C. V. Blattler of White Lake, S. Dak.; three nephews: W. L. Zesiger, a civil engineer in Lakewood, Ohio, H. A. Zesiger of Fairview Village, Ohio, and Milton Christiansen of Chicago; four nieces: Mrs. A. Helfrich and Mrs. Jack Harris, both of Cleveland, Martha Christiansen of Detroit, Mich., and Mrs. Louis Kruger of Chicago; also numerous nieces and nephews throughout the west.

Mr. Zesiger was elected a Member of the American Society of Civil Engineers on August 31, 1909. He became a Life Member in January, 1939.

ROY RICHARD ZIPPRODT, M. ASCE[1]

Died June 21, 1947

Roy Richard Zipprodt was born in Highland, Ill., in 1890. His parents were Fred A. and Elisa (Combe) Zipprodt. He received his early schooling at the Highland High School where he was valedictorian of his class and in 1915 was graduated from the University of Illinois, in Urbana, with the degree of Bachelor of Science in architectural engineering.

Mr. Zipprodt's first professional engagement was engineer of tests with the Emergency Fleet Corporation in Washington, D. C.; and from 1915 to 1919 he was associated with W. A. Slater,[2] M. ASCE, of Lehigh University (Bethlehem, Pa.), conducting research and comparative tests on different types of concrete floor designs. He did further research at the Bureau of Standards, also in Washington, on the same type of work. From 1920 to 1927 Mr. Zipprodt was engaged in the field of architectural engineering with the firm of Beuttler and Arnold, Architects and Engineers, in Sioux City, Iowa.

During 1928 he accepted an engagement with the Chicago (Ill.) Structural Bureau of the Portland Cement Association, and in 1929 he became the first regional structural engineer in the New York (N. Y.) office of the association, in charge of all structural activities in New York, New Jersey, and New England. In 1936 he was called to Columbia University in New York as associate professor of civil and architectural engineering, remaining there until 1940 when he resigned to work in Washington as principal civil and research engineer, serving with distinction in the Office of the Quartermaster General, United States Army.

At the time of his death Mr. Zipprodt was consulting engineer to the Iron and Steel Institute in New York, and he also was engaged in rewriting a series of textbooks (by George A. Hool, S. B. Kinne, and W. S. Kinne) entitled: "Foundations, Abutments and Footings," "Reinforced Concrete and Masonry Structures," "Structural Members and Connections," "Stresses in Framed

---

[1] Memoir prepared by Stewart S. Neff, M. ASCE.

[2] For memoir, see *Transactions*, ASCE, Vol. 96, 1932, p. 1555.

Structures," "Steel and Timber Structures," and "Movable and Long-Span Steel Bridges."

In his work as consulting engineer for the Iron and Steel Institute, he initiated and gave technical direction to a number of important research projects—notably on footings and on improved types of reinforcing bars. He exerted wise leadership in encouraging and sponsoring research of a thorough and fundamental nature.

He is probably best known for his long service as secretary of committee No. 318 of the American Concrete Institute. By his inspiration, his sound technical knowledge, and his careful and accurate editing of all committee reports, he contributed largely in this capacity to the "Standard Building Code Requirements for Reinforced Concrete," which has had nationwide influence on design and construction practice in reinforced concrete buildings.

Mr. Zipprodt was respected and admired by all who knew him for his honesty, integrity, and thoroughness. He had a kindly and unassuming personality and an unlimited capacity for work. He accepted responsibilities willingly, carring them out with remarkable energy and zest.

He is survived by his widow, the former Edna Bradley to whom he was married in 1916, and by a son, Elliott.

Mr. Zipprodt was elected a Member of the American Society of Civil Engineers on February 15, 1937.

Associate Members

ERNEST SAMUEL ALDERMAN, Assoc. M. ASCE[1]

DIED JANUARY 9, 1947

Ernest Samuel Alderman was born at Grass Valley, Calif., on June 30, 1886. He obtained the degree of Bachelor of Science in Civil Engineering in 1911 from the University of California in Berkeley, where he was a member of the Alpha Kappa fraternity.

His chosen profession was engineering, like that of his father, Samuel J. Alderman, who was a civil and mining engineer in Grass Valley. As a boy in his "teens" he assisted his father during vacations and served as transitman on surveys in Nevada and Sierra counties in California. During the summers of his college vacations he was transitman in general mining surveys in California and after graduation he went into private practice as engineer and surveyor on miscellaneous surveys and on design of water supply systems, one of the first being for the small resort townsite on Lake Tahoe, California.

After a year of practice Mr. Alderman found that highway work offered many opportunities and he accepted an appointment with the United States Office of Public Roads and Rural Engineering. Sixteen months were spent in studies of field and laboratory methods; and then as a junior highway engineer he worked on surveys, plans, and construction of various types of post roads and drainage structures. He progressed rapidly and during 1915 and 1916 made investigations of bituminous treatment of local materials in Palm Beach County, Florida. During World War I he served in the United States Army as engineer officer in charge of highway work at Camp Travis in Texas.

---

[1] Memoir prepared by Leo L. Laine, Assoc. M. ASCE.

From 1918 to 1920 he represented the Bureau of Public Roads in Oklahoma and during 1921–1922 he accepted the position of chief engineer for the state highway department. Perhaps his greatest contribution to society as an engineer came at this time when he was instrumental in establishing the federal aid system in Oklahoma on a sound basis and in drafting state highway specifications in conformity with government standards.

With this work completed he went into private practice as a contractor dealing mainly in drainage structures. His interest in state affairs continued and during 1931 to 1935 he served on the water resources board for the State of Oklahoma. After this time he was again in the service of the highway department and worked on research analysis until a sudden heart attack ended his career.

Mr. Alderman was a pioneer highway engineer and a man of high principles and integrity. He was noted for his dependability and accuracy and was held in high esteem by his associates. Although his work was his hobby, he still found time for the activities of the First Presbyterian Church, the Oklahoma Society of Professional Engineers, and other organizations.

He is survived by his widow; four daughters, Patsy, Sally, Mrs. R. O. Bell (Houston, Tex.), and Mrs. W. A. Doty, Jr. (Bartlesville, Okla.); and a son, Ernest Alderman, Jr. A brother, E. T. Alderman, of Portola, Calif., and two sisters, Mrs. Ethel Read and Mrs. Marguerite Bell—both of Los Angeles—also survive.

Mr. Alderman was elected a Junior of the American Society of Civil Engineers on October 3, 1911, and an Associate Member on January 15, 1917.

ROBERT MORRISON ANDREW, Assoc. M. ASCE[1]

DIED NOVEMBER 27, 1945

Robert Morrison Andrew was born at Columbia, Tenn., on Nevember 7, 1907. He was the son of John Morrison and Sally (Jones) Andrew. During his early boyhood his parents moved to St. Petersburg, Fla., where he was graduated from high school in 1928.

---

[1] Memoir prepared by J. T. Hallett, M. ASCE.

While regularly pursuing his high school education, he showed an intense interest in engineering as a result of holding minor positions with various engineering firms during this time, and serving as transitman for the City of St. Petersburg on street and sewer construction. In September, 1929, Mr. Andrew entered Purdue University in West Lafayette, Ind., from which he was graduated in June, 1933, with the degree of Bachelor of Science in Civil Engineering.

From January to September, 1931, he served as office assistant with the state highway commission of Indiana, and on completing his college training in June, 1933, he returned to serve as a project engineer until September, 1942. During this service as a project engineer he was in responsible charge of several highway projects, which embraced phases of construction that included both rural and municipal work. In October, 1942, Mr. Andrew entered the employ of Shumaker Brothers of Indianapolis, Ind., as a promotional engineer for ready-mixed concrete, and he was employed by this firm until his death, after an illness of about one year.

Mr. Andrew was well liked among his associates, within and without the engineering field, and he was an upright member of any community in which he lived. He will be remembered by all with whom he came in contact for his high standards of ethics and morals.

Mr. Andrew was married to Margaret Inman of Plainfield, Ind., on December 13, 1929. He is survived by his widow; a daughter, Roberta Jane; and two sisters, Viola (Mrs. C. M. Garney) of Columbia, and Margaret (Mrs. Jack Cole) of Chicago, Ill.

He was a licensed professional engineer and land surveyor in the State of Indiana. He held memberships in Tau Beta Pi and Chi Epsilon fraternities, and also an associate membership in the Society of Sigma Xi.

Mr. Andrew was elected a Junior of the American Society of Civil Engineers on October 30, 1933, and an Associate Member on August 16, 1937.

MURRAY CHASE AYERS, Assoc. M. ASCE[1]

DIED NOVEMBER 24, 1947

Murray Chase Ayers, the son of Henry and Ellen (Chase) Ayers, was born in Hartsgrove, Ashtabula County, Ohio, on December 19, 1883. After secondary school he attended the Case School of Applied Science (Cleveland, Ohio) and later received the degree of Bachelor of Arts from Stanford University (Stanford University, Calif.) in 1912.

Mr. Ayers' first engineering position was with the Wailuku Sugar Company in Wailuku, Maui, Hawaii, where he surveyed and mapped land, in addition to designing and supervising construction of irrigation systems and railways for the sugar plantations. After spending four years in the Hawaiian Islands he returned to the United States and worked as resident engineer for the California Division of Highways in the Sacramento office and then became inspector of construction for the Pacific Electric Railway Company in Los Angeles (Calif.).

---

[1] Memoir prepared by Howard O. Borrowdale, Dept. of Building and Safety, County of Los Angeles, Los Angeles, Calif.

Turning to structural design in 1922, Mr. Ayers worked in the architectural offices of Ross Montgomery, Edwin Bergstrom, John Austin, M. ASCE, and C. R. Johnson in Los Angeles, and as structural engineer for B. D. Kronnick (construction) Company. Notable among the buildings on which Mr. Ayers did design work are the Los Angeles General Hospital and the Carmelite Mission in Alhambra, Calif.; the Calvary Mausoleum, the Shrine Auditorium, sections of the Los Angeles County Museum, the Southern California Athletic Club, St. Vincent's Hospital, and the Guaranty Building —all in Los Angeles; also, the Los Angeles Chamber of Commerce Building, the Los Angeles Patriotic Hall, Hollywood Bowl, Sylvania Convent, Loyola Retreat, and the enlargement of St. Vibiana's Cathedral. Outside Los Angeles he did engineering work for St. Clare's School (Oxnard, Calif.), St. Andrew's Church, and the Pasadena Municipal Auditorium (Pasadena, Calif.), the high school auditorium (Redlands, Calif.), and the restoration of the Franciscan Mission (Santa Barbara, Calif.).

With the organization of the Los Angeles County Department of Building and Safety in 1933 Mr. Ayers became the senior structural engineer, holding the position until his death. His duties in the building department were both technical and administrative. During his more than fourteen years of service in the department, all structural plans for buildings erected in the unincorporated area of the county were processed and checked under his supervision.

His knowledge, experience, and integrity did much to gain for this department its reputation for soundness and efficiency. The respect and friendship which Mr. Ayers enjoyed among his colleagues may be taken for granted; a greater tribute to his abilities and personality is evident in the number of his friends in the Los Angeles area.

For many years Mr. Ayers made his home in Alhambra, near Los Angeles, where he was active in fraternal and civic work. He was a member of the Alhambra Lodge of the F. and A. M. and the Granada Lodge of the Order of the Eastern Star, and belonged to the Presbyterian Church of Alhambra.

In 1917 Mr. Ayers was married to Elsie L. Whittemore of Alhambra. He is survived by his widow and by two sons, Alden Ayers, a petroleum engineer, and Donald Ayers, a graduate student in the classical language and literature department at Princeton University (Princeton, N. J.).

Mr. Ayers was elected an Associate Member of the American Society of Civil Engineers on May 15, 1917.

THOMAS GEORGE GORDON BECK, Assoc. M. ASCE[1]

DIED JANUARY 6, 1948

Thomas George Gordon Beck was born at Palmerston South, Otago, New Zealand, on August 2, 1901. His parents were Thomas Smith and Roslyn (Noy) Beck. He completed his secondary school education at Otago Boys' High School there.

In 1919 he joined the New Zealand Public Works Department, working for two years in the North Auckland District, from which he was transferred to Arthur's Pass, where he acted as assistant engineer in charge of the electrification of Otira tunnel for three years. After completing his engineering course at Canterbury University at Christchurch he was transferred to the Dunedin District in 1925 where his work gave promise of the great heights which he was to attain.

Awarded a Commonwealth Fund Service Scholarship in 1931, he spent two years abroad—in the United States—studying irrigation, drainage, and river control. On his return he commenced investigations and preliminary surveys on irrigation in Canterbury. Under the direction of J. Wood, engineer-in-chief, he prepared a complete report and his recommendations were approved and adopted by the New Zealand Government. Work was soon under way with the Redcliff project in Canterbury to irrigate 5,000 acres and shortly after the Levels scheme also in Canterbury was undertaken to irrigate 12,000 acres of farming lands hitherto too dry for intensive utilization.

Simultaneously, comprehensive investigations were organized by Mr. Beck on the subjects of rainfall, soil moisture, runoff, evaporation, humidity, wind velocity, river flow, and response to irrigation, with the result that the need for irrigation became well defined in Mid-Canterbury. Work was then commenced on the Ashburton-Lyndhurst scheme and the Rangitata diversion race. On the latter project, Mr. Beck pioneered the use of a modern earth-moving plant (bulldozers, carryalls, and so on), and Public Works Department workshops were erected at Temuka to service this plant, which comprised 70% of all such equipment in New Zealand. The construction of the 42 miles of diversion race involved the excavation of 5,000,000 cubic yards of material.

At a later stage Mr. Beck conceived the idea of using the 1,000 cubic feet per second of water in the diversion race for electric power generation when it was not needed for irrigation. A power station generating 25,000 kilowatts was installed to help supply peak loads, and thus assure maximum use of this great flow of water. The workshops, established to service mechanical equipment, were later used to fabricate equipment and installations for the irrigation plan and Tekapo hydroelectric scheme, thus saving the New Zealand Government many thousands of pounds in the cost of repair and construction work.

During the active stages of development of the diversion race in 1937, the biggest rural water supply project in New Zealand was undertaken—the Downlands water supply scheme—which proved a boon to farmers over a large area of good land. While construction work proceeded on the diversion race, the Mayfield-Hinds irrigation plan was commenced and rapid progress made. The envisaged development in Mid-Canterbury will irrigate 420,000 acres distributed over five separate schemes, 30,000 acres of which will draw on the Rangitata diversion race.

---

[1] Memoir prepared by Harry L. Hume, M. ASCE.

In 1938, at Tekapo, Mr. Beck pioneered the shield method of tunnel construction and supervised the design and construction of the first shield to be used in New Zealand. Mr. Beck became district engineer at Christchurch in 1940 and, in addition to irrigation responsibilities, shouldered those of war construction. In 1942 he was transferred to Wellington to assist in the defense program as second assistant engineer-in-chief. Later he became assistant engineer-in-chief and finally engineer-in-chief, prior to taking over his last position as deputy commissioner of works in the Ministry of Works in 1946.

During his sojourn in the head office Mr. Beck was an active member, and chairman for a period, of the Soil Conservation and Rivers Control Council. To these duties he gave enthusiasm and zeal in his characteristic way, leaving an indelible impression in yet another field of progress. During this period he was closely associated with the late A. P. Grant, engineer to the council.

Mr. Beck had a brilliant career and won meteoric promotion by a rare combination of sheer brilliance and ability, coupled with a dynamic personality and resourceful nature. His imagination and perspicacity were brought to bear with forceful artistry on New Zealand engineering problems—particularly with irrigation in Mid-Canterbury. His loss to the department and to New Zealand is not only untimely but of the first magnitude.

His has been a proud record of achievement as one of the master architects of New Zealand's destiny, and it is an inspiration to his colleagues; as a man capable of molding the greater future of New Zealand, his loss will be keenly felt by the New Zealand Government and by all with whom he came in contact.

On February 11, 1931, Mr. Beck was married to Francis Ayson, who survives him. Four children, Pamela, Alistair, Margery, and Elizabeth also survive.

Mr. Beck was elected an Associate Member of the American Society of Civil Engineers on August 15, 1932.

ROYALL DOUGLAS BRADBURY, Assoc. M. ASCE[1]

Died March 21, 1947

Royall Douglas Bradbury was born in Jefferson City, Mo., on October 4, 1883. He attended the University of Missouri at Columbia, and the Massachusetts Institute of Technology, then in Boston, receiving the degree of Bachelor of Science in Civil Engineering from the latter school.

---

[1] Memoir prepared by C. D. Curtiss and C. N. Conner, Members, ASCE, and A. R. Rankin, Assoc. M. ASCE.

Following his graduation in 1906, he returned to the Massachusetts Institute of Technology as an instructor in civil engineering, teaching structural theory and bridge design until 1913. The clarity and preciseness of his teaching and his ability to simplify a complicated problem so that it could be readily understood were so much appreciated by his students that he won their lifelong respect and friendship. His friendly, helpful, and earnest manner demonstrated the fact that he was a born teacher. Mr. Bradbury was one of the first to introduce the theories being developed in Germany on continuous structures in the classroom. During this period, he also engaged in outside engineering consultation for the Boston Elevated Railways and the Clinton Wire Cloth Company (Boston), as well as others. In 1913 he gave up teaching to devote full time to a consulting practice in Boston.

In 1916 he became manager of the structural products department of the Clinton Wire Cloth Company and for a time was general sales manager of the company. In 1923 this company merged with others to form the Wickwire Spencer Steel Company, and Mr. Bradbury was made manager of the structural products department in Buffalo, N. Y. During this period he was very active in the design of reinforced concrete structures, and prepared a loose-leaf manual on the use of wire mesh reinforcing in such structures. In 1925 Mr. Bradbury took the position of contract manager with the Aberthaw Company of Boston and during the next five years was engaged in contract work on many types of construction.

Mr. Bradbury was selected as engineer-director of the Wire Reinforcement Institute in 1930, a position he held until 1945. It was in this position that he will be best remembered by his engineering associates, for he became recognized as a national (and even international) authority on the design of reinforced concrete pavements. In 1938 he published a treatise which was widely used as a textbook in many engineering colleges and as a reference book in the libraries of practicing engineers.[2] He was frequently called upon by state highway departments for assistance in recommending and preparing standards in pavement design. At his death he was chairman of the committee on rigid pavement design of the Highway Research Board of the National Research Council.

Although probably best known for his work in the highway field, Mr. Bradbury continued his interest in building construction until his death. For more than seventeen years he was a member of the building code committee of the American Concrete Institute, and was chairman of two subcommittees of this group which prepared standards for detailing buildings. He was active in technical committee work for the American Society for Testing Materials and the American Road Builders' Association.

As special assignment engineer, Mr. Bradbury joined the staff of the Public Roads Administration in January, 1946, and was actively engaged in work on the theory of design for reinforced concrete pavements at his death. Participation in pavement design standardization by the American Association of State Highway Officials was an important aspect of his work.

---

[2] "Reinforced Concrete Pavements," by Royall D. Bradbury, Wire Reinforcement Inst., Washington, D. C., 1938.

Because of his prior teaching experience and his many years of public relations work, Mr. Bradbury was selected to arrange study courses and field inspection tours for the many foreign highway engineers coming to the United States to study and to observe modern highway design and construction methods. His technical ability and interest in people made an indelible impression on a group of twenty-two engineers from India in 1946, for whom he had arranged a comprehensive course of instruction. He was in the midst of planning for a second group of engineers from India, when he died of a sudden heart attack.

Mr. Bradbury was married to Agatha C. Carney of Boston, on June 14, 1916. He is survived by his widow; a son, John D. Bradbury (Major, United States Marine Corps); a daughter, Martha (Mrs. Long D. Chambliss); a granddaughter, Diana D. Bradbury; and his mother, Mrs. Mattie C. Bradbury (Jefferson City, Mo.).

In his younger days Mr. Bradbury was a member of the Boston Athletic Club and the Wollaston (Mass.) Golf Club, where, at one time, he took golf lessons from Walter Hagen. However, as he grew older he retired from these activities to devote more time to his family, since he was basically a family man. His work was always his hobby, and he found relaxation in the complicated mathematical calculations of pavement design.

Mr. Bradbury was elected an Associate Member of the American Society of Civil Engineers on April 2, 1912. He became a Life Member in Jaunary 1947.

HARRY CUSTER DIESEM, Assoc. M. ASCE[1]

DIED JANUARY 5, 1948

Harry Custer Diesem, the son of Louis Israel and Margaret (Cook) Diesem, was born on July 11, 1877, in Galion, Ohio. He attended the public schools in Galion and was graduated from the University of Kansas at Lawrence with the degree of Bachelor of Arts in 1899.

Mr. Diesem, through choice or circumstance, entered the field of civil engineering and soon after graduation was employed as a deputy county surveyor of Finney County, Kansas, until 1903, at which time he became county surveyor. He occupied this post until the spring of 1905 when he was employed as office engineer of the Standard Beet Sugar Company on an irrigation project in Culbertson, Nebr.

During the academic year of 1905–1906, Mr. Diesem attended the graduate school of the University of Nebraska in Lincoln, and in 1906–1907 he was employed by the Tri-State Land Company on irrigation work at Scottsbluff, Nebr. In the latter part of 1907, as an employee of J. G. White and Company, he was in Denver, Colo., and in June, 1908, was transferred to Richfield, Idaho. The excellent background of professional knowledge and experience which he had gained on these irrigation projects resulted in Mr. Diesem's employment by the United States Department of Agriculture on irrigation investigations in the semi-arid states.

---

[1] Memoir prepared by George H. Hudson, M. ASCE.

In May, 1918, Mr. Diesem answered the call to the colors and was commissioned in the Corps of Engineers, United States Army. Captain Diesem served as commanding officer of the engineer depot at Fort Leavenworth, Kans., until his release from service in October, 1919. He retained a reserve commission and made the affairs of the organized reserve one of his most important hobbies. In 1937 he was graduated from the Command and General Staff School at Fort Leavenworth. He earned promotions through all intermediate grades and attained the grade of Colonel in the Corps of Engineers Reserve prior to his retirement in 1941 because of age.

After his release from the Army in 1919, Mr. Diesem was employed as an engineer-appraiser by the Federal Land Bank at Wichita, Kans. He remained with that office, utilizing his knowledge of hydraulics and land development in evaluating drainage projects until 1929. At that time he was transferred to the Federal Land Bank in New Orleans, La., remaining with that agency until his retirement at the age of seventy in 1947.

Mr. Diesem was an active member of the Presbyterian faith. He retained membership in the Society of American Military Engineers.

On September 22, 1909, Mr. Diesem was married to Bessie Louise Paine of Topeka, Kans. He is survived by his widow and two daughters, Margaret (Mrs. Estill Roberts) and Dorathy (Mrs. R. T. DeLouche, Jr.).

Mr. Diesem was elected an Associate Member of the American Society of Civil Engineers on March 5, 1912. He became a Life Member in January, 1947.

JAMES DOUGAN, Assoc. M. ASCE[1]

DIED DECEMBER 16, 1946

James Dougan, the son of Felix and Margaret (Leslie) Dougan, was born in New York, N. Y., on August 8, 1874. He had one sister, Margaret.

Mr. Dougan was educated in the public schools of New York; then he attended the Cooper Union Institute at night while working for Clinton and Russell, Architects, in the daytime. Mr. Dougan was graduated from Cooper Union in 1901, with the degree of Bachelor of Science; and in 1902 he received the degree of Civil Engineer from the same institution.

---

[1] Memoir prepared by T. Kennard Thomson, M. ASCE, and William Young, Pres., William Young Co., New York, N. Y.

His association with Clinton and Russell lasted from 1895 to 1939. He was first a draftsman, then an assistant engineer, and finally, in 1919, chief engineer, a position which he held until his retirement, due to heart trouble, in 1939. As assistant engineer and then chief engineer for that firm, he designed, specified, and directed work on foundations, structural design, construction details, heating, ventilation, and plumbing for the buildings which Clinton and Russell erected.

Some of the buildings in New York of which he supervised construction include: The Rhinelander printing buildings; the Mutual Life Insurance Company and the Atlantic Mutual Insurance Company buildings; the Broad Exchange Building, the Apthorp, or Astor, apartment; and the Hotel Astor and its additions (1909). In 1903 and 1904 he undertook construction of the Seventy-First Regiment Armory, the Mutual Assurance Society Building in Richmond, Va. (its addition in 1909), and the International Bank. Other outstanding buildings in lower Manhattan constructed under his supervision were: The United States Express Building, the American Exchange National Bank, the Hudson Terminal buildings, and the Whitehall Building. From 1911 to 1913 he was engaged on the same type of work on the Life Insurance Company Building of Richmond (its addition in 1923) and the First National Bank of Richmond. In 1914 he was engaged on the building at 1337 Broadway in New York, and several years later on the Colonial Bank of San Juan (Puerto Rico).

Other examples of Mr. Dougan's work are: The New Bedford Hotel (New Bedford, Mass.), the Trust Company of Jersey City (N. J.), the State and City Bank of Richmond, the New Amsterdam Casualty Company Building (New York), 60 Wall Street Tower Building (New York), the Elks Memorial Building (Chicago, Ill.), and the Elks Home (New Bedford, Mass.).

At one time Mr. Dougan was special engineer for the Department of the Interior in Washington, D. C., as chief engineer for Clinton and Russell, on heating, ventilating, and so on. During World War I, Mr. Dougan also did much work for the government.

He was a member of the F. and A. M. (Thirty-Second Degree), and Past Grand Patron of the Order of Eastern Star of New Jersey.

From acquaintance of almost half a century, since Mr. Dougan's work on the Mutual Life Building, the writers have a keen realization of his engineering ability and of the fact of his eligibility for full membership in the Society for most of that time. They are indebted to his daughter, Mrs. J. Edward Schierloh, for much information as to details of his engagements.

Everything that he did was well and cheerfully carried out. He was absolutely reliable, a true friend, and a citizen of the highest caliber in every respect.

He was married on June 25, 1902, to Mrs. George Moore Wilmot (a widow with a daughter, Minna) at Haddonfield, N. J. They moved to Hackensack, N. J., where a daughter, Agnes, was born on August 15, 1903. He is survived by his widow; two daughters, Minna (Mrs. L. H. Ackerman) and Agnes (Mrs. J. Edward Schierloh); four grandsons, Robert, Donald, and William Ackerman, and James E. Schierloh; two granddaughters, Katherine and Elizabeth Ackerman; two great-grandchildren, Leslie and Allan Ackerman; and a brother, Peter Dougan.

Mr. Dougan was elected a Junior of the American Society of Civil Engineers on March 5, 1901, and an Associate Member on March 2, 1904.

DOUGLASS HEWITT FERRY, Assoc. M. ASCE[1]

DIED NOVEMBER 25, 1947

Douglass Hewitt Ferry was born at Elkhart, Ind., on March 29, 1886. He was the son of John Howard and Clara (Smith) Ferry.

His elementary education was received in the schools of San Diego, Calif., where his parents moved when he was only a few months old. Part of his boyhood was spent on several ranches which his father owned and operated in San Diego County. It was at that time that he learned how to take care of himself and his horse on trips into the back country and also how to do the many tasks connected with operating a ranch. This experience proved to be most useful to him in his work as an engineer in unsettled country.

Later, the family moved to Chicago and Lake Forest in Illinois, the home of his grandparents. He attended Armour Academy (Chicago), from which he was graduated in 1903.

Following his graduation, Mr. Ferry spent a year in California at one of the aforementioned ranches assisting in the management of three thousand acres of fruit and cattle land. He became an expert horseman and learned all phases of caring for cattle.

Mr. Ferry used to say that when he was a small boy he chanced to pass an office in a building in San Diego and saw on the door the words, "Civil Engineer." He liked the looks of the title and determined then and there to become an engineer. When he started his college course at Stanford University (Stanford University, Calif.) in the fall of 1904 it was, therefore, as a civil engineering student that he enrolled. He took an active part in campus activities, especially as a charter member of Hammer and Coffin, the Journalistic Society, and as a contributing artist and, finally, as editor of *Chaparral,* the Stanford humorous publication. In 1907 he was art editor of *The Quad,* the Stanford yearbook, and also art editor of *Sequoia,* the literary magazine published at that time by the English Club, of which he was a member.

While attending Stanford he worked during the summer vacations on various engineering jobs, surveying and assisting in mapping and field work in Siskiyou and San Diego counties.

Immediately following his graduation in May, 1909, Mr. Ferry left for the Yukon, to join the engineering force of the Yukon Gold Company, a Guggenheim corporation which at that time was operating nine gold dredges in the Klondike Basin as well as numerous hydraulic mines. Mr. Ferry worked on mapping and drafting in the offices of this company and was given the opportunity to make an extensive research of the gold-saving devices used on the dredges and to design new and improved equipment. In March, 1912, he was made superintendent of the thawing department, in full charge of nine camps and four hundred men engaged in thawing the frozen ground ahead of the dredges. Part of his work was the locating and delivering of thirty thousand cords of wood to the steam boilers ahead of the dredges.

---

[1] Memoir prepared by Leon B. Reynolds, M. ASCE.

In the fall of 1914, Mr. Ferry brought his wife and son, John, to the United States for a three-month vacation, intending to return to Canada in the spring. He decided to remain, however, and spent the following year in San Diego. In the spring of 1916 he renewed his association with the Yukon Gold Company and was placed in charge of prospecting crews working in Trinity County in California. He continued in this type of work until the spring of 1917 when he supervised the building of a dredge on the American River, near Auburn, Calif. After operating the dredge for a year, he was sent to Murray, Idaho, to take charge of the operation of a dredge on Pritchard Creek. He remained there as manager for eight years.

After the Murray operation was finished, the dredge was sent to the Malay States, where the Yukon Gold Company was inaugurating an extensive tin mining operation. Mr. Ferry was offered the management of the project but preferred to remain in the United States, so as not to be separated from his family.

For two years he conducted examinations for a small syndicate composed of former Yukon Gold Company officials, since the company had concluded all its mining operations in the United States. In the spring of 1927 Mr. Ferry started an examination of ground on Foots Creek, in Jackson County in southern Oregon, later building and operating a dredge there for this same syndicate. In 1935 the dredge was torn down, rebuilt, and moved to Grave Creek in Josephine County, about forty miles away. There operation continued with Mr. Ferry as manager until the fall of 1940.

By that time Mr. Ferry had developed a property in Idaho, near Boise, where the Oregon dredge was moved during the winter of 1940–1941. Operations were started in the spring of 1941, at Twin Springs, Idaho, the property being known as Boise King Placers. The dredge continued to operate until forced to close by World War II.

After traveling in California for some time, Mr. Ferry retired and settled down as consulting engineer at San Jose, Calif. His health had not been good for some time and he devoted himself largely to his garden and orchard. His death occurred in his sleep, from a heart attack, on November 25, 1947.

Mr. Ferry was known as a man who was hard to work for because he exacted a high type of excellence from his men, but also as a man who was absolutely just. His men responded by giving him such respect and loyalty that some of them had worked for him from ten to twenty-five years and refused to work for anyone else.

He was a man of the highest personal integrity, giving his entire thought and concern to his work, denying himself recreations and friendships ordinarily enjoyed by men in his position. He was devoted to his family and took them with him whenever it was possible to do so. Another consistent loyalty was his alma mater, Stanford University.

Mr. Ferry's main hobby was horticulture, in which he excelled as a grower of fine fruit and flowers. A talented artist, he never had time for much work in that field. As a writer he was prolific and witty, and one friend told him that he should have followed that profession exclusively. Many of his friends thought that a good doctor was also lost to the world when he became an engineer.

Mr. Ferry's natural aptitude for mathematics, his thoroughness, and quick grasp of detail, as well as his ability to handle men and large-scale operations, made him a successful engineer. His training in civil engineering was invaluable to him in surveying and mapping; in building roads, dams, and bridges; and in other problems connected with opening and operating mining properties. The opportunity afforded him in the Yukon to enter the mining field, as well as the field of civil engineering, enlarged his scope of operations and gave him opportunities for advancement far ahead of most men of his years. He became an authority on prospecting for gold and estimating the values of gold-bearing properties, and his advice and opinions were highly considered. His ability to handle current problems and his foresight in anticipating and preparing for future ones enabled him to make a signal success in the operating field as well.

On August 5, 1911, in Dawson, Yukon Territory, Canada, he was married to Eudora Bundy, a Stanford classmate. He is survived by his widow; a son, John; and twin daughters, Elizabeth F. Morris and Frances F. Perske.

Mr. Ferry was elected an Associate Member of the American Society of Civil Engineers on March 2, 1915.

ALEXANDER SYLVESTER FORSTER, Assoc. M. ASCE[1]

DIED JANUARY 25, 1947

Alexander Sylvester Forster, the eldest son of Alexander Edward Forster and Laura (Brown) Forster, was born in Toledo, Ohio, on June 10, 1879.

He was educated in the Toledo public schools and was graduated from Toledo Central High School. He studied engineering under a private tutor during his employment in the field of engineering.

In June, 1898, he was employed in the Toledo city engineers' office, being clerk, inspector, draftsman, and assistant engineer, successively. In March, 1901, Mr. Forster entered the employ of the American Bridge Company in the

---

[1] Memoir prepared by George A. Taylor, M. ASCE.

Toledo plant, where his employment continued until October, 1915. Here, he obtained a broad experience in structural engineering, having been frequently sent to other plants of this company to assist in planning various structures on important engineering works.

He accepted a position in October, 1915, in the engineering department of the Toledo Bridge and Crane Company, as estimator and chief draftsman, later becoming superintendent of the plant. His employment continued there until the end of 1919.

In January, 1920, Mr. Forster was employed by the DeVore Company of Toledo, an engineering firm specializing in the design of industrial plants, such as glass manufacturing, paper mills, foundries, and power plants.

He became a partner in the architectural and engineering firm of Rheinfrank and Forster in May, 1924. This association continued until January, 1926, when he organized an engineering firm which became the partnership of Forster, Wernert, and Taylor, in 1928, specializing in structural engineering and factory design. He was active in this business until his death.

Mr. Forster had a wide acquaintance, and was an active member in many organizations. He was an Elder in the Collingwood Presbyterian Church, a member of the Kiwanis Club, serving as chairman of various committees, and as director of the club for three terms. He was a member of Sylvania Country Club, the Masonic Order, and the National and Ohio Societies of Registered Professional Engineers.

In addition, Mr. Forster was interested in civic affairs, being a member of the Toledo Port Commission and the Toledo Chamber of Commerce and was on the committee appointed by the city manager to revise the Toledo City Building Code. With the adoption of the code in 1946, he was made chairman of the Board of Appeals, whose duties were to interpret the provisions of the code.

In 1903 Mr. Forster was married to Ruth Jean Campbell; they had two children, a son, John, of Toledo, and a daughter, Jean, of New York, N. Y. He is survived by his widow; two children; two grandchildren; two brothers, Frank H. and Erle H.; and a sister, Eva Fern (Mrs. Merle Planck).

Mr. Forster was elected an Associate Member of the American Society of Civil Engineers on May 31, 1916.

MAGNUS LAWRENCE HYDLE, Assoc. M. ASCE[1]

DIED JANUARY 29, 1947

Magnus Lawrence Hydle, the son of Lars Magnus Hydle and Kathrine (Hakestad) Hydle, was born at Ulvik, Hardanger, Norway, on September 19, 1883. After receiving his elementary education in Ulvik, he attended Trondheim (Norway) Institute of Technology, being graduated with the degree of

---

[1] Memoir prepared by Fred D. Pyle, M. ASCE.

Civil Engineer in 1905. He then attended Christiania Institute at Oslo, Norway, majoring in design of indeterminate structures.

After a short period on the design of the Empire State Building in New York, N. Y., he went with the Northern Pacific Railway Company in 1906. He progressed through the various steps from draftsman at Saint Paul, Minn., to resident engineer on construction in the State of Washington. He then returned to Saint Paul as inventory engineer from 1915 to 1917. From May, 1917, to August, 1919, he was with the Oregon State Highway Department at Salem, Ore., as office engineer, efficiency engineer, and bridge designer, except for a leave of absence from September, 1917, to January, 1918, when he served as a First Lieutenant in the Engineer Corps of the United States Army.

With the exception of six months when he was with the Southern Pacific Railroad Company at Los Angeles, Calif., as head draftsman and field engineer, Mr. Hydle was employed from 1919 to 1923 by the California State Highway Department at Sacramento, Calif., as designer and resident engineer on bridge construction. The following year he was with the Pan American Petroleum Company at Los Angeles, in charge of harbor development, and at the Lorain, Ohio, plant of United States Steel Corporation, supervising the manufacture of 150 miles of heavy oil pipe for the Elk Hills project. From November, 1924, to October, 1929, Mr. Hydle served on various work, generally in the employ of consulting engineers on design of irrigation projects, storm drains, subdivisions, and highway bridges.

From October, 1929, to January, 1942, he was with the California Division of Highways on designs, estimates, and reports. From January, 1942, to October, 1945, he became structural engineer for the Consolidated Aircraft Corporation at San Diego, Calif., on design of foundations and of concrete, steel, and wooden buildings, also making critical reviews of plant as to safety and adequacy. From October 15, 1945, until his death, Mr. Hydle was with the Harbor Department of the City of San Diego on structural design.

On September 29, 1919, he was married to Alice Rosalyee Bahr at San Francisco. He is survived by his widow; two daughters, Mildred (Mrs. Edward Getz) and Jacqueline (Mrs. Howard C. Guy); three brothers, Olaf, Lars, and Odmund Hydle; and a sister, Eva Standley.

Mr. Hydle was elected an Associate Member of the American Society of Civil Engineers on May 22, 1944.

ROY WRENSHAW JABLONSKY, Assoc. M. ASCE[1]

DIED JUNE 13, 1947

In writing the biography of a man of accomplishment it is interesting to note his background, as contributing to the character and ideals of the man himself.

---

[1] Memoir prepared by Raymond E. Julius, Asst. Highway Engr., Clayton, Mo.

Roy Jablonsky was born on May 24, 1891, in St. Louis County, Missouri. The home of his parents was located on the eastern side of Kienlen Avenue at its intersection with what was then called Hamburger Avenue (later changed to St. Louis Avenue). His father, August Jablonsky, was born in Germany and his mother, Priscilla (Saunders) Jablonsky, was born in England. Both his parents immigrated to America at an early age. August Jablonsky at the time of his son's birth was engaged in the florist business. Roy Jablonsky, however, at no time evinced any interest in or desire for this line of work.

He attended the Wellston public schools and when he had finished his grade school studies, he entered a business school to study bookkeeping and stenography. On the completion of his commercial course he secured employment in a bond brokerage house which position he retained until accepting employment with a friend of his father, William Elbring, county surveyor and highway engineer of St. Louis County. In this employment he soon realized a particular ambition to study engineering and to undertake an engineering career. Thus began a life devoted to the study of professional engineering, and it may be said that he continued as a student of that profession until his death. For the most part his study of engineering was informal and somewhat unorthodox. Being an insatiable reader was of immeasurable aid to him in the methods he pursued in the study of his profession.

After procuring a working knowledge of algebra, geometry, and trigonometry, he secured books by eminent authors on the technical aspects of the various branches of the civil engineering profession. In stating that he pursued his studies in an unorthodox manner, particular reference is made to the fact that he became proficient in the design of reinforced concrete structures before he undertook the design of structural steel. Thus it was that in his own way he proceeded relentlessly to master the various branches of civil engineering required in the design and construction of county highways and bridges.

While studying engineering he was enabled through his employment with the county highway engineer to give practical application to theory, which was, indeed, an ideal arrangemtent. As quickly as his ability permitted he advanced in his work; in November, 1911, he became instrumentman in a survey party and in April, 1914, he was promoted to chief of the survey party.

A little more than two years later, in September, 1916, Mr. Jablonsky was made assistant county highway engineer, a position he held until July, 1919, when he was appointed by the county court as a special engineer of St. Louis County, as well as engineer in charge of construction on Missouri highway project S 18.39. From September, 1920, until April, 1921, during the construction of federal aid highways under the Centennial Road Law of Missouri, Mr. Jablonsky was engineer in charge of project No. 79.

On the completion of this project he engaged in the construction business until his appointment in February, 1924, by Arthur M. Hyde (who was then governor of Missouri), as county surveyor and ex officio highway engineer of St. Louis County to complete the unexpired term of William Elbring who had resigned. At the elections held in 1924 and 1928 Mr. Jablonsky was elected and re-elected county surveyor and ex officio highway engineer, remaining in that office until January 1, 1933. From that date until January 1, 1941, he engaged in private engineering business, specializing in street and sewer design for subdivision development.

From the time of his appointment by former Governor Hyde until the relinquishment of this office on January 1, 1933, Mr. Jablonsky supervised the expenditure of the funds derived from $3,000,000 and $10,000,000 road bond issues. It was his responsibility to design and construct a system of modern and convenient highways in St. Louis County. His acomplishments during this period brought him nationwide acclaim and they will remain as lasting momuments to his ability.

It was during this period that his forthright character manifested itself. When he assured himself that he was right, neither political expediency nor personal gain could effect a compromise—a most unusual characteristic in one holding a political office. His honesty, integrity, and ability could not be denied even by his political opponents. Many times intimate friends would complain that their friendship never influenced him in the least in attempting to secure official favors. That he planned wisely and that he directed efficiently is attested by the fact that the engineering costs in all the projects undertaken in the expenditure of $13,000,000 did not exceed five per cent.

During 1928 a cash refund from the State of Missouri to the County of St. Louis was authorized by the legislature, for roads taken over by the state. This money, under the provisions of the law, was required to be expended in the establishment of other roads by the county. The sums thereby derived were sizable and sufficiently so to inspire Mr. Jablonsky to plan a north and south highway through the center of St. Louis County with a right-of-way width of 100 feet. At the time rights of way 100 feet wide were unheard of in this area and the establishment of such a highway was bitterly opposed, not only by property owners and taxpayers, but by county officials as well. His keen foresight, however, revealed to him that this artery had potentialities and would in the not distant future become a very important and heavily traveled highway. Therefore, notwithstanding objection and criticism, he refused to compromise, and in his characteristic forthright manner he pressed his point until the county court finally acquiesed. This highway became a major artery just as he had predicted, and the state highway commission, which now controls it, has praised him publicly for his vision and foresight.

Later he envisioned a need for another major highway running southwestwardly through the southern half of St. Louis County. Not having the authority nor the funds to carry out this project, he proceeded to sell his idea to the Missouri State Highway Commission. Again he met opposition; he refused, however, to compromise, and undaunted, he promoted his idea with greater vigor. Finally, the state highway commission conceded the point and established and constructed the road which is now route No. 66

running southwestwardly from the City of St. Louis. He has often been acclaimed for his persistence in promoting this project which anyone traveling westwardly from St. Louis knows is an important and convenient traffic artery. The establishment of this project also served another purpose, that of developing real estate along its entire length. On January 1, 1941, Mr. Jablonsky again became county surveyor and ex officio highway engineer of St. Louis County and was re-elected to that office for a four-year term beginning on January 1, 1945.

During World War II governmental regulations restricted the use of materials almost entirely to maintenance and repair, permitting very little for construction and reconstruction. While this was an expedient most revolting to Mr. Jablonsky, he nevertheless subordinated his feelings and submitted in full measure. During those dark days traffic had to be accommodated and without hesitation he cooperated to that end. Many war industries were located within the County of St. Louis and every plant without a single exception was adequately served despite the difficulty in securing materials and labor for maintenance of highways.

As testimony to his loyalty it is only necessary to mention the fact that he organized and actively participated in the Office of Civilian Defense in St. Louis County. It can be said without fear of contradiction that he put forth more effort and spent more of his own time toward maintaining the Office of Civilian Defense than any other member of that organization in this county. He served as secretary from its inception until its dissolution after the war.

What might be considered the greatest tribute to his ability was his appointment on September 21, 1945, by the late Thomas H. MacDonald, Hon. M. ASCE, Commissioner of Public Roads of the Public Roads Administration, as a member of the Board of Highway Consultants. The board consisted of eleven engineers from various parts of the United States who met with the commissioner of public roads to discuss the various features of the secondary federal aid highway program. On January 11 and 12, 1946, he was called to Washington, D. C., to attend a meeting of this board. That he impressed the members of the board is evidenced by the number of letters which he received seeking information and counsel on the various phases of the secondary road program.

During November, 1946, Mr. Jablonsky was invited to lecture on secondary highways in several midwestern states but after reviewing the agenda he realized that the lecture tour could not be undertaken without interfering with his work as county highway engineer. His inordinate devotion to the office to which he was elected by the people of St. Louis County is best illustrated in a letter to Mr. MacDonald tendering his resignation from the board of consultants. It reads as follows:

> "On November 18th, 1946, Mr. Fuller of your Division Office called on me and outlined my agenda as highway consultant for the coming months. The conferences and discussions I am sure would be very instructive and conducive of sincere cooperation amongst the various agencies, but I have

> concluded because of certain circumstances that I should not undertake the assignment.
>
> "Since the adoption of new laws my functions as Highway Engineer have been considerably enlarged and this factor, coupled with my limited technical personnel, requires me to devote all of my time to the official duties of the office. At this juncture I might add that attending the hearing would be one thing and thoroughly studying all matters involved in preparation for the meetings would be another, and for neither of which could I devote the necessary time.
>
> "That I should terminate my appointment as highway consultant I am sure you will agree, thereby leaving a vacancy to be filled by someone who is in a position to devote the necessary time to this important office.
>
> "Please be assured that I have enjoyed working with your Department, limited as my work was, and I regret very much that I find it necessary to discontinue that work."

The foregoing letter also bespeaks of Mr. Jablonsky's sincerity in the conduct of official matters.

He is survived by his widow, Frieda (Wagner) Jablonsky, to whom he was married on August 21, 1912, at the Emmanuel Evangelical Lutheran church in St. Louis County, and by three children, August W., Roy D., and Betty Jane.

His passing has been a distinct loss, not only to his family, who revered him greatly, but to all the people of St. Louis County whose welfare was his guiding star. No one can gainsay that he was devoted to the people of St. Louis County and that his limitless efforts in their behalf contributed to his untimely death.

Mr. Jablonsky was elected an Associate Member of the American Society of Civil Engineers on June 6, 1921.

ROLAND WINTHROP LEFAVOUR, Assoc. M. ASCE[1]

DIED JULY 15, 1947

Roland Winthrop Lefavour was born in Beverly, Mass., on July 7, 1890. His parents were Howard Richard and Rebecca Ober (Perry) Lefavour. Among his relatives and forebears were ship captains, soldiers and sailors in the War for Independence and in the Civil War, town officials, and college presidents.

He attended school in Beverly, and continued his education in art school and under tutors, for a time making a serious study of music. Upon taking a position with a firm of landscape architects he discovered his vocation—surveying, mapping, drawing, and allied subjects.

From 1909 to 1911 Mr. Lefavour was assistant city engineer of Beverly. In 1918 he was engaged by Tufts College in Medford, Mass., as an instructor in surveying, and in 1922 was made assistant professor of civil engineering, a post which he held at his death.

---

[1] Memoir prepared by Frederic N. Weaver, M. ASCE.

He was a member of the American Society for Engineering Education, the Society of American Military Engineers, the American Association for the Advancement of Science, and was affiliated with the Odd Fellows and the National Grange. For years he belonged to the Society of the Sons of the American Revolution.

Professor Lefavour's life did not fall into any set pattern—in fact, it was an unusual combination of the practical and the artistic. As an avocation he painted in oils and water colors, and excelled in etchings. He was unusually competent in his own field and was an excellent teacher, a genial friend, and an educated and cultured gentleman.

He is survived by two brothers, Howard L. Lefavour of Northboro, Mass., and W. Russell Lefavour of Beverly.

Professor Lefavour was elected an Associate Member of the American Society of Civil Engineers on August 4, 1924.

PAUL WRIGHT LONGSDORF, Assoc. M. ASCE[1]

DIED AUGUST 28, 1946

Paul Wright Longsdorf was born on October 6, 1882, in Floradale, Adams County, Pa. He was the son of Charles S. and Elizabeth (Wright) Longsdorf.

His early education was obtained in the public schools of Floradale and in the private schools conducted by the Society of Friends. Later, he attended George School in Newtown, Pa., and was graduated in 1902. Subsequently, he entered Pennsylvania State College at State College, where he studied civil engineering for two years.

After leaving college Mr. Longsdorf was employed by the Pennsylvania Railroad Company. This engagement lasted from April, 1905, to June, 1908, when he was employed in making surveys for a low-grade freight line from Morrisville, N. J., to Newark, N. J.; he made property and construction surveys and was also masonry inspector in charge of the construction train on elevated track work.

From 1908 to 1915 Mr. Longsdorf was occupied primarily in real estate development and in landscaping private estates. During this period he was engaged by M. R. Yerkes in Bryn Mawr, Pa. (from June to November, 1908); the Schuylkill Stone Company in Monocacy, Pa. (from January to March, 1909); Thomas Meehan and Sons in Philadelphia, Pa. (from March, 1909, to February, 1910, and from September, 1910, to June, 1913); and C. C. Worthington in Shawnee-on-Delaware, Monroe County, Pa. (from February to September, 1910).

---

[1] Memoir prepared by George B. Mebus, M. ASCE.

In May, 1915, he was appointed township engineer of Cheltenham Township, Montgomery County, Pa., a position he held until June, 1923. When Mr. Longsdorf was township engineer rapid development took place in the township due to the unusual influx of residents from Philadelphia. The township initiated a very comprehensive road and bridge construction program, and its present excellent highway system is the result of Mr. Longsdorf's foresight as a planner and his ability as an organizer.

In June, 1923, he formed a partnership with George Meyers Patterson to engage in construction work, consisting primarily of grading and road and sewer construction. During the seven years of this partnership Mr. Longsdorf handled the construction of the majority of many excellent suburban land subdivisions, in addition to doing a great quantity of municipal construction. This partnership was terminated in 1930. He continued in the construction business after 1930, having organized the firm of Paul W. Longsdorf, Incorporated. He remained active in this business until his death.

Mr. Longsdorf completed the following works: The Oak Lane Manor, Carl Metz, Getler and Helwig, and Joshua M. Holmes developments; several projects at Lehigh University (Bethlehem, Pa.), including Richards Dormitory, Eugene Grace Hall, Drinker House (another dormitory), with all grading, walks, and drives, in addition to the new track and athletic field for the Department of Athletics; Sunset Memorial Park with new roads; two estates, one for James Elliott in Pottstown, Pa., and another for Archibal Johnston in Bethlehem; Jeanes Hospital in Philadelphia; the Thomas Williams School for the Cheltenham Township School Board; and playgrounds for Girard College in Philadelphia.

He was always a sincere citizen and a good neighbor. He was very active in a multitude of civic and historical organizations, and was a very generous person, who contributed to many worthy enterprises, almost always anonymously. Membership in the following organizations is indicative of his many interests: The Pennsylvania Horticultural Society, the Historical Society of Pennsylvania, the Philadelphia Zoological Society, the Pennsylvania Alpine Club, the American Road Builders Association, the American Public Works Association, the Society of American Military Engineers, the Engineers and the Union League clubs of Philadelphia, and the Old York Road Country Club and the Rotary Club of Jenkintown, Pa.

On October 28, 1909, he was married to Annie Michener at Huntingdon Valley, Pa. He is survived by his widow; a daughter, Alice E. (Mrs. J. Linden Heacock, Jr.); and a son, Charles M. Longsdorf.

Mr. Longsdorf was elected an Associate Member of the American Society of Civil Engineers on April 18, 1927.

BURTON LOWTHER, Assoc. M. ASCE[1]

DIED SEPTEMBER 7, 1945

Burton Lowther was born on June 15, 1874, at Saugerties, N. Y. He was the son of James Lowther, a native of Northern Ireland, and Eliza (Allen) Lowther, a native of Devonshire, England. He received his engineering education at Rensselaer Polytechnic Institute in Troy, N. Y.

He began his professional career in 1900 as instrumentman and second assistant engineer in railway and mill construction, with the T. A. Edison Cement Company in Stewartsville, N. J., and then proceeded to other professional assignments. From 1900 to 1901 he made field drawings in the office of the resident bridge engineer of the New York Central Railway Company, becoming an instrumentman in 1901 and 1902 with the American Railways Company of Philadelphia, Pa., on survey and construction of the Chicago and Joliet Electric Railway, and an engineer inspector of masonry and bridge construction. In the resident engineer's office at Dayton, Ohio, he was engaged on the improvement of railway properties, and later was made assistant resident engineer in entire charge of reconstruction and strengthening of city bridges, in addition to survey and computation for city trackwork and pile driving.

The next year he was first assistant resident engineer with the Western Ohio Electric Railway, in the building of 20 miles of track, taking charge on the resignation of the resident engineer soon after work was begun. From 1903 to 1904 as draftsman with the Minneapolis (Minn.) Steel and Machinery Company, he detailed steelwork for bridges and buildings.

In the spring of 1904 he surveyed 20 miles of electric railway in Jackson and Cass counties in Missouri; also, he became mechanical engineer for the Prouty-Pierce Gasoline Locomotive Company in Kansas City, Kans. After this Mr. Lowther was engineer of tests and materials for the Bell Telephone Company in Kansas City, Mo., in charge of acceptance of material, electrolytic surveys in Missouri and Kansas, and design of special apparatus in telephone construction.

In 1907, as resident construction engineer with the Freeborn Engineering and Construction Company in Kansas City (Mo.), Mr. Lowther supervised the building of a 2,000-barrel Portland cement plant at Independence, Mo., for the Kansas City (Mo.) Portland Cement Company. The same year he became vice-president of the Moechel and Lowther Engineering Company, engineers, chemists, and metallurgists in Kansas City (Mo.), engaging in consulting work in mining for cement companies, with personal direction of physical tests in the laboratory, and having entire charge of the civil engineering work. He also made the design for developing the gold district in Phonolite, Nye County, Nev.

---

[1] Memoir prepared by Dwight D. Gross and Robert Follansbee, Members, ASCE, and John McGowan, Assoc. M. ASCE.

As construction engineer (from 1908 to 1914) for the Kansas City (Mo.) Municipal Water Works, Mr. Lowther had charge of the construction of a 45,000,000-gallon reservoir, as well as the repair and remodeling of another reservoir; the installation of steam driven pumps of a total capacity of 80,000,000 gallons per day with boilers and general powerhouse construction; and the building of a $250,000 river improvement project on the Missouri River to protect the city water supply.

From 1914 to 1915 Mr. Lowther was chief engineer of the Kansas City Municipal Water Works by civil service appointment, having entire charge of this municipal utility valued at more than $12,000,000. Three years later he became superintendent, as well, of the same organization, with full executive authority in conducting the affairs of this utility under the board of fire and water commissioners.

He was chief engineer and general superintendent of the Denver (Colo.) Municipal Water Works from 1918 to 1926. In August, 1918, he was employed as engineer by the Denver Union Water Company (the Denver water plant being privately owned at that time), and on November 1, 1918, when the city acquired the water plant, Mr. Lowther was engaged as chief engineer and general superintendent. He at once recommended extensive improvements to the plant and had charge of the planning and installation of improvements valued at $10,000,000.

In 1926 Mr. Lowther set up his own office as consulting hydraulic and sanitary engineer in Denver, specializing in design and construction of water works and sewage disposal plants, continuing until 1945. Some of his more notable engagements as consulting engineer included: The Water System for Boulder City, Nev., at the time of construction of the Hoover Dam; the construction of the arms plant at Denver; the water and sewage plants for the cities of Colorado Springs and Montrose in Colorado; the water works systems of Johnstown, Longmont, Louisville, Craig, Walsenburg, and Oak Creek in Colorado; and the sewage disposal systems of Brighton, Colo.

Before studying engineering Mr. Lowther studied medicine, including pathology, physiology, and water-borne diseases, and he also took a special laboratory course in bacteriology. During his career of water works engineering this training was of great value to him. He was a member of the American Water Works Association and the American Public Health Association, and he served as collaborator in the preparation of the material for the *Water Works Practice Manual.* He was a pioneer in the use of chlorine in water treatment, and in reinforced concrete pressure pipe with expansion joints. In addition to his other activities he belonged to the F. and A. M.

Mr. Lowther was an outstanding engineer in the fields of water works and sanitation. He was a gifted conversationalist and a man with the courage of his convictions. He made many friends, but he would not compromise his opinions in order to make friends.

On January 5, 1924, he was married to Dr. Kay R. LaCroix. He is survived by his widow, and by two daughters by a previous marriage, Mrs.

Elizabeth Campbell of Washington, D. C., and Mrs. Lillian Porter of California.

Mr. Lowther was elected an Associate Member of the American Society of Civil Engineers on October 7, 1908.

JOHN JOSEPH McGINLEY, Assoc. M. ASCE[1]

DIED DECEMBER 9, 1940

John Joseph McGinley, the son of James and Mary Elizabeth (Brennan) McGinley, was born in New York, N. Y., on March 2, 1889.

After completing his preparatory education in the New York City schools, Mr. McGinley attended the College of the City of New York for four years and was graduated in 1909 with the degree of Bachelor of Science in Mechanical Engineering. Later, he did graduate work in the School of Civil Engineering at Columbia University in New York, and, after completing his studies in three years, he was awarded the degree of Civil Engineer in 1912.

Mr. McGinley's initial professional training was received in his native city, where for two years, from 1912 to 1913, he was associated with major highway and bridge construction projects, including the Hell Gate Bridge. In 1914 he joined the engineering staff of the New York State Public Service Commission and until the advent of World War I supervised engineering work in the field on the extensive subway construction program then under way in New York.

Shortly after the outbreak of World War I, Mr. McGinley enlisted in the air service of the United States Army. After completing the courses at the Officers Training School at Kelly Field in San Antonio, Tex., and the School of Military Aeronautics at the Georgia School of Technology in Atlanta, he received a Second Lieutenant's commission in the Air Service. He served consecutive assignments with the Headquarters Staff at Hazelhurst Field in Mineola, N. Y.; the 361st and the 1,107th Aero-Squadrons in Winchester, England; and the Headquarters Staff of the Air Service in London, England. In June, 1919, he was honorably discharged.

After a three-year service with the engineering division of the Texas Company in a supervisory capacity, Mr. McGinley, in 1922, became attached to the Department of Plant and Structures, later the Department of Public Works of the City of New York, and for eighteen years, until his death in 1940, served in the Bridge Division as an assistant engineer. During this period he was associated with the construction and maintenance of all bridges over navigable waters in the metropolitan area of New York and came to be recognized as an authority on bridgework, particularly the movable bridge type.

---

[1] Memoir prepared by James I. Rooney, M. ASCE.

Mr. McGinley was known as a man of high principles, sound judgment, and fine character—truly, a cultured gentleman, respected by his associates, and loved by his friends.

On October 8, 1934, Mr. McGinley was married to Cecilia A. Coyle of Huntsville, Ala., who survives him.

He was a registered professional engineer in the State of New York, and a member of the Newman Club of Columbia University and Post No. 1,222 of the American Legion in New York.

Mr. McGinley was elected an Associate Member of the American Society of Civil Engineers on September 10, 1923.

WILLIAM CANON MULDROW, Assoc. M. ASCE[1]

DIED JANUARY 14, 1947

William Canon Muldrow, the son of Simon O. and Annie L. Muldrow, was born at Paducah, Ky., on August 31, 1882. He was educated at the Mississippi Agricultural and Mechanical College at Starkville, which he attended from September, 1899, until March, 1903, majoring in agriculture.

Mr. Muldrow (Bill, to his many friends) had practiced engineering since 1903, when he started work as rodman with a field party of the Bureau of Reclamation at Phoenix, Ariz. In 1904 he transferred within the Bureau to the State of Washington where he continued in engineering work with advancing responsibilities and where he was engaged on locally financed irrigation projects, and in private practice in the Pasco, Yakima, and Okanogan districts in Washington until 1918.

In March, 1918, he was employed by the Inter-Ocean Barge and Transport Company of Seattle, Wash., as chief engineer, in full charge of design and construction of two five hundred-ton concrete barges, the first of their kind to be constructed in the United States. At last report, as of about 1945, they were still in active service and in excellent condition. Because of a lack of precedent and information on this type of construction, it was necessary to devise methods for the accurate placement of the reinforcing steel and for the settlement of the concrete around the steel to meet the rigid requirements of inspection and safety, all of which were satisfactorily solved. This work was terminated in 1918, at the end of World War I.

From 1920 to 1941 Mr. Muldrow served as engineer-manager and secretary of the Lake Chelan Reclamation District and the Columbia and Sunnyside irrigation districts in Washington. His studies while in the latter position furnished the basis and initial impetus for the passage of the National Repayment Act of 1939. These studies contributed much to the knowledge and practice of irrigation project maintenance and water use.

---

[1] Memoir prepared by Mark L. Nelson, M. ASCE.

In 1941 he was employed by the Corps of Engineers, United States Army, on the construction of military facilities in Alaska and the Aleutians, where he earned a citation from the War Department for his work during World War II. Since 1943, he was employed by the Portland District of the Corps of Engineers on flowage appraisals and as specialist on irrigation and water use studies for multiple-purpose project planning. He died on January 14, 1947, at Midland, Tex., where he was visiting with relatives while on annual leave from the Engineer Corps.

Mr. Muldrow was a registered professional engineer in the State of Washington and a member of the following organizations: The Washington State Reclamation Association, the Oregon Professional Engineering Employees Association, the Benton Lodge, No. 277 of the F. and A. M., and the Scottish Rite of Kennewick, Wash.

He was married at Yakima, Wash., on February 3, 1906, to Magdalene Caroline Hearle, who died at Portland, Ore., on August 11, 1945. He is survived by a sister, Annie Lida (Mrs. John Perkins) of Midland, and three nephews, Maxwell L. Larson, Ronald E. Larson, and Eugene H. Larson, all of Portland.

Mr. Muldrow was elected an Associate Member of the American Society of Civil Engineers on September 12, 1921.

LEMUEL BYNUM OGILVIE, Assoc. M. ASCE[1]

Died December 4, 1946

Lemuel Bynum Ogilvie was born in Crutchfield, Ky., on September 3, 1901. He was the son of Leslie Eugene and Ethelyn (Farrel) Ogilvie. At an early age he moved with his family to the Panhandle district of Texas and Oklahoma.

His education was informal. After attending high school in this district, he turned his attention to the rapidly developing oil industry and to its possibilities for engineering and construction. His first job in September, 1917, at the age of sixteen, was with the Texas Company, in Houston, Tex., in the Refinery Engineering Department, where he was employed as tracer in the drafting room and, later, as rodman in a field survey party. From this time, until the end of his life, Mr. Ogilvie was identified with the southwest and its oil production activity.

In October, 1920, Mr. Ogilvie left the Texas Company and took a position with the Humble Oil and Refining Company in Houston in the Production Engineering Department. He started as draftsman, then worked as transitman on land surveys and field locations, and later became chief of party. He spent three years with the Humble Oil and Refining Company.

For the next three years he held various jobs as draftsman, instrumentman, or chief of party. He was employed by J. M. Moskowitz, Surveyor, of Houston; the American Petroleum Company in Houston; the City of Corpus Christi, Tex.; and the Pure Oil Company in Mexia, Tex.; in addition to engaging in private engineering work in Fort Worth and Dallas, Tex.

---

[1] Memoir prepared by David R. Williams, Williams Brothers' Co., Engineers-Contractors, Tulsa, Okla.

Mr. Ogilvie returned to Houston in 1926 to take a position with the Harris County Engineering Department, in whose employ he remained for the next ten years. During this time he worked as chief of party on highway, bridge, and grade separation locations. He was county engineer on the Harris County Drainage Report of 1930—a $33,000,000 program. As office engineer of the county's bond program for highways and bridges, he did considerable design on drainage structures and was in charge of preparation of estimates.

From 1936 to 1938, Mr. Ogilvie was superintendent on highway and bridge construction for Parker Brothers and Company, Incorporated, in Houston. In January, 1938, he accepted the position of assistant chief engineer with the Mené Grande Oil Company, Compania Anonima (a subsidiary of the Gulf Oil Corporation), in Venezuela and became chief engineer in February, 1940. While employed by the Mené Grande Oil Company, Mr. Ogilvie was actively in charge of design and construction of docks, terminals, roads, bridges, pipe lines, and terminal structures—together with water, sewer, electric, and gas systems serving two large terminals. The first major oil pipe line in Venezuela was completed under his supervision in 1939.

In September, 1942, he requested, and was granted, a leave of absence from the company for the duration of World War II. It was during the early part of 1942 that submarine warfare in the Caribbean Sea was at its most critical period. Too young for World War I, Mr. Ogilvie now became fully occupied in construction projects which were a part of the over-all strategy for the prosecution of World War II. He worked for the Bechtel-Price-Callahan Company in Edmonton, Alberta, Canada, as principal engineer in charge of all planning and scheduling for the Canol Project. His work was completed on April 17, 1943, at which time Mr. Ogilvie resigned and returned to the United States as superintendent of construction for the Williams Brothers' Corporation on a water line for the United States Navy from Homestead to Key West, Fla., and later on for the Tennessee Gas Transmission line.

In 1944 he went to the Canal Zone, still in the employ of Williams Brothers' Corporation, as chief engineer on the construction of transisthmian Navy fuel lines, linking the Atlantic and Pacific terminals, complete with terminal facilities, loading racks, supplementary water plants, and pumping stations necessary to operate these pipe lines. The outstanding ability and devotion to duty displayed by Mr. Ogilvie, together with his unceasing hard work, contributed substantially to the success of these projects.

In September, 1945, Mr. Ogilvie returned to Venezuela for the Williams Brothers' Corporation. All of 1945 and half of 1946 he spent in eastern Venezuela on the construction of roads. From July, 1946, until his death he was in Maracaibo as chief engineer in charge of all Williams Brothers' activities in western Venezuela.

Mr. Ogilvie had filled all the junior grades in his profession and was devoting his later years to projects demanding mature and seasoned judgment. Possessed of imagination, quick sympathies, professional integrity, rare broad-gaged vision, he put to good use these qualities which are so essential in Americans representing our country abroad.

He is survived by his wife and two children, a daughter and a son, by his first marriage, and two small daughters by his second marriage.

Mr. Ogilvie was elected an Associate Member of the American Society of Civil Engineers on October 19, 1942.

ARTHUR GEORGE READY, Assoc. M. ASCE[1]

Died August 14, 1947

Arthur George Ready, the son of George A. and Katherine (Brennan) Ready, was born in Albany, N. Y., on December 28, 1895. His preliminary education was received at St. John's Preparatory School in Brooklyn, N. Y.

For his professional studies he enrolled at St. John's College in Brooklyn, N. Y., and after completing a four-year course was graduated in 1917 with the degree of Bachelor of Science in Civil Engineering.

Meanwhile, he had spent his summers to good advantage in obtaining engineering training; during 1915 he was timekeeper for the Tremper Construction Company in Brooklyn, and during 1916 he worked as transitman on a government project in Newport News, Va., for Westinghouse, Church, Kerr and Company. After graduation Mr. Ready continued with this latter association and immediately was appointed assistant engineer in the civil engineering department, engaged in office and field work at the embarkation station where he had charge of septic tank construction. When he completed one year of this work, he joined the United States Navy and served for a year.

A civilian once again, Mr. Ready was engaged from May, 1919, to December, 1920, by his previous employer, Westinghouse, Church, Kerr and Company —this time as field engineer on industrial construction in and near New York, N. Y. There followed a short employment with Henry R. Kent and Company of Rutherford, N. J., for whom he served as field engineer on a power plant at Lock Haven, Pa.

Mr. Ready's next employer was Walter E. Truesdell, a consulting engineer in New York City. While working for Mr. Truesdell he spent a short period on surveys and designs for paper mill improvements and then a year as resident engineer in Pearl River, N. Y., on construction of a high school and factory. For a time in 1921 he was engaged by George L. Watson, M. ASCE, consulting engineer, as estimator on the design of a sewerage system for Clifton, N. J.

---

[1] Memoir prepared by R. G. Macdonald, Secy.-Treas., Technical Assn. of the Pulp and Paper Industry, New York, N. Y.

Commencing in January, 1923, Mr. Ready began an association with the late George F. Hardy, M. ASCE, consulting engineer, which continued, with the exception of short interruptions, for many years and which developed his career as a construction and operations engineer in the paper industry. First employed as draftsman, he was soon promoted to positions of greater importance, such as assistant resident engineer and resident engineer in charge of pulp and paper mill construction. This work took him first to Moss Point, Miss., in 1924; then to Everett, Wash., for a year with the Everett Pulp and Paper Company; and again to the south, where he spent a number of years in Monroe, La., in connection with the plant of the Brown Paper Mill Company.

During this time Mr. Ready was in responsible charge of not only the construction of buildings, but also the process and power equipment. His duties, in addition, included design work on a hydroelectric station. This engagement was interrupted at the end of 1932 by the depression.

For a few months in 1933, Mr. Ready was superintendent with Beckerle and Wright, engineers and contractors of Pearl River, N. Y., being responsible for the construction of a residence, a garage, and a studio at New City, N. Y. From August, 1934, to July, 1935, he was employed by the Robert Gair Company in Piermont, N. Y. This engagement was a return to his interest in the paper manufacturing industry, and comprised the engineering charge of layout and installation of paper mill machinery.

Mr. Ready again resumed work with the George F. Hardy organization, this time for five and a half years. He supervised the construction of the mills of the Union Bag and Paper Corporation in Savannah, Ga. On completion of this undertaking he was transferred in 1938 to Lufkin, Tex., where he was similarly engaged for the Southland Paper Mills—an important development in the industry in which southern pine was first used in manufacturing newsprint.

His familiarity with the construction of paper mills led naturally to his being charged with the operation of such enterprises. On completion of the work at Lufkin, he was sent to St. Marys, Ga., where he made his home for the rest of his life. After starting construction of a new mill, he was induced to take charge of its operation as manager on behalf of the St. Marys Kraft Corporation, the construction of which plant he had handled for that company and Mr. Hardy.

Mr. Ready continued in this work until 1943, at which time he entered the employment of Rayonier, Incorporated, in its engineering department. Toward the end of 1944, he resumed his work on paper manufacture, when he accepted a position with Cia Industrial de Atenquique, Société Anonyme of Guadalajara, Jalisco, Mexico. The work included selection and purchase of equipment for a kraft pulp and paper mill as well as the hiring and training of personnel for operating the new mill. Shortly before his death, he became consulting engineer for the National Container Corporation in Jacksonville, Fla., in March, 1947. His death occurred in a Savannah hospital, to which he had been transferred from his home in St. Marys.

His long experience qualified him as an expert in paper manufacture in the southern states. Many years of work made Mr. Ready familiar with all the phases of the industry in the varied geographic fields. He was a member of the Technical Association of the Pulp and Paper Industry. He also belonged to the Knights of Columbus and St. Marys Catholic Church.

Surviving are his widow who, before their marriage in May 28, 1927, was Florence Berg; a sister, Mrs. Harry Patton, of Brooklyn; and two uncles, James Ready, of New York, and Frank Ready, of Hackensack, N. J.

Mr. Ready was elected a Junior of the American Society of Civil Engineers on July 6, 1920, and an Associate Member on March 5, 1928.

JOHN GEORGE SCHMIDT, Assoc. M. ASCE[1]

Died April 11, 1945

John George Schmidt was born in 1885 in Logansport, Ind. His education consisted in part of a correspondence course in mechanical engineering, courses in drafting at the Logansport Young Men's Christian Association, and a mathematics course at the University of Cincinnati (Cincinnati, Ohio), in addition to studies at night school.

Mr. Schmidt began his engineering career as draftsman in the shops of the Pennsylvania Railroad Company at Logansport. From 1907 until 1910 he was city engineer for Logansport, where he was engaged in designing and constructing sewerage systems, street improvements, and other general municipal projects.

He joined the Chicago Great Western Railroad Company in January, 1910, as draftsman, and in that same year was appointed city engineer of Oelwein, Iowa, building sewerage extensions and an addition to a water works plant, as well as laying pavements. In addition, he was elected surveyor of Fayette County, Iowa, in charge of large (100,000-yard) pavement projects. His time was equally divided between the work on the railroad and the latter positions.

In November, 1911, Mr. Schmidt went to Jacksonville, Fla., and, as principal assistant engineer, was in charge of topographical survey and sewer design. He supervised the investigation and redesign of sewage plants in that city, making topographical surveys and maps. Following this position, Mr. Schmidt went to Cincinnati in May, 1913, where he became assistant engineer in the sewerage department, improving some of the sewage facilities and designing many others. From January to March, 1914, he became a junior engineer with the United States Engineering Department, Corps of Army Engineers, on the design of steel dump scows and other steel construction during his stay there.

In March, 1914, Mr. Schmidt became chief engineer of the Department of Public Service in the city of Norwood, Ohio, a position which he held until 1919, except for a period during World War I when he was an inspector in the Department of Naval Ordnance. From 1919 until 1943 he spent his time in private practice, specializing in subdivision work, including architectural, street, sewer, and survey projects.

---

[1] Memoir prepared by Fred R. McMinn and Hunter W. Hanly, Members, ASCE.

Recovering from a serious illness in 1943, Mr. Schmidt was appointed, for a second time, city engineer of Norwood, a position he held until his death. He had charge of street paving, a water works plant and water mains, an electric light plant, and sewerage and garbage disposal, as well as other municipal projects. He was making a topographical survey of the city to eliminate railroad grade crossings by constructing undergrade crossings, in addition to his other work, when recurrence of his earlier illness brought an end to his career.

He is survived by his widow, Signe Nelson Schmidt, of Norwood.

Mr. Schmidt was elected an Associate Member of the American Society of Civil Engineers on September 12, 1916.

EDWARD SIMEON SKILLIN, Assoc. M. ASCE[1]

DIED JULY 1, 1947

Edward Simeon Skillin was born in 1866 in Hempstead, Long Island, N. Y. His parents were Edward Skillin and Mrs. Georgiana Dallas (Hewlett) Skinner. He was educated in private schools in Orange, N. J., entering New York University in New York, N. Y., as a sophomore in 1884. He was graduated in 1887 with the degree of Bachelor of Science in Civil Engineering.

Shortly after leaving college, Mr. Skillin started his business career as inspector for a water supply line, working under the direction of Charles B. Brush,[2] M. ASCE. In 1888 he joined the engineering staff of the New York Telephone Company, as assistant engineer, working on the underground electric conduit system installed by that company in New York. Three years later, in 1891, he joined Post and McCord, structural steel contractors, as draftsman and later became a contracting engineer for that company. In 1900 he joined the Snead and Company Iron Works, structural and ornamental iron and steel contractors.

Mr. Skillin was employed by this company as contracting engineer until 1902, when he left to join the Snare and Triest Company, contracting engineers, in New York, serving first as a contracting engineer; in 1906 he was elected a vice-president of that company, retaining this office until his death. During the forty-six years in which he was connected with the Snare and Triest Company (the name of the company being changed in 1921 to Frederick Snare Corporation), he was actively engaged in the supervision of the numerous contracts which the company carried out in the United States, the West Indies, and South America.

---

[1] Memoir prepared by A. W. Buttenheim, Pres., Frederick Snare Corp., New York, N. Y.

[2] For memoir, see *Transactions*, ASCE, Vol. LXXXIV, 1921, p. 818.

Since his principal activity was securing new business, he traveled extensively for the company and was very active and helpful in solving the problems involved in carrying out contracts after they had been awarded. Although he was stationed on various occasions in the Havana (Cuba) office of the corporation, his principal duties were in connection with work in the New York and Philadelphia (Pa.) areas.

Mr. Skillin actively assisted his company in carrying out important construction contracts for the United States Army and the United States Navy during World War I. They included the Chickamauga (Georgia) Cantonments, the Raritan (N. J.) Arsenal, and the Quartermaster Terminal in Philadelphia. During World War II, the company with which he was associated built the Caven Point Terminal, in Jersey City, N. J., for the Engineer Corps, U. S. Army; the naval bases at Guantánamo (Cuba), Jamaica, and the Bahamas; also a new breakwater, marine railways, and graving docks in the Canal Zone.

His active interest in the construction industry did not prevent his taking a keen interest in his home town, Glen Ridge, N. J., where he was a member of the Borough Council from 1911 to 1914, and active in other civic work. During World War I, he was a member of the Home Guard. He made many trips to Europe and spent a number of summers in Nova Scotia. He greatly enjoyed golf and fishing. His kindly and sympathetic nature brought him a host of friends, many of whom were members of the Society.

Mr. Skillin was an Episcopalian, and a regular attendant at Christ Church in Glen Ridge. He was a patron and member of the Montclair (N. J.) Art Museum. Greatly interested in welfare work, he was a regular contributor to the Bonnie Brae Farm for Boys in Millington, N. J.

Two of Mr. Skillin's ancestors were Sir Henry Skilings, Lord High Sheriff of Glasgow in the seventeenth century, and Simeon Skillin of Philadelphia, who carved the first figurehead on "Old Ironsides," the *U.S.S. Constitution*. He was an active member of the Sons of the American Revolution, the Pilgrims of the United States, and the Society of Colonial Wars. In addition, he was a member of the Pan-American Society, the Commerce and Industry Association of New York, Incorporated, and the Peruvian-American Chamber of Commerce, of which he was a director. For many years he was treasurer of the Engineers' Club of New York. He was also a member of the Railroad-Machinery Club, the University Club, and the Glen Ridge Country Club.

On May 6, 1903, Mr. Skillin was married to Geraldine Fearons, who died in 1914. He is survived by two children, a daughter, Elizabeth (Mrs. James D. Andrew), and a son, Edward S. Skillin, Jr.

Two of his associates, with whom he was actively engaged in business, Frederick Snare, Affiliate, ASCE, and Wolfgang Triest,[3] M. ASCE, both died within ten months of the date of Mr. Skillin's death. He will be greatly missed by the friends and business associates who survive him.

Mr. Skillin was elected an Associate Member of the American Society of Civil Engineers on December 1, 1908.

---

[3] For memoir, see *Transactions*, ASCE, Vol. 112, 1947, p. 1538.

JEROME BRANCH STOCKING, Assoc. M. ASCE[1]

DIED JULY 9, 1947

Jerome Branch Stocking, one of the seven children of Myron T. and Elizabeth (Thompson) Stocking, was born on November 4, 1873, at Lisbon, N. Y.

He attended the rural school in Lisbon and then spent three years as a student at Ogdensburg Free Academy (Ogdensburg, N. Y.); he was graduated from the academy and entered Amherst College (Amherst, Mass.), where he was a student for two years. To earn money to further his education he then spent several terms as a rural school teacher in New York State, commuting on horseback from his home. One such teaching position was in Amagansett, Long Island, N. Y.

Deciding to change his course of study, he entered the law college at the University of Michigan, in Ann Arbor, from which he was graduated in 1899 with the degree of Bachelor of Law. While at the university the Spanish-American War was declared, and he enlisted, serving from April 26 to September 28, 1898. At Amherst he joined the Delta Kappa Epsilon fraternity.

On being graduated he went to Albany, N. Y., where he passed the state bar examinations and subsequently established a law office in Buffalo, N. Y. There, while practicing law, he was stricken with a severe illness, diagnosed by the attending physicians as pleurisy. Fearful that this would develop into tuberculosis his physicians recommended that he go west and follow some kind of outdoor activity.

Accordingly, in 1904 he went to Dalhart, Tex., where his brother, Forest B. Stocking, was employed. After some months his health improved and, becoming restless, he went to Colorado in 1905. In Colorado Mr. Stocking learned that some railroad survey parties were being organized and he secured a position as chainman on such a party and was employed from June to September, 1905, by the Great Northern Railroad Company and the Northern Pacific Railroad Company. From September, 1905, to November, 1907, he worked for the Oregon Short Line Railroad Company (O.S.L.R.R.), part of the Union Pacific Railroad Company system, successively, as head chainman, rodman, topographer, leveler, and transitman. During this period he was engaged on the survey of a proposed railroad from Huntington, Ore., to Lewiston, Idaho, through the famous Hell's Canyon of Snake River. He also worked on the survey and construction of the Rupert to Bliss cutoff in Idaho for the O.S.L.R.R.

From November, 1907, to April, 1909, he was field assistant and chief of party with the United States Reclamation Service on the Boise-Payette project in Idaho, in charge of the design and construction of the canal system in a division of about 18,000 acres. He was engineer with the Conrad (Mont.) Land and Water Company on canal surveys from April to July, 1909.

---

[1] **Memoir prepared by Lynn Crandall, M. ASCE.**

From that date until 1916 Mr. Stocking was employed on the Salmon River irrigation project at Hollister, Idaho—first, as division engineer and, later, as assistant chief engineer. From September, 1916, to April, 1917, he was engineer (Carey Act) in the office of the Idaho state engineer. From 1917 until his death he was watermaster and engineer for the North Side Canal Company of Jerome, Idaho, on a 170,000-acre irrigation project in Jerome and Gooding counties in Idaho. He died while observing the effect of a prairie grass fire on the telephone line of the North Side Canal Company on Sugar Loaf Butte (east of Jerome).

"J. B.," as he was known to hundreds of friends, never married. His life was characterized by unremitting and steadfast industry and attention to the details of his job. As a young man he earned his way through college and paid all the expenses of his education.

R. J. Newell, M. ASCE, director, Region No. 1, United States Bureau of Reclamation, writes, as follows, concerning Mr. Stocking's early years in Idaho:

> "My close acquaintance with Mr. Stocking began with the construction of the Salmon Dam southwest of Twin Falls. This dam of considerable size was located over 30 miles from a railhead and the connecting road through the sagebrush was almost entirely unimproved. On this account, the hauling of a large volume of construction materials and supplies (principally cement) to the dam proved to be an important feature of the construction job. It was done by traction engines and trailer wagons and a large number of freight teams.
>
> "Mr. Stocking was in charge of the hauling during 1910 and 1911. He worked tremendously, hiring and organizing the crew, and galloping back and forth on a saddle horse, to hurry the loading at one end and the unloading at the other and assist with breakdowns along the road. There was rarely any delay at the dam on account of shortage of material while Stocking was on the job."

As watermaster for thirty years with the North Side Canal Company at Jerome he lived in the canal company office building. There he had a battery of telephones beside his desk and bed, keeping constant watch over the water levels in every section of the thousand miles of canals and laterals embraced within that project.

Mr. Stocking lived much to himself, finding his pleasure in the successful accomplishment of his daily work. He seldom talked about his personal affairs, and friends and associates of many years never heard him mention his early experiences, except in reply to specific questions. Trained in a rigorous school where long hours of hard work—beginning not later than 7 a.m. each day—were the accepted procedure, he had scant sympathy with such modern innovations as forty-hour weeks, time and a half for overtime, and so on.

His only consideration was to get the job done well, as economically and as quickly as possible. A few days before his death he remarked to a fellow worker that he took more pride in his membership in the Society that in anything he possessed.

He is survived by two sisters, Venila S. (Mrs. L. C. Heptonstall) and Emma L. Stocking, and by one brother, Forest B. Stocking.

Mr. Stocking was elected an Associate Member of the American Society of Civil Engineers on January 2, 1912.

ROBERT RAYMOND TWAY, Assoc. M. ASCE[1]

DIED AUGUST 21, 1946

Robert Raymond Tway, the son of Harry and Clara (Cartlich) Tway, was born in London, Ohio, on September 4, 1888. He attended the public schools in Springfield, Mo., and after completing two years of work at the School of Mines at Rolla, Mo., he moved to Oklahoma where he attended the Oklahoma State School of Mines at Wilburton, from which he received the degree of Bachelor of Science in Civil and Mining Engineering in 1912.

With three years of practical experience as city engineer for Wilburton and county engineer for Latimer County, Oklahoma, he returned to educational work to serve as instructor in mining engineering in the extension department of the State School of Mines at Henryetta, Okla., while continuing some private practice in engineering. In 1917 he began a five-year period as county engineer for Okmulgee County, Oklahoma, where he supervised the expenditure of $500,000 annually. In 1922–1923 Mr. Tway served as project engineer on federal aid project No. 53 in Okmulgee County, which was a million dollar undertaking

After six months in 1924 as division engineer for the State Highway Department, he entered the contracting field in which he prospered for twenty-two years before succumbing to a heart attack while on vacation in Buena Vista, Colo. His work consisted largely of bridge and drainage structures. His projects included: An underpass at Dalhart, Tex.; an overpass at El Reno and Cushing in Oklahoma; a bridge over the Trinity River north of Dallas, Tex.; and two bridges over the Arkansas River near Fort Smith, Ark. Mr. Tway was ever alert to new methods and was responsible for developing a patented pneumatic tamping roller.

He was a man of high principles and integrity. He has been described as one of the best contractors that Oklahoma has had and was noted for his sense of fair play and honest dealing. In addition to his activities in the Oklahoma Society of Professional Engineers, he belonged to the Chamber of Commerce, and was a Shriner, a Rotarian, and Thirty-Second Degree Mason. He was also active in the First Christian Church.

In Wilburton, on July 30, 1913, he was married to Lora Collier. He is survived by his widow; two daughters, Betty Gene Tway of Oklahoma City,

---

[1] Memoir prepared by Leo L. Laine, Assoc. M. ASCE.

Okla., and Mrs. John Charles Herndon of Salmon City, Idaho; and a son, Jack C. Tway. A son, Robert, Jr., has since passed away.

Mr. Tway was elected an Associate Member of the American Society of Civil Engineers on November 26, 1923.

BENJAMIN WEISS, Assoc. M. ASCE[1]

Died December 30, 1946

Benjamin Weiss, the son of Abraham and Lotte (Gruenwald) Weiss, was born on May 14, 1886, in Russia. He studied engineering at Cooper Union in New York, N. Y., receiving the degree of Bachelor of Science in 1909 and the postgraduate degree of Civil Engineer in 1921.

After several years as draftsman with the New York Telephone Company, as a result of a competitive civil service examination, Benjamin Weiss was appointed to the Queens Borough Topographic Bureau in 1907. For forty years he served on the civil service engineering staff of the City of New York—for almost all his professional career in the office of the President of the Borough of Queens, one of the five semiautonomous governmental subdivisions of the Greater City.

Successively, he held positions of increasing responsibility, finally serving on the staff of the Consulting Engineer to the Borough President and as engineering representative on the Queens Board of Local Improvements. This board initiates or approves all borough projects that are paid for entirely or in part by assessments on property to be benefited. The board's opinion also has considerable influence on city-wide improvements and private enterprises in so far as they affect the interests of the Borough of Queens—the population of which increased from less than 500,000 to more than 1,500,000 during the decades covered by Mr. Weiss' service.

One of his collateral activities was to supply radio publicity about the present needs and future requirements of Queens Borough for the benefit of the local taxpayers and to further Borough developments. Among the outstanding projects of national importance which he publicized were the site of the New York World's Fair, the temporary home for the United Nations activities, La Guardia Field, and Idlewild Airport. Of local importance was publicity on the transition of Queens from farm and suburban communities into an integrated urban unit involving street openings, highway paving, parkway developments, and provisions for building some of the largest sewers in the city—in all of which Mr. Weiss had some part.

On May 12, 1932, in New York, he was married to Lillian Fleischman. Mr. and Mrs. Weiss lived in the Jamaica Estates section of Queens where he indulged in music and painting as spare time hobbies.

---

[1] Memoir prepared by Alfred Brahdy, M. ASCE.

Mr. Weiss was a director of the Municipal Engineers of the City of New York and had been chairman of the Lecture and Publication Committee and editor of the quarterly, *Municipal Engineers Journal.* He was a licensed professional engineer and land surveyor in the State of New York.

Mr. Weiss was elected an Associate Member of the American Society of Civil Engineers on August 27, 1928.

Juniors

WILLIAM GEORGE CLARK, Jun. ASCE[1]

Died July 11, 1946

William George Clark was born on December 22, 1858, on the ancestral estate of his parents near Wattsburg, Erie County, Pa. He was the eldest of the six children born to Lorenzo Dow and Jane (Mitchell) Clark and was the last of the family to die.

Mr. Clark attended school in Wattsburg until 1875, and high school in Ann Arbor, Mich., for two years. In 1879 he entered the University of Michigan, also in Ann Arbor, from which he was graduated with the degree of Bachelor of Science in Civil Engineering in 1884, having been absent from college one year. He was a member of Delta Upsilon fraternity.

During the period from August, 1884, to June, 1887, Mr. Clark served, successively, as instructor in mathematics and chemistry, assistant professor of civil engineering, and acting professor of civil engineering at the Missouri School of Mines and Metallurgy at Rolla. From 1887 until 1892, he was first assistant city engineer of Toledo, Ohio, and for the following three years he was junior member of the firm of Wickenden and Clark, engineers, in Toledo. In 1895, he was elected city civil engineer of Toledo, remaining until 1897, when he became chief engineer for the American Sanitary Engineering Company of Detroit, Mich., holding that position until 1900. Mr. Clark engaged in private practice as a consulting engineer in Toledo from 1900 until his death—specializing in water supply, water purification, water softening, sewers, sewage disposal, and hydraulic developments.

He acquired recognition as one of the leading sanitary engineers in the Middle West. In addition to the many plant designs done exclusively by his office, he was frequently consulted, and collaborated with others, on water works improvements. Since the turn of the twentieth century, he made, or assisted in making, reports on a number of water supplies, including those at Toledo, Elyria, and Piqua, in Ohio; Decatur, Ill.; Rocky Ford, Colo.; and others. His firm built, rebuilt, or enlarged water purification and softening plants in Toledo, Sandusky, Port Clinton, Fremont, Napoleon, Lima, Barberton, and Piqua, in Ohio and in Flat Rock, Dearborn Ford Plant, Flint, and Alpena, in Michigan—as well as those in Decatur and Rocky Ford.

---

[1] Memoir prepared by William Clark Jones, Div. Engr., Water Dept., City of Toledo, Toledo, Ohio, and Owen E. Clark, Div. Engr. of Constr., Water Dept., City of Toledo, Toledo, Ohio.

Mr. Clark was one of the pioneers of rapid-sand filtration working along in the early days with George W. Fuller[2] and J. W. Ellms, Members, ASCE. He was designing engineer on some of the earliest plants of this type, notably the old Maumee River Works in the City of Toledo, which was first placed in operation on February 12, 1910. (This plant was abandoned and superseded by the Lake Erie Plant in 1942.)

He also did much of the early work in the development of technique and plant design for lime-soda softening. He worked closely with Charles P. Hoover of Columbus, Ohio, in this pioneering activity. Early softening plants of his design, in Piqua and Fremont, are still in successful operation.

Besides being an old and active member of the American Water Works Association, Mr. Clark was a member of the National Society of Professional Engineers and the Ohio Society of Professional Engineers. In his later years, Mr. Clark took pride in the distinction of being the oldest Junior in the Society, having acquired that membership prior to the adoption of age limitation rules. He never transferred his Society membership from Junior to Associate Member.

On December 21, 1886, at Plainfield, N. J., Mr. Clark was united in marriage to Annie Beasley, a Virginian of unusual cultural attainment and gracious personality, who died in Toledo in June, 1906. In June, 1914, Mr. Clark was married to Mrs. Arrilla Richter who died in 1943.

Mr. Clark's work was his hobby and absorbing interest. His ideals were high and he gave unswerving loyalty to what he believed to be right. He possessed a fine personality and no day was so crowded with work that there was not time for a story or a joke. He was an omnivorous reader. His extensive library bore ample testimony to the fact. Many of his technical books and periodicals will continue to serve in the Toledo Public Library and the library of the University of Toledo.

Since he had long been a member of the Masonic Order in which he had received the Thirty-Second Degree of the Scottish Rite, his funeral was conducted under the auspices of this order. He also belonged to the Toledo Chamber of Commerce and the Exchange Club of Toledo.

During the lifetime of the first Mrs. Clark, both Mr. and Mrs. Clark were active charter members of the Collingwood Presbyterian Church, but in later years Mr. Clark's religious connection was with the Baptist Church. No citizen was ever more interested in and loyal to the city of his chosen residence and occupation than William G. Clark was to Toledo and it is certain the word was graven in his heart.

Mr. Clark was elected a Junior of the American Society of Civil Engineers on April 4, 1888.

---

[2] For memoir, see *Transactions*, ASCE, Vol. 100, 1935, p. 1653.

LUDLOW VANDERBURG CLARK DEICHLER, Jun. ASCE[1]

DIED JUNE 27, 1945

Ludlow Vanderburg Clark Deichler was born on September 9, 1909, in Philadelphia, Pa., the son of L. Waller Deichler and Bessie L. (Clark) Deichler. Many years of usefulness were ahead of him when he was killed on June 27, 1945, on Okinawa. Seldom has a young man gone so far in his duties and so far in his rank in the United States Navy Civil Engineer Corps Reserve, at so early an age.

Before he entered college, Mr. Deichler spent more than two years in practical engineering work. For more than a year (1928–1929), he was job engineer with the Frank Stewart Construction Company in Philadelphia, on apartment house construction. The following year, August, 1929, to September, 1930, he was employed as draftsman for the United Engineers and Constructors, Incorporated, in Philadelphia, in connection with the electrification system of the Pennsylvania Railroad.

His activities are a matter of record, unusually complete—in the Navy Department; in the registrar's office at Georgia School of Technology in Atlanta; and in the office of its National Alumni Association's secretary in Atlanta. Abstracts from those records are given as furnished by the offices mentioned, and a few are selected and given herein from among the commendatory letters sent to the Board of Examiners for the Civil Engineer Corps of the United States Navy in 1936.

His scholastic and athletic activities are shown in the following letter from the registrar:

> "Mr. Deichler was admitted to the Georgia School of Technology on September 17, 1930. He was awarded the degree of Bachelor of Science in Civil Engineering on June 4, 1934, 'with honor'; on June 8, 1936, the degree of Master of Science in Civil Engineering; and on June 3, 1940, the Professional Degree of Civil Engineer.
>
> "As an undergraduate he was a member of the honor societies of Phi Kappa Phi, Tau Beta Pi, Phi Eta Sigma, and Scabbard and Blade. His name appeared upon the Honor Roll during the four years as an undergraduate. He received the president's Gold 'T' for scholastic excellence in his junior year. He was also a member of the Bulldog Club, Civil Crew, A.S.C.E., and Yellow Jacket Club, and the social fraternity of Phi Gamma Delta. He participated in football during his freshman year, and track during his sophomore year, was a member of the Rifle Team during his four years as an undergraduate, and was a Captain in the Reserve Officer's Training Corps."

Commencing in the fall of 1937, he was active on a number of construction projects as superintendent or engineer. For a few months he was in the employ of the Tellepsen Construction Company in Houston, Tex., on construction at the Baytown Refinery of the Humble Oil and Refining Company. The following year (1938) he was with the Standard Oil Company of California in San Francisco, designing steel and concrete refinery structures. For the first eight months of 1939, he worked with E. B. Badger and Sons in Boston, Mass., erecting cracking plants for high-octane gasoline.

---

[1] Memoir prepared by L. F. Bellinger, M. ASCE, and Carl W. Porter, Capt. (C.E.C.), U. S. N.

His early years and subprofessional work are covered in the following letter from H. R. Jeward, Superintendent of Irwin and Leighton, Contractors, of Philadelphia. The period covered by the letter is from 1921 to 1936.

"Philadelphia, 24th, August, 1936

"Dear Captain Allen:

> "Relative to your letter of the 21st instant, please be advised that I have known the applicant for the past fifteen years, and can honestly state that his character is of the best. He graduated from the Georgia School of Technology two years ago, an honour man, in the Engineering Class.
>
> "On several occasions he has been employed as an instrumentman, and superintendent of construction, on work of which I was the General Sup't.
>
> "At all times I found him to be reliable, and painstaking to a high degree. His work always gave entire satisfaction, to all with whom he came in contact.
>
> "I am certain that if he passes the examinations, he will if appointed uphold the traditions of the service."

His work for and with a contractor, from about 1932 to 1936, his carrying on the work of that contractor as manager before and after said contractor's decease, and a high opinion of his character and business ability, are given in this letter by his mother-in-law, Mrs. Allan Artley:

"September 1, 1936

"Dear Capt. Allen:

> "I am very pleased to give you the following information concerning Mr. Clark Deichler as requested by yourself in your letter of Aug. 21st.
>
> "My only way of judging his engineering ability would be to accept the very high opinion that I know my husband, the late Mr. Allan Artley, had of him.
>
> "Clark Deichler first worked for Mr. Artley while he was in school at Ga. Tech. He designed concrete and did some estimating in a part time position during the school year, and in summer worked both in the office and superintended work outside.
>
> "After his graduation from Ga. Tech. Clark accepted a position with Montgomery Ward & Co.
>
> "I can remember Mr. Artley's keen disappointment at losing Clark, but he was unable to compete with northern wages.
>
> "However, at Mr. Artley's repeated invitation, Clark returned to Atlanta several years ago, and joined the firm.
>
> "During the latter months of my husband's life, Mr. Artley sufficiently trusted Clark with partial control of his business, and just before his death the boy was put in complete charge.
>
> "Since Mr. Artley's death, Clark has handled all of my affairs and taken care of a large part of the business of the estate."

It is of interest to note that for the examination for the Civil Engineer Corps United States Navy, there were 850 applicants, and only about a dozen passed the examination. Mr. Deichler did well enough to receive an invitation with others near the top, to join the Civil Engineer Corps Reserve as Lieutenant (jg) (C.E.C.), U.S.N.R., which offer was accepted on July 8, 1938. He went on active duty on October 2, 1939.

His service duties are summarized in the following official record:

> "Subj.: Comdr. L. V. C. Deichler, (CEC), USNR—Abstract of Record of.
>
> 1. Comdr. Deichler was appointed Lt (jg), (CEC), USNR, rank from 3 June 1938, which appointment was accepted on 8 July 1938.
> 2. He went on active duty 2 October 1939 and since that date has had the following duties which are listed chronologically:
>   - (a) From 2 October 1939 to March 1941—Assistant Public Works Officer, Navy Yard, Boston; Resident Officer in Charge of Construction, Naval Air Station, Squantum; Naval Hospital, Chelsea; Naval Ammunition Depot, Hingham, Mass.
>   - (b) April, 1941—in the Bureau of Yards and Docks War Plans Section.
>   - (c) May, 1941 to January, 1942—Public Works Office, Naval Operating Base, Norfolk, additional duty Fifth Naval District.
>   - (d) February, 1942—Duty in the Bureau of Yards and Docks with temporary additional duty under Officer in Charge of Construction, Naval Air Station, Quonset Point, R. I.
>   - (e) March, 1942 to October, 1942—Naval Construction Training Command, Camp Allen, Va., as Ordnance Officer and Assistant Training Officer.
>   - (f) November, 1942—Naval Construction Training Command, Camp Peary as Ordnance Officer.
>   - (g) 30 November 1942—Assigned as Officer in Charge of 36th Naval Construction Battalion; sailed overseas in January, 1943, and returned September 18, 1944. He was detached from the 36th Construction Battalion 22 November 1944.
>   - (h) December, 1944 and January 1945—at Camp Parks.
>   - (i) 24 January 1945—was assigned as Officer in Charge of 42nd Naval Construction Regiment; sailed overseas in February 1945. Was killed 27 June 1945 on Okinawa.
> 3. (a) He was promoted to Lt. Comdr. by spot promotion 12 March 1943; was given an out-of-line promotion to Lt. Comdr. on 1 May 1943.
>   - (b) He was spot promoted to Comdr. on 1 September 1943 and promoted to Comdr. on 1 February 1944."

His advancement from Lieutenant (jg) to Commander in about three years indicates the strenuousness of his Sea Bee duties and the appreciation of his superior officers. His record, both in civilian and in service activity, would indicate that his was a strong character. He was notable for his clearness in speech, boldness in conception of ideas, independence in arriving at conclusions, and forcefulness in seeing that orders were carried out.

His outstanding capacity was recognized while he was still in college. At Georgia "Tech" he was president of the Student Chapter of the Society, and received the competitive prize granted by the Georgia Section for the best special thesis. He was quite a regular attendant at the monthly meetings of the section while he was in Atlanta.

Besides his family, Commander Deichler leaves his numerous college classmates and friends, all with deeply respectful regrets that he is not permitted to carry on into old age his engineering energies and his creative capabilities.

He was married on June 29, 1934 to Miss Emogene Artley, in Atlanta. Surviving him are the following near relatives:

Next of kin:	Mrs. Emogene Artley Deichler (wife), San Francisco, California.
Children:	Emogene Clark Deichler (daughter), born 12 February 1935. Ludlow Vandenburg Clark Deichler, II (son), born 2 October 1937.
Parents:	L. Waller Deichler (father), Philadelphia, Mrs. Bessie L. (Clark) Deichler (mother).

Commander Deichler was elected a Junior of the American Society of Civil Engineers on October 29, 1934, and an Associate Member on July 9, 1945, before the Society was advised of his death on June 27, 1945.

HUGH JOSEPH NOONAN, Jun. ASCE[1]

Died December 13, 1944

Hugh Joseph Noonan, the son of John J. and Margaret (O'Brien) Noonan, was born in Peabody, Mass., on October 24, 1919.

Mr. Noonan received his early schooling at St. John's grade school and St. Mary's high school. He attended Northeastern University in Boston, Mass., for five years, being graduated with the degree of Bachelor of Science in Civil Engineering.

In June, 1942, Mr. Noonan enlisted in the United States Army Air Corps, and was called to active duty in October, 1942. He was assigned, as an aviation cadet, to the Army Air Corps Technical Training Command at Chanute Field, Ill. Upon completion of his training, in January, 1943, he was commissioned a Second Lieutenant, and assigned to the Army Air Corps Engineering Division as an aircraft engineering officer.

In June, 1944, Lieutenant Noonan was ordered to the Pacific Theater of Operations where he valiantly performed the duties demanded of an Air Corps engineering officer. He was fatally injured while ably directing maintenance on an aircraft, and later died on December 13, 1944. He is buried somewhere on the Island of Leyte, in the Philippines.

Lieutenant Noonan was an engineer of great promise with ambition for success in the field of his chosen profession; above all he had an ardent desire to serve the United States to the fullest of his ability.

---

[1] Memoir prepared by F. J. Lariviere, Jun. ASCE.

In addition to his parents, Lieutenant Noonan is survived by one sister, Mrs. Joseph Deneen, and two brothers, John and William.

Lieutenant Noonan was elected a Junior of the American Society of Civil Engineers on October 19, 1942.

BERNARD WICK SHRUBB, Jun. ASCE[1]

DIED JULY 13, 1947

Bernard Wick Shrubb, the son of Bernard Zietz Shrubb and Elsie (Wick) Shrubb, was born in Brooklyn, N. Y., on March 16, 1921. A year later the family moved near to Freehold, N. J., where Mr. Shrubb attended a little two-room country school; he had the same teacher until he entered the Freehold High School from which he was graduated in 1937. While in school he won many prizes in 4H Club competition.

He was only sixteen years old when he went to Tuscaloosa, Ala., to attend the University of Alabama. He earned a good part of his tuition during vacations, but was forced to miss one year because of insufficient finances. After he received the degree of Bachelor of Science in Civil Engineering in 1943, he went to work for the Douglas Aircraft Company at Santa Monica, Calif.

There Mr. Shrubb did layout and design work on the hydraulic systems of the C–54 and C–74 type airplanes. His work at this time was valuable in building a vitally important air transport service.

In July, 1944, he received his commission as Ensign in the United States Naval Reserve and attended the indoctrination schools at Tucson, Ariz., and Hollywood, Fla. After receiving his amphibious training at Little Creek, Va., he went to Pearl Harbor, Hawaii, on the Landing Ship Medium (L.S.M.) 298, as executive officer. Returning to Astoria, Ore., he assisted in decomissioning the L.S.M. 298 and preparing her for the "Moth Ball Fleet."

Promoted to the rank of Lieutenant (jg), he was sent to Green Cove Springs, Fla., to do the same type of work with the L.S.M. 108 and L.S.M. 109. He was commanding officer of the latter ship.

After Mr. Shrubb had returned to inactive status in December, 1946, he secured a position as junior engineer with the Real Estate Department of the American Express Company in New York, N. Y. He was to have been advanced to a higher position in this company the following September.

He was elected chaplain of the Veterans of Foreign Wars Post in Freehold. Since he loved to sing, he became a member of the choir of the Ardena Baptist Church, which he attended nearly all his life—whenever at home. He was a member of the church from his early school days.

Mr. Shrubb died in an automobile accident of a fractured skull; he was killed instantly. The cause of the accident was never determined. He will be remembered for his kindness, for his sense of humor, and for his marked determination in all undertakings. Everyone he met became a friend—because he himself was friendly. His interests were many—among them was his love of sports. His detailed records of the major league baseball teams (also college football) showed that he was a worthy statistician. The engineering profession has lost an engineer; his associates have lost a friend; but all have benefited from having known him.

---

[1] **Memoir prepared by Harold Rozelle, formerly Stress Analyst at Douglas Aircraft Co., Santa Monica, Calif.**

He was married to Jane Gulledge of Tuscaloosa, on July 31, 1942. A son, Richard Gordon Shrubb, survives.

Mr. Shrubb was elected a Junior of the American Society of Civil Engineers on October 18, 1943.

DEAN EDWIN SWIFT, Jun. ASCE[1]

Died January 7, 1946

Dean Edwin Swift, the son of Edwin Dean and Bessie (Minor) Swift, was born in Blackburn, Mo., on December 1, 1911. In 1921, his family moved to Olathe, Kans., where he attended high school, being graduated in 1930 with an outstanding scholastic and athletic record .

In 1930 he entered the College of Emporia in Emporia, Kans., but a year later transferred to Kansas State College at Manhattan. At "K-State" he distinguished himself as a student and earned an enviable record as an athlete. His extracurricular activities included the Sigma Tau fraternity, the Acacia society fraternity, and the glee club, as well as football and wrestling. He was also president of the Student Chapter of ASCE, and held the rank of Lieutenant Colonel in the Reserve Officers Training Corps (R.O.T.C.). As a result of being selected the "Outstanding R.O.T.C. Student in the Seventh Corps Area," he was awarded a trip to Washington, D. C., where he received the Pershing Gold Medal for Military Merit. He was graduated with honors in 1935, with the degree of Bachelor of Science in Civil Engineering.

Following graduation he was engaged for a short time in highway and dam construction in Kansas, and on February 12, 1937, entered the United States Army as a Second Lieutenant in the Corps of Engineers. After a two-year assignment with the Second Engineers at Fort Logan, Colo., he entered Massachusetts Institute of Technology at Cambridge, and in 1940 received the degree of Master of Science in Civil Engineering.

From 1940 to 1944, he held various assignments, and, while in command of an armored engineer battalion at Camp Chaffee in Arkansas, he received the Legion of Merit award,

> "* * * for exceptionally meritorious conduct in the performance of outstanding service in organizing and supervising rescue work in connection with the all-time high water flood of the Arkansas River, 11 May to 27 May 1943, in which his efforts and professional knowledge saved many lives and

---

[1] Memoir prepared by C. L. Erb, Jr., and I. P. Hanson, Assoc. Members, ASCE.

> much property. He planned and directed the construction of two pontoon bridges over dangerous flood waters in order to provide support for temporary water mains to carry water into Fort Smith, Arkansas, when that city's water supply was completely cut off. By his zeal and energy, Colonel Swift efficiently controlled his partially trained battalion in these efforts, thereby displaying high professional and leadership qualities."

In March, 1944, Dean Swift, then a Lieutenant Colonel, went overseas to command the Forty-Eighth Combat Engineers and with this unit was in the battle for Cassino in Italy. His unit built the twin bridges across the Rapido River and roads through the mountains to Rome, and later took part in the invasion of southern France. In January, 1945, he assumed command of the 1,175th Engineer Combat Group and with that unit built several major bridges across the Rhine at Ludwigshafen, Mannheim, and at Heidelberg.

He was commissioned a full Colonel in May, 1945, and in June sailed directly for the Pacific theater of operations. In the Philippines he was in command of a port construction and repair group, engaged in the construction of gasoline storage and pumping facilities for Clark Field and in reconstruction of dock facilities at Manila.

On December 28, 1945, while in the Philippine Islands, Colonel Swift was stricken with poliomyelitis and died on January 7, 1946. He is buried in the United States Armed Forces Cemetery in Manila.

In addition to the Legion of Merit award, he received the Purple Heart and the Bronze Star. From the French Government, he received the Croix de Guerre with Palm "for exceptional war services rendered in the course of operations for the liberation of France."

He was married to Rita Brown at Manhattan, Kans., on June 6, 1936. He is survived by his widow; three daughters, Marilee, Carol, and Deanna; his parents, Mr. and Mrs. Edwin Swift; and two brothers, Harry and Philip.

Colonel Swift was elected a Junior of the American Society of Civil Engineers on October 14, 1935.

WILLIAM FRASER TOMPKINS, JR., Jun. ASCE[1]

DIED MARCH 13, 1945

William Fraser Tompkins, Jr., was born in Washington, D. C., on December 28, 1919. He was the son of an Army engineer, William F. Tompkins, then a Major, stationed at Fort Humphreys (later Fort Belvoir), Virginia. His mother was Elizabeth Hamilton, daughter of Mr. and Mrs. George E. Hamilton of Washington. On the paternal side of his family in Virginia, and on the maternal side in Maryland, he had many ancestors who, since early colonial days, had distinguished themselves in the service of their country. This background, combined with an intense loyalty for every person and every organization with whom he was associated, was the cause of his development into a young American citizen growing to manhood with the highest ideals of patriotism and devotion to duty.

---

[1] Memoir prepared by William F. Tompkins, M. ASCE.

Because of the constant moves of his Army father he was obliged to attend many different schools in boyhood. His principal education, prior to college, was received in the Bethesda-Chevy Chase High School at Bethesda, Md., from which he was graduated with distinction in 1936. Since his father had been ordered to New Orleans, La., he entered Tulane University there in the fall of 1936, taking the course in civil engineering. Although he was then but sixteen years old, he was unusually well developed, both physically and mentally, and for several years had determined to become an engineer. He was graduated with honors from Tulane in June, 1940, receiving the degree of Bachelor of Science in Civil Engineering. While at Tulane, Mr. Tompkins was elected to the honorary engineering society, Tau Beta Pi; he was a member of the Phi Kappa Sigma fraternity and of the boxing and baseball squads during his college years. His principal hobby, however, was hunting, and, as a natural corollary, he collected guns of all types. During summer vacations he worked with surveying parties; to illustrate his natural inclination for engineering, he always spent much of his spare time reading any engineering articles, technical magazines, or books he could lay his hands upon.

After graduation, Mr. Tompkins entered the office of the chief engineer of the Southern Pacific Lines at Houston, Tex., and almost immediately thereafter he enlisted in the 111th Engineers, a regiment of the 36th Division of the Texas National Guard. When the National Guard was called to duty in September, 1940, the 111th Engineers, with the 36th Division, were sent for training to Camp Bowie in Brownwood, Tex. He remained with the 111th Engineers for more than a year and was promoted from a Private to a Corporal; he was then commissioned a Second Lieutenant, and later promoted to First Lieutenant.

Early in 1942 he was ordered to Washington (D. C.) for duty with the Operations Division of the War Department General Staff. His work was connected with that of the logistics group, and one of his principal tasks was to allot ammunition to the entire United States Army. This was a most difficult and trying assignment, requiring not only an intimate knowledge of the subject, but much patience and tact, for ammunition was then in very short supply. His success is shown by his promotion first to Captain, and then to Major, and by his being awarded the Legion of Merit. He was Commissioned a Major when only twenty-two years old and certainly one of the youngest Majors of the Corps of Engineers—and, at that time, very possibly the youngest.

Like most young officers Major Tompkins was not satisfied with desk duty and his repeated requests (almost demands) to be assigned to a combat unit finally led to a transfer to Camp Breckinridge, Kentucky, in the spring of 1943, for training with an engineer combat group scheduled for assignment overseas. He accompanied this group to England in December, but subsequently was transferred to the 181st Engineers, a heavy pontoon battalion. As executive officer of this unit, he landed in Normandy shortly after "D"

Day and, on the drive to Paris, assisted in the construction of the first heavy pontoon bridge constructed in France over the Seine River.

On December 21, 1944, he was relieved from duty with the 181st Engineers and assigned to another heavy pontoon battalion, the 552d Engineers, as commanding officer. He remained with this unit to the end; with it Major Tompkins participated in the erection of the first heavy pontoon bridge constructed in Belgium, over the Meuse River, and, some time thereafter, of the first bridge of a similar type over the Rhine River. This was the bridge constructed from Kripp to Linz in Germany, some ten miles south of the famous Remagen Bridge; it was completed on March 11, 1945.

On March 13, at approximately 3:30 p.m., he was standing in an anchor barge just upstream of the bridge, when a German plane flew over and bombed the bridge; a bomb fragment struck him and pierced his helmet, causing his death. Major Tompkins was buried in the Henri Chappelle Cemetery near Aachen, Germany. The bridge was named after him, and the history of the 552d Engineers was dedicated to Major Tompkins with a full page photograph of him as the frontispiece. After the fixed bridge at Remagen collapsed, this pontoon bridge proved to be a vital necessity. More than 57,000 vehicles crossed this bridge while it was under the maintenance of the 552d Engineers. He received the Silver Star posthumously, the citation reading:

> "For gallantry in action. From 10 March 1945 to 13 March 1945, at ***. [The location was then secret information.] Major Tompkins, a Battalion Commander, displayed fearless and unflinching devotion to duty, planned and supervised the technical phase of the difficult construction of the first heavy ponton bridge over the treacherous Rhine River. Unmindful of his personal safety, he repeatedly exposed himself to aerial bombardment and constant enemy-observed shellfire to direct many technical details on the bridge construction. He so imbued and inspired his men by his gallant actions and extraordinary leadership that successful and timely completion of a hazardous project was achieved. Major Tompkins was mortally wounded while engaged in this vital task."

A person's character can best be judged by what his companions think of him. A few short excerpts are given from the many letters received after his death. The army engineer, First Army, wrote that he had seen him early that fatal afternoon, very happy in a job well done, and anxious to get on to the next one. The corps engineer, Seventh Corps, wrote that he was a fine battalion commander and that he did an outstanding job on this bridge, and, according to all present, more than anyone else; also, that at the time of the bombing he was checking the guy lines to make sure that the bridge was secure. His successor in command of the 552d Engineers wrote:

> "Major Tompkins was held in the highest esteem by all members of his command. He was a splendid leader, much admired by all his subordinates and was an outstanding personality. Those of us who have had the good fortune to serve under him feel that we have learned much and benefitted greatly by our associations with him. His loss will be deeply felt by his many friends."

The commanding officer of the 181st Engineers, the unit with which he served in Normandy, wrote:

"His passing was a great shock to us and has been keenly felt by everyone in the unit. His association with this organization will never be forgotten, for he was a friend and an inspiration to all, and to me personally, one of the finest officers I have ever been associated with."

His former professor of civil engineering at Tulane University wrote that he was an "original thinker," of one of the few he had ever taught; and the vice-president (formerly the chief engineer) of the Southern Pacific Lines at Houston wrote to the effect that, although Major Tompkins had been with the railroad but a short time before the war, he had won their regard and that they regretted he was not returning to fill the place which had been reserved for him.

Major Tompkins was married on December 6, 1941, the day before Pearl Harbor, to Evelyn Knopp, daughter of Dr. and Mrs. Thomas B. Knopp of Hondo, Tex., whom he met when stationed at Brownwood. He is survived by his widow and by two sons, William F. Tompkins, III, and Thomas K. Tompkins. He is also survived by his parents, Maj.-Gen. and Mrs. William F. Tompkins of Richmond, and by a brother, Christopher Tompkins, of New Orleans. Another brother, George Hamilton Tompkins, a night pursuit flyer, U. S. Army Air Forces, was killed in Normandy.

Major Tompkins was elected a Junior of the American Society of Civil Engineers on October 9, 1940.

JOHN MERLIN WILSON, Jun. ASCE[1]

DIED MAY 4, 1945

John Merlin Wilson, the son of John and Lorena (Woltzen) Wilson, was born in Guthrie Center, Iowa, on July 22, 1920. John, the eldest of three sons, spent his early life in Guthrie Center and Guthrie County. He attended Guthrie Center High School, from which he was gaduated in 1937. In 1937 he won a First in American history and a Second in economics in the Iowa Brain Derby. He entered Iowa State College at Ames, in 1938, being graduated in June, 1942, with the degree of Bachelor of Science in Civil Engineering. Immediately after graduation he started work for North American Aviation, Incorporated, of Kansas as an engineering trainee. Later he became an analyst in the production control department.

On November 16, 1942, Mr. Wilson joined the United States Marine Corps, shortly thereafter receiving a commission as Second Lieutenant. From November, 1942, until April, 1943, Leiutenant Wilson received his military training at Camp Le Jeune, New River, N. C., and Camp Pendleton, Oceanside, Calif. In February, 1942, he was promoted to First Lieutenant, and six weeks later he embarked for the South Pacific. He participated in the battle of Peleliu and, when not in combat, his duties at various times included operating a saw mill, building roads, and other engineering jobs.

---

[1] Memoir prepared by Keith R. Jeffers, Lt. (j.g.), U. S. Naval Reserve, Torrance, Calif.

He died while helping his platoon repulse a Japanese landing in the battle of Okinawa. A baseball diamond, named Wilson Field, has been constructed on Okinawa as a memorial to Lieutenant Wilson. He was awarded the Bronze Star and citation, posthumously, for meritorious achievement in action during the battle on Okinawa.

Lieutenant Wilson was sincere and diligent in all his work. These traits won the admiration of his superiors and associates; they also won him many friends. He was always interested in sports—his two favorites being hunting and fishing.

He is survived by his parents and two brothers, Robert and Thomas.

Lieutenant Wilson was elected a Junior of the American Society of Civil Engineers on October 19, 1942.

Affiliates

FREDERICK SNARE, Affiliate, ASCE[1]

DIED SEPTEMBER 22, 1946

Frederick Snare was born in Huntingdon, Pa., on December 4, 1862, the son of Edmund and Andria M. (Ziegler) Snare. His father was a physician. He was educated in private and public schools.

In 1879, he started his business career with the Huntingdon Car Works, after which he was employed by James G. Lindsey and Company, iron and steel commission merchants. From 1884 to 1899, he was with the Pencoyd Iron Works, during which time he became its vice-president. This experience, prior to the forming of his own construction company, admirably fitted him for the prominent part he was to take in later years in bridge building as the Pencoyd Iron Works, later incorporated in the American Bridge Company, was one of the leading rolling mills and fabricators of steel bridges.

On January 1, 1900, Frederick Snare, in association with W. G. Triest,[2] M. ASCE, organized an engineering and construction corporation known as Snare and Triest, Incorporated. The name of the corporation was later changed to the Snare and Triest Company, and finally to the Frederick Snare Corporation. Mr. Snare was president of the corporation until 1927, when he became the chairman of its board of directors. Under his leadership, foundations for some of the outstanding bridges in the New York (N. Y.) district were built by his corporation, and numerous other complete bridges including foundations were constructed. The Bronx-Whitestone Bridge across the East River; the Rip Van Winkle Bridge across the Hudson River; and the Outer Bridge and Goethals Bridge connecting Staten Island with New Jersey are examples of the type of work successfully carried out.

---

[1] Memoir prepared by A. W. Buttenheim, Pres., Frederick Snare Corp., New York, N. Y.

[2] For memoir, see *Transactions*, ASCE, Vol. 112, 1947, p. 1538.

In 1902 Mr. Snare went to Cuba to supervise some pier work under a contract awarded to his company by Gen. Leonard Wood, who was then in charge of construction work for the Cuban government. This started a career of port, bridge, and other heavy construction work in Latin American countries, which was later to include Colombia, Chile, Ecuador, Peru, Puerto Rico, Venezuela, and many other countries in Central America and the West Indies. Such well-known steamship terminals as the United Fruit Company and the Ward Line in Havana (Cuba), and the North Coast Railway piers at Nuevitas, Cuba, as well as the ports of Callao and Matarani in Peru; Cartagena, Colombia; and Cruz Grande, Chile, are a few of the many ports which were constructed under Mr. Snare's supervision. In many cases the contracts included design as well as construction.

During World War I, while his company was carrying out important construction contracts for the United States government, Mr. Snare, with the rank of Colonel, went to France to head some of the important activities of the Red Cross. During World War II, because of his wide experience his corporation had in construction work in the West Indies, his organization was selected to build the many new facilities that were needed at the United States Naval Base at Guantánamo, Cuba. His corporation did similar work in Panama, Jamaica, and Great Exuma Island in the Bahamas, and also handled all the field work in connection with the nickel oxide plant, adjacent to Levisa Bay on the north coast of Cuba. Additional war work included the design and building of the plant for storage and loading of bauxite which was constructed at Trinidad, and many other contracts for the United States Army and Navy.

One of the outstanding accomplishments of Frederick Snare was the service he rendered the cause for better understanding between the citizens of Latin America and the citizens of the United States. He spent many years in Cuba, and was twice decorated by the Cuban government for his many contributions to the social and economic life of that country. At the time of Mr. Snare's death, Grau San Martin, President of Cuba, gave public recognition that Frederick Snare was the outstanding representative of the foreign colony in Cuba, and cited the many services he had rendered to the Cuban people.

An ardent golfer, Mr. Snare organized the Havana Country Culb in 1911, and was its president from that time until his death. By organizing this beautiful club, which was the first golf club in Cuba and which is patronized by leading Cubans and by Americans who visit Havana, a common meeting ground was made available. This probably contributed more than any other single factor toward better understanding between the citizens of these two countries.

Mr. Snare's interest in golf gave him other opportunities to help international relations. He assisted in the founding of the Canadian Seniors' Golf Association and he was instrumental in the organization of the Great Britain Seniors Golfers' Society. As captain of the team of the United States Senior Golf Association, he frequently visited England and Canada, accompanied by a delegation of Americans who met and played with the leading citizens of these two countries. While Mr. Snare was president, the United States Seniors Golf Association also entertained many prominent British and Canadian golfers, when their seniors' associations were visiting United States.

Wherever Mr. Snare's construction activities brought him into contact with the citizens of Latin America, he did much to aid inter-American relations. An example of this was the important part he played in the organization of the Anglo-American Hospital in Havana. This hospital is used extensively by Cubans as well as by Americans, and has been very successful in serving the purpose for which it was organized.

Among the many clubs of which he was a member were the Engineers Club, the Metropolitan Club, the Union League Club, and India House, all of New York City.

On December 13, 1883, Mr. Snare was married to Ellen N. Brown at Huntingdon, Pa. They had four children. He is survived by two daughters, Elsie Lawrence (Mrs. Hugh Peters) and Helen Jeannette (Mrs. Frederic A. Stanwood).

It is interesting that his death on September 22, 1946, occurred within a few hours of, and at the same age as, that of his old friend and former associate, Mr. Triest.

Mr. Snare was elected an Affiliate of the American Society of Civil Engineers on October 6, 1903.

MAXIMILIAN TOCH, Affiliate, ASCE[1]

DIED MAY 28, 1946

Maximilian Toch, a learned man in many fields—chemistry, law, and art—was born in New York, N. Y., on July 17, 1864. His parents were Moses and Caroline (Levy) Toch.

In 1882 he completed a special course in chemistry at New York University in New York City, and in 1886 he received the degree of Bachelor of Legal Law from this university. Later he did graduate work in chemistry and engineering in New York City at Columbia University, and at the College of the City of New York. Mr. Toch lectured on organic chemistry at Columbia University in 1905 and 1906 and was the municipal lecturer on paint at the College of the City of New York during 1909. From 1919 to 1924, he was adjunct professor of industrial chemistry at Cooper Union, also in New York City.

Many of Mr. Toch's important contributions to paint technology were associated with his work for the United States government. In World War I, he developed the Toch camouflage system for the entire eastern seaboard. In addition to inventing the battleship-gray paint formula for the United States Navy, he perfected the cement filler used in the Panama Canal (1904 to 1914) as well as the integral waterproofing incorporated in the construction of numerous United States Navy dry docks. Mr. Toch was largely responsible for the introduction of China wood oil in the United States. His interest in the defense problems of the United States was shown by his long membership in the Army Ordnance Association and the Society of American Military Engineers. Mr. Toch was also a frequent adviser to fellow chemists and engineers—Leo Baekeland once remarked that he never would have perfected Bakelite if it had not been for the help and encouragement of Maximilian Toch.

---

[1] Memoir prepared by the late Lawrence H. Singer, Esq., New York, N. Y.

Mr. Toch's major activity throughout his life was in the field of chemistry and chemical engineering. For many years he was president and chief chemist of Toch Brothers, Incorporated, of Staten Island, N. Y. He served as chairman of the board of the Standard Varnish Works and was a past-president and an honorary member of both the Chemists' Club of New York and the American Institute of Chemists.

In 1924, he was appointed an honorary professor of chemical engineering and industrial chemistry at the University of Peking and lectured at the National Institute of Technology, both in Peking, China. During this same year he was awarded the degree of Doctor of Science by the University of Peking.

Among the several books, which Maximilian Toch wrote in connection with chemical engineering and industrial chemistry, were "The Chemistry and Technology of Paint" and "The Protection and Decoration of Concrete."

Dr. Toch carried a camera almost everywhere, and he became recognized as one of the world's leading photographers. His prints were exhibited in countless international shows, and he was constantly collecting medals and prizes for lens work. He was a fellow of the Royal Photographic Society of London, England, and of the Photographic Society of America; a past-president of the Camera Club of New York; and treasurer of the Oval Table, a leading camera group in the United States.

Allied with his work in photography was his deep interest in the world of art. For many years Dr. Toch held the position of professor of artistic painting at the National Academy of Design. In 1931 he provoked an international storm by charging, at a meeting of the New York Microscopical Society, that only one of the approximately thirty Rembrandts at the Metropolitan Museum of Art in New York City was genuine. He was the author of several book on art, including "Materials for Permanent Painting," "How to Paint Permanent Pictures," and "Paint, Painting, and Restoration." His last article pertaining to art, completed shortly before his death, was entitled "Hell-Bent for Art."

In addition to the many organizations already listed, Dr. Toch was a fellow of the American Association for the Advancement of Science, the New York Microscopical Society, and the Chemical Society of London. He was a member of the American Association of Arts and Sciences, the American Chemical Society, the American Institute of Chemical Engineers, and the Society of Chemical Industry in London.

Willam Howard Taft, when President of the United States, appointed Dr. Toch to the Second International Congress of Chemists.

On the social side, Dr. Toch was a member of the Cosmos Club of Washington, D. C., and the Bungalow Island Club of Portland, Ont., Canada. Also, among his wide range of interests was his active membership in the Society of American Magicians. A great sportsman, he rode horseback almost daily until he was eighty. He was an expert with a rifle and a lifetime devotee of fishing.

Mr. Toch was a warmhearted individual with a genial, vivacious personality outstanding in his professional attainments, and diligent in the pursuit of his hobbies. He will long be remembered with affection by all his friends and acquaintances.

On October 14, 1891, he was married to Hermine E. Levy. He is survived by his widow and four daughters, Elaine (Mrs. M. Pinco), Constance (Mrs. Arthur D. Robson), Alma (Mrs. Morris Lober), and Maxine (Mrs. Richard Frakenfelder).

Dr. Toch was elected an Affiliate of the American Society of Civil Engineers on September 1, 1903.

TRANSACTIONS

OF THE

AMERICAN SOCIETY OF CIVIL ENGINEERS

INDEX

VOLUME 113

1948

SUBJECT INDEX, PAGE 1635
AUTHOR INDEX, PAGE 1661

Titles of papers are in quotation marks when given with the author's name.

VOLUME 113

SUBJECT INDEX

ABUTMENTS

See DAMS

ADDRESSES

See AMERICAN SOCIETY OF CIVIL ENGINEERS—Addresses

AGRICULTURE

See IRRIGATION; LAND . . .; SOILS

AIR ENTRAINMENT

See WATER, FLOW OF . . .

AIRFIELDS

See AIRPORTS

AIRPLANES

"Stability of Thin Cylindrical Shells in Torsion," R. G. Sturm (with discussion), 681.

AIRPORTS

See also FOUNDATIONS, AIRPORT

"Classification and Identification of Soils," Arthur Casagrande (with discussion), 901.

"Drainage of Airport Surfaces—Some Basic Design Considerations," Stifel W. Jens, 785. *Discussion:* H. M. Williams, W. I. Hicks, E. J. A. Gain, E. F. Brater, Carl F. Izzard, and Stifel W. Jens, 810.

"New Formulas for Stresses in Concrete Pavements of Airfields," H. M. Westergaard (with discussion), 425.

ALINEMENT CURVES

See CURVES (alinement curves)

AMERICAN SOCIETY OF CIVIL ENGINEERS

Addresses

1948—Annual Presidential Address, R. E. Dougherty, 1391.

Memoirs of Members. *See* name of member in Author Index

APPARATUS

See under relative subject; *also* cross references under INSTRUMENTS

AQUIFERS

See RESERVOIRS; WELLS

BASINS, STILLING

See WATER, FLOW OF, IN OPEN CHANNELS

BEAMS (General)

See also GIRDERS; STRESS AND STRAIN—Beams; STRUCTURES, THEORY OF—Beams and Girders

"Beam Deflections by Second and Third Moments," Hsu Shih-Chang, 445. *Discussion:* A. Floris, Robert B. B. Moorman, Frank J. McCormick, William A. Conwell, and Hsu Shih-Chang, 465.

"Theory of Inelastic Bending with Reference to Limit Design," Alexander Hrennikoff (with discussion), 213.

Transverse tests of beams cut from dome fragments, 306.

BEAMS, CONTINUOUS

See BRIDGES; STRUCTURES, THEORY OF—Frames, Continuous

BENCH MARKS

See SURVEYS AND SURVEYING

BENDING

See BUCKLING; MOMENTS; STRESS AND STRAIN

BIBLIOGRAPHY

See subheading Bibliography under relative subject. (Comprehensive bibliographical footnotes existing in individual papers in which books and other material are cited)

BIOGRAPHIES OF DECEASED MEMBERS

See cross reference under MEMOIRS OF DECEASED MEMBERS

BOLTS

See JOINTS

BORINGS

"Cleaning and Grouting of Limestone Foundations, Tennessee Valley Authority": A Symposium, Arthur L. Pauls, T. F. Taylor (with discussion), 79.

BRIDGES (General)

See also BEAMS; COLUMNS; STRESS AND STRAIN—Bridges; STRUCTURES, THEORY OF—Bridges; TRUSS . . .

Live load for medium and long-span highway bridges, 276.

BRIDGES, TRUSS

See STRESS AND STRAIN—Bridges, Truss; TRUSSES

BUCKLING

"A Rational Explanation of Column Behavior," Frederick L. Ryder (with discussion), 40.

"Stability of Thin Cylindrical Shells in Torsion," R. G. Sturm (with discussion), 681.

"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, Harold B. Hammill, Henry J. Degenkolb, Henry D. Dewell, John J. Gould, and Leon H. Nishkian (with discussion), 1123.

BUILDINGS

See STRUCTURES, THEORY OF; TRUSSES, BUILDING; *also* under type of building, for example, MILLS

CAISSONS

"River Infiltration As a Source of Ground Water Supply," Raphael G. Kazmann (with discussion), 404.

CEMENT

See also GROUTING

"Experimental Observations on Grouting Sands and Gravels," Alfred Machis (with discussion), 181.

CHANNELS (waterways)

See RIVERS; SILT AND SILTING, CHANNEL; WATER, FLOW OF, IN OPEN CHANNELS

CHARTS

See HYDROGRAPHS; *also* under relative subject

CLAY

See SOILS

COLUMNS

See also STRESS AND STRAIN—Columns; STRUCTURES, THEORY OF—Columns

"Analysis of Stepped-Column Mill Bents," Daniel S. Ling, 1077. *Discussion:* Ralph E. Spaulding, E. I. Fiesenheiser, Walter J. Gray, L. J. Mensch, Arthur R. Graves, Erik Sollid, A. Floris, Charles W. Dohn, and Daniel S. Ling, 1099.

Factor of safety for columns, 54.

"A Rational Explanation of Column Behavior," Frederick L. Ryder, 40. *Discussion:* R. Robinson Rowe, George Winter, Robert M. Mains, Chi-Teh Wang, Ernest L. Robinson, and Frederick L. Ryder, 71.

"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, Harold B. Hammill, Henry J. Degenkolb, Henry D. Dewell, John J. Gould, and Leon H. Nishkian (with discussion), 1123.

COMPRESSION TESTS, SOIL

See SOILS—Tests and Testing

CONCRETE

Slabs. *See* SLABS.

CONCRETE DAMS

See DAMS, MASONRY AND CONCRETE

CONDUITS

See also FLUMES; TUNNELS

"Underground Conduits—An Appraisal of Modern Research," M. G. Spangler, 316. *Discussion:* Wilson V. Binger, Jacob Feld, Zdeněk Bažant, Jr., Anson Marston, George E. Shafer, E. F. Kelley, Bailey Tremper, and M. G. Spangler, 346.

Weights of cast-iron conduit sections, 997.

CONNECTORS AND CONNECTIONS

See JOINTS

CONSOLIDATION TESTS, SOIL

See SOILS—Tests and Testing

CONSUMPTIVE USE (water)

See WATER, CONSUMPTIVE USE OF

CONTINUOUS BEAMS
See BRIDGES; STRUCTURES, THEORY OF—Frames, Continuous

CONTINUOUS FRAMES
See STRUCTURES, THEORY OF—Frames, Continuous

CONTINUOUS STRUCTURES
See STRUCTURES, THEORY OF

COST INDEXES
See COSTS

COSTS, RESERVOIR GATE
"Estimating Data for Reservoir Gates," Frank L. Boissonnault (with discussion), 992.

COSTS, TUNNEL
Rock content of earth in relation to estimated and actual cost of Moffat Tunnel, in Colorado, 1330.

CULVERTS
"Underground Conduits—An Appraisal of Modern Research," M. G. Spangler (with discussion), 316.

CURVES (alinement curves)
"The General Principles of Highway Transition Curve Design," John Joseph Leeming, 868. *Discussion:* Carl F. Meyer, Ralph L. Fisher, William R. Welty, Elmer R. Haile, Charles M. Noble, and John Joseph Leeming, 881.

CURVES (elastic curves)
See STRUCTURES, THEORY OF

CURVES (graphic curves)
See cross reference under GRAPHICAL CHARTS

CURVES (transition curves)
See CURVES (alinement curves)

CUTOFFS
See RIVERS

CYLINDRICAL SHELLS
See STRUCTURES, THEORY OF—Shell Structures

DAMS (General)
See also FOUNDATIONS, DAM; GROUND WATER; MODELS, HYDRAULIC; RESERVOIRS; SPILLWAYS; WATER PRESSURE
Weights of radial crest gates used in power dams and water supply dams, 1013.

DAMS, EARTH
Application of drain wells to the design of earth dams, 747.
"Investigation of Drainage Rates Affecting Stability of Earth Dams," F. H. Kellogg, 1261. *Discussion:* Harry R. Cedergren, Ross M. Riegel, T. W. Lambe, W. L. Shannon, and F. H. Kellogg, 1285.
Soil classification for rolled earth dams (United States Engineer Department Airfield Classification System), 943.

DAMS, GRAVITY
See DAMS, MASONRY AND CONCRETE

DAMS, MASONRY AND CONCRETE
"Cleaning and Grouting of Limestone Foundations, Tennessee Valley Authority": A Symposium, Arthur L. Pauls, T. F. Taylor (with discussion), 79.

DEFINITIONS
See TERMINOLOGY

DEFLECTIONS
See MOMENTS; STRESS AND STRAIN; *also* under relative structure or structural part, for example, COLUMNS

DEFORMATION
See STRESS AND STRAIN; STRUCTURES, THEORY OF; *also* under specific type of strain, for example, TORSION; *also* under type of material, for example, TIMBER

DENTATED SILLS
See SILLS

DEPARTURES
See cross reference under LATITUDES AND DEPARTURES

DESIGN
See under relative subject; *see also* SPECIFICATIONS; STRUCTURES, THEORY OF

DISCHARGE
See WATER, FLOW OF . . .

DOMES
"Thin-Shelled Domes Loaded Eccentrically," Walter C. Voss, Dean Peabody, Jr., Howard R. Staley, and Albert G. H. Dietz, 293.

DRAINAGE
See also IRRIGATION; LAND RECLAMATION; PIPES AND PIPING; RAINFALL; RUNOFF; SEWERS; WELLS; *also* under type of drainage, for example, AIRPORTS
"Investigation of Drainage Rates Affecting Stability of Earth Dams," F. H. Kellogg (with discussion), 1261.

DRAIN WELLS
See WELLS

DRAWDOWN
See RESERVOIRS; WELLS

DRILLS AND DRILLING
"Cleaning and Grouting of Limestone Foundations, Tennessee Valley Authority": A Symposium, Arthur L. Pauls, T. F. Taylor (with discussion), 79.

EARTH . . .
See GROUND . . .; LAND . . .; SOIL . . .

EARTH DAMS
See DAMS, EARTH

EARTH MOVEMENTS

See LAND SUBSIDENCE

EARTH PRESSURE

"Subsidence of the Terminal Island-Long Beach Area, California," Frederic R. Harris and Eugene H. Harlow (with discussion), 375.

"Underground Conduits—An Appraisal of Modern Research," M. G. Spangler (with discussion), 316.

EARTHWORK

See also DAMS, EARTH; FOUNDATIONS

"Stability of Soil Slopes," Ek-Khoo Tan (with discussion), 139.

ELASTICITY

See also PHOTOELASTICITY; STRESS AND STRAIN

"Continuous Frame Analysis by Elastic Support Action," J. Charles Rathbun and C. W. Cunningham (with discussion), 577.

"A Rational Explanation of Column Behavior," Frederick L. Ryder (with discussion), 40.

"Thin-Shelled Domes Loaded Eccentrically," Walter C. Voss, Dean Peabody, Jr., Howard R. Staley, and Albert G. H. Dietz, 293.

EMBANKMENTS

See also DAMS, EARTH

"Stability of Soil Slopes," Ek-Khoo Tan (with discussion), 139.

"Underground Conduits—An Appraisal of Modern Research," M. G. Spangler (with discussion), 316.

ENERGY, LOSS OF

See FRICTION . . .; HYDRAULIC JUMP; WATER, FLOW OF . . .

ENGINEERING GLOSSARIES

See TERMINOLOGY

ENGINEERS AND ENGINEERING

See also AMERICAN SOCIETY OF CIVIL ENGINEERS (for memoirs of deceased members, *see* name of member in Author Index); RESEARCH (cross references thereunder); TERMINOLOGY

Function and cooperation of the geologist and engineer in tunneling operations, 1311, 1344.

"Reflections on Standard Specifications for Structural Design," Alfred M. Freudenthal (with discussion), 269.

EQUATIONS

See also relative subject

"Variation of Coefficients of Simultaneous Linear Equations," Bernard L. Weiner, 1349. *Discussion:* T. F. Hickerson, A. Floris, Frederick S. Merritt, I. F. Morrison, Sigmund Roos, and Bernard L. Weiner, 1375.

EROSION, STREAM

"Development and Hydraulic Design, Saint Anthony Falls Stilling Basin," Fred W Blaisdell (with discussion), 483.

EROSION, TAILWATER

"Development and Hydraulic Design, Saint Anthony Falls Stilling Basin," Fred W. Blaisdell (with discussion), 483.

EXCAVATION

See EARTH PRESSURE; FOUNDATIONS; *also* under type of specific substructure, for example, TUNNELS

EXPLORATION

See BORINGS; EXCAVATION (cross references thereunder); FOUNDATIONS

EXPOSITIONS

See under technical subject, for example, TIMBER AND TIMBERING

EXPRESSWAYS

See HIGHWAYS AND ROADS

FACTOR OF SAFETY

See relative subject

FATIGUE

See relative subject

FLOODS (Geographical)

Mississippi River

"Mississippi River Cutoffs," Gerard H. Matthes (with discussion), 1.

FLOW

See FLOODS; GROUND WATER; HEAT AND HEAT TRANSMISSION; RUNOFF; SEEPAGE; SEWERS; WATER, FLOW OF . . .

FLUMES

"Development and Hydraulic Design, Saint Anthony Falls Stilling Basin," Fred W. Blaisdell (with discussion), 483.

FORMULAS

See relative subject

FOUNDATIONS (General)

See also GROUTING; SOILS; WATER PRESSURE

Major benefits in drain well usage in weak compressible foundations, 743.

FOUNDATIONS, AIRPORT

United States Engineer Department Airfield Classification System, with physical characteristics of soil groups tabulated, 902, 911.

FOUNDATIONS, DAM

"Cleaning and Grouting of Limestone Foundations, Tennessee Valley Authority": A Symposium, Arthur L. Pauls, T. F. Taylor (with discussion), 79.

FRAMES

See STRUCTURES, THEORY OF—Frames; *see also* BEAMS; COLUMNS; GIRDERS

FREQUENCY DISTRIBUTION

See under topic, for example, WELDS AND WELDING

FRICTION COEFFICIENTS

"Friction Coefficients in a Large Tunnel," G. H. Hickox, A. J. Peterka, and R. A. Elder, 1027. *Discussion:* Weston Gavett, W. R. Barrows, Hunter Rouse, Karl R. Kennison, E. J. K. Chapman, Julian Hinds, William P. Creager and Stephen H. Haybrook, G. S. Tapley, J. N. Bradley and S. P. Wing, and G. H. Hickox, A. J. Peterka and R. A. Elder, 1047.

GAGES (General)

See also METERS AND METERING; *also* cross references under INSTRUMENTS

GAGES, PRESSURE

Manometer installations, Apalachia Tunnel of TVA, 1032.

GATES

See cross reference under RESERVOIR GATES

GEODETIC SURVEYS AND SURVEYING

See SURVEYS AND SURVEYING, GEODETIC

GEOLOGY

"Application of Geology to Tunneling Problems," Ernest E. Wahlstrom (with discussion), 1310.

"Cleaning and Grouting of Limestone Foundations, Tennessee Valley Authority": A Symposium, Arthur L. Pauls, T. F. Taylor (with discussion), 79.

"Subsidence of the Terminal Island-Long Beach Area, California," Frederic R. Harris and Eugene H. Harlow (with discussion), 375.

GEOMETRY

See MATHEMATICS; *also* relative subject, for example, MOMENTS

GIRDERS

See also BEAMS; STRUCTURES, THEORY OF—Beams and Girders

"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, Harold B. Hammill, Henry J. Degenkolb, Henry D. Dewell, John J. Gould, and Leon H. Nishkian (with discussion), 1123.

GLOSSARIES

See TERMINOLOGY

GRADING

See EARTHWORK

GRAPHICAL CHARTS

See HYDROGRAPHS; *also* under relative subject

GRAVEL

See also GROUTING; SEEPAGE; SOILS; STONE

"Experimental Observations on Grouting Sands and Gravels," Alfred Machis (with discussion), 181.

GRAVITY DAMS

See DAMS, MASONRY AND CONCRETE

GROUND . . .
See also EARTH . . .; LAND . . .; SOIL . . .

GROUND WATER
See also SEEPAGE; WATER SUPPLY
"Cleaning and Grouting of Limestone Foundations, Tennessee Valley Authority": A Symposium, Arthur L. Pauls, T. F. Taylor (with discussion), 79.
"Experimental Observations on Grouting Sands and Gravels," Alfred Machis (with discussion), 181.
"Investigation of Drainage Rates Affecting Stability of Earth Dams," F. H. Kellogg (with discussion), 1261.
Prediction of ground-water conditions in tunneling, 1319.
"River Infiltration As a Source of Ground Water Supply," Raphael G. Kazmann (with discussion), 404.
"Subsidence of the Terminal Island-Long Beach Area, California," Frederic R. Harris and Eugene H. Harlow (with discussion), 375.

GROUNDWORK
See EARTHWORK; FOUNDATIONS

GROUTING
"Cleaning and Grouting of Limestone Foundations, Tennessee Valley Authority": A Symposium, Arthur L. Pauls, T. F. Taylor (with discussion), 79.
"Experimental Observations on Grouting Sands and Gravels," Alfred Machis (with discussion), 181.

GUSSET PLATES
See PLATES

HEAT AND HEAT TRANSMISSION
See also TEMPERATURE
Analogy between seepage and heat flow, 1267.

HEAT FLOW
See HEAT AND HEAT TRANSMISSION

HIGHWAY BRIDGES
See BRIDGES

HIGHWAY CURVES
See CURVES (alinement curves)

HIGHWAYS AND ROADS (General)
See also BRIDGES; CURVES (alinement curves); PAVEMENT AND PAVING; WHEEL LOADS
United States Bureau of Public Roads soils classification system, 902, 909.

Bibliography
Soils classification and identification, 928.

Planning and Design
"The General Principles of Highway Transition Curve Design," John Joseph Leeming (with discussion), 868.

HIGHWAYS AND ROADS (Geographical)
  Texas
    Texas Highway Department soil classification system, 902, 910.
HYDRAULIC JUMP
  "Development and Hydraulic Design, Saint Anthony Falls Stilling Basin," Fred W. Blaisdell (with discussion), 483.
  Bibliography
    Development and hydraulic design of stilling basins, 517.
HYDRAULIC MODELS
  See MODELS, HYDRAULIC
HYDRAULICS
  See also CONDUITS; COSTS . . .; DAMS; DRAINAGE; EROSION; FLOODS; FLOW (cross references thereunder); FLUMES; FOUNDATIONS . . .; FRICTION . . .; GROUND WATER; HYDRAULIC . . .; HYDRO- . . .; IRRIGATION; LAND RECLAMATION; MODELS, HYDRAULIC; PIPES AND PIPING; RAINFALL; RESERVOIRS; RIVER . . .; RUNOFF; SEA WATER; SEEPAGE; SEWERS; SILLS; SILT AND SILTING . . .; SPILLWAYS; TERMINOLOGY; TUNNELS, WATER; WATER . . .; WELLS
  "Friction Coefficients in a Large Tunnel," G. H. Hickox, A. J. Peterka, and R. A. Elder (with discussion), 1027.
  "Mississippi River Cutoffs," Gerard H. Matthes (with discussion), 1.
  Bibliography
    Development and hydraulic design of stilling basins, 517.
HYDRODYNAMICS
  "Development and Hydraulic Design, Saint Anthony Falls Stilling Basin," Fred W. Blaisdell (with discussion), 483.
HYDROGRAPHS, RUNOFF
  Airport drainage hydrographs, 798.
HYDROGRAPHS, WATER STORAGE
  Drainage of sand model resulting from drawdown of impounded water, 1264, 1266.
HYDROSTATIC UPLIFT
  See WATER PRESSURE
INDUSTRIAL PLANTS
  See under type of plant, for example, MILLS
INDUSTRIAL WATER SUPPLY
  See IRRIGATION; WATER SUPPLY
INDUSTRY
  See under relative technical classification, for example, LAND RECLAMATION: *also* under type of industry or industrial plant, for example, MILLS
INFILTRATION
  See GROUND WATER; RUNOFF; SEEPAGE

INFLUENCE LINES
See MOMENTS

INSTRUMENTS
See GAGES; METERS AND METERING; *also* under usage

IRRIGATION (General)
See also WATER SUPPLY

IRRIGATION (Geographical)
Western States
"Forecasting Productivity of Irrigable Lands," W. C. Muldrow (with discussion), 562.

JOINTS (General)
See also WELDS AND WELDING
Bolt failure by shear, 1153.
Characteristics of nails in plywood, 1228.
"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, Harold B. Hammill, Henry J. Degenkolb, Henry D. Dewell, John J. Gould, and Leon H. Nishkian (with discussion), 1123.

JUMP, HYDRAULIC
See HYDRAULIC JUMP

LAND . . .
See also EARTH . . .; GROUND . . .; SOIL . . .; SURVEYS AND SURVEYING . . .

LAND RECLAMATION
"Forecasting Productivity of Irrigable Lands," W. C. Muldrow (with discussion), 562.

LANDSLIDES
"Stability of Soil Slopes," Ek-Khoo Tan (with discussion), 139.

LAND SUBSIDENCE
"Subsidence of the Terminal Island-Long Beach Area, California," Frederic R. Harris and Eugene H. Harlow (with discussion), 375.
Bibliography
Land subsidence, stressing Terminal Island, Long Beach, California, area, 395.

LATITUDES AND DEPARTURES
See SURVEYS AND SURVEYING

LEAST SQUARES, THEORY OF
See PROBABILITY, THEORY OF

LIMESTONE
"Cleaning and Grouting of Limestone Foundations, Tennessee Valley Authority": A Symposium, Arthur L. Pauls, T. F. Taylor (with discussion), 79.

LIMIT DESIGN, THEORY OF

"Theory of Inelastic Bending with Reference to Limit Design," Alexander Hrennikoff, 213. *Discussion:* George Winter, D. C. Drucker, Michael R. Horne, E. P. Popov, Bernard A. Vallerga, and Alexander Hrennikoff, 248.

LOAD

See STRESS AND STRAIN; WHEEL LOADS; *also* under structure, structural member or part, for example, BEAMS; CONDUITS; JOINTS

LOAD, SUSPENDED

See SILT AND SILTING

LOSS OF ENERGY

See FRICTION; HYDRAULIC JUMP; WATER, FLOW OF . . .

LUMBER AND LUMBERING

See TIMBER AND TIMBERING

MANOMETERS

See GAGES, PRESSURE

MASONRY DAMS

See DAMS, MASONRY AND CONCRETE

MATERIALS OF CONSTRUCTION

See also CEMENT; EARTHWORK; EROSION; GRAVEL; PLYWOOD; SAND; SEEPAGE; SOILS; STEEL; STONE; STRESS AND STRAIN . . .; STRUCTURES, THEORY OF—Materials and Tests; TIMBER AND TIMBERING; *also* cross references under PERMEABILITY OF MATERIALS; STRENGTH OF MATERIALS (cross references thereunder)

"Reflections on Standard Specifications for Structural Design," Alfred M. Freudenthal (with discussion), 269.

MATHEMATICS

See also EQUATIONS; PROBABILITY, THEORY OF; *also* under relative subject

Flow net computations for drainage flow rates in relation to stability of earth dams, 1279.

"Influence Lines for Continuous Structures by Geometrical Computation," Dean F. Peterson, Jr. (with discussion), 755.

Matrix algebra methods is relation to the solving of simultaneous linear equations, 1380, 1386.

"A Rational Explanation of Column Behavior," Frederick L. Ryder (with discussion), 40.

Spiral functions in equations for transition spirals, 877.

MATRIX ALGEBRA

See MATHEMATICS

MECHANICS, SOIL

See SOILS

MEMBERS, STRUCTURAL

See BEAMS; COLUMNS; GIRDERS; STRUCTURES, THEORY OF

MEMOIRS OF DECEASED MEMBERS
See name of member in Author Index

METERS AND METERING (General)
See also cross references under INSTRUMENTS
Schematic diagram of constant-head permeameter, 191.

MILLS
"Analysis of Stepped-Column Mill Bents," Daniel S. Ling (with discussion), 1077.

MINERALOGY
See relative technical subject, for example, SOILS

MODELS, HYDRAULIC
"Development and Hydraulic Design, Saint Anthony Falls Stilling Basin," Fred W. Blaisdell (with discussion), 483.
"Investigation of Drainage Rates Affecting Stability of Earth Dams," F. H. Kellogg (with discussion), 1261.
Bibliography
Development and hydraulic design of stilling basins, 517.

MODELS, STRUCTURAL
"Stability of Soil Slopes," Ek-Khoo Tan (with discussion), 139.
"Thin-Shelled Domes Loaded Eccentrically," Walter C. Voss, Dean Peabody, Jr., Howard R. Staley, and Albert G. H. Dietz, 293.

MOMENTS
See also STRESS AND STRAIN; STRUCTURES, THEORY OF; *also* under specific type of strain, for example, TORSION
"Analysis of Stepped-Column Mill Bents," Daniel S. Ling (with discussion), 1077.
"Beam Deflections by Second and Third Moments," Hsu Shih-Chang (with discussion), 445.
"Continuous Frame Analysis by Elastic Support Action," J. Charles Rathbun and C. W. Cunningham (with discussion), 577.
"Influence Lines for Continuous Structures by Geometrical Computation," Dean F. Peterson, Jr. (with discussion), 755.
"A Rational Explanation of Column Behavior," Frederick L. Ryder (with discussion), 40.
"Stability of Thin Cylindrical Shells in Torsion," R. G. Sturm (with discussion), 681.
"Theory of Inelastic Bending with Reference to Limit Design," Alexander Hrennikoff (with discussion), 213.

MOTORWAYS
See HIGHWAYS AND ROADS

NAILS
See JOINTS

NOMENCLATURE
See TERMINOLOGY

OBITUARIES OF MEMBERS

See cross reference under MEMOIRS OF DECEASED MEMBERS

OIL, FLOW OF

See WELLS

OPEN CHANNELS

See cross references under CHANNELS

PAVEMENT AND PAVING

See also STRESS AND STRAIN—Pavement and Paving; STRUCTURES, THEORY OF—Pavement and Paving; WHEEL LOADS

"Drainage of Airport Surfaces—Some Basic Design Considerations," Stifel W. Jens (with discussion), 785.

Group classification of highway subgrade materials, 982, 983.

"New Formulas for Stresses in Concrete Pavements of Airfields," H. M. Westergaard, 425. *Discussion:* Robert Horonjeff, Evan P. Bone, and H. M. Westergaard, 440.

Thickness as related to bearing ratios, 934.

PERCOLATION

See GROUND WATER; SEEPAGE

PERMEABILITY OF MATERIALS

See SEEPAGE; STRENGTH OF MATERIALS (cross references thereunder); *also* under type of material, for example, SAND; SOILS

PERMEAMETERS

See METERS AND METERING (General)

PHOTOELASTICITY

Studies of gelatin models of soil slopes, 147.

PIPES AND PIPING

See also CONDUITS; SEWERS; TUNNELS, WATER; WATER, FLOW OF, IN PIPES

Comparison of test data on large concrete pipe, Apalachia Tunnel, TVA, and other structures, 1068.

"Drainage of Airport Surfaces—Some Basic Design Considerations," Stifel W. Jens (with discussion), 785.

"Stability of Thin Cylindrical Shells in Torsion," R. G. Sturm (with discussion), 681.

"Underground Conduits—An Appraisal of Modern Research," M. G. Spangler (with discussion), 316.

PLANNING

See HIGHWAYS AND ROADS—Planning and Design

PLASTICITY

"Stability of Soil Slopes," Ek-Khoo Tan (with discussion), 139.

"Theory of Inelastic Bending with Reference to Limit Design," Alexander Hrennikoff (with discussion), 213.

PLASTIC THEORY
See PLASTICITY

PLATES
See also SLABS
"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, Harold B. Hammill, Henry J. Degenkolb, Henry D. Dewell, John J. Gould, and Leon H. Nishkian (with discussion), 1123.

PLYWOOD
Characteristics of plywood used in timber structures, 1223.

POLLUTION, STREAM
See WATER POLLUTION

PORE WATER PRESSURE
See WATER PRESSURE

POROSITY
See SEEPAGE

PORTS
See AIRPORTS

PRECIPITATION
See RAINFALL

PRESIDENTIAL ADDRESSES (ASCE)
See AMERICAN SOCIETY OF CIVIL ENGINEERS—Addresses

PRESSURE
See EARTH PRESSURE; STRESS AND STRAIN; WATER PRESSURE

PRESSURE GAGES
See GAGES, PRESSURE

PRESSURE TUNNELS
See TUNNELS, WATER

PROBABILITY, THEORY OF
"Least Squares Adjustment of Triangulation Net Between Geodetic Stations," E. F. Coddington and O. C. J. Marshall (with discussion), 837.

PROTOTYPES AND MODELS
See MODELS

PUBLIC WORKS
See RIVER VALLEY AUTHORITIES; *also* under relative type of structure or project

RAINFALL
See also DRAINAGE; FLOODS; RUNOFF; SEEPAGE
"Drainage of Airport Surfaces—Some Basic Design Considerations," Stifel W. Jens (with discussion), 785.

RECLAMATION, LAND
See LAND RECLAMATION

RESEARCH
See MODELS; PROBABILITY, THEORY OF; STRUCTURES, THEORY OF; TESTS AND TESTING (cross references thereunder); *also* under relative subject

RESERVOIR GATES
See RESERVOIRS

RESERVOIRS (General)
See also COSTS, RESERVOIR . . .; HYDROGRAPHS, WATER STORAGE
Data on weights of reservoir gates and gate hoists, 992.
"Estimating Data for Reservoir Gates," Frank L. Boissonnault, 992. *Discussion:* Theodore B. Rights, George R. Latham, D. A. Buzzell, A. E. Niederhoff, Joseph R. Bowman, and Frank L. Boissonnault, 1012.

RESERVOIRS, WATER STORAGE
Effects of increased rates of reservoir drawdown, 1261.

RIGID FRAMES
See STRUCTURES, THEORY OF—Frames, Rigid

RIVERS (General)
See also SILT AND SILTING, CHANNEL

RIVERS (Geographical)
Mississippi River
"Mississippi River Cutoffs," Gerard H. Matthes, 1. *Discussion:* W. E. Elam, C. L. Hall, H. D. Vogel, Harry N. Pharr, Lytle Brown, Anson Marston, E. J. Williams, Jr., and Eugene A. Graves, and Gerard H. Matthes, 16.
Performance statistics, artificial cutoffs on the Mississippi River, years 1934–1945, 7.

RIVER VALLEY AUTHORITIES
"Cleaning and Grouting of Limestone Foundations, Tennessee Valley Authority": A Symposium, Arthur L. Pauls, T. F. Taylor, 79. *Discussion:* Berlen C. Moneymaker, C. E. Blee, Robert A. Laurence, James B. Ward, J. C. Nowell, John E. Partridge, and Portland P. Fox, 120.
"Friction Coefficients in a Large Tunnel," G. H. Hickox, A. J. Peterka, and R. A. Elder (with discussion), 1027.
Investigation of drainage rates affecting stability of earth dams performed under auspices of the Tennessee Valley Authority, 1261, 1277.

ROADS
See HIGHWAYS AND ROADS

ROCK
See FOUNDATIONS; LIMESTONE; SHALE; STONE

ROOFS AND ROOFING
See DOMES; TRUSSES

ROUGHNESS FACTOR
See FRICTION COEFFICIENTS; TUNNELS, WATER

RUNOFF

See also HYDROGRAPHS, RUNOFF; RAINFALL

"Drainage of Airport Surfaces—Some Basic Design Considerations," Stifel W. Jens (with discussion), 785.

SAFETY FACTOR

See relative subject

SALT WATER

See SEA WATER

SAND

See also SILT AND SILTING . . .; SOILS

"Experimental Observations on Grouting Sands and Gravels," Alfred Machis (with discussion), 181.

SANITATION

See DRAINAGE; PIPES AND PIPING; SEWERS; TANKS; WATER . . .

SCOUR

See EROSION . . .

SEA WATER

"Experimental Observations on Grouting Sands and Gravels," Alfred Machis (with discussion), 181.

SEEPAGE

See also GROUND WATER

"Drainage of Airport Surfaces—Some Basic Design Considerations," Stifel W. Jens (with discussion), 785.

"Experimental Observations on Grouting Sands and Gravels," Alfred Machis (with discussion), 181.

"Investigation of Drainage Rates Affecting Stability of Earth Dams," F. H. Kellogg (with discussion), 1261.

SETTLEMENT OF STRUCTURES

See EARTH PRESSURE; EMBANKMENTS; FOUNDATIONS . . .; SOIL . . .

SEWERS

See also PIPES AND PIPING

"Underground Conduits—An Appraisal of Modern Research," M. G. Spangler (with discussion), 316.

SHALE

"Cleaning and Grouting of Limestone Foundations, Tennessee Valley Authority": A Symposium, Arthur L. Pauls, T. F. Taylor (with discussion), 79.

SHEAR

Shear correction factors in relation to continuous frame analysis by elastic support action, 586.

"Stability of Soil Slopes," Ek-Khoo Tan (with discussion), 139.

"Stability of Thin Cylindrical Shells in Torsion," R. G. Sturm (with discussion), 681.

"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, Harold B. Hammill, Henry J. Degenkolb, Henry D. Dewell, John J. Gould, and Leon H. Nishkian (with discussion), 1123.

"Theory of Inelastic Bending with Reference to Limit Design," Alexander Hrennikoff (with discussion), 213.

SHELL STRUCTURES

See DOMES; FLUMES; STRUCTURES, THEORY OF—Shell Structures

SILLS

"Development and Hydraulic Design, Saint Anthony Falls Stilling Basin," Fred W. Blaisdell (with discussion), 483.

SILT AND SILTING, CHANNEL

Silt clogging of intake area in relation to infiltration of river water, 412.

SLABS

See also PLATES

"New Formulas for Stresses in Concrete Pavements of Airfields," H. M. Westergaard (with discussion), 425.

SOCIETIES, TECHNICAL

See AMERICAN SOCIETY OF CIVIL ENGINEERS

SOIL MECHANICS

See SOILS

SOIL PRESSURE

See EARTH PRESSURE

SOILS

See also EARTH PRESSURE; EROSION . . .; GRAVEL; GROUND . . .; SAND; SEEPAGE; SHALE; SILT AND SILTING . . .; STRESS AND STRAIN—Soils

"Stability of Soil Slopes," Ek-Khoo Tan, 139. *Discussion:* Paul Baumann, Gregory P. Tschebotarioff, Jacob Feld, Earl M. Buckingham, K. B. Hirashima, Donald M. Burmister, and Ek-Khoo Tan, 159.

Bibliography

Classification and identification of soils, 928.

Soils in Southern California and their relation to land subsidence, 395.

Classification

Agricultural classification of surface soils, 903, 908.

"Classification and Identification of Soils," Arthur Casagrande, 901. *Discussion:* Ralph E. Fadum, James H. Stratton, Donald J. Belcher, J. A. Haine and J. W. Hilf, Jacob Feld, Kenneth S. Lane, George F. Sowers, René S. Pulido y Morales, Raymond F. Dawson, D. F. Glynn, L. F. Cooling and A. W. Skempton and R. Glossop, Milton Vargas, Donald M. Burmister, M. G. Spangler, E. W. Lane, B. K. Hough, Jr., and Arthur Casagrande, 931.

Crop production on various classes of soils, 571.

Suggested subdivision of soil groups according to mineralogical composition, 970.

Tests and Testing

"Consolidation of Fine-Grained Soils by Drain Wells," Reginald A. Barron (with discussion), 718.

"Subsidence of the Terminal Island-Long Beach Area, California," Frederic R. Harris and Eugene H. Harlow, 375. *Discussion:* James Gilluly and U. S. Grant, Jacob Feld, and Hyde Forbes, 397.

SPACE STRUCTURES

See DOMES

SPECIFICATIONS

"Reflections on Standard Specifications for Structural Design," Alfred M. Freudenthal (with discussion), 269.

SPILLWAYS

See also DAMS

"Development and Hydraulic Design, Saint Anthony Falls Stilling Basin," Fred W. Blaisdell (with discussion), 483.

SPIRALS

See CURVES (alinement curves)

STATIONS

See relative type of station, for example, geodetic stations under SURVEYS AND SURVEYING, GEODETIC

STATISTICS

See relative subject

STEEL

See also under special structure or structural part

Observations to determine the fatigue strength of butt welds stressed from zero to tension, 282.

STILLING BASINS

See WATER, FLOW OF, IN OPEN CHANNELS

STONE

See also GRAVEL; LIMESTONE

Rocks in their relation to tunneling problems, 1310.

STORAGE

See RESERVOIRS . . .; WATER STORAGE (cross references thereunder)

STORMS

See RAINFALL

STRAIN

See STRESS AND STRAIN

STREAM EROSION

See EROSION, STREAM

STREAM FLOW

See WATER, FLOW OF, IN OPEN CHANNELS

STREAMS

See RIVERS and cross reference thereunder.

STRENGTH OF MATERIALS

See STRESS AND STRAIN; *also* under specific material (*see* list of materials under MATERIALS OF CONSTRUCTION); *also* under fabricated structure or structural part

STRESS AND STRAIN

See also EARTH PRESSURE; ELASTICITY; MOMENTS; PHOTOELASTICITY; STRENGTH OF MATERIALS (cross references thereunder); STRUCTURES, THEORY OF; WATER PRESSURE; WHEEL LOADS; *also* under specific type of stress, for example, BUCKLING; SHEAR; TORSION

Beams

"Theory of Inelastic Bending with Reference to Limit Design," Alexander Hrennikoff (with discussion), 213.

Bridges, Truss

"Influence Lines for Continuous Structures by Geometrical Computation," Dean F. Peterson, Jr. (with discussion), 755.

Columns

"Analysis of Stepped-Column Mill Bents," Daniel S. Ling (with discussion), 1077.

"A Rational Explanation of Column Behavior," Frederick L. Ryder (with discussion), 40.

Domes

"Thin-Shelled Domes Loaded Eccentrically," Walter C. Voss, Dean Peabody, Jr., Howard R. Staley, and Albert G. H. Dietz, 293.

Pavement and Paving

"New Formulas for Stresses in Concrete Pavements of Airfields," H. M. Westergaard (with discussion), 425.

Soils

"Consolidation of Fine-Grained Soils by Drain Wells," Reginald A. Barron (with discussion), 718.

Timber

"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, Harold B. Hammill, Henry J. Degenkolb, Henry D. Dewell, John J. Gould, and Leon H. Nishkian (with discussion), 1123.

STRUCTURAL ANALYSIS

See EQUATIONS; STRUCTURES, THEORY OF

STRUCTURAL MATERIALS

See MATERIALS OF CONSTRUCTION

STRUCTURAL MEMBERS

See BEAMS; COLUMNS; GIRDERS; STRUCTURES, THEORY OF

STRUCTURAL MODELS

See MODELS, STRUCTURAL

STRUCTURES

See BUILDINGS (cross references thereunder); FOUNDATIONS; MATERIALS OF CONSTRUCTION; MODELS, STRUCTURAL; STRESS AND STRAIN; STRUCTURES, THEORY OF; *also* under type of structure or related subject; *also* cross references under SHELL STRUCTURES; SPACE STRUCTURES; SUBSTRUCTURES; UNDERGROUND STRUCTURES

STRUCTURES, THEORY OF

See also STRESS AND STRAIN; *also* under type of structure or structural part

Beams and Girders

"Beam Deflections by Second and Third Moments," Hsu Shih-Chang (with discussion), 445.

"Theory of Inelastic Bending with Reference to Limit Design," Alexander Hrennikoff (with discussion), 213.

Bridges

"Influence Lines for Continuous Structures by Geometrical Computation," Dean F. Peterson, Jr. (with discussion), 755.

Columns

"Analysis of Stepped-Column Mill Bents," Daniel S. Ling (with discussion), 1077.

Elastic curve in columns, 40.

Elastic curves as related to stepped-column mill bents, 1077.

Principle of superposition as it applies to the analysis of stepped-column mill bents, 1078.

"A Rational Explanation of Column Behavior," Frederick L. Ryder (with discussion), 40.

Frames (General)

"Beam Deflections by Second and Third Moments," Hsu Shih-Chang (with discussion), 445.

Frames, Continuous

"Continuous Frame Analysis by Elastic Support Action," J. Charles Rathbun and C. W. Cunningham, 577. *Discussion:* L. J. Mensch, Frederick S. Merritt, I. Oesterblom, Thomas C. Kavanagh, A. Floris, Tao King, Leroy A. Beaufoy, A. A. Eremin, Robert B. B. Moorman, Eduardo Agramonte, Stephen J. Fraenkel and Robert L. Janes, Phil M. Ferguson, Ralph W. Stewart, Maurice Barron, Edwin H. Gaylord, William A. Conwell, Harry Garfinkel, and J. Charles Rathbun and C. W. Cunningham, 614.

"Influence Lines for Continuous Structures by Geometrical Computation," Dean F. Peterson, Jr., 755. *Discussion:* Ralph E. Spaulding, Ralph W. Stewart, Yi-Mai Yao, Otto Gottschalk, and Dean F. Peterson, Jr., 772.

"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, Harold B. Hammill, Henry J. Degenkolb, Henry D. Dewell, John J. Gould, and Leon H. Nishkian (with discussion), 1123.

Frames, Rigid

"Analysis of Stepped-Column Mill Bents," Daniel S. Ling (with discussion), 1077.

"Continuous Frame Analysis by Elastic Support Action," J. Charles Rathbun and C. W. Cunningham (with discussion), 577.

"Influence Lines for Continuous Structures by Geometrical Computation," Dean F. Peterson, Jr. (with discussion), 755.

Materials and Tests

"Reflections on Standard Specifications for Structural Design," Alfred M. Freudenthal, 269. *Discussion:* I. Oesterblom, John M. Hayes, Alfred M. Freudenthal, 288.

Pavement and Paving

"New Formulas for Stresses in Concrete Pavements of Airfields," H. M. Westergaard (with discussion), 425.

Shell Structures

"Stability of Thin Cylindrical Shells in Torsion," R. G. Sturm, 681. *Discussion:* S. B. Batdorf and Manuel Stein, Glenn Murphy, L. H. Donnell, R. L. Moore, and R. G. Sturm, 706.

SUBSIDENCE

See LAND SUBSIDENCE

SUBSTRUCTURES

See EARTHWORK; FOUNDATIONS; *also* under type of substructure

SUPERPOSITION, PRINCIPLE OF

See STRUCTURES, THEORY OF

SURFACING

See AIRPORTS

SURVEYS AND SURVEYING

See also TRIANGULATION

Subsidence of typical bench marks in period 1931–1947, in and around Terminal Island, Long Beach, California, 378.

SURVEYS AND SURVEYING, GEODETIC

Latitude and departure equations of sample net in relation to geodetic surveys, 846.

"Least Squares Adjustment of Triangulation Net Between Geodetic Stations," E. F. Coddington and O. C. J. Marshall (with discussion), 837.

SYMBOLS

See relative subject

TAILWATER EROSION

See EROSION, TAILWATER

TANKS (General)

"Stability of Thin Cylindrical Shells in Torsion," R. G. Sturm (with discussion), 681.

TECHNICAL SOCIETIES

See AMERICAN SOCIETY OF CIVIL ENGINEERS

TEMPERATURE
See also HEAT AND HEAT TRANSMISSION
Temperature characteristics of infiltrated water, 411.
Thermographs in relation to river infiltration as a source of ground water supply, 410.

TENNESSEE VALLEY AUTHORITY
See RIVER VALLEY AUTHORITIES

TENSION
See STRESS AND STRAIN

TERMINOLOGY (Arranged hereunder by specific or comprehensive subject word when possible)
"Cement slurry" or "cement grout," 182.
Conduit beddings differentiated, 335, 336.
Conduits divided into two main types and four subclassifications, 317, 324.
Soil consolidation, 718.
Soils classification and identification, 901.
Soil texture, 979.
Timber structure testing terms, 1135.

TESTS AND TESTING
See MODELS; STRENGTH OF MATERIALS (cross references thereunder); STRESS AND STRAIN; STRUCTURES, THEORY OF—Materials and Tests; *also* under material, structure, or structural part tested, for example, DOMES; TIMBER

THEORIES
See cross references hereunder; *see also* under relative subject or its relative science, for example, MATHEMATICS

THEORY OF LEAST SQUARES
See PROBABILITY, THEORY OF

THEORY OF LIMIT DESIGN
See LIMIT DESIGN, THEORY OF

THEORY OF PLASTICITY
See PLASTICITY

THEORY OF PROBABILITY
See PROBABILITY, THEORY OF

THEORY OF STRUCTURES
See STRUCTURES, THEORY OF

THEORY OF SUPERPOSITION
See STRUCTURES, THEORY OF

THERMOGRAPHS
See TEMPERATURE

TIMBER AND TIMBERING
Factors *R* introduced to replace the conventional factor of safety, 1133.

"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, Harold B. Hammill, Henry J. Degenkolb, Henry D. Dewell, John J. Gould, and Leon H. Nishkian, 1123. *Discussion:* James H. Carr, Jr., Howard J. Hansen, E. G. Stern, Melvin W. Jackson, Charles Mackintosh, Sidney Novick, Frank J. Hanrahan, Walter J. Ryan, Walter M. Pratt, and Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, 1238.

TORSION

"Stability of Thin Cylindrical Shells in Torsion," R. G. Sturm (with discussion), 681.

TRANSITION CURVES

See CURVES (alinement curves)

TRANSITION SPIRALS

See CURVES (alinement curves)

TRIANGULATION

See also SURVEYS AND SURVEYING

"Least Squares Adjustment of Triangulation Net Between Geodetic Stations," E. F. Coddington and O. C. J. Marshall, 837. *Discussion:* Julius L. Speert, C. H. Ney, H. W. Hemple, Lansing G. Simmons, and E. F. Coddington and O. C. J. Marshall, 855.

TRIANGULATION NETS

See TRIANGULATION

TRUSS BRIDGES

See BRIDGES, TRUSS (cross references thereunder)

TRUSSES, BUILDING

Tests of timber roof trusses, 1131.

"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, Harold B. Hammill, Henry J. Degenkolb, Henry D. Dewell, John J. Gould, and Leon H. Nishkian (with discussion), 1123.

TRUSSES, VIERENDEEL

Analysis by elastic support action of Vierendeel trusses, 613, 627, 630, 645.

TUBES

See STRUCTURES, THEORY OF—Shell Structures; TUNNELS; WATER, FLOW OF, IN PIPES

TUNNELS (General)

See also COSTS, TUNNEL

"Application of Geology to Tunneling Problems," Ernest E. Wahlstrom, 1310. *Discussion:* Berlen C. Moneymaker, Robert S. Mayo, F. A. Nickell, R. H. Keays, Jacob Feld, Thomas W. Fluhr, Portland P. Fox, Hyde Forbes, Roger Rhoades and A. B. Reeves and W. H. Irwin, and Ernest E. Wahlstrom, 1322.

TUNNELS, PRESSURE

See TUNNELS, WATER

TUNNELS, WATER

See also COSTS, TUNNEL

"Friction Coefficients in a Large Tunnel," G. H. Hickox, A. J. Peterka, and R. A. Elder (with discussion), 1027.

Garzas Tunnel in Puerto Rico geologically discussed, 1322.

UNDERGROUND STRUCTURES

See under type of structure, for example, CONDUITS

UNDERGROUND WATER

See GROUND WATER

UPLIFT, HYDROSTATIC

See WATER PRESSURE

VALLEY AUTHORITIES

See RIVER VALLEY AUTHORITIES

VIERENDEEL TRUSSES

See TRUSSES, VIERENDEEL

WATER . . .

See also CONDUITS; COSTS . . .; DAMS; DRAINAGE; FLOODS; GROUND WATER; HYDRAULIC . . .; HYDRO- . . .; IRRIGATION; LAND RECLAMATION; METERS AND METERING; MODELS, HYDRAULIC; PIPES AND PIPING; RAINFALL; RESERVOIRS; RUN-OFF; SEA WATER; SEEPAGE; SEWERS; TUNNELS, WATER; WELLS

WATER, CONSUMPTIVE USE OF

"Forecasting Productivity of Irrigable Lands," W. C. Muldrow, 562. *Discussion:* Arthur F. Johnson, H. W. Van Loo, Will H. Noble, Charles Kirby Fox, M. R. Lewis, Harry F. Blaney, R. L. Parshall, and Howard N. Magness, 567.

WATER, FLOW OF, IN OPEN CHANNELS

See also SILT AND SILTING, CHANNEL

"Development and Hydraulic Design, Saint Anthony Falls Stilling Basin," Fred W. Blaisdell, 483. *Discussion:* Donald E. Blotcky, M. M. Culp, Paul Baumann, A. J. Peterka, Louis M. Laushey, J. H. Douma, Robert F. Ewald, and Fred W. Blaisdell, 521.

Effect of entrained air on scour pattern, 515.

"Mississippi River Cutoffs," Gerard H. Matthes (with discussion), 1.

Bibliography

Development and hydraulic design of stilling basins, 517.

WATER, FLOW OF, IN PIPES

See also PIPES AND PIPING; SEWERS; WATER PRESSURE

"Drainage of Airport Surfaces—Some Basic Design Considerations," Stifel W. Jens (with discussion), 785.

"Friction Coefficients in a Large Tunnel," G. H. Hickox, A. J. Peterka, and R. A. Elder (with discussion), 1027.

WATER POLLUTION

"Experimental Observations on Grouting Sands and Gravels," Alfred Machis (with discussion), 181.

WATER PRESSURE

See also GAGES, PRESSURE

"Consolidation of Fine-Grained Soils by Drain Wells," Reginald A. Barron (with discussion), 718.

"Friction Coefficients in a Large Tunnel," G. H. Hickox, A. J. Peterka, and R. A. Elder (with discussion), 1027.

"Investigation of Drainage Rates Affecting Stability of Earth Dams," F. H. Kellogg (with discussion), 1261.

WATER STORAGE

See HYDROGRAPHS, WATER STORAGE; RESERVOIRS . . .

WATER SUPPLY

See also DAMS; GROUND WATER; IRRIGATION; PIPES AND PIPING; TUNNELS, WATER; WATER . . .; WELLS

"River Infiltration As a Source of Ground Water Supply," Raphael G. Kazmann, 404. *Discussion:* C. V. Youngquist, and Raphael G. Kazmann, 421.

WATER TUNNELS

See TUNNELS, WATER

WATER, UNDERGROUND

See GROUND WATER

WEIGHTS

See under relative subject

WELDS AND WELDING

Frequency distribution in relation to fatigue strength of butt welds in bridge steel, 281.

WELLS

See also GROUND WATER

"Consolidation of Fine-Grained Soils by Drain Wells," Reginald A. Barron, 718. *Discussion:* Kenneth S. Lane, Philip Keene, Walter Kjellman, and Reginald A. Barron, 743.

"Experimental Observations on Grouting Sands and Gravels," Alfred Machis, 181. *Discussion:* James B. Hays, William S. Foster, Frederick J. Smith, and Alfred Machis, 206.

"River Infiltration As a Source of Ground Water Supply," Raphael G. Kazmann (with discussion), 404.

"Subsidence of the Terminal Island-Long Beach Area, California," Frederic R. Harris and Eugene H. Harlow (with discussion), 375.

WHEEL LOADS

New Formulas for Stresses in Concrete Pavements of Airfields," H. M. Westergaard (with discussion), 425.

WOOD

See PLYWOOD; TIMBER AND TIMBERING; TRUSSES . . .

AUTHOR INDEX

AGRAMONTE, EDUARDO
Frame analysis, 636.
ALDERMAN, ERNEST SAMUEL
Memoir of, 1580.
ALFRED, FRANK HOOKER
Memoir of, 1414.
ANDERBERG, EDWARD
Memoir of, 1418.
ANDEREGG, RUPERT ANDREW
Memoir of, 1421.
ANDREW, ROBERT MORRISON
Memoir of, 1581.
AYERS, MURRAY CHASE
Memoir of, 1582.
BARBOUR, FRANK ALEXANDER
Memoir of, 1422.
BARRON, MAURICE
Frame analysis, 649.
BARRON, REGINALD A.
"Consolidation of Fine-Grained Soils by Drain Wells," 718.
BARROWS, W. R.
Tunnel coefficients, 1048.
BATDORF, S. B.
Cylindrical shells, 706.
BAUMANN, PAUL
Saint Anthony Falls stilling basin, 530.
Soil slopes, 159.
BAŽANT, ZDENĚK, JR.
Underground conduits, 349.
BEAUFOY, LEROY A.
Frame analysis, 630.
BECK, THOMAS GEORGE GORDON
Memoir of, 1584.
BELCHER, DONALD J.
Soils classification, 937.
BINGER, WILSON V.
Underground conduits, 346.

BLAISDELL, FRED W.
"Development and Hydraulic Design, Saint Anthony Falls Stilling Basin," 483.

BLANEY, HARRY F.
Irrigable lands, 573.

BLEE, C. E.
Tennessee Valley, 123.

BLOTCKY, DONALD E.
Saint Anthony Falls stilling basin, 521.

BOISSONNAULT, FRANK L.
"Estimating Data for Reservoir Gates," 992.

BONE, EVAN P.
Pavements of airfields, 441.

BOWMAN, JOSEPH R.
Reservoir gates, 1018.

BOYD, WALTER LACY
Memoir of, 1424.

BRADBURY, ROYALL DOUGLAS
Memoir of, 1585.

BRADLEY, J. N.
Tunnel coefficients, 1066.

BRATER, E. F.
Airport drainage, 819.

BROCKWAY, PAUL LEMON
Memoir of, 1426.

BROWN, LYTLE
Cutoffs, 21.

BUCKINGHAM, EARL M.
Soil slopes, 167.

BUNDY, ORA
Memoir of, 1428.

BURMISTER, DONALD M.
Soils classification, 971.
Soil slopes, 171.

BUTLER, ARTHUR GRAY
Memoir of, 1430.

BUZZELL, D. A.
Reservoir gates, 1016.

CAMPBELL, BENJAMIN LUCIEN
Memoir of, 1433.

CARPENTER, EDWARD EMERY
Memoir of, 1433.

CARR, JAMES H., JR.
Timber tests, 1238.

CASAGRANDE, ARTHUR
"Classification and Identification of Soils," 901.

CEDERGREN, HARRY R.
Drainage rates, 1285.

CHAMBERS, JOHN TAYLOR
Memoir of, 1436.

CHAPMAN, E. J. K.
Tunnel coefficients, 1057.

CLARK, WILLIAM GEORGE
Memoir of, 1616.

CODDINGTON, E. F.
"Least Squares Adjustment of Triangulation Net Between Geodetic Stations," 837.

COE, CLEVELAND BEACH
Memoir of, 1437.

CONARD, CLARENCE KNIGHT
Memoir of, 1439.

CONWELL, WILLIAM A.
Beam deflections, 469.
Frame analysis, 666.

COOLING, L. F.
Soils classification, 966.

CREAGER, WILLIAM P.
Tunnel coefficients, 1063.

CROWE, FRANCIS TRENHOLM
Memoir of, 1397.

CULP, M. M.
Saint Anthony Falls stilling basin, 529.

CUNNINGHAM, C. W.
"Continuous Frame Analysis by Elastic Support Action," 577.

DAVIS, JOHN ROSE WILSON
Memoir of, 1442.

DAWSON, RAYMOND F.
Soils classification, 963.

DEGENKOLB, HENRY J.
"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, 1123.

DEICHLER, LUDLOW VANDERBURG CLARK
Memoir of, 1618.

DERR, ORVILLE VOSBURY
Memoir of, 1445.

DEVELIN, RICHARD GRIFFITH
Memoir of, 1447.

DEWELL, HENRY D.
"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, 1123.

DIESEM, HARRY CUSTER
Memoir of, 1587.

DIETZ, ALBERT G. H.
"Thin-Shelled Domes Loaded Eccentrically," 293.

DOHN, CHARLES W.
Mill bents, 1114.

DONNELL, L. H.
Cylindrical shells, 709.

DOUGAN, JAMES
Memoir of, 1588.

DOUGHERTY, R. E.
"The Society's Future and Its Economic Limitations," Annual Presidential Address, 1391.

DOUMA, J. H.
Saint Anthony Falls stilling basin, 545.

DRESSER, ELBERT HERMON
Memoir of, 1449.

DRUCKER, D. C.
Inelastic bending, 248.

DUNNELLS, CLIFFORD GEORGE
Memoir of, 1450.

ELAM, W. E.
Cutoffs, 16.

ELDER, R. A.
"Friction Coefficients in a Large Tunnel," 1027.

EREMIN, A. A.
Frame analysis, 634.

EWALD, ROBERT F.
Saint Anthony Falls stilling basin, 550.

FADUM, RALPH E.
Soils classification, 931.

FELD, JACOB
Soil slopes, 164.
Soils classification, 946.
Subsidence, 399.
Tunneling geology, 1332.
Underground conduits, 347.

FERGUSON, PHIL M.
Frame analysis, 644.

FERRY, DOUGLASS HEWITT
Memoir of, 1590.

FIESENHEISER, E. I.
Mill bents, 1100.

FISHER, RALPH L.
Transition curve, 884.

FITTERER, JOHN CONRAD
Memoir of, 1452.

FLORIS, A.
Beam deflections, 465.
Frame analysis, 625.
Mill bents, 1112.
Simultaneous equations, 1377.

FLUHR, THOMAS W.
Tunneling geology, 1334.

FORBES, HYDE
Subsidence, 400.
Tunneling geology, 1336.

FORSTER, ALEXANDER SYLVESTER
Memoir of, 1592.

FOSTER, WILLIAM S.
Sands and gravels, 206.

FOX, CHARLES KIRBY
Irrigable lands, 570.

FOX, PORTLAND P.
Tennessee Valley, 137.
Tunneling geology, 1335.

FRAENKEL, STEPHEN J.
Frame analysis, 643.

FRENCH, JAMES BENTON
Memoir of, 1453.

FREUDENTHAL, ALFRED M.
"Reflections on Standard Specifications for Structural Design," 269.

FURBER, PIERCE POWERS
Memoir of, 1457.

GAIN, E. J. A.
Airport drainage, 817.

GARFINKEL, HARRY
Frame analysis, 667.

GAVETT, WESTON
Tunnel coefficients, 1047.

GAYLORD, EDWIN H.
Frame analysis, 660.

GIBSON, JAMES EDWIN
Memoir of, 1458.

GILLULY, JAMES
Subsidence, 397.

GLOSSOP, R.
Soils classification, 966.

GLYNN, D. F.
Soils classification, 964.

GODFREY, EDWARD
Memoir of, 1461.

GOTTSCHALK, OTTO
Influence lines, 779.

GOULD, JOHN J.
"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, 1123.

GRANT, U. S.
Subsidence, 397.

GRAVES, ARTHUR R.
Mill bents, 1107.

GRAVES, EUGENE A.
Cutoffs, 24.

GRAY, WALTER J.
Mill bents, 1102.

GREGORY, ALFRED COOKMAN
Memoir of, 1462.

GRUNER, HEINRICH EDUARD
Memoir of, 1464.

HAILE, ELMER R., JR.
Transition curve, 889.

HAINE, J. A.
Soils classification, 941.

HALL, C. L.
Cutoffs, 17.

HALL, JULIUS REED
Memoir of, 1466.

HALL, LESLIE STANDISH
Memoir of, 1468.

HAMMILL, HAROLD B.
"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, 1123.

HANRAHAN, FRANK J.
Timber tests, 1248.

HANSEN, HOWARD J.
Timber tests, 1240.

HARLOW, EUGENE H.
"Subsidence of the Terminal Island-Long Beach Area, California," 375.

HARRIS, BORDEN BAKER
Memoir of, 1470.

HARRIS, FREDERIC R.
"Subsidence of the Terminal Island-Long Beach Area, California," 375.

HARVEY, HERBRAND
Memoir of, 1472.

HAYBROOK, STEPHEN H.
Tunnel coefficients, 1063.

HAYES, JOHN M.
Design specifications, 289.

HAYS, JAMES B.
Sands and gravels, 206.

HEMPLE, H. W.
Triangulation net, 863.

HENRY, PHILIP WALTER
Memoir of, 1475.

HICKERSON, T. F.
Simultaneous equations, 1375.

HICKOX, G. H.
"Friction Coefficients in a Large Tunnel," 1027.

HICKS, W. I.
Airport drainage, 813.

HILF, J. W.
Soils classification, 941.

HINDS, JULIAN
Tunnel coefficients, 1059.

HIRASHIMA, K. B.
Soil slopes, 169.

HOGUE, CHESTER JAMES
Memoir of, 1480.

HORNE, MICHAEL R.
Inelastic bending, 250.

HORONJEFF, ROBERT
Pavements of airfields, 440.

HOUGH, B. K., JR.
Soils classification, 985.

HRENNIKOFF, ALEXANDER
"Theory of Inelastic Bending with Reference to Limit Design," 213.

HSU, SHIH-CHANG
"Beam Deflections by Second and Third Moments," 445.

HYDLE, MAGNUS LAWRENCE
Memoir of, 1593.

IRWIN, W. H.
Tunneling geology, 1345.

IZZARD, CARL F.
Airport drainage, 825.

JABLONSKY, ROY WRENSHAW
Memoir of, 1594.

JACKSON, MELVIN W.
Timber tests, 1243.

JANES, ROBERT L.
Frame analysis, 643.

JENS, STIFEL W.
"Drainage of Airport Surfaces—Some Basic Design Considerations," 785.

JOHNSON, ARTHUR F.
Irrigable lands, 567.

JONES, HENRY MACY
Memoir of, 1481.

KAVANAGH, THOMAS C.
Frame analysis, 623.

KAYS, MARION REED
Memoir of, 1483.
KAZMANN, RAPHAEL G.
"River Infiltration As a Source of Ground Water Supply," 404.
KEAYS, R. H.
Tunneling geology, 1329.
KEENE, PHILIP
Drain wells, 747.
KELLEY, E. F.
Underground conduits, 364.
KELLEY, JOSEPH GRANT
Memoir of, 1487.
KELLOGG, F. H.
"Investigation of Drainage Rates Affecting Stability of Earth Dams," 1261.
KEMP, JOHN EDWARD
Memoir of, 1488.
KENNISON, KARL R.
Tunnel coefficients, 1053.
KING, TAO
Frame analysis, 626.
KITTREDGE, GEORGE WATSON
Memoir of, 1403.
KJELLMAN, WALTER
Drain wells, 748.
LAMBE, T. W.
Drainage rates, 1294.
LANE, E. W.
Soils classification, 984.
LANE, KENNETH S.
Drain wells, 743.
Soils classification, 949.
LATHAM, GEORGE R.
Reservoir gates, 1015.
LAURENCE, ROBERT A.
Tennessee Valley, 124.
LAUSHEY, LOUIS M.
Saint Anthony Falls stilling basin, 539.
LEA, RICHARD SMITH
Memoir of, 1491.

LEA, WILLIAM SCHURMAN
Memoir of, 1493.
LEE, GEORGE ALLEN
Memoir of, 1495.
LEEMING, JOHN JOSEPH
"The General Principles of Highway Transition Curve Design," 868.
LEFAVOUR, ROLAND WINTHROP
Memoir of, 1598.
LEWIS, M. R.
Irrigable lands, 571.
LING, DANIEL S.
"Analysis of Stepped-Column Mill Bents," 1077.
LOBO, CARLOS
Memoir of, 1496.
LOCKWOOD, JAMES BUTTON CLYDE
Memoir of, 1497.
LONGSDORF, PAUL WRIGHT
Memoir of, 1599.
LOWTHER, BURTON
Memoir of, 1601.
LUIGGI, LUIGI
Memoir of, 1406.
MacCULLOCH, CHARLES HARVEY
Memoir of, 1499.
MACHIS, ALFRED
"Experimental Observations on Grouting Sands and Gravels," 181.
MACKINTOSH, CHARLES
Timber tests, 1244.
MAGNESS, HOWARD N.
Irrigable lands, 576.
MAINS, ROBERT M.
Column behavior, 73.
MANEY, GEORGE ALFRED
Memoir of, 1500.
MARSHALL, O. C. J.
"Least Squares Adjustment of Triangulation Net Between Geodetic Stations," 837.
MARSTON, ANSON
Cutoffs, 22.
Underground conduits, 351.

MATTHES, GERARD H.
"Mississippi River Cutoffs," 1.

MAXWELL, WAYNE DICKSON
Memoir of, 1504.

MAYO, ROBERT S.
Tunneling geology, 1325.

McCLURE, THOMAS MERO
Memoir of, 1505.

McCORMICK, FRANK J.
Beam deflections, 469.

McCORMICK, ROBERT LEE
Memoir of, 1508.

McDONALD, WILLIAM NAYLOR
Memoir of, 1509.

McGINLEY, JOHN JOSEPH
Memoir of, 1603.

McKEEN, BENJAMIN
Memoir of, 1511.

McNEW, JOHN THOMAS LAMAR
Memoir of, 1512.

MENSCH, L. J.
Frame analysis, 614.
Mill bents, 1103.

MERRITT, FREDERICK S.
Frame analysis, 616.
Simultaneous equations, 1378.

MEYER, CARL F.
Transition curve, 881.

MILLER, RUDOLPH PHILIP
Memoir of, 1517.

MIXER, CHARLES ADAM
Memoir of, 1524.

MONEYMAKER, BERLEN C.
Tennessee Valley, 120.
Tunneling geology, 1322.

MOORE, R. L.
Cylindrical shells, 713.

MOORMAN, ROBERT B. B.
Beam deflections, 466.
Frame analysis, 635.

MORRISON, I. F.
Simultaneous equations, 1379.

MULDROW, W. C.
"Forecasting Productivity of Irrigable Lands," 562.
Memoir of, 1604.

MURPHY, GLENN
Cylindrical shells, 707.

NACE, ROBERT REGNAULT
Memoir of, 1526.

NEY, C. H.
Triangulation net, 856.

NICKELL, F. A.
Tunneling geology, 1326.

NIEDERHOFF, A. E.
Reservoir gates, 1017.

NISHKIAN, LEON H.
"Tests of Timber Structures from Golden Gate International Exposition," by Committee of the San Francisco (Calif.) Section, ASCE, on Timber Test Program, 1123.

NOBLE, CHARLES M.
Transition curve, 893.

NOBLE, WILL H.
Irrigable lands, 568.

NOONAN, HUGH JOSEPH
Memoir of, 1621.

NOVICK, SIDNEY
Timber tests, 1246.

NOWELL, J. C.
Tennessee Valley, 128.

OESTERBLOM, I.
Design specifications, 288.
Frame analysis, 621.

OGILVIE, LEMUEL BYNUM
Memoir of, 1605.

PARRY, CHARLES EDWARD
Memoir of, 1527.

PARSHALL, R. L.
Irrigable lands, 575.

PARTRIDGE, JOHN F.
Tennessee Valley, 130.

PAULS, ARTHUR L.
"Cleaning and Grouting of Limestone Foundations, Tennessee Valley Authority (symposium): Fort Loudoun Dam," 82.

PEABODY, DEAN, JR.
"Thin-Shelled Domes Loaded Eccentrically," 293.

PETERKA, A. J.
"Friction Coefficients in a Large Tunnel," 1027.
Saint Anthony Falls stilling basin, 534.

PETERSON, DEAN F., JR.
"Influence Lines for Continuous Structures by Geometrical Computation," 755.

PHARR, HARRY N.
Cutoffs, 20.
Memoir of, 1529.

POPOV, E. P.
Inelastic bending, 258.

PORTER, HENRY CYRUS
Memoir of, 1533.

PRATT, WALTER M.
Timber tests, 1251.

PUGH, De WITT PAWLING
Memoir of, 1535.

PULIDO y MORALES, RENÉ S.
Soils classification, 960.

RAMIREZ, REINALDO
Memoir of, 1536.

RATHBUN, J. CHARLES
"Continuous Frame Analysis by Elastic Support Action," 577.

READY, ARTHUR GEORGE
Memoir of, 1607.

REEVES, A. B.
Tunneling geology, 1345.

RHOADES, ROGER
Tunneling geology, 1345.

RIEGEL, ROSS M.
Drainage rates, 1293.

RIGHTS, THEODORE B.
Reservoir gates, 1012.

ROBBINS, ALLAN APPLETON
Memoir of, 1537.

ROBINSON, ERNEST L.
Column behavior, 76.

ROOS, SIGMUND
Simultaneous equations, 1381.

ROUSE, HUNTER
Tunnel coefficients, 1049.

ROWE, R. ROBINSON
Column behavior, 71.

ROWLAND, WILLIAM JOSEPH
Memoir of, 1539.

RYAN, WALTER J.
Timber tests, 1250.

RYDER, FREDERICK L.
"A Rational Explanation of Column Behavior," 40.

SAWYER, JOSHUA STEWART
Memoir of, 1540.

SCHMIDT, JOHN GEORGE
Memoir of, 1609.

SEERY, FRANCIS JOSEPH
Memoir of, 1541.

SHAFER, GEORGE E.
Underground conduits, 354.

SHANNON, W. L.
Drainage rates, 1302.

SHRUBB, BERNARD WICK
Memoir of, 1622.

SIEMS, VALENTINE BERNARD
Memoir of, 1543.

SIESEL, SIDNEY MONTEFIORE
Memoir of, 1544.

SIMMONS, LANSING G.
Triangulation net, 865.

SIRRINE, JOSEPH EMORY
Memoir of, 1546.

SKEMPTON, A. W.
Soils classification, 966.

SKILLIN, EDWARD SIMEON
Memoir of, 1610.

SMITH, FREDERICK J.
Sands and gravels, 208.

SMITH, LEONARD SEWALL
Memoir of, 1548.

SNARE, FREDERICK
Memoir of, 1628.

SNOW, JESSE BAKER
Memoir of, 1550.

SOLLID, ERIK
Mill bents, 1108.

SOWERS, GEORGE F.
Soils classification, 959.

SPANGLER, M. G.
Soils classification, 978.
"Underground Conduits—An Appraisal of Modern Research," 316.

SPAULDING, RALPH E.
Influence lines, 772.
Mill bents, 1099.

SPEERT, JULIUS L.
Triangulation net, 855.

STALEY, HOWARD R.
"Thin-Shelled Domes Loaded Eccentrically," 293.

STANDER, ISAAC JOSHUA
Memoir of, 1555.

STANFORD, HOMER REED
Memoir of, 1557.

STEIN, MANUEL
Cylindrical shells, 706.

STERN, E. G.
Timber tests, 1242.

STEWART, RALPH W.
Frame analysis, 645.
Influence lines, 772.

STOCKING, JEROME BRANCH
Memoir of, 1612.

STRATTON, JAMES H.
Soils classification, 933.

STURM, R. G.
"Stability of Thin Cylindrical Shells in Torsion," 681.

SWIFT, DEAN EDWIN
Memoir of, 1623.

TAN, EK-KHOO
"Stability of Soil Slopes," 139.

TAPLEY, G. S.
Tunnel coefficients, 1064.

TAYLOR, T. F.
"Cleaning and Grouting of Limestone Foundations, Tennessee Valley Authority (symposium): Douglas Dam," 102.

TOCH, MAXIMILIAN
Memoir of, 1630.

TOMPKINS, WILLIAM FRASER, JR.
Memoir of, 1624.

TREMPER, BAILEY
Underground conduits, 365.

TSCHEBOTARIOFF, GREGORY P.
Soil slopes, 163.

TWAY, ROBERT RAYMOND
Memoir of, 1614.

VALLERGA, BERNARD A.
Inelastic bending, 261.

VAN ALSTYNE, HENRY ARTHUR
Memoir of, 1559.

VAN LOO, H. W.
Irrigable lands, 568.

VAN VLECK, ALBION NOYES
Memoir of, 1563.

VARGAS, MILTON
Soils classification, 970.

VOGEL, H. D.
Cutoffs, 18.

VOSS, WALTER C.
"Thin-Shelled Domes Loaded Eccentrically," 293.

WAHLSTROM, ERNEST E.
"Application of Geology to Tunneling Problems," 1310.

WANG, CHI-TEH
Column behavior, 74.

WARD, JAMES B.
Tennessee Valley, 126.

WARD, WILLIAM BERNARD
Memoir of, 1566.

WEGNER, WILLIAM GODFREY
Memoir of, 1567.

WEINER, BERNARD L.
"Variation of Coefficients of Simultaneous Linear Equations," 1349.

WEISS, BENJAMIN
Memoir of, 1615.

WELTY, WILLIAM R.
Transition curve, 888.

WESTERGAARD, H. M.
"New Formulas for Stresses in Concrete Pavements of Airfields," 425.

WICKLINE, GEORGE GROVER
Memoir of, 1568.

WIDMER, ARTHUR JACOB
Memoir of, 1571.

WILLIAMS, CLEMENT CLARENCE
Memoir of, 1573.

WILLIAMS, E. J., JR.
Cutoffs, 24.

WILLIAMS, H. M.
Airport drainage, 810.

WILSON, JOHN MERLIN
Memoir of, 1627.

WING, S. P.
Tunnel coefficients, 1066.

WINTER, GEORGE
Column behavior, 71.
Inelastic bending, 248.

YAO, YI-MAI
Influence lines, 776.

YOUNGQUIST, C. V.
River infiltration, 421.

ZESIGER, ALBERT WILLIAM
Memoir of, 1576.

ZIPPRODT, ROY RICHARD
Memoir of, 1579.